1. (a)

$$\dot{g} = k_1 S_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

→ $k_1 S_0$ is the source term. It describes the production of ~~chemical signal by th~~ gene product by the biochemical signal

→ $-k_2 g$ is the decay term. It describes the spontaneous decay of the gene product itself

→ $\frac{k_3 g^2}{k_4^2 + g^2}$ is the feedback term. The creation of gene product induces a positive feedback and enhances the production.

(b) $$\frac{dg}{dt} = k_1 S_0 - k_2 g + \frac{k_3 (g/k_4)^2}{1 + (g/k_4)^2}$$

let $\frac{g}{k_4} = x$ then $k_4 \frac{d(g/k_4)}{dt} = k_1 S_0 - k_2 k_4 \left(\frac{g}{k_4}\right) + \frac{k_3 \left(\frac{g}{k_4}\right)^2}{1 + (g/k_4)^2}$

$$\rightarrow k_4 \frac{dx}{dt} = k_1 S_0 - k_2 k_4 x + \frac{k_3 x^2}{1 + x^2}$$

$$\frac{1}{k_3} \rightarrow \frac{k_4}{k_3} \frac{dx}{dt} = \frac{k_1 S_0}{k_3} - \frac{k_2 k_4}{k_3} x + \frac{x^2}{1 + x^2}$$

let $\underline{\tau = \frac{k_3 t}{k_4}}$, $\underline{s = \frac{k_1}{k_3} S_0}$, $\underline{r = \frac{k_2 k_4}{k_3}}$

$$\rightarrow \underline{\frac{dx}{d\tau} = s - rx + \frac{x^2}{1 + x^2}}$$

(c) If $s=0$, then

$$\frac{dx}{dt} = -rx + \frac{x^2}{1+x^2}$$, for fixed points $\frac{dx}{dt} = 0$

$$\therefore \quad -rx + \frac{x^2}{1+x^2} = 0 \quad \Rightarrow \quad \left(\frac{x}{1+x^2} - r\right)x = 0$$

$$\Rightarrow \quad x\left(x^2 - \frac{1}{r}x + 1\right) = 0 \qquad (r>0)$$

For 2 distinct positive fixed points need

$$\left(\frac{1}{r}\right)^2 - 4 > 0 \quad \therefore \quad \frac{1}{r^2} > 4 \quad \to \quad r^2 < \frac{1}{4}$$

$$\because r > 0 \quad \therefore \quad r < \frac{1}{2} = r_c$$

$$\Rightarrow \quad \boxed{r_c = \frac{1}{2}}$$

For $r < r_c = \frac{1}{2}$, $\quad x^* = \dfrac{\frac{1}{r} \pm \sqrt{\frac{1}{r^2} - 4}}{2}$, $x^* = 0$

For $r = r_c = \frac{1}{2}$ $\quad \cancel{x^* =} \quad x^* = 1,\ x^* = 0$

For $r > r_c = \frac{1}{2}$ $\quad$ ~~two fixed point~~ $x^* = 0$

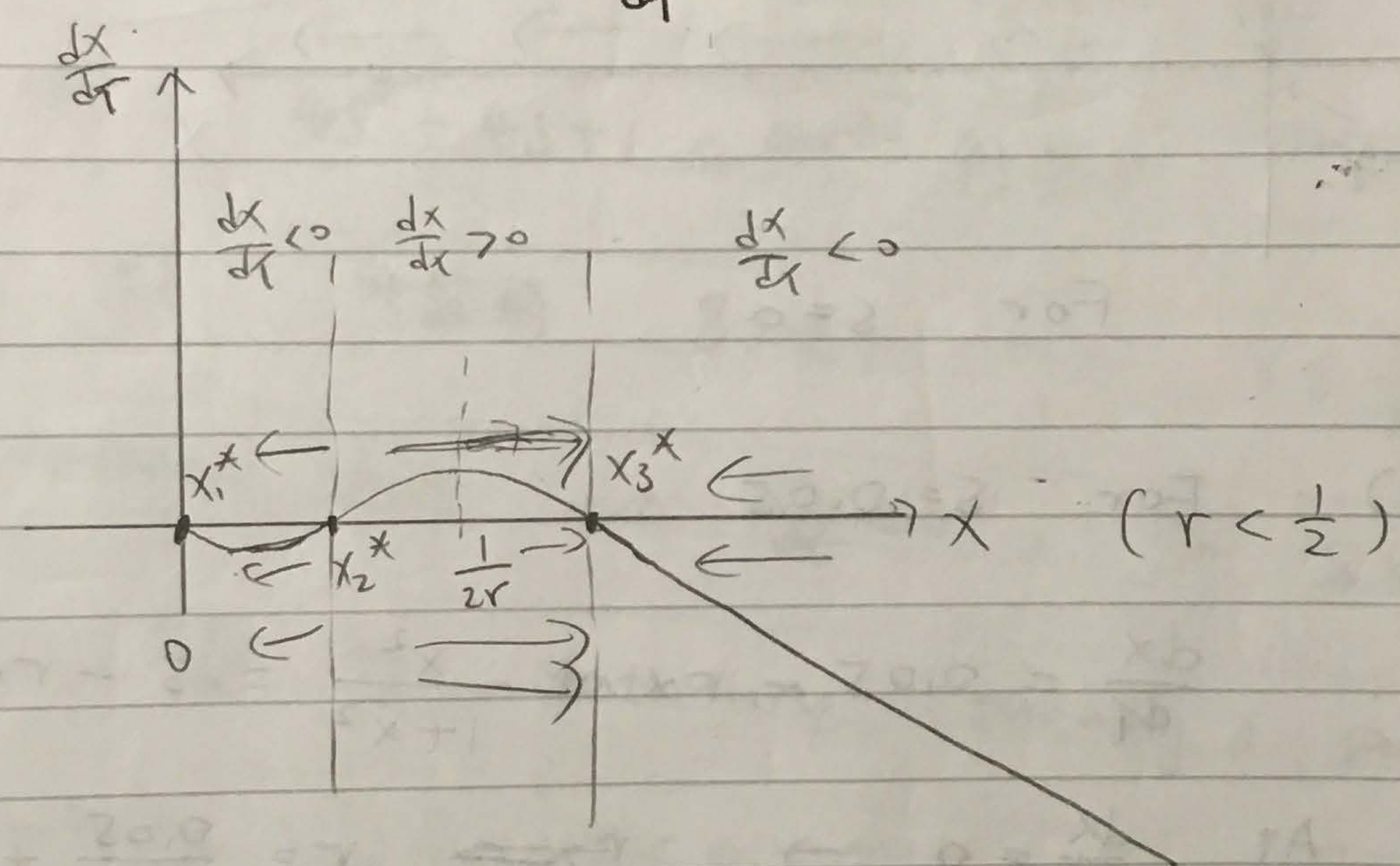
As $x \to \infty$ $\quad \frac{dx}{dt} = -rx + 1 < 0$

$\therefore$ ~~The~~ For $r < r_c$, the ~~larger~~ fixed point with largest r should be stable

As $x \to -\infty$ (although this is physically impossible, it helps draw diagram)

$$\frac{dx}{dt} = -rx + 1 > 0 \Rightarrow r = 0 \text{ is stable}$$

$\because r > 0 \quad \therefore$ ~~the~~ with the help of 2 asymptotes we can draw $\frac{dx}{dt}$ vs. x



Hence $x_1^* = 0$ is stable

$x_2^* = \frac{1}{2r} - \sqrt{\frac{1}{4r^2} - 1}$ is unstable

$x_3^* = \frac{1}{2r} + \sqrt{\frac{1}{4r^2} - 1}$ is stable

When $r = r_c$

$x_1^* = 0$ is stable

$x_{23}^* = \frac{1}{2r}$ is marginal

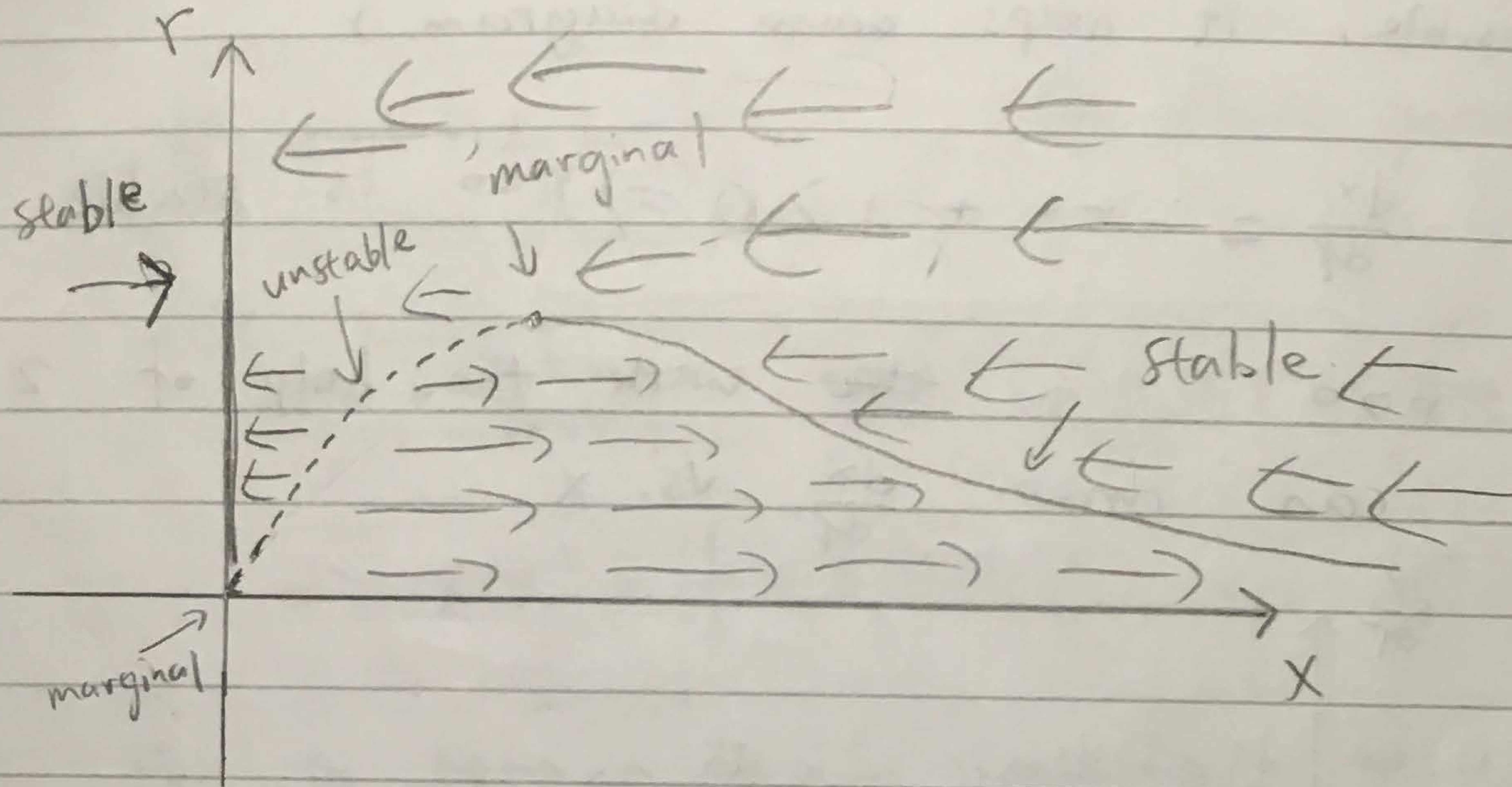
when $r > r_c$

$x_1^* = 0$ is stable

when $r = 0$ $x_1^* = 0$ is marginal

At $\frac{dx}{dt}=0 \rightarrow r=\frac{x}{1+x^2}$ or $x=0$

So the (r, x) plot is



For $s=0$

(d) For $S = 0.05$

$$\frac{dx}{dt} = 0.05 - rx + \frac{x^2}{1+x^2} = S - rx + \frac{x^2}{1+x^2}$$

At $\frac{dx}{dt}=0 \rightarrow \cancel{rx}\ r = \frac{0.05}{x} + \frac{x}{1+x^2}$

$$\frac{dr}{dx} = -\frac{0.05}{x^2} + \frac{1+x^2-x(2x)}{(1+x^2)^2}$$

For $\frac{dr}{dx}=0 \Rightarrow \frac{0.05}{x^2} = \frac{1-x^2}{(1+x^2)^2}$

$\Rightarrow 0.05(1+2x^2+x^4) = x^2 - x^4$

let $y=x^2 \rightarrow 0.05+0.1y+0.05y^2 = y-y^2$

(in general) $S(1+2x^2+x^4) = x^2 - x^4$

→ let $y = x^2$

$s + \cancel{2s}\, 2sy + sy^2 = y - y^2$

$\Rightarrow \quad (s+1)y^2 + (2s-1)y + s = 0$

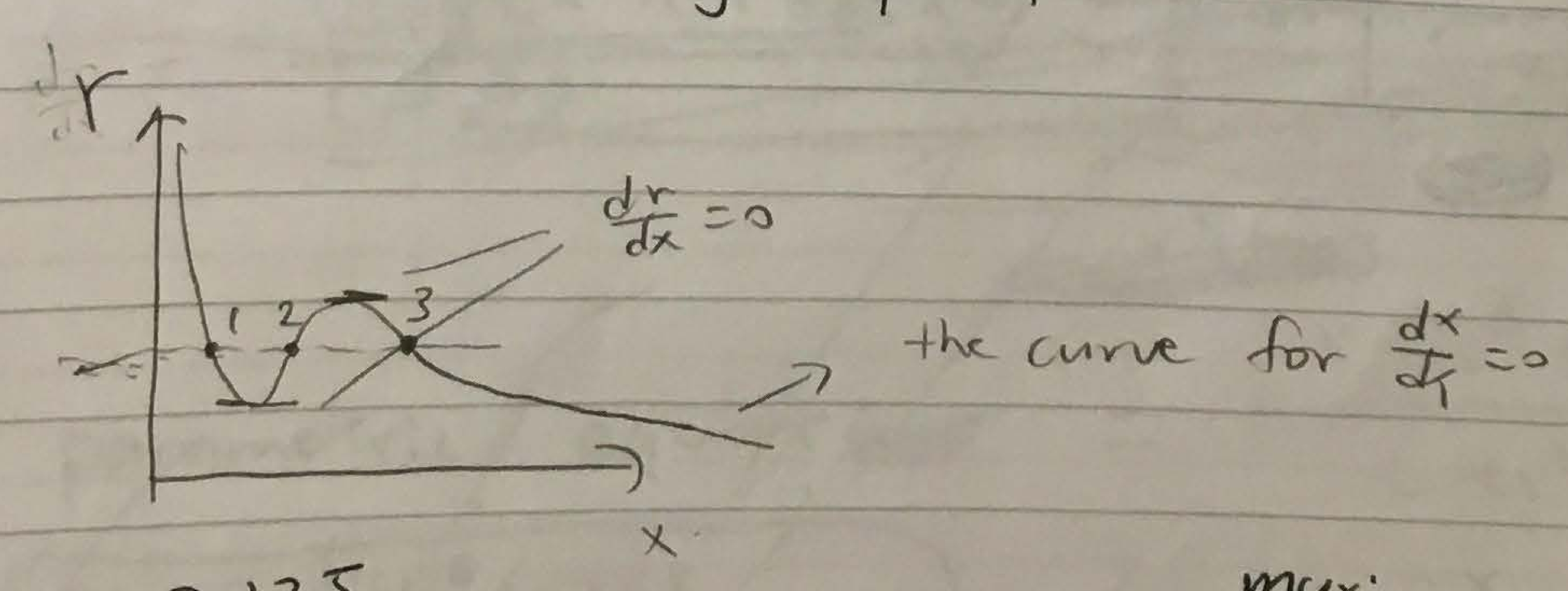
For real y, $\quad (2s-1)^2 - 4s(s+1) \geq 0$

$\Rightarrow \quad \cancel{4s^2} - 4s + 1 - \cancel{4s^2} - 4s \geq 0$

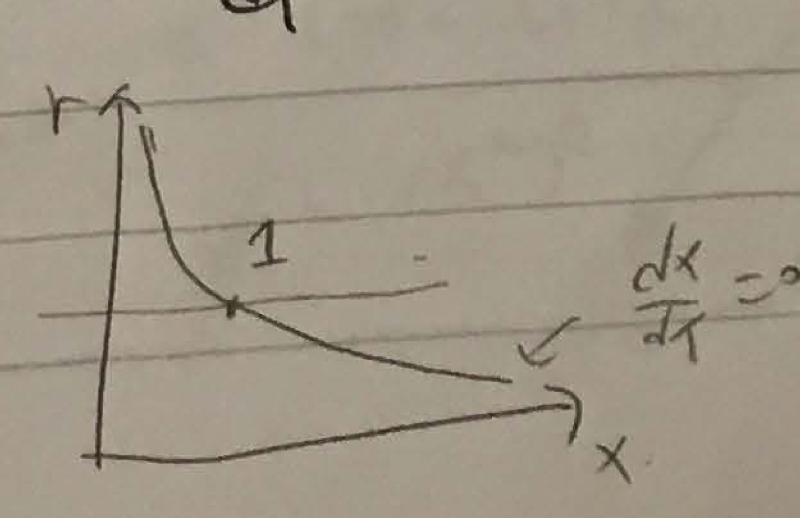
$\Rightarrow \quad \cancel{4s} \quad 8s \leq 1 \quad \Rightarrow \quad s \leq \frac{1}{8} = 0.125$

This means that at $\frac{dx}{dt} = 0$, if $s \leq \frac{1}{8}$, then $\frac{dr}{dx}$ has stationary (turning) points.

→ particularly if $s < 0.125$, then there will be 3 maximum x-values correspond to $\frac{dx}{dt} = 0$ because there are 2 turning points

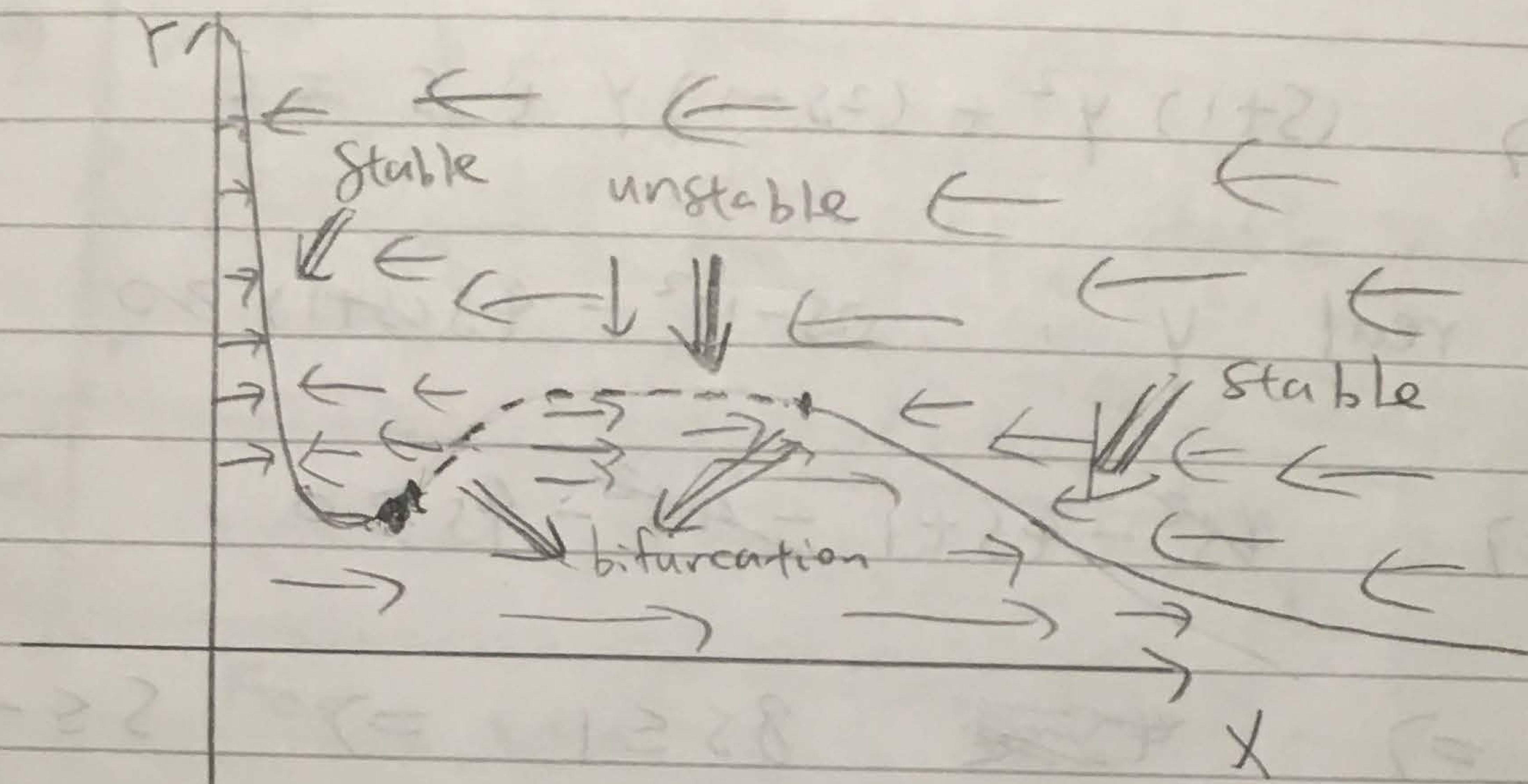


If ~~$s \leq 0.125$~~ $s \geq 0.125$, ~~then~~ then only 1 maximum x-value correspond to $\frac{dx}{dt} = 0$

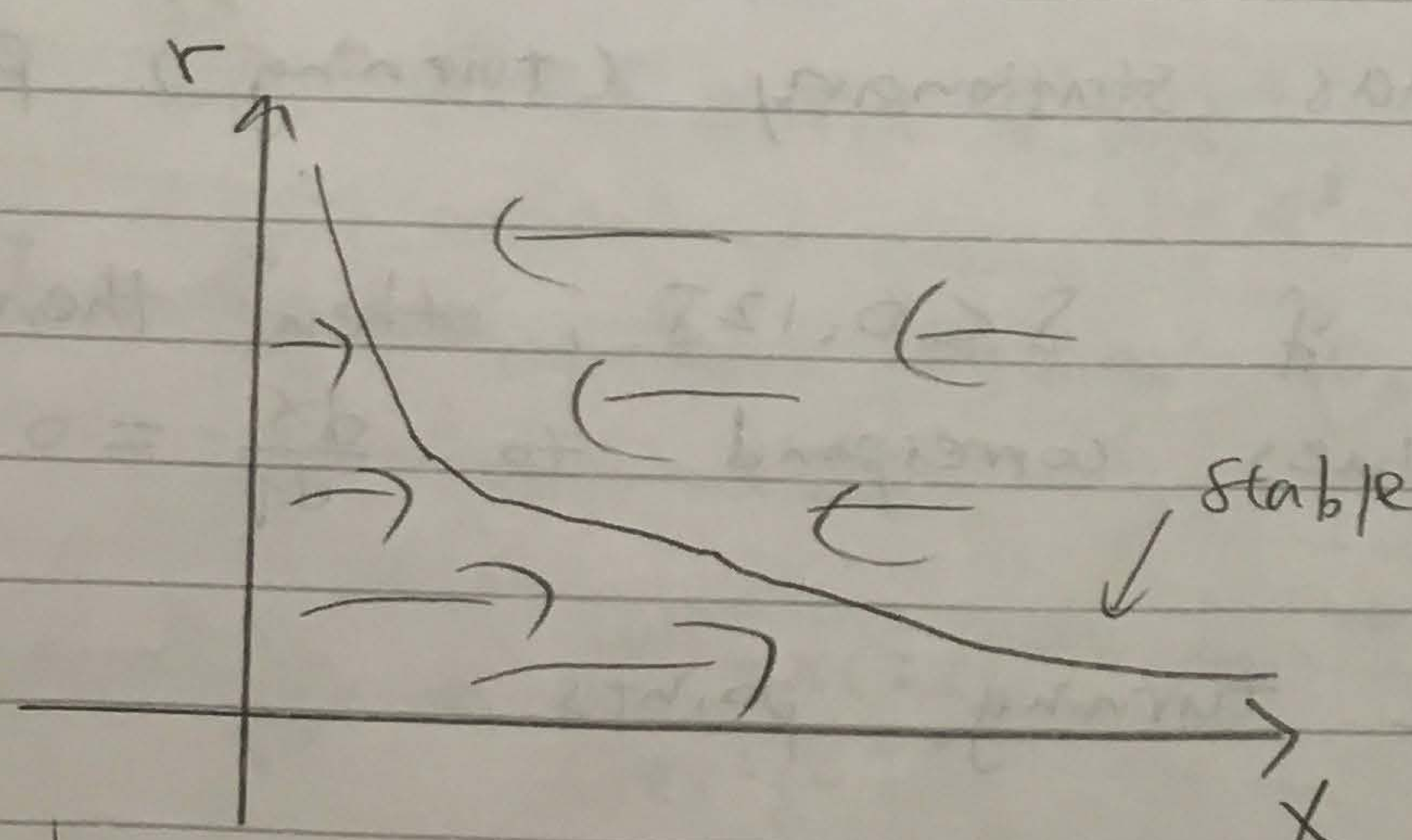


$\therefore \quad 0.05 < 0.125$

$\therefore \quad S = 0.05$:



$S = 0.15$:



(e) ~~At~~

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

fixed points $\Rightarrow \frac{dx}{d\tau} = 0$

Bifurcation $\Rightarrow \frac{dr}{dx} = 0$ when $\frac{dx}{d\tau} = 0$

$\Rightarrow \cancel{(s+1)x^4 + (2s+1)x^2 + s = 0}$

$\Rightarrow \cancel{(x^4 + 2x^2)s = 2x^2 - x^4}$

$$\Rightarrow (x^4 + 2x^2 + 1)s = x^2 - x^4$$

$$\Rightarrow s = \frac{x^2 - x^4}{x^4 + 2x^2 + 1}$$

$$\Rightarrow s = \frac{x^2(1+x)(1-x)}{(1+x^2)^2}$$

then $r = \frac{s}{x} + \frac{x}{1+x^2} = \frac{x(1+x)(1-x)}{(1+x^2)^2} + \frac{x(1+x^2)}{(1+x^2)^2}$

$$= \cancel{\frac{(1+x)(x - x^2 + x^3)}{(1+x^2)^2}} \cancel{\frac{x^2(1+x)}{(1+x^2)^2}} \cancel{\frac{x(1+x)}{(1+x^2)^2}}$$

$$= \frac{x - x^3 + x + x^3}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

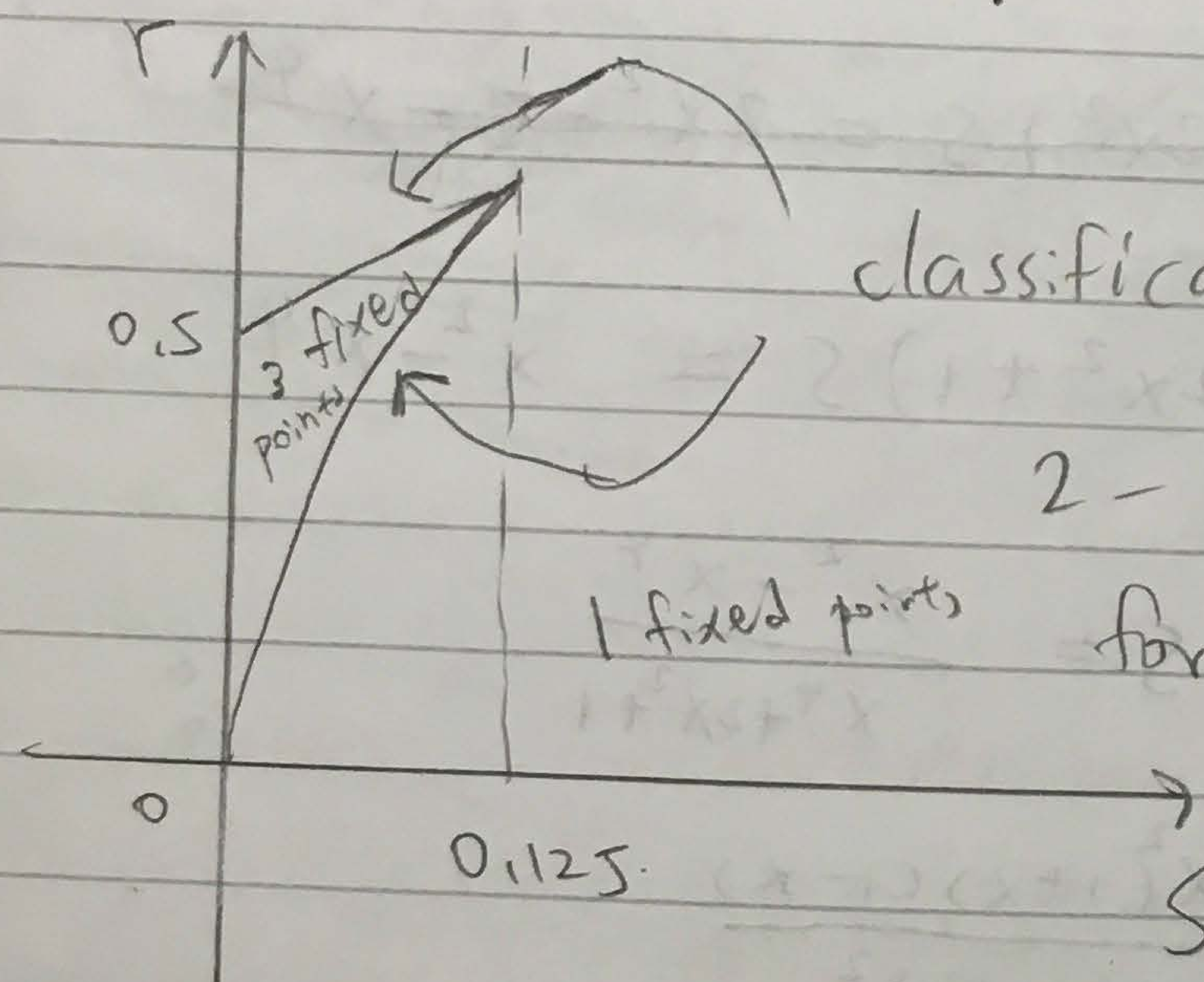
$\therefore$ The parametric equations

$$\boxed{\begin{aligned} r &= \frac{x^2(1+x)}{(1+x^2)^2} \\ s &= \frac{x^2(1+x)(1-x)}{(1+x^2)^2} \end{aligned}}$$

x is parameter

→ parametric equation is

$$\boxed{\begin{aligned} r &= \frac{2x}{(1+x^2)^2} \\ s &= \frac{x^2(1+x)(1-x)}{(1+x^2)^2} \end{aligned}}$$



classification :

2 - saddle node bifurcations for a given s

For (r, s) out-side this curve, no bifurcation occurs

2. $\dot{x} = \epsilon + rx - x^3 = f(x)$

(a) for $\epsilon = 0$, $\dot{x} = rx - x^3 = f(x)$

Fixed points $\dot{x} = 0 \Rightarrow rx - x^3 = 0$

$\rightarrow x(r - x^2) = 0$

$\therefore x_1^* = 0$ If $r > 0$, $x_2^* = \sqrt{r}$, $x_3^* = -\sqrt{r}$

If $r \le 0$, no other fixed points

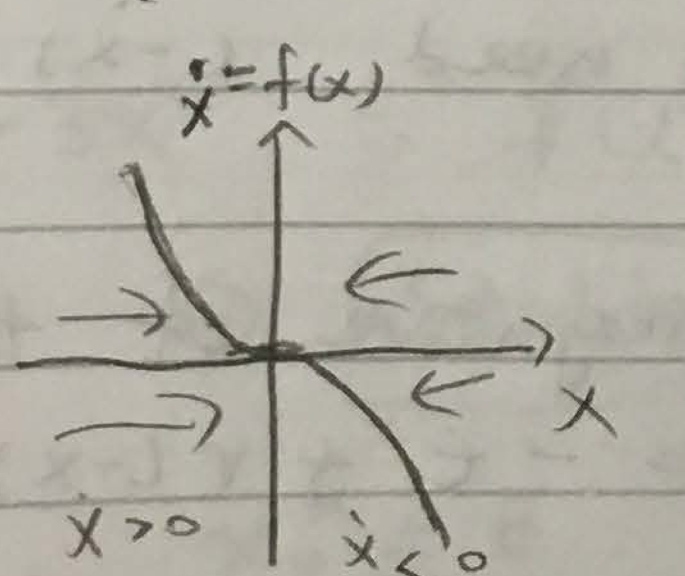
$\rightarrow$ If $r > 0$, for $x_1^* = 0$, $f'(0) = r > 0 \rightarrow \boxed{\text{unstable}}$

for $x_2^* = \sqrt{r}$, $f'(\sqrt{r}) = r - 3x^2 \big|_{x=\sqrt{r}} = r - 3r = -2r < 0 \rightarrow \boxed{\text{stable}}$

for $x_3^* = -\sqrt{r}$, $f'(-\sqrt{r}) = r - 3r = -2r < 0 \rightarrow \boxed{\text{stable}}$

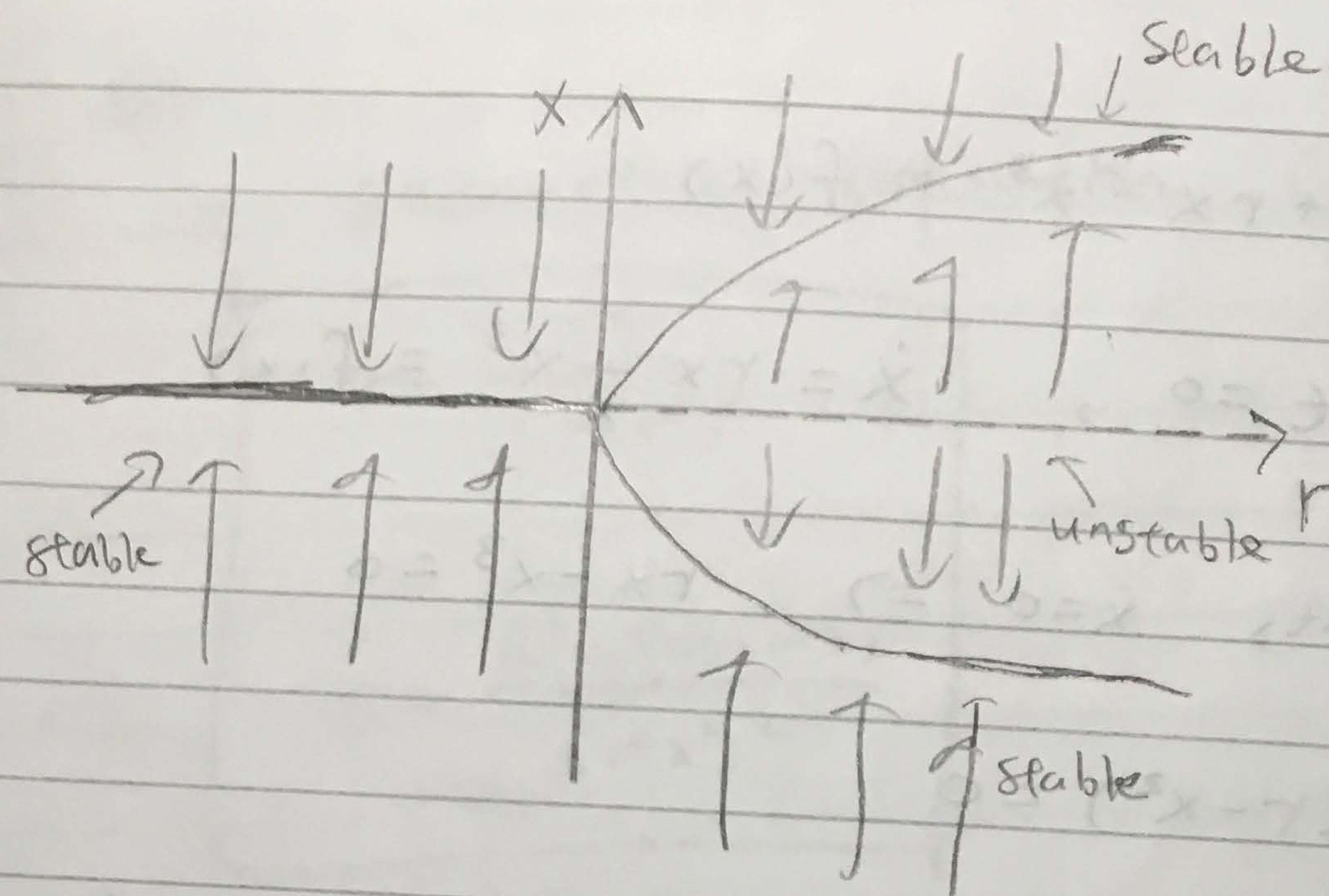
If $r < 0$. $\boxed{x_1^* = 0}$ $f'(0) = r < 0 \rightarrow \boxed{\text{stable}}$

If $r = 0$,



the fixed point $\boxed{x_1^* = 0}$ is still $\boxed{\text{Stable}}$

At $r = 0$, the fixed point $x = 0$ goes from stable to unstable, and two more stable fixed points emerge. So supercritical pitchfork bifurcation occurs at $r = 0$



(b) If $\epsilon = 0$, $\dot{x} = rx - x^3$

If $x \Rightarrow -x$, then $\dot{x} \Rightarrow -\dot{x}$, $x^3 \Rightarrow -x^3$

$\therefore$ $\cancel{(\dot{-x}) = f(-x)}$ $(\dot{-x}) = r(-x) - (-x)^3$

$\Rightarrow$ system symmetric under $x \Rightarrow -x$

But if $\epsilon \neq 0$, $\dot{x} = \epsilon + rx - x^3$

$x \Rightarrow -x$

If symmetrical, we need $(\dot{-x}) = \epsilon + r(-x) - (-x)^3$

but taking the negative of the original equation we get $(\dot{-x}) = -\epsilon + r(-x) - (-x)^3$

This means that the symmetry is broken unless $\epsilon = 0$

$0 = \epsilon + rx - x^3$

(c) for ϵ small ($|\epsilon| \ll |r|$), we assume the new fixed points $x^* = x_0^* + \delta x$, where x_0^* is the solution to $\dot{x} = rx - x^3 = 0$, ~~and~~ $\delta x \sim \epsilon$

then at $x = x^*$, $\dot{x} = \epsilon + rx - x^3 = 0$

$$\therefore \epsilon + r(x_0^* + \delta x) - (x_0^* + \delta x)^3 = 0$$

$$\rightarrow \epsilon + r x_0^* + r\delta x - x_0^{*3} - 3x_0^{*2}\delta x + \underbrace{O(\delta x^2)}_{\text{neglected}} = 0$$

$$\rightarrow (3x_0^* - r)\,\delta x \simeq \underbrace{(r x_0^* - x_0^{*3})}_{=0 \text{ by definition}} + \epsilon$$

$$\rightarrow \delta x \simeq \frac{\epsilon}{3x_0^* - r}$$

$\rightarrow$ For $r < 0$, $x_0^* = 0 \Rightarrow \boxed{x^* = -\frac{\epsilon}{r}}$

$f'(x) = r - 3x^2 \quad \therefore f'(x^*) = r - 3\left(\frac{\epsilon}{r}\right)^2 < r < 0$

$\rightarrow \boxed{\text{Stable}}$ fixed point

$\rightarrow$ For $r > 0$, $x_0^* = 0 \Rightarrow \boxed{x^* = -\frac{\epsilon}{r}}$

$$f'(x^*) = r - \underbrace{3\left(\frac{\epsilon}{r}\right)^2}_{|\epsilon| \ll |r|} \approx r > 0$$

$\rightarrow \boxed{\text{unstable}}$

$$x_0^* = \sqrt{r} \Rightarrow \boxed{x^* = \sqrt{r} + \frac{\epsilon}{2r}}$$

$$f'(x^*) = r - 3\left(\sqrt{r} + \frac{\epsilon}{2r}\right)^2 \simeq -2r < 0$$

$\rightarrow \boxed{\text{stable}}$

$x_0^* = -\sqrt{r} \Rightarrow \boxed{x^* = -\sqrt{r} + \frac{\epsilon}{2r}}$

$f'(x^*) = r - 3(-\sqrt{r} + \frac{\epsilon}{2r})^2 \simeq -2r < 0$

$\rightarrow \boxed{\text{Stable}}$

As $r \to 0$ the approximation breaks down, ~~need to~~ ~~we~~

(d) For $r = 0$, $\dot{x} = \epsilon - x^3 = f(x)$

fixed point $\dot{x} = 0 \Rightarrow x^{*3} = \epsilon$

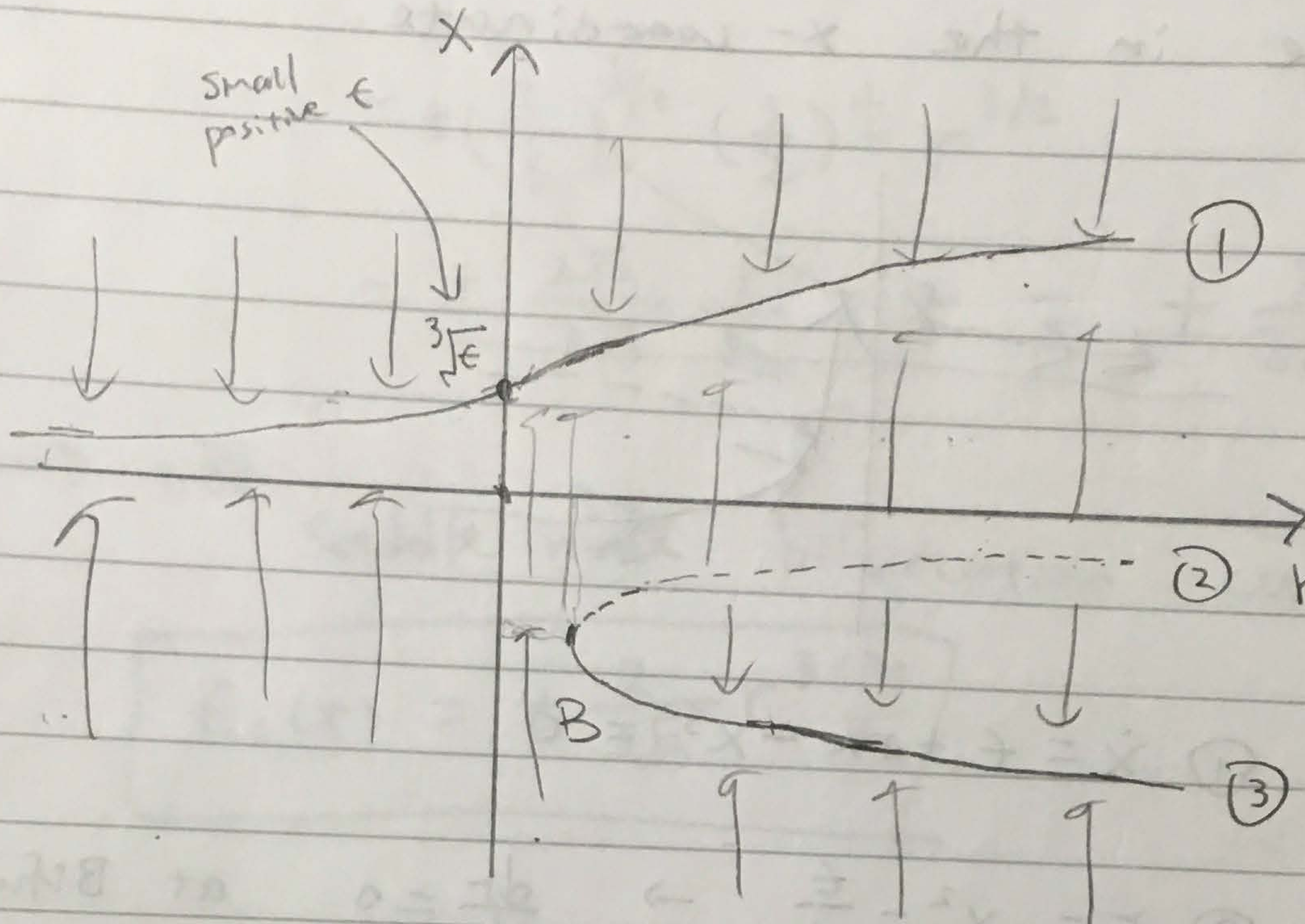
$\rightarrow \boxed{x^* = \sqrt[3]{\epsilon}}$

$f'(x) = -3x^2 \rightarrow f'(x^*) = -3x^{*2} < 0$ for $\epsilon \neq 0$

$\rightarrow \boxed{\text{stable}}$ fixed point

(e) $\dot{x} = \epsilon + rx - x^3$, fixed points $0 = \epsilon + rx - x^3$

$\rightarrow \quad r = x^2 - \frac{\epsilon}{x}$



At ~~Bifurca~~ Bifurcation point B. $x \approx -\frac{\epsilon}{r}$

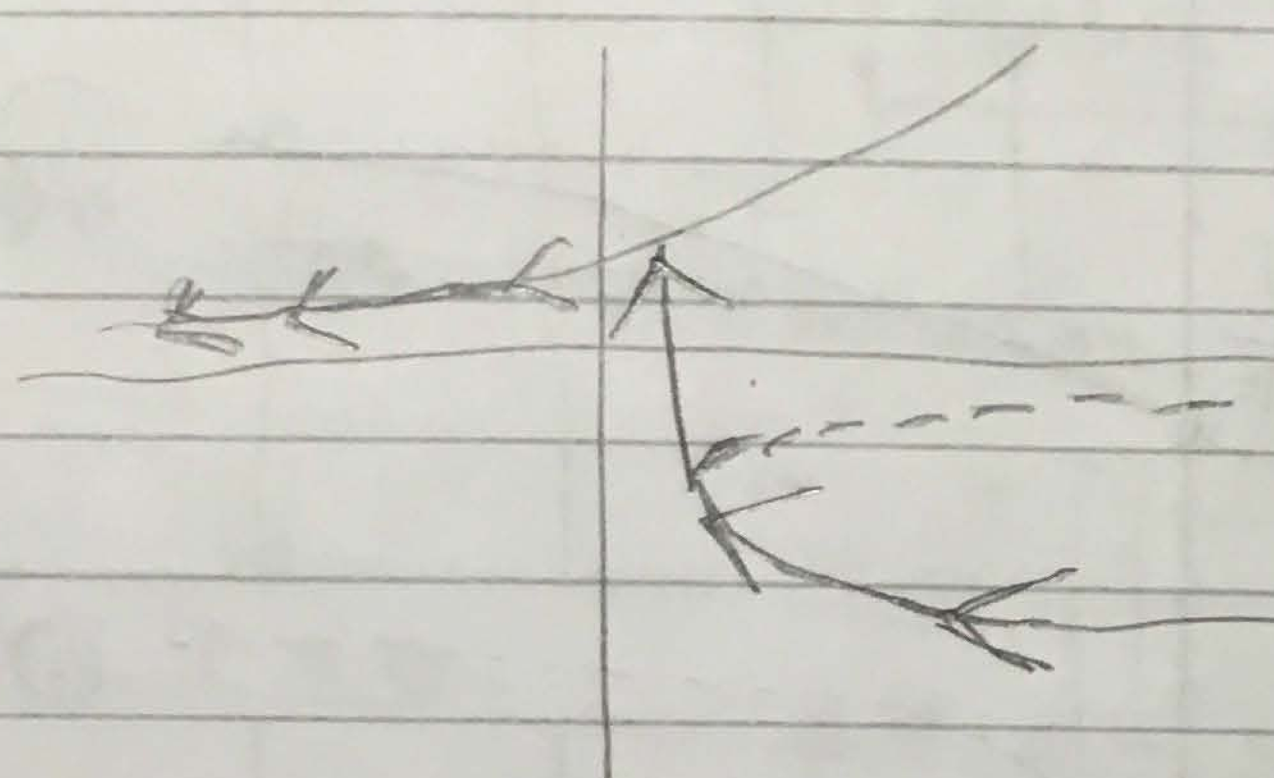
$0 = \frac{d}{dx}(\epsilon + rx - x^3) = r - 3x^2 \Rightarrow r \simeq \frac{3\epsilon^2}{r^2}$

$\Rightarrow \begin{cases} r \simeq \sqrt[3]{3\epsilon^2} \\ x \simeq -\frac{\epsilon}{\sqrt[3]{3\epsilon^2}} \end{cases}$

As r approaches from negative values to positive values, The system stays along the ~~fixed point~~ stable fixed points branch ① ~~, However, it can also make a sudden jump~~

→ However, as r approaches from positive values to negative values along the ~~fix~~ stable fixed points branch ③, when r reaches $\simeq \sqrt[3]{3\epsilon^2}$

(system at point B), if r continues to decrease, then the system will fall to the stable fixed points branch ① abruptly, with an abrupt change in the x-coordinate.



(f) ① $\dot{x} = \epsilon + rx - x^3 = 0$

② $r = x^2 - \frac{\epsilon}{x} \rightarrow \frac{dr}{dx} = 0$ at Bifurcation point B

$\Rightarrow 2x + \frac{\epsilon}{x^2} = 0$ ~~$\epsilon = x = (\frac{\epsilon}{2})^{1/3}$~~

$\Rightarrow x = (-\frac{\epsilon}{2})^{\frac{1}{3}}$ ③

substitute ③ into ① gives

$\epsilon + r(-\frac{\epsilon}{2})^{\frac{1}{3}} - \left((-\frac{\epsilon}{2})^{1/3}\right)^3 = 0$

$\rightarrow r(-\frac{\epsilon}{2})^{\frac{1}{3}} + \epsilon - (-\frac{\epsilon}{2}) = 0$

$\Rightarrow r(\frac{\epsilon}{2})^{\frac{1}{3}} = \frac{3}{2}\epsilon$

~~$\Rightarrow \epsilon = \frac{2}{3}r$~~

$r = (\frac{3}{2}\epsilon)(\frac{\epsilon}{2})^{-\frac{1}{3}} \rightarrow \epsilon^{2/3} = \frac{2}{3}(\frac{1}{2})^{\frac{1}{3}} r$

$= (\epsilon^{\frac{1}{3}})^2$

(r cannot be negative on $\epsilon_c(r)$)

$$\Rightarrow \quad \epsilon^{1/3} = \pm\left(\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{1}{3}} r\right)^{\frac{1}{2}}$$

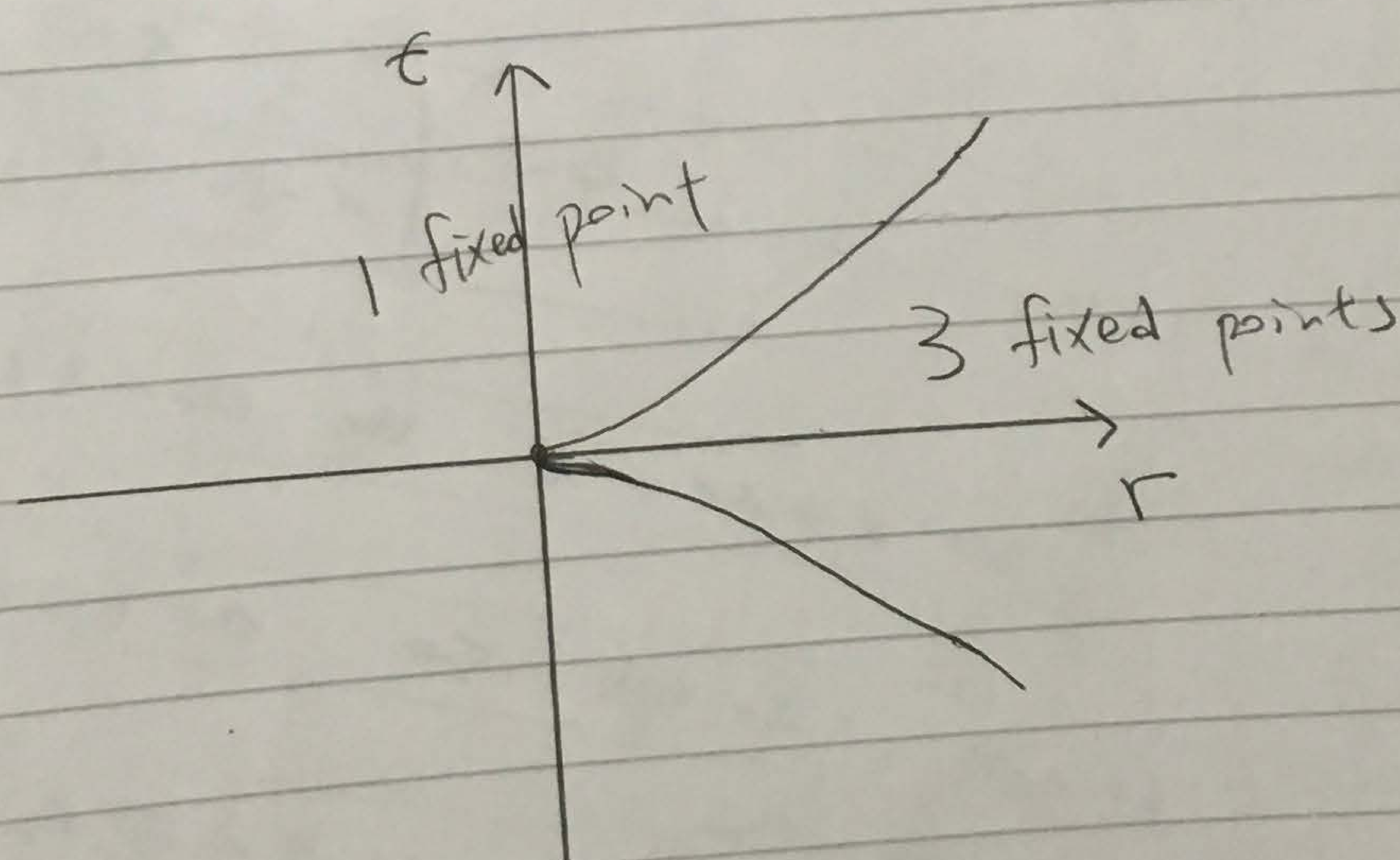
$$\Rightarrow \quad \epsilon = \pm\left(\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{1}{3}} r\right)^{\frac{1}{2}\cdot 3}$$

$$= \pm\left(\frac{2}{3}\right)^{3/2}\left(\frac{1}{2}\right)^{\frac{1}{2}} r^{3/2}$$

$$= \pm \frac{2\cancel{\sqrt{2}}}{3\sqrt{3}} \frac{1}{\cancel{\sqrt{2}}} r^{3/2} = \pm \frac{2}{3\sqrt{3}} r^{3/2}$$

$\Rightarrow$ ~~B~~ Saddle node bifurcation curves

$$\boxed{\epsilon_c(r) = \pm \frac{2}{3\sqrt{3}} r^{3/2}}$$



3. (i) $\dot{x} = y \qquad \dot{y} = -2x - 3y$

$$\therefore \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\dot{x}} = A \underline{x}$$

Eigenvalues $\quad \det(A - \lambda I) = 0$

$$\rightarrow \quad \det \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda(\lambda+3) + 2 = 0 \quad \Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow \quad (\lambda+2)(\lambda+1) = 0 \quad \Rightarrow \quad \lambda_1 = -2 \,, \quad \lambda_2 = -1$$

$\rightarrow$ stable node

For $\lambda_1 = -2 \qquad A \underline{v_1} = \lambda_1 \underline{v_1}$

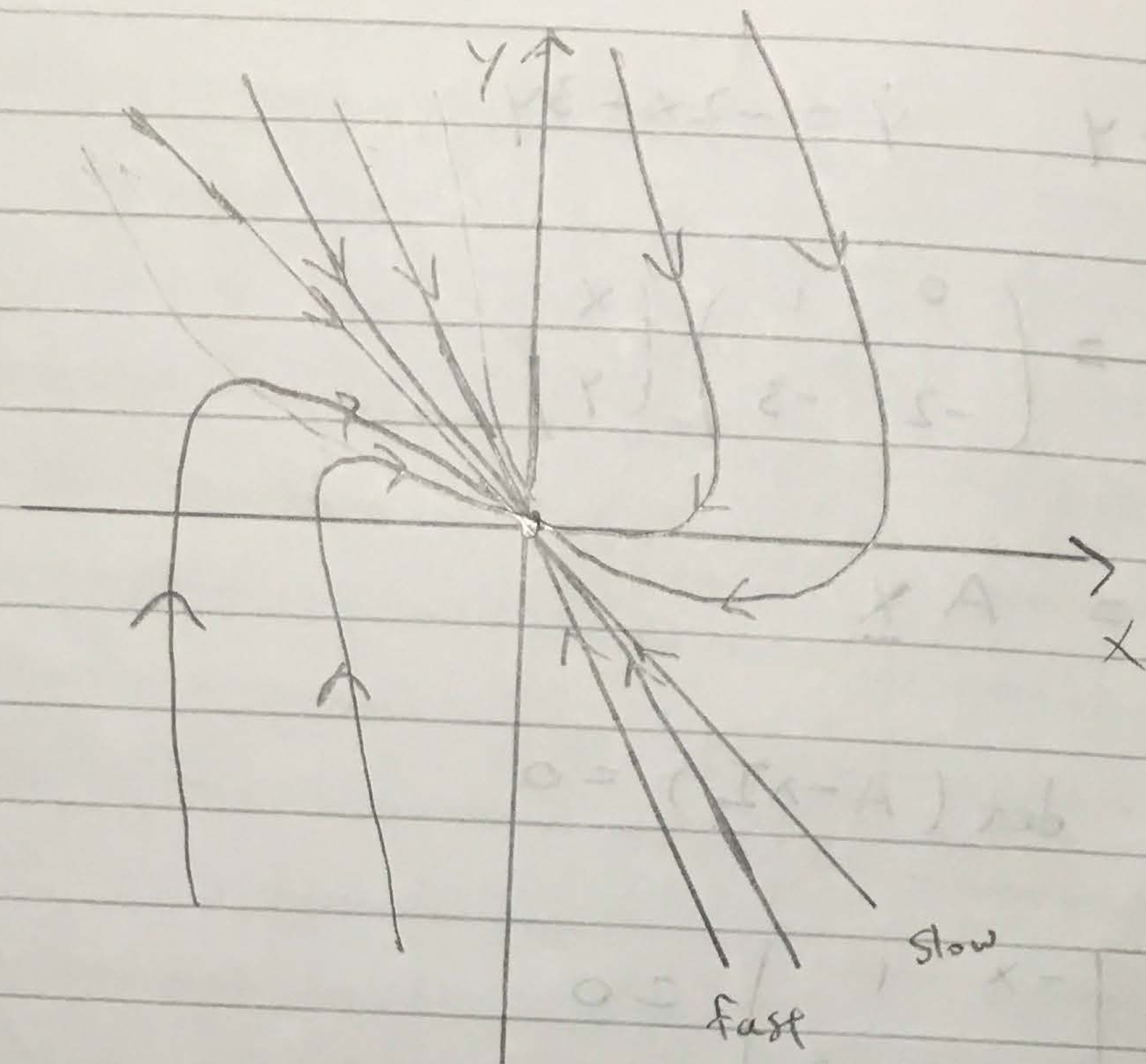
$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2c_1 + c_2 = 0$$

$$\Rightarrow \quad \underline{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \rightarrow \text{ fast direction}$$

For $\lambda_2 = -1 \qquad A \underline{v_2} = \lambda_2 \underline{v_2}$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad c_1 + c_2 = 0$$

$$\Rightarrow \quad \underline{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \rightarrow \text{ slow direction}$$



fixed points : $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

(ii) Fixed point $(0,0)$

$$\dot{x} = 5x + 2y \qquad \dot{y} = -17x - 5y$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -17 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\underline{x}} = A \underline{x}$$

Eigenvalues $\begin{vmatrix} 5-\lambda & 2 \\ -17 & -5-\lambda \end{vmatrix} = 0$

$$(\lambda - 5)(\lambda + 5) + 34 = 0$$

$$\rightarrow \cancel{\lambda^2 - 25 + 3} \quad \lambda^2 - 25 + 34 = 0$$

$$\lambda^2 + 9 = 0$$

$$\lambda_1 = 3i \ , \quad \lambda_2 = -3i$$

The fixed point is a <u>Centre</u>

If $\lambda = 3i$:

$\underline{v} = (a, b)^T$

$$\begin{pmatrix} 5-3i & 2 \\ -17 & -5-3i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$a(5-3i) + 2b = 0$

$-17a - (5+3i)b = 0$

$\rightarrow$ $\cancel{\underline{v_1} = \begin{pmatrix} -17 \\ 5+3i \end{pmatrix}}$ $\underline{v_1} = \begin{pmatrix} 5+3i \\ -17 \end{pmatrix}$

If $\lambda = -3i$

$$\begin{pmatrix} 5+3i & 2 \\ -17 & -5+3i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\rightarrow$ $-17a - (5-3i)b = 0$ $\Rightarrow$ $\underline{v_2} = \begin{pmatrix} 5-3i \\ -17 \end{pmatrix}$

$\rightarrow$ $\underline{x}(t) = c_1 e^{3it} \begin{pmatrix} 5+3i \\ -17 \end{pmatrix} + c_2 e^{-3it} \begin{pmatrix} 5-3i \\ -17 \end{pmatrix}$

$t=0$ $\Rightarrow$ $(5+3i)c_1 + (5-3i)c_2 = 0$

$\rightarrow$ $c_1 = c_2$ if c_1, c_2 are both real

let $c_1 = c_2 = c$

$$x(t) = ce^{3it}(5+3i) + ce^{-3it}(5-3i)$$

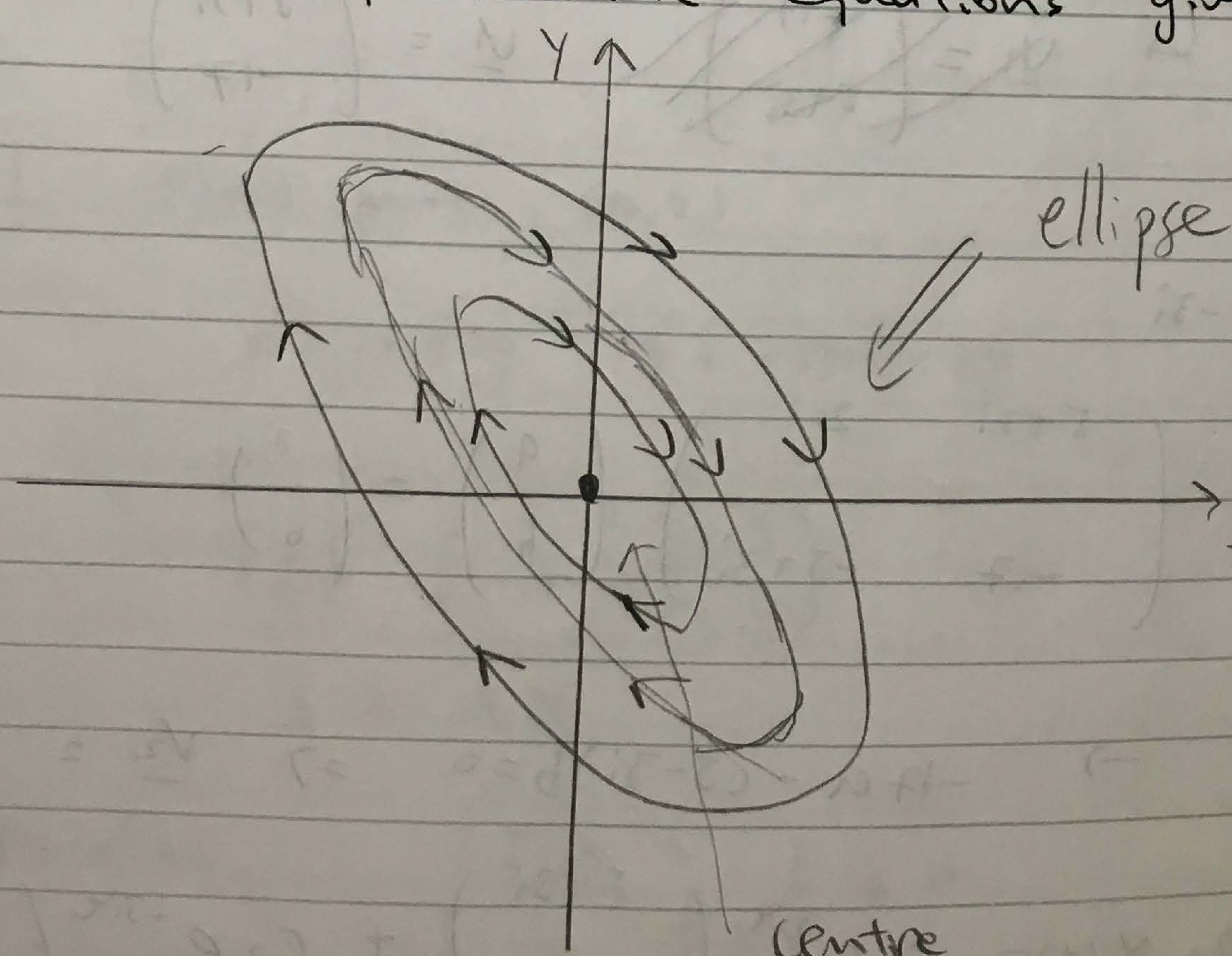
$$= 10c(\underbrace{e^{3it}+e^{-3it}}_{2\cos(3t)}) + 6ic(\underbrace{e^{3it}-e^{-3it}}_{2i\sin(3t)})$$

$$= \underline{20c\cos(3t) + -12c\sin(3t)}$$

$$y(t) = ce^{3it}(-17) + ce^{-3it}(-17)$$

$$= -34c(\underbrace{e^{3it}+e^{-3it}}_{2\cos(3t)}) = \underline{-c68\cos(3t)}$$

Plot this parametric equations gives :



(iii) $\dot{x} = x - y \quad \dot{y} = x^2 - 4$

fixed points : $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$

$\Rightarrow$ $\underline{\underline{(2, 2)}}$ or $\underline{\underline{(-2, -2)}}$

~~→ For (2,2)~~

Jacobian matrix $J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$

→ For (2,2)

$$J = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$$

Eigenvalues $(1-\lambda)(-\lambda) + 4 = 0$

$\rightarrow \quad \lambda^2 - \lambda + 4 = 0$

$\rightarrow \quad \lambda = \frac{1}{2}(1 \pm i\sqrt{15})$

$\because Re(\lambda) > 0 \quad \therefore \rightarrow$ $\underline{\text{unstable spiral}}$

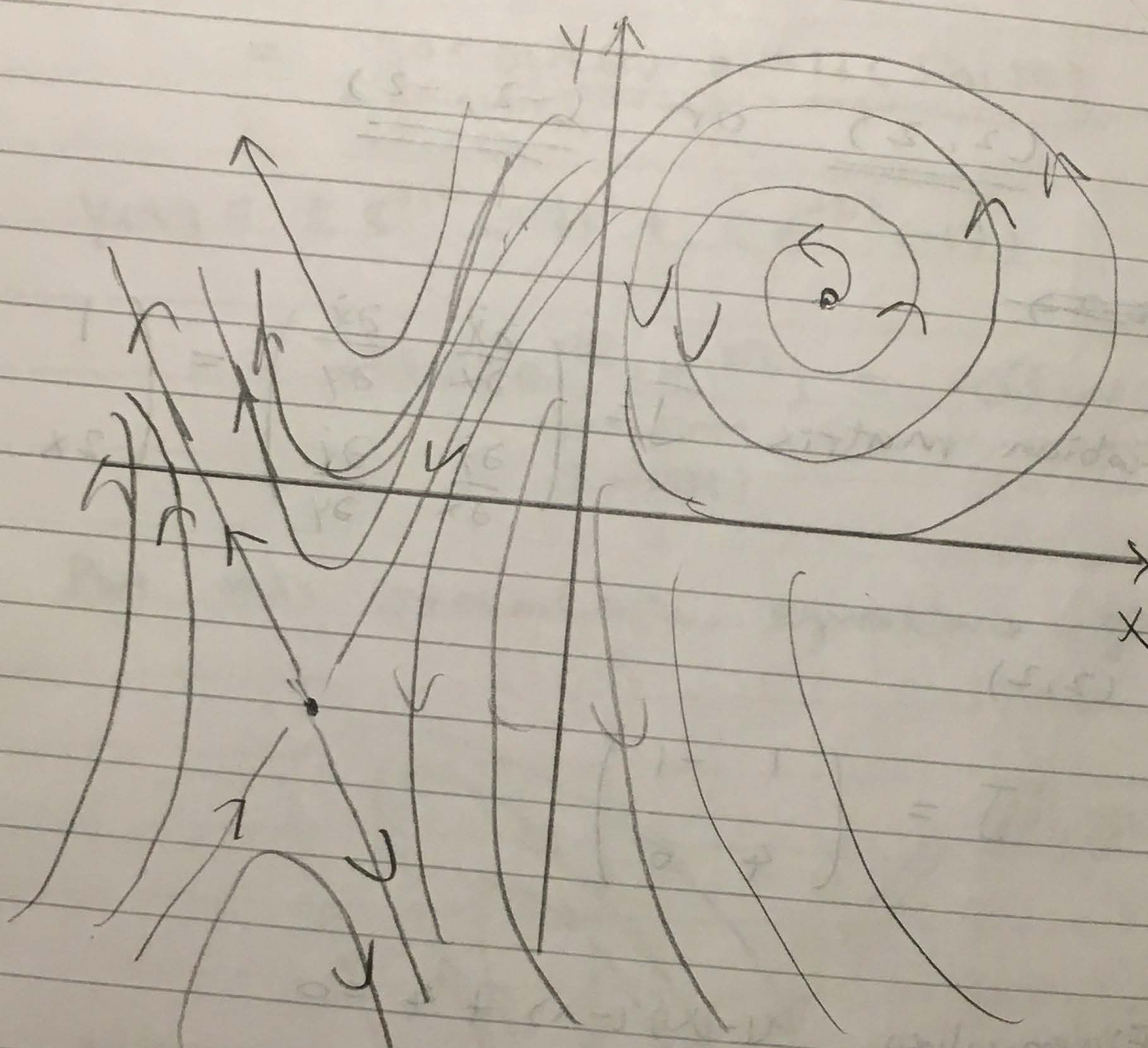
→ For $(-2, -2)$ $\quad J = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$

Eigenvalues $\lambda^2 - \lambda - 4 = 0$

$\rightarrow \quad \lambda = \frac{1}{2}(1 \pm \sqrt{17})$

One is positive and the other is negative

→ Saddle ~~point~~ node



4. $\ddot{\theta} + b\dot{\theta} + \sin\theta = 0$

(a)

let $\boxed{\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -bv - \sin\theta \end{aligned}}$

(b) $\rightarrow$ $\begin{pmatrix} \dot{\theta} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos\theta & -b \end{pmatrix} \begin{pmatrix} \theta \\ v \end{pmatrix}$

Fixed points $\begin{cases} \dot{\theta} = 0 \\ \dot{v} = 0 \end{cases} \Rightarrow \begin{cases} \theta = k\pi & (k = \text{integer}) \\ v = 0 \end{cases}$

$\rightarrow$ The Jacobine matrix for $(0, 0)$ (and $(0, 2\pi)$, $(0, 4\pi)$ ....)

$$J = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}$$

Eigenvalues $(b+\lambda)\lambda + 1 = 0 \rightarrow \lambda^2 + b\lambda + 1 = 0$

$$\rightarrow \lambda = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4}\right)$$

$\rightarrow$ Jacobine for $(0, \pi)$ (and $(0, 3\pi)$, $(0, 5\pi)$, ...)

$$J = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}$$

Eigenvalues $(b+\lambda)\lambda - 1 = 0 \rightarrow \lambda^2 + b\lambda - 1 = 0$

$$\rightarrow \lambda = \frac{1}{2}\left(-b \pm \sqrt{b^2 + 4}\right)$$

→ For $(0,0), (0,2\pi), (0,4\pi), \ldots$

If $b<2$, then both λ are real and negative

→ Stable node

If $b>2$ then both λ are complex and $Re(\lambda) < 0$

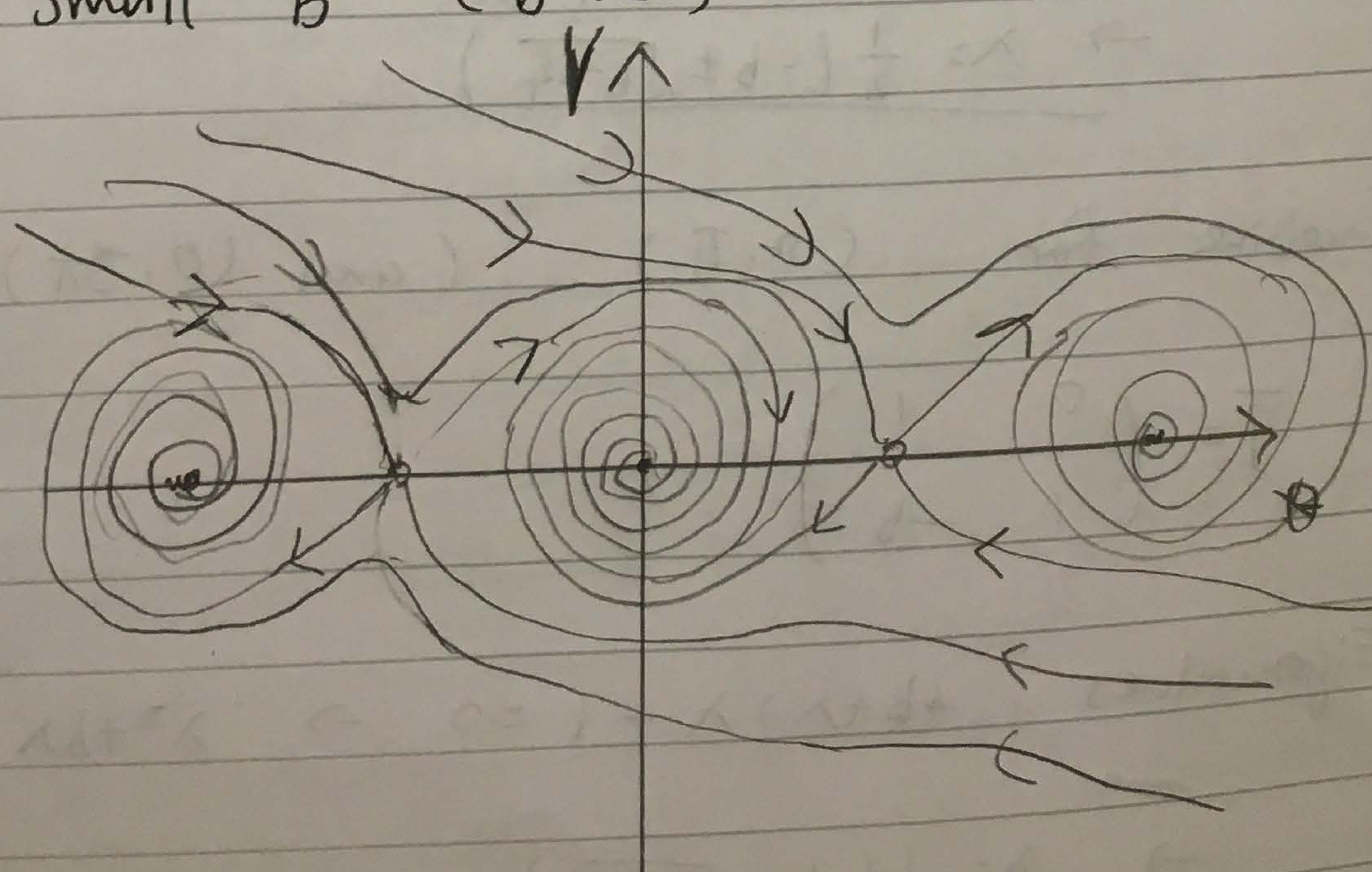
→ Stable spiral

→ For $(0,\pi), (0,3\pi), (0,5\pi), \ldots$

Both λ are real, one positive, one negative

∴ Saddle node

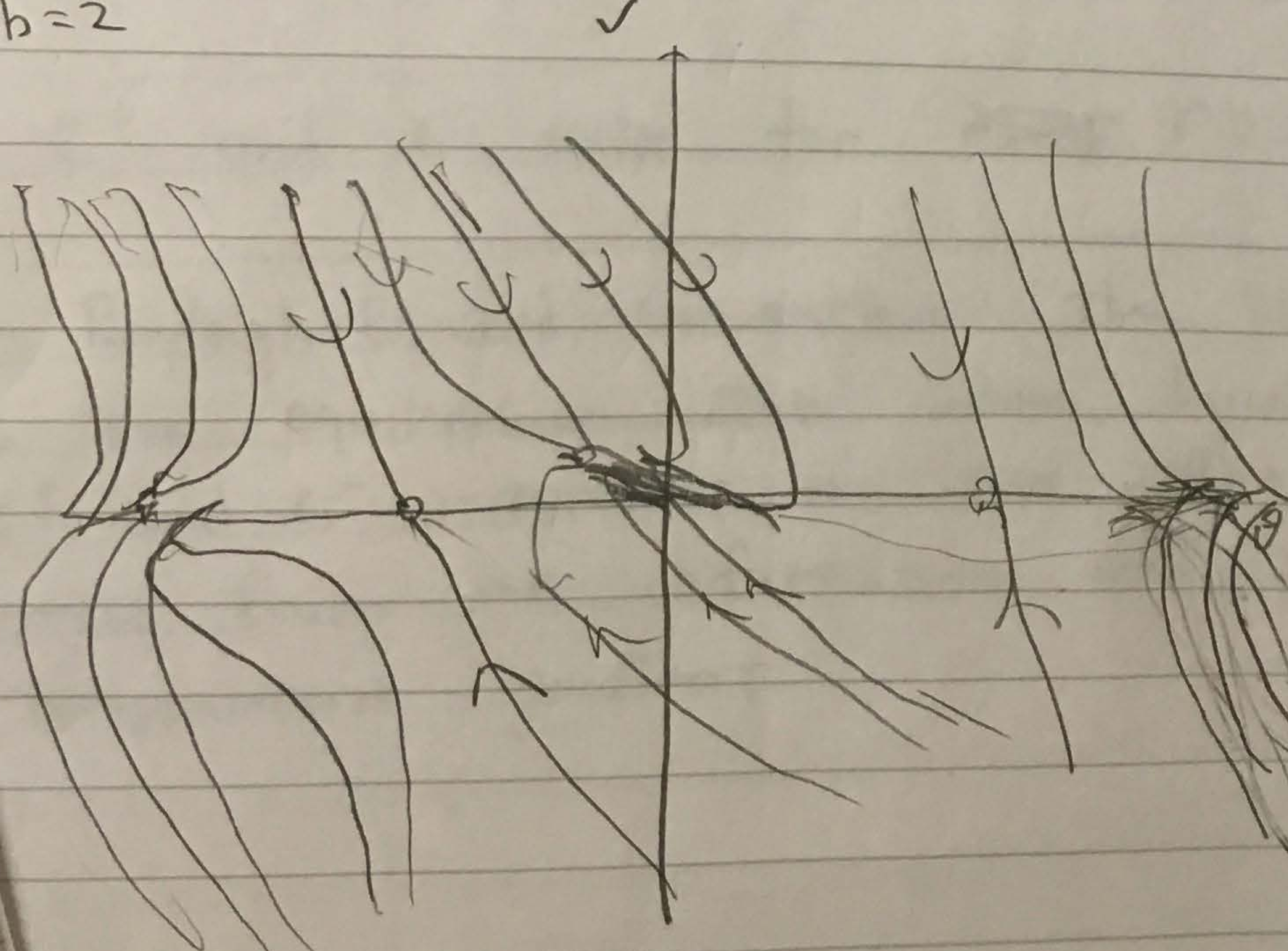
For small b $(b<2)$:



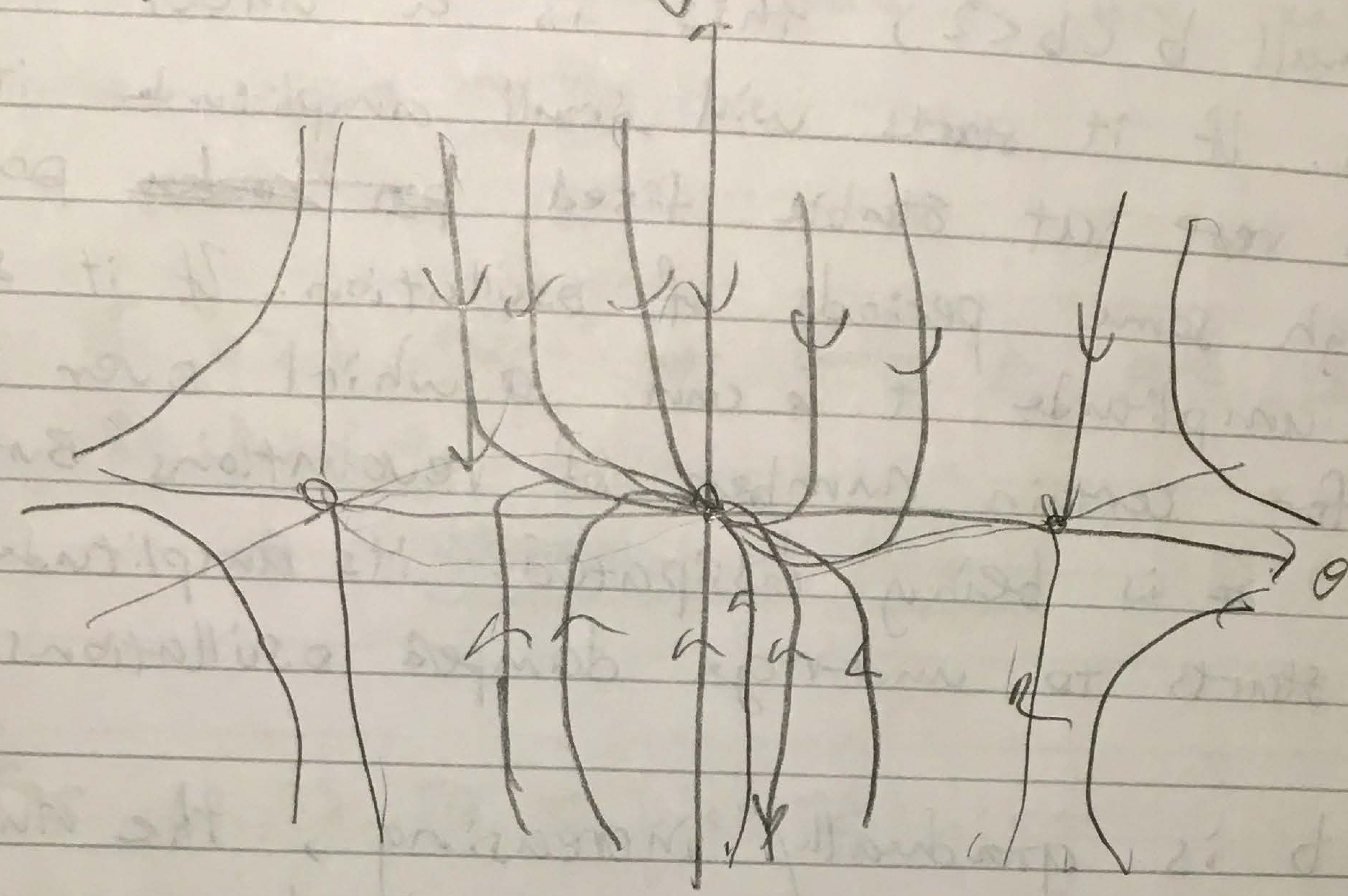
→ For small b (b<2) this is a under-damped pendulum. If it starts will small amplitude it will come to rest at stable fixed ~~nodes~~ points. ~~If~~ ~~it~~ through some periods of oscillation. If it starts with larger amplitude it can whirl over ~~to~~ the top for certain number of revolutions. But as energy is being dissipated, its amplitude decreases and starts to undergo damped osillations.

→ As b is gradually increasing, the number of loops of the spiral on phase portrait will decreases, and finally as b>2, there will be no loops. The stable spiral is replaced by a stable fixed point (stable node). The pendulum will not oscillate for more than one period. ~~The~~ This is over-damped ~~oscillations~~ pendulum

b=2



b>2



As b → 2, the spiral collapses.

5. (a) Lorenz equations

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

Fixed points need $\dot{x}=0$, $\dot{y}=0$, $\dot{z}=0$

$\therefore$ $y=x$, $(r-1)x = xz$, $x^2 = bz$

If $x=0$, then $y=0$, $z=0$ $\Rightarrow$ $(0,0,0)$ is a solution

If $x \neq 0$, $z = r-1$ $\Rightarrow$ $x^2 = b(r-1)$

$\therefore$ $x = y = \pm\sqrt{b(r-1)}$ $\quad\therefore$ $\boxed{\begin{array}{l}(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1) \Rightarrow C^+ \\ \text{or } (-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1) \Rightarrow C^-\end{array}}$

are also solutions.

$(0,0,0)$ exists for all r

C^+ and C^- exist for $r>1$

$\rightarrow$ In Rayleigh-Benard Convection, the origin corresponds to the equilibrium state when liquid is uniform. C^+ and C^- represent the two rolling states that arise from the bifurcation due to change in temperature gradient

(b) ~~Is~~ Jacobine for Lorenz equations

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

At origin $(x, y, z) = (0, 0, 0)$

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

Find eigenvalues

$$(-\sigma-\lambda)(-1-\lambda)(-b-\lambda) + \cancel{\sigma r b} \; \sigma r(b+\lambda)$$

$$\therefore \quad (\lambda+\sigma)(\lambda+1)(\lambda+b) = \sigma r(b+\lambda)$$

$$\therefore \quad \cancel{\lambda^3 + (\sigma+1+b)\lambda^2 + (\sigma b+\sigma+b)\lambda + \sigma b = \sigma r(b+\lambda)}$$

$$\therefore \quad \cancel{\lambda^3 + (\sigma+1+b)\lambda^2 + (\sigma b+\sigma+b)\lambda + \sigma b(1-r) = 0}$$

$$\cancel{\lambda^3 + (\sigma+1+b)\lambda^2 + (\sigma b+\sigma+b+\sigma r)}$$

$$\Rightarrow \quad \underline{\underline{\lambda = -b}} \quad . \quad \text{or} \quad (\lambda+\sigma)(\lambda+1) = \sigma r$$

$$\cancel{\Rightarrow \lambda^2 + \sigma\lambda + \sigma b}$$

$$\lambda^2 + \sigma\lambda + (\sigma - \sigma r) = 0$$

$$\therefore \quad \lambda = \frac{1}{2}\left(-\sigma \pm \sqrt{\sigma^2 - 4(\sigma-\sigma r)}\right)$$

$$= \frac{1}{2}\left(-\sigma \pm \sqrt{(\sigma^2 - 4\sigma(1-r)}\right)$$

$\rightarrow \quad \lambda^2 + (\sigma+1)\lambda + \sigma(1-r) = 0$

$\therefore \quad \lambda = \frac{1}{2}\left(-\sigma - 1 \pm \sqrt{(\sigma+1)^2 - 4\sigma(1-r)}\right)$

If $r<1$, $\sqrt{(\sigma+1)^2 - 4\sigma(1-r)} < \sqrt{(\sigma+1)^2} = \sigma+1$

$\therefore \lambda < 0$

Also $\because \lambda = -b < 0$

$\therefore$ If $r<1$, all eigenvalues <0.

$\rightarrow$ stable node

If $r>1$ ~~one~~ for the 2 eigenvalues other than $-b$, one is positive and the other is negative.

$\rightarrow$ saddle node

For C^+ and C^- :

they are stable for $1<r<r_H = \dfrac{\sigma(\sigma+b+3)}{\sigma-b-1}$

~~At $r>r_H$ they have unstable limit cycles~~

And they are unstable for $r>r_H$

(c) For $r=28$, $\sigma=10$, $b=8/3$

and plane $z = r - 2\sigma = 28 - 20 = 8$

$$\dot{z} = xy - bz = xy - \left(\tfrac{8}{3}\right)(8) = xy - \tfrac{64}{3}$$

$$\simeq xy - 21.33$$

We see crosses have larger ~~xy~~ xy product values than diamonds ~~hat~~ have

$\rightarrow$ crosses are upwards
diamonds are downwards

(d) $\underline{\delta x} = (\delta x, \delta y, \delta z)$

$\frac{dx}{dt} = \dot{x} = f(x, y, z)$, $\frac{d}{dt}(x + \delta x) = f(x+\delta x, y+\delta y, z+\delta z)$

$$\rightarrow \frac{d}{dt}(\delta x) = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z$$

$$\rightarrow \frac{d}{dt}(\delta y) = \frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial y}\delta y + \frac{\partial g}{\partial z}\delta z$$

$$\rightarrow \frac{d}{dt}(\delta z) = \frac{\partial h}{\partial x}\delta x + \frac{\partial h}{\partial y}\delta y + \frac{\partial h}{\partial z}\delta z$$

$$\rightarrow \frac{d}{dt}(\underline{\delta x}) = J(\underline{\delta x})$$

where $\boxed{J_{ij} = \frac{\partial \dot{x}_i}{\partial x_j}}$

$$\frac{d}{dt}(|\underline{\delta x}|^2) = \frac{d}{dt}(\underline{\delta x}^T \underline{\delta x}) = \underline{\delta x}^T \underbrace{\frac{d}{dt}(\underline{\delta x})}_{J(\underline{\delta x})} + \frac{d}{dt}(\underline{\delta x}^T)\underline{\delta x}$$

$$= \underline{\delta x}^T J \underline{\delta x} + \left(\frac{d}{dt}\underline{\delta x}\right)^T \underline{\delta x}$$

$$= \underline{\delta x}^T J \underline{\delta x} + (J\underline{\delta x})^T \underline{\delta x} = \underline{\delta x}^T J \underline{\delta x} + \underline{\delta x}^T J^T \underline{\delta x}$$

$$= \underline{\delta x}^T (J + J^T)\underline{\delta x} = 2\underline{\delta x}^T \underbrace{\left(\frac{J+J^T}{2}\right)}_{L} \underline{\delta x}$$

$$= 2\underline{\delta x}^T L \underline{\delta x} \qquad \left(L = \frac{J+J^T}{2}\right)$$

If $\underline{\delta x}$ is an eigenvector of L with eigenvalue λ.

then $\quad L\underline{\delta x} = \lambda \underline{\delta x}$

$$\therefore \frac{d}{dt}(|\underline{\delta x}|^2) = 2\underline{\delta x}^T L \underline{\delta x} = 2\lambda \underline{\delta x}^T \underline{\delta x} = 2\lambda |\underline{\delta x}|^2$$

$$\Rightarrow \frac{d(|\underline{\delta x}|^2)}{|\underline{\delta x}|^2} = 2\lambda\, dt$$

$$\Rightarrow |\underline{\delta x}|^2(t) = |\underline{\delta x}|^2(0)\, e^{2\lambda t}$$

$$\rightarrow \boxed{|\underline{\delta x}|(t) = |\underline{\delta x}|(0)\, e^{\lambda t}}$$

If $\lambda > 0 \rightarrow$ grows exponentially
If $\lambda < 0 \rightarrow$ decays exponentially

(e)

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} & \frac{\partial \dot{x}}{\partial z} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} & \frac{\partial \dot{y}}{\partial z} \\ \frac{\partial \dot{z}}{\partial x} & \frac{\partial \dot{z}}{\partial y} & \frac{\partial \dot{z}}{\partial z} \end{pmatrix}$$

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

$$\therefore J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

At $(x,y,z) = (0,0,r-2\sigma)$

$$J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 2\sigma & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$L = \frac{J + J^T}{2} = \begin{pmatrix} -\sigma & \frac{3\sigma}{2} & 0 \\ \frac{3\sigma}{2} & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

Eigenvalue equation

$$-(\sigma+\lambda)(1+\lambda)(b+\lambda) \cancel{+ \frac{9b\sigma^2}{4} = 0}$$
$$+ \frac{9}{4}\sigma^2(b+\lambda) = 0$$

$\therefore \quad \underline{\underline{\lambda = -b}}$, or

$$(1+\lambda)(\sigma+\lambda) = \frac{9}{4}\sigma^2$$

$$\therefore \lambda^2 + (\sigma+1)\lambda + (\sigma - \frac{9}{4}\sigma^2) = 0$$

$$\lambda = \frac{1}{2}\left(-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - (4\sigma - 9\sigma^2)}\right)$$

$$(\sigma+1)^2 - (4\sigma - 9\sigma^2) = \sigma^2 + 2\sigma + 1 - 4\sigma + 9\sigma^2$$
$$= (\sigma-1)^2 + (3\sigma)^2$$

$$\cancel{= (\sigma+3\sigma)(\sigma+\sigma-1+3\sigma)}$$

$$\cancel{= (-2\sigma-1)(2\sigma-1)}$$

$\rightarrow \quad \cancel{\lambda = b, \frac{1}{2}}$

$$\boxed{\begin{aligned} \lambda_1 &= -b \\ \lambda_2 &= \frac{1}{2}\left(-(\sigma+1) + \sqrt{(\sigma-1)^2 + (3\sigma)^2}\right) \\ \lambda_3 &= \frac{1}{2}\left(-(\sigma+1) - \sqrt{(\sigma-1)^2 + (3\sigma)^2}\right) \end{aligned}}$$

For $r = 28$, $\sigma = 10$, $b = 8/3$

$\lambda_1 = -8/3$ eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{z}$

$= \underline{-2.67}$

$\lambda_2 = 10.16$

$\lambda_3 = -21.16$ → their eigenvectors are on x-y plane

$\because \lambda_1 < 0$, $\lambda_3 < 0$, $\lambda_2 > 0$, and $|\lambda_3|, |\lambda_2| > |\lambda_1|$

$|\lambda_3|, |\lambda_2|$ ↓ xy plane

$|\lambda_1|$ ↓ z-direction

$\therefore$ we know that dominant grow and decay happen in the x-y plane.

decay in z is slow.

(f)

$$\delta V(t+dt) = \delta V(t) + \oint_{\partial V} \underline{u} \cdot d\underline{s}\, dt$$

$$(\underline{u} = (\dot{x}, \dot{y}, \dot{z}) = \text{velocity in phase space})$$

$$\frac{d}{dt}(\delta V) = \oint_{\partial V} \underline{u} \cdot d\underline{s} \underbrace{=}_{\text{divergence theorem}} \int_V \nabla \cdot \underline{u}\, dV$$

if $\nabla \cdot \underline{u} = \text{const.}$ → $\delta \dot{V} = \delta V(\nabla \cdot \underline{u})$

For Lorenz equations.

$$\nabla \cdot \underline{u} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(\sigma + 1 + b) = \text{const.}$$

$$\therefore \quad \delta \dot{V} = -(\sigma + 1 + b)\,\delta V \quad \Rightarrow \boxed{\delta V = \delta V_0 e^{-(\sigma+1+b)t}}$$

$\because$ Eigenfunctions orthogonal

$\therefore \quad \delta V = \delta x \, \delta y \, \delta z$

orthogonality

$$= \left[\delta x_0 \exp\left(\frac{t}{2}\left(-(\sigma+1) + \sqrt{(\sigma-1)^2 + (3\sigma)^2}\right)\right)\right]$$

$$\times \left[\delta y_0 \exp\left(\frac{t}{2}\left(-(\sigma+1) - \sqrt{(\sigma-1)^2 + (3\sigma)^2}\right)\right)\right]$$

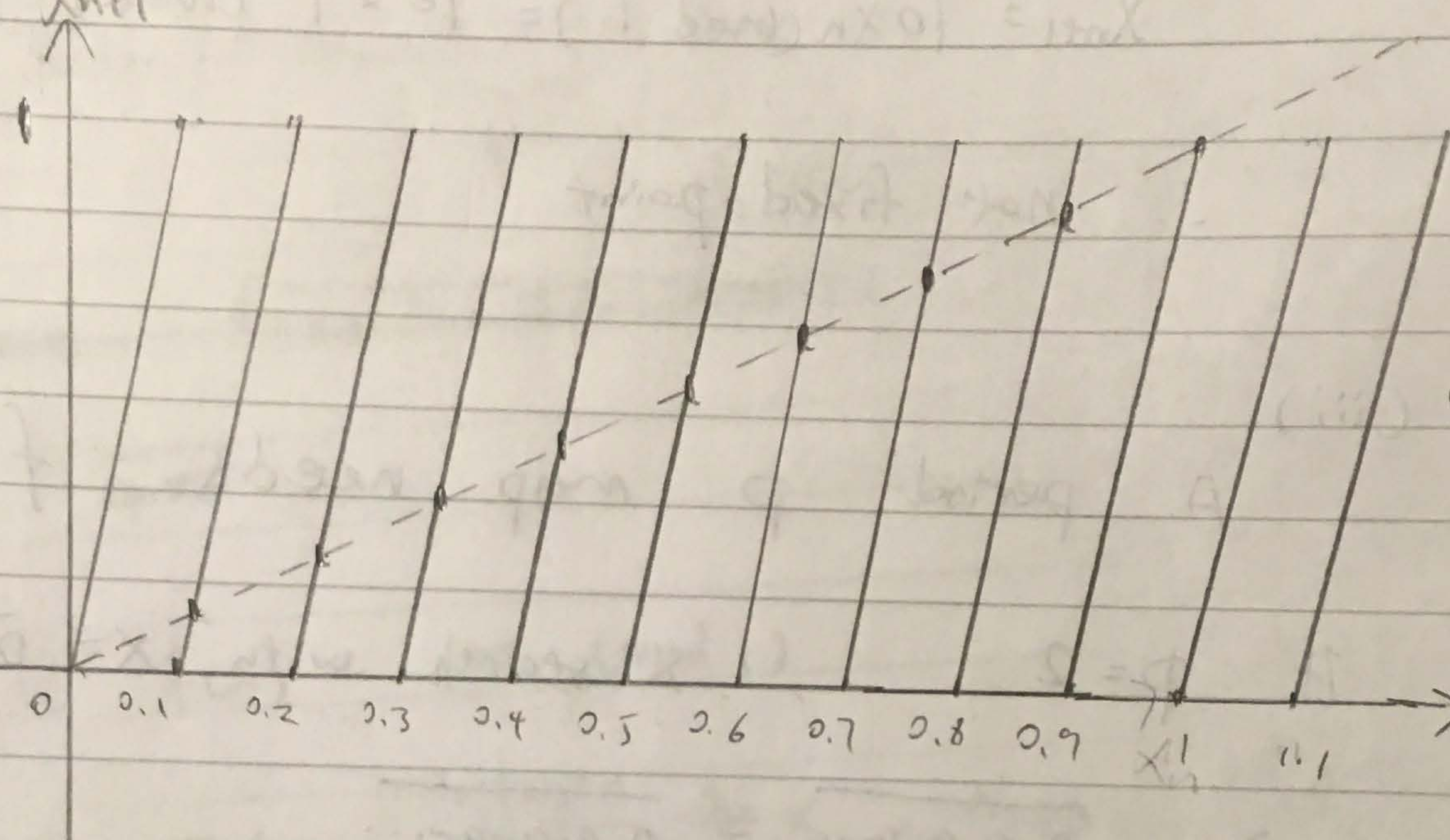
$$\times \left[\delta z_0 \exp(-bt) \right].$$

$$= \delta x_0 \, \delta y_0 \, \delta z_0 \exp(-(\sigma+1+b)t)$$

$$= \delta V_0 \exp(-(\sigma+1+b)t) \implies \text{confirmed.}$$

6. $x_{n+1} = 10 x_n \pmod 1$

(i)



(ii) the fixed points : $x_{n+1} = x_n = x^*$

$\because$ $x_{n+1} = 10 x_n \pmod 1 < 1$

$\therefore$ $x^* < 1$

Assume $x^* = \overline{0.a_1 a_2 a_3 a_4 a_5 \cdots}$

then $x^* = 10 x^* \pmod 1$

$\Rightarrow$ $\overline{0.a_1 a_2 a_3 a_4 a_5 \cdots} = \overline{a_1.a_2 a_3 a_4 a_5 a_6 \cdots} \pmod 1$

$= \overline{0.a_2 a_3 a_4 a_5 a_6 \cdots}$

$\Rightarrow a_1 = a_2 = a_3 = a_4 = a_5 = \cdots$

$\Rightarrow$ fixed points are $\boxed{x^* = 0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9},}$

NB If $x_n^* = 0.9999\ldots = \frac{9}{9} = 1$

$x_{n+1} = 10x_n \pmod 1 = 10 \times 1 \pmod 1 = 0$

$\therefore$ not fixed point

(iii)

A period p map needs $f^p(x) = x$

If $p = 2$, x starts with $x = \overline{0.a_1a_2a_3\ldots}$

$\rightarrow \overline{0.a_1a_2a_3\ldots} = \overline{0.a_3a_4a_5\ldots}$

$\therefore a_1 = a_3 = a_5 = a_7 \ldots$

$a_2 = a_4 = a_6 = a_8 \ldots$

set $a_2 = 0$, $a_1 = 1$

$\rightarrow 0.10101010\ldots$ is a period 2 point

For general p, we have

$a_i = a_{i+p}$

Set $a_1 = a_{1+p} = a_{1+2p} = a_{1+3p} = \cdots = 1$

$a_2 = a_3 = \cdots = a_p = 0$

we get a period p solution

$\therefore$ periodic points for all periods exist.

stability of p-cycle determined by

$$\frac{d}{dx} f^p(x)\Big|_{x=x_0} = \frac{d}{dx}\Big[\underbrace{f(f(f(\cdots x)\cdots))}_{p^{th}}\Big]\Big|_{x=x_0}$$

$$= \underbrace{f'(x_{p-1})\, f'(x_{p-2}) \cdots f'(x_0)}_{p^{th}}$$

(where $x_{i+1} = f(x_i)$)

~~where $f(x) = 10x \pmod 1$~~

~~and~~

~~For~~ if $f(x) = 10x \pmod 1$

If ~~x~~ $\frac{\text{integer}}{10} \neq x$, then. $f'(x) = 10$

for any $x \neq \frac{\text{integer}}{10}$

$\therefore \quad \frac{d}{dx} f^p(x) = 10^p > 1 \Rightarrow$ unstable

(iv) If we initially start with an irrational number, then orbit is aperiodic because for irrational numbers digits never repeat periodically.

$\because$ There are infinitely many irrational numbers

$\therefore$ There are infinitely many aperiodic solutions

(V)

The Liapunov exponent

$$\lambda = \lim_{n\to\infty} \left[\frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \underbrace{f'(x_i)}_{10} \right| \right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot n \cdot \ln 10 = \ln 10 > 1$$

$\rightarrow$ system chaotic

sensitive dependence on initial conditions