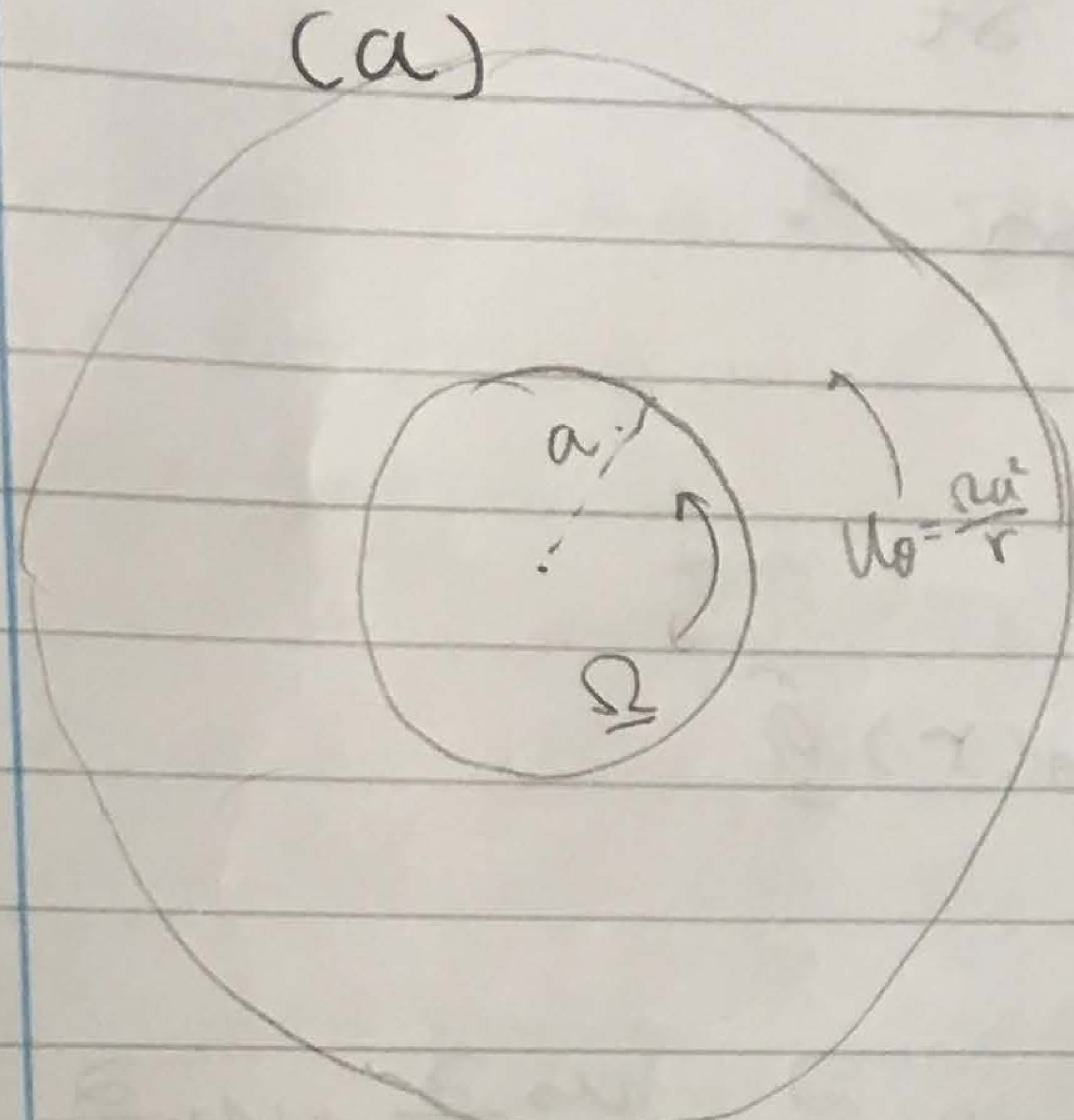
To: Caroline Terquem

B1 Problem Set 2 Ziyan Li

1.

(a)



For $r<a$ ~~$u = \Omega r$~~

$u_\theta = \Omega r$

$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r}\cancel{\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ 0 & \Omega r & 0 \end{vmatrix}}$

$= \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ A_r & rA_\theta & A_z \end{vmatrix}.$

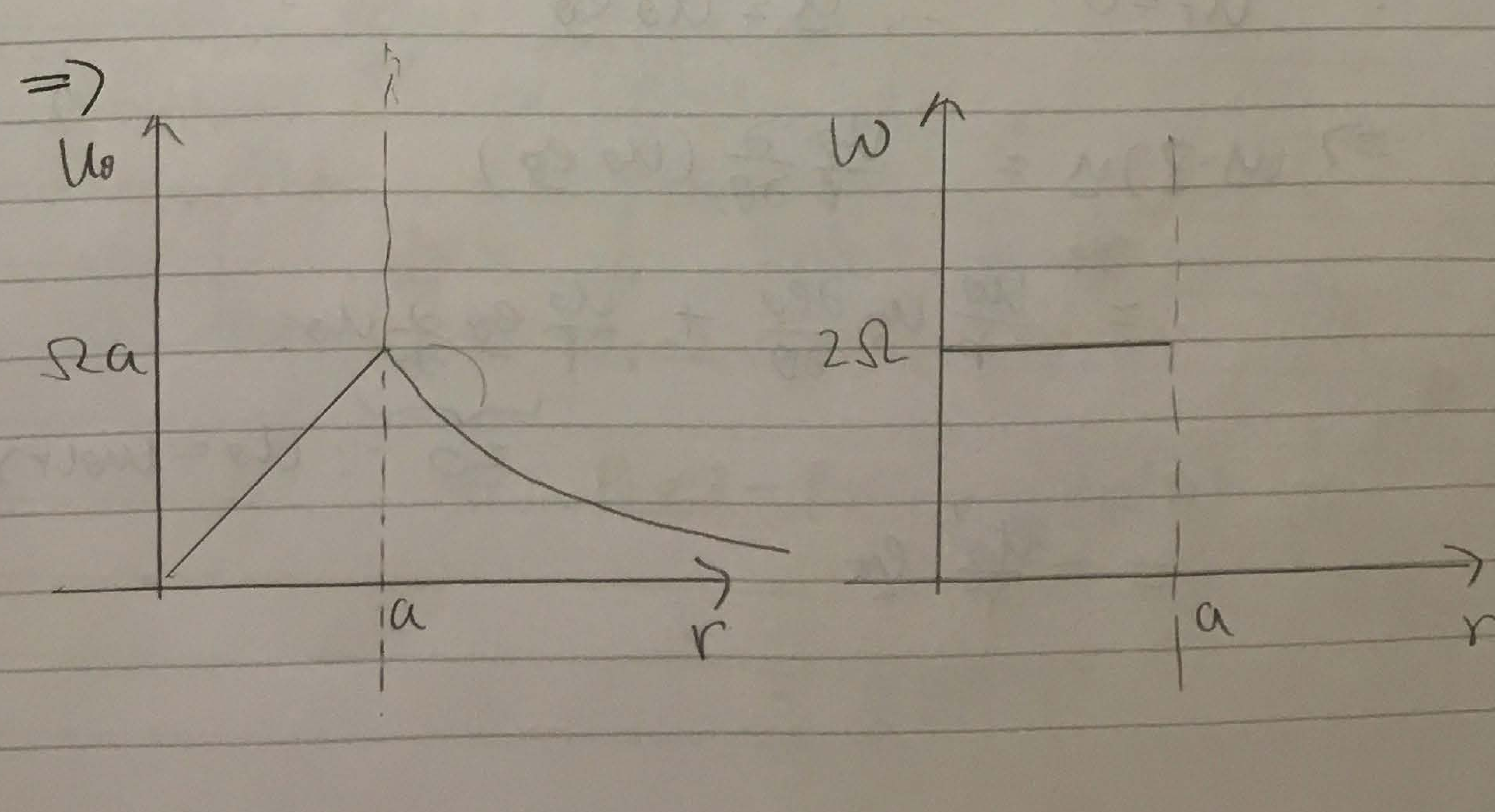
$= \frac{\Omega}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ 0 & r^2 & 0 \end{vmatrix} = 2r \cdot \frac{\Omega}{r}\hat{\underline{z}} = 2\Omega\hat{\underline{z}}$

For $r \geq a$ $u_\theta = \frac{\Omega a^2}{r}$

$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \partial_r & \partial_\theta & \partial_z \\ 0 & \frac{\Omega a^2}{r}\cdot r & \end{vmatrix} = 0$

~~(except at)~~

$\Rightarrow$



(b) neglect viscous damping $\Rightarrow \nu = 0$

steady flow $\frac{\partial \underline{u}}{\partial t} = 0$

$\Rightarrow$ Naiver-stokes equation :

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla}P.$$

$\cancel{\underline{u} = u_\theta \hat{\underline{\theta}}}$ $\underline{u} = u_\theta(r)\,\hat{\underline{\theta}}$

in cylindrical polar $\underline{u}\cdot\underline{\nabla} = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}$.

$(u_z = 0)$

$$\Rightarrow (\underline{u}\cdot\underline{\nabla})\underline{u} = \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r e_r + u_\theta e_\theta)$$

~~$\because u_r = 0$~~

$$\cancel{\Rightarrow (\underline{u}\cdot\underline{\nabla})\underline{u} = \left(\frac{u_\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r e_r)}$$

$$\cancel{= \frac{u_\theta}{r}\frac{\partial}{\partial\theta}(u_r e_r)}$$

$$\cancel{= \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} + \frac{u_\theta}{r}u_r\frac{\partial e_r}{\partial\theta}}$$

$\because u_r = 0 \quad \therefore \underline{u} = u_\theta \underline{e_\theta}$

$$\Rightarrow (\underline{u}\cdot\underline{\nabla})\underline{u} = \frac{u_\theta}{r}\frac{\partial}{\partial\theta}(u_\theta \underline{e_\theta})$$

$$= \frac{u_\theta}{r}u_\theta\frac{\partial \underline{e_\theta}}{\partial\theta} + \underbrace{\frac{u_\theta}{r}\underline{e_\theta}\frac{\partial}{\partial\theta}u_\theta}_{=0 \ \because u_\theta = u_\theta(r)}.$$

$$= -\frac{u_\theta^2}{r}\underline{e_r}$$

$$\Rightarrow \quad -\frac{u_\theta^2}{r}\,\underline{\hat{e_r}} = -\frac{1}{\rho}\underline{\nabla} p$$

$$\rightarrow \quad \underline{\nabla} p = \frac{\rho u_\theta^2}{r}\,\underline{\hat{e_r}}$$

$\Rightarrow$ $\underline{\nabla} p$ is along $\underline{\hat{e_r}}$ only, so $p = p(r)$

$$\Rightarrow \quad \frac{dp}{dr} = \frac{\rho u_\theta^2}{r}$$

For $0 < r < a$, $u_\theta = \Omega r$

$$\frac{dp}{dr} = \frac{\rho\Omega^2 r^2}{r} = \rho\Omega^2 r$$

$$\int_{p(0)}^{p(a)} dp = \int_0^a \rho\Omega^2 r\,dr \Rightarrow p(a) - p(0) = \frac{1}{2}\rho\Omega^2 a^2$$

For $r \geq a$, $u_\theta = \frac{\Omega a^2}{r}$

$$\Rightarrow \quad \frac{dp}{dr} = \frac{\rho\Omega^2 a^4}{r^3}$$

$$\int_{p(a)}^{p(\infty)} dp = \int_{r=a}^{r=\infty} \rho\Omega^2 a^4 \frac{1}{r^3}\,dr$$

$$\Rightarrow \quad p(\infty) - p(a) = -\frac{1}{2}\rho\Omega^2 a^4 \left.\frac{1}{r^2}\right|_a^\infty$$

$$= \frac{1}{2}\rho\Omega^2 a^2$$

$\rightarrow \quad P(0) - P(\infty) = [P(0) - P(a)] + [P(a) - P(\infty)]$

$= -\frac{1}{2}\rho\Omega^2 a^2 - \frac{1}{2}\rho\Omega^2 a^2 = \underline{\underline{-\rho\Omega^2 a^2}}$

If use Bernoulli's theorem

$P(0) - P(\infty) = 0$ because $u_\theta(0) = 0$
$u_\theta(\infty) = 0$

$P(0) + \frac{1}{2}\rho u_\theta^2(0) = P(\infty) + \frac{1}{2}\rho u_\theta^2(\infty)$

$\Rightarrow P(0) = P(\infty)$

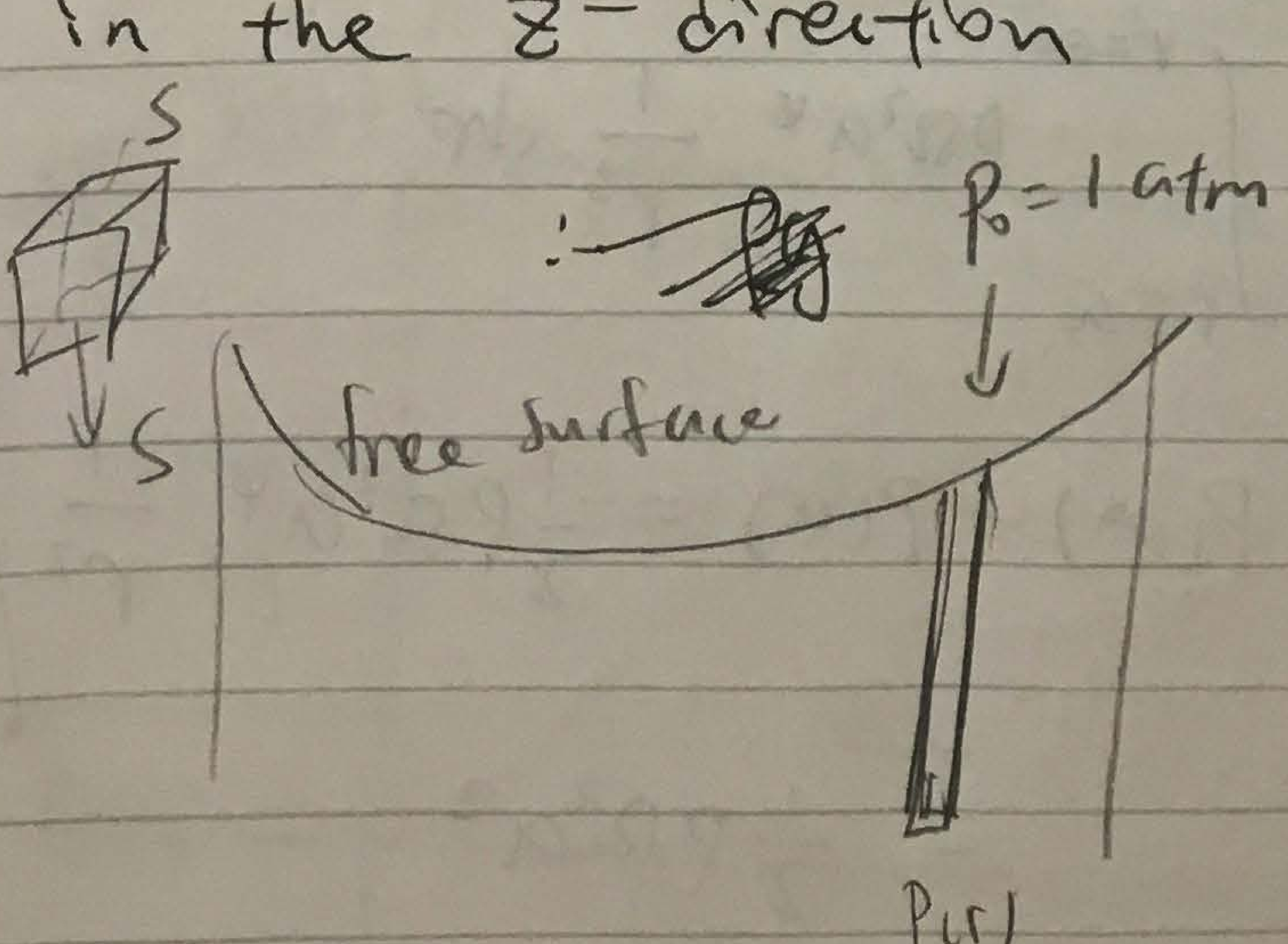
This is not true because the Bernoulli function

$H = \frac{P}{\rho} + \frac{u^2}{2}$

is only constant if the flow is irrotational, but for $r < a$ we have $\underline{\omega \neq 0}$

$\rightarrow$ Cannot use Bernoulli.

(c) There is no acceleration of any fluid particle in the z-direction



At radius r, the pressure $P(r)$ can support height $h(r)$ of water, then $S\rho g h(r) = P(r) S$

$\rightarrow \quad h(r) = \dfrac{P(r)}{\rho g}$

For $r<a$ $\quad \dfrac{dP}{dr} = \rho \Omega^2 r$.

$\rightarrow \quad P(r) = P(0) + \frac{1}{2}\rho\Omega^2 r^2$. $\quad$, let $h(0) = \dfrac{P(0)}{\rho g}$

$\rightarrow \quad \boxed{h(r) = h(0) + \dfrac{\Omega^2 r^2}{2g}}$

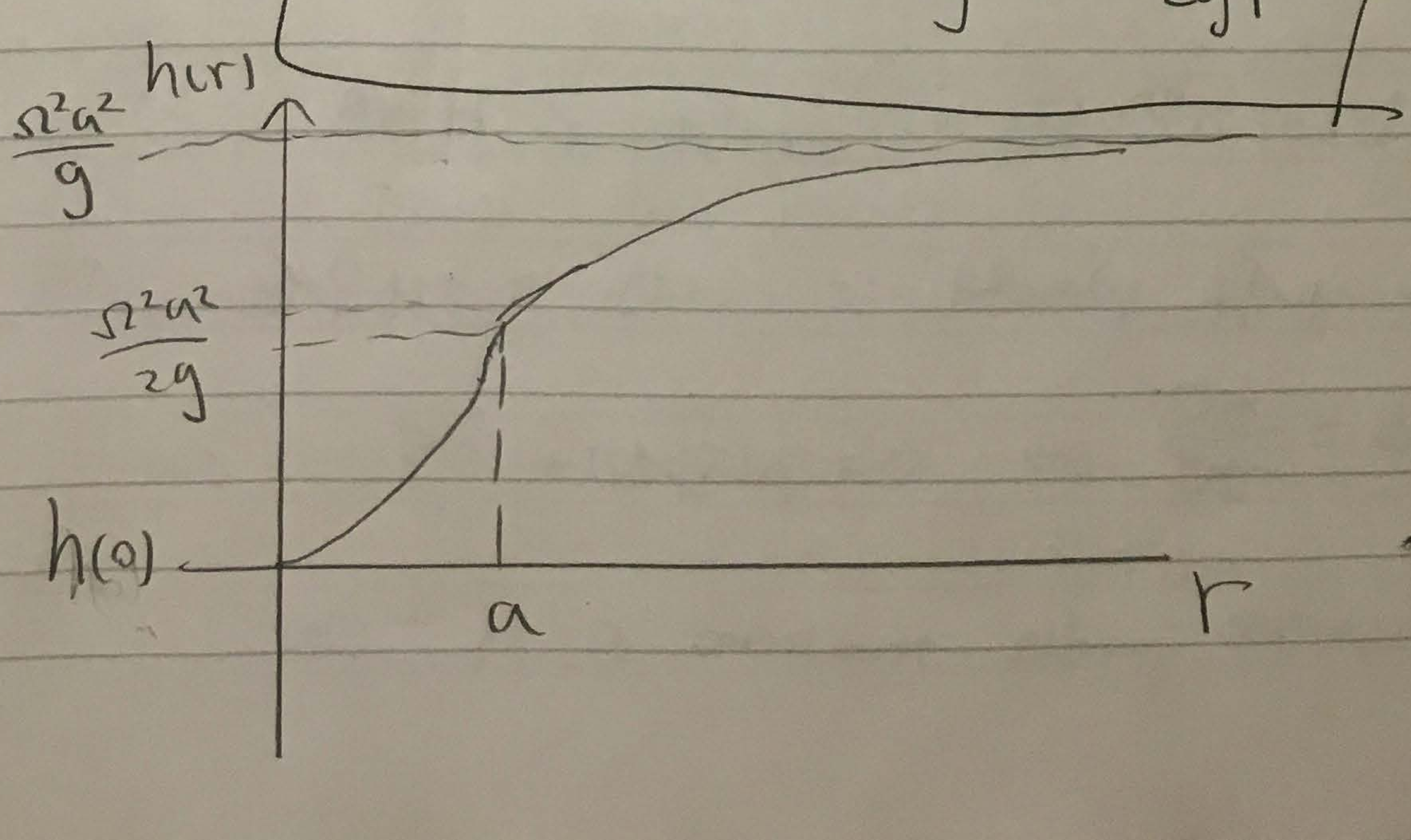
For $r \geq a$, $\quad h(a) = h(0) + \dfrac{\Omega^2 a^2}{2g}$, $P(a) = P(0) + \frac{1}{2}\rho\Omega^2 a^2$

$\dfrac{dP}{dr} = \dfrac{\rho\Omega^2 a^4}{r^3} \rightarrow P(r) = P(a) + \displaystyle\int_a^r \dfrac{\rho\Omega^2 a^4}{r^3}$

$\rightarrow \quad P(r) = P(a) + \frac{1}{2}\rho\Omega^2(a^2 - \frac{a^4}{r^2})$.

$\rightarrow \quad P(r) = P(0) + \rho\Omega^2 a^2 - \frac{1}{2}\rho\Omega^2 \frac{a^4}{r^2}$

$\Rightarrow \quad \boxed{h(r) = h(0) + \dfrac{\Omega^2 a^2}{g} - \dfrac{\Omega^2 a^4}{2 g r^2}}$



ii Between vortex core ($r \to 0$) and large r ($r \to \infty$), change in height Δh is

$$\Delta h = \frac{\Omega^2 a^2}{g}$$

if $a = 5cm = 5 \times 10^{-2} m$, $\Omega = 120$ rpm

$$= \frac{120 \times 2\pi}{60 s} = 4\pi \ s^{-1}$$

then $$\Delta h = \frac{(4\pi)^2 (5 \times 10^{-2})^2}{(9.8)} = 0.04 \ m$$

If container is 1.5 m deep, for upper surface to not exposed to air.

$\to \Delta h \leq 1.5m$,

$\therefore$ maximum $\Omega \to \Delta h = 1.5m$

$$\Rightarrow \Omega^2 = \frac{g\Delta h}{a^2} \to \Omega = \frac{1}{a}\sqrt{g\Delta h}$$

$$\Omega = \frac{1}{5 \times 10^{-2}} (9.8 \times 1.5)^{\frac{1}{2}} = 76.7 \ s^{-1}$$

2.

(a) Navier Stokes equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \underline{u}$$

$$(\underline{u}\cdot\underline{\nabla})\underline{u} = (\underline{\nabla}\times\underline{u})\times\underline{u} + \nabla\left(\frac{u^2}{2}\right)$$ (vector identity)

$$\rightarrow \quad \frac{\partial \underline{u}}{\partial t} + (\underline{\nabla}\times\underline{u})\times\underline{u} + \nabla\left(\frac{u^2}{2}\right) = -\frac{1}{\rho}\underline{\nabla}P + \nu\nabla^2\underline{u}$$

$$\rightarrow \quad \frac{\partial \underline{u}}{\partial t} + \nabla\left(\frac{u^2}{2} + \frac{P}{\rho}\right) = \underline{u}\times(\underline{\nabla}\times\underline{u}) + \nu\nabla^2\underline{u}$$

$$\because \quad H = \frac{u^2}{2} + \frac{P}{\rho}, \quad \underline{\omega} = \underline{\nabla}\times\underline{u}$$

$$\therefore \quad \frac{\partial \underline{u}}{\partial t} + \underline{\nabla}H = \underline{u}\times\underline{\omega} + \nu\nabla^2\underline{u}$$

If flow is steady and inviscid, then

$$\frac{\partial \underline{u}}{\partial t} = 0, \quad \nu = 0 \quad \Rightarrow \quad \underline{\nabla}H = \underline{u}\times(\underline{\nabla}\times\underline{u})$$

taking dot product with $\underline{u}$:

$$\underline{u}\cdot\underline{\nabla}H = \underline{u}\cdot\underline{u}\times(\underline{\nabla}\times\underline{u}) = \underline{u}\cdot\underline{u}\times\underline{\omega} = 0$$

$\because$ H is scalar $\therefore$ $\underline{u}\cdot\underline{\nabla}H = (\underline{u}\cdot\underline{\nabla})H$

$\Rightarrow \quad (\underline{u}\cdot\underline{\nabla})H = 0 \quad \because$ steady flow $\therefore \frac{\partial H}{\partial t} = 0$

$$\Rightarrow \quad \frac{\partial H}{\partial t} + (\underline{u}\cdot\underline{\nabla})H = 0 \quad \Rightarrow \quad \frac{DH}{Dt} = 0$$

$\Rightarrow$ H is constant along streamlines.

If flow is steady, inviscid, and irrotational,

then $\underline{\omega} = 0$, $\frac{\partial \underline{u}}{\partial t} = 0$, $\nu = 0$

$\rightarrow$ $\nabla H = 0$, also steady $\rightarrow$ $\frac{\partial H}{\partial t} = 0$

$\rightarrow$ H is constant everywhere

(b) $\frac{\partial \underline{u}}{\partial t} + \nabla H = \underline{u} \times \underline{\omega} + \nu \nabla^2 \underline{u}$ (1)

take curl of (1): use $\underline{\omega} = \nabla \times \underline{u}$

lines.

$$\frac{\partial \underline{\omega}}{\partial t} + \underbrace{\nabla \times \nabla H}_{0} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \underbrace{\nabla \times \nabla^2 \underline{u}}_{= \nabla^2 (\nabla \times \underline{u}) = \nabla^2 \underline{\omega}}$$

$$\rightarrow \frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$$

is the vorticity equation

$$\nabla \times (\underline{u} \times \underline{\omega}) = (\underline{\omega} \cdot \nabla) \underline{u} - (\underline{u} \cdot \nabla) \underline{\omega} + \underline{u} (\nabla \cdot \underline{\omega}) - \underline{\omega} (\nabla \cdot \underline{u})$$

$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) = 0$ $\nabla \cdot \underline{u} = 0$ due to incompressibility.

$$\rightarrow \frac{\partial \underline{\omega}}{\partial t} = (\underline{\omega} \cdot \nabla) \underline{u} - (\underline{u} \cdot \nabla) \underline{\omega} + \nu \nabla^2 \underline{\omega}$$

In 2D flow $\underline{u} = (u_x(x,y), u_y(x,y), 0)$

$$\underline{\omega} = \nabla \times \underline{u} = \underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & 0 \end{vmatrix}}_{\partial_z u_x = \partial_z u_y = 0} = \omega \hat{z} = (0, 0, \omega_z)$$

$\rightarrow \quad (\underline{\omega} \cdot \underline{\nabla})\underline{u} = \omega_z \partial_z \underline{u} = 0$

$\underline{u} = \underline{u}(x, y)$

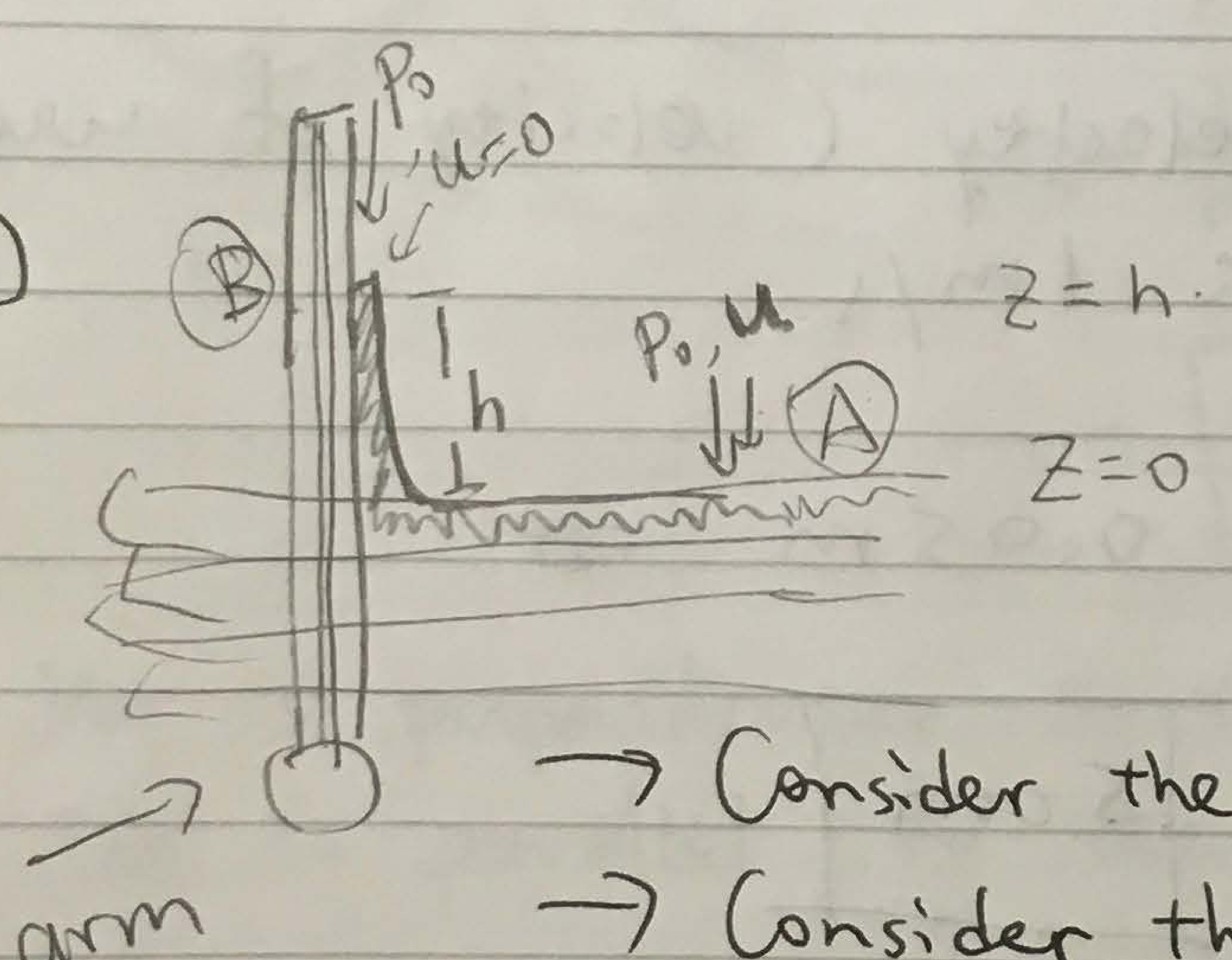
$\rightarrow \quad$ 2D :
$$\frac{\partial \omega}{\partial t} + (\underline{u} \cdot \underline{\nabla})\omega = \nu \nabla^2 \omega$$

For inviscid flow : $\nu = 0$

$$\Rightarrow \quad \frac{\partial \omega}{\partial t} + (\underline{u} \cdot \underline{\nabla})\omega = 0$$

$$\Rightarrow \quad \frac{D\omega}{Dt} = 0 \quad \Rightarrow \quad \omega \text{ is constant along streamlines.}$$

(c)



$\rightarrow$ Consider the reference frame of the punt

$\rightarrow$ Consider the surface streamline

At point A, pressure $= P_0 = 1\,atm$

flow velocity $= u$

height $z = 0$

At point B (water top on the arm) :

pressure $= P_0 = 1\,atm$

flow velocity $= 0$

height $z = h$

This case the gravity is the body force

So we modify H to be $\quad H = \frac{u^2}{2} + \frac{P}{\rho} + gz$

(So take ∇H and put into (1) can recover the N-S equation with the body force).

assume inviscid ~~flow~~, incompressible flow,

H is constant along surface stream line.

$$\therefore \quad \cancel{\frac{P_o}{\rho}} + gh = \frac{u^2}{2} + \cancel{\frac{P_o}{\rho}}$$

$$\Rightarrow \quad h = \frac{u^2}{2g}$$

assume flow velocity (velocity of water relative to the point) $\approx 1 m/s$

$$\rightarrow \quad h = \frac{(1)^2}{2\times 9.8} = 0.05m$$

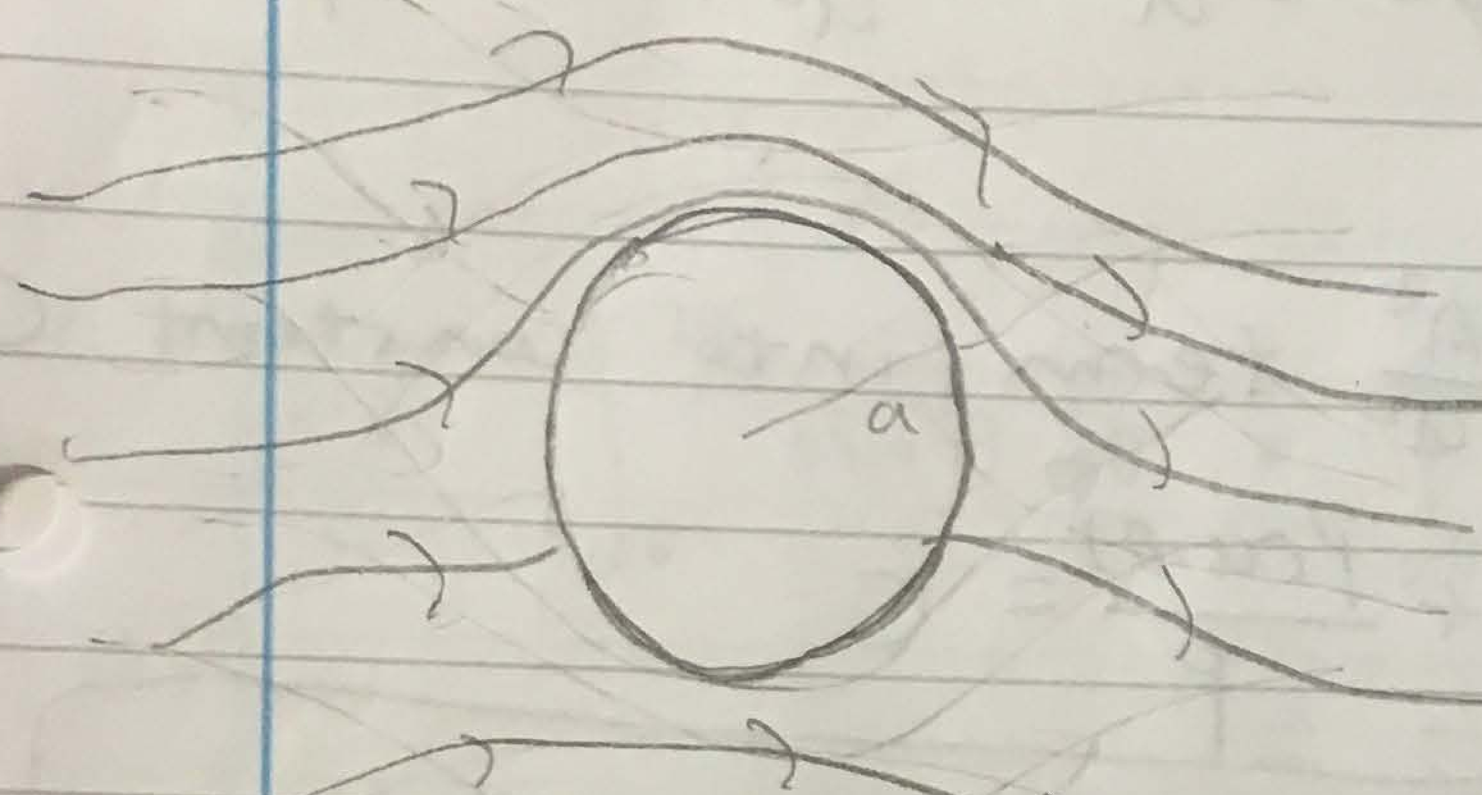
$$= \boxed{5cm}$$

3. (a)

The velocity potential $\varphi = U(r+\frac{a^2}{r})\cos\theta - A\theta$

$$\Rightarrow \quad \underline{u} = \nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\underline{\theta}}$$

$$= u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}$$



$$u_r = \frac{\partial\varphi}{\partial r} = \cancel{U\cos\theta} \quad \boxed{U\cos\theta\left(1-\frac{a^2}{r^2}\right)} \quad ①$$

$$u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \left[-U\left(r+\frac{a^2}{r}\right)\sin\theta - A\right]\frac{1}{r}$$

$$= \boxed{-U\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{A}{r}} \quad ②$$

u_r is perpendicular to the wall of cylinder
u_θ is parallel to the wall of cylinder.

(b) ~~—~~

Flow is irrotational, inviscid, and incompressible
so H is constant ~~along~~ everywhere.

$\Rightarrow$ on surface of ~~a~~ cylinder

$$\frac{u^2(a,\theta)}{2} + \frac{P(a,\theta)}{\rho} = \text{constant} = C$$

$u_r(a,\theta) = 0$ as a result of ①

$\therefore$ $\cancel{u^2 = u_\theta^2}$ $u^2(a,\theta) = u_\theta^2(a,\theta) = \left[-U(1+1)\sin\theta - \frac{A}{a}\right]^2$

$= (2U\sin\theta + \frac{A}{a})^2$

$\Rightarrow \frac{(2U\sin\theta+\frac{A}{a})^2}{2} + \frac{P(a,\theta)}{\rho} = \cancel{con}\ C$

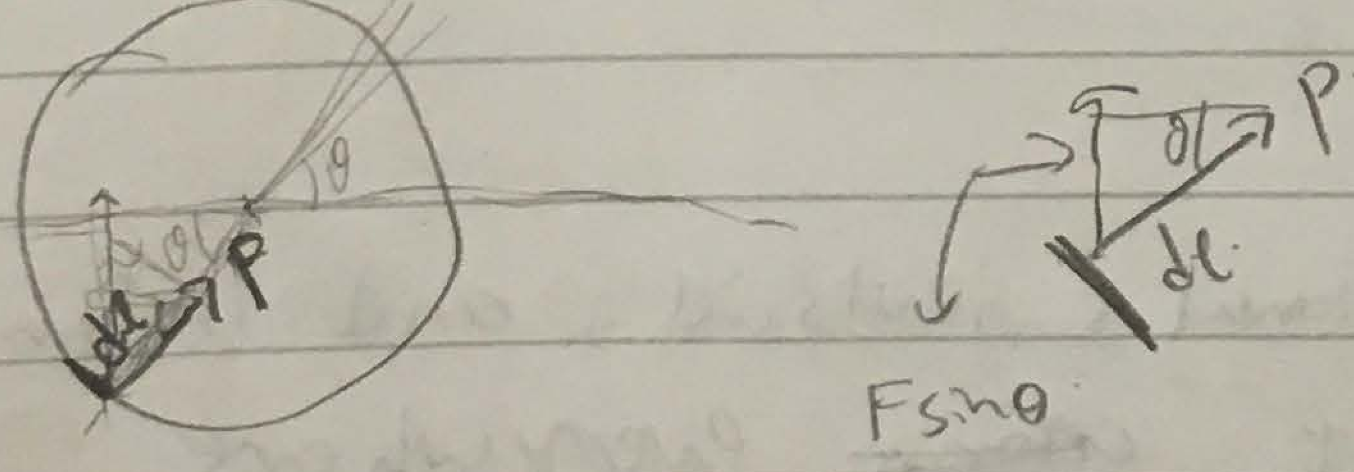
$\Rightarrow \frac{1}{2}(4U^2\sin^2\theta + 4U\sin\theta\frac{A}{a} + \frac{A^2}{a^2}) + \frac{P(a,\theta)}{\rho} = C$

We can absorb the $\frac{1}{2}\cdot\frac{A^2}{a^2}$ term into constant C

$\Rightarrow 2U^2\sin^2\theta + 2U\sin\theta\frac{A}{a} + \frac{P(a,\theta)}{\rho} = C$

$\Rightarrow \boxed{P(a,\theta) = const - 2\rho U^2\sin^2\theta - \frac{2\rho UA}{a}\sin\theta}$

the const $= \cancel{P_\infty}\ P(a,\theta=0)$



$F = p\,dl$

$\theta > \pi$ $\quad \sin\theta < 0 \quad -\sin\theta > 0 \quad p\uparrow$

$\theta < \pi$ $\quad \sin\theta > 0 \quad -\sin\theta < 0 \quad p\downarrow$

(c) The Lift per unit span L is given by

$L = -\oint_{r=a} P(a,\theta)\sin\theta\, dl$

$= -\int_0^{2\pi} const\cdot\sin\theta\cdot a\,d\theta + \int_0^{2\pi} 2\rho U^2 a\sin^3\theta\, d\theta$

$+ \int_0^{2\pi} 2\rho UA\sin^2\theta\, d\theta$

$$\int_0^{2\pi} \sin\theta \, d\theta = -\cos\theta \Big|_0^{2\pi} = 0$$

$$\int_0^{2\pi} \sin^3\theta \, d\theta = \int_{0 \,\to\, x=0}^{2\pi \,\to\, x=0} \underbrace{(1-\cos^2\theta)}_{1-x^2} \underbrace{\sin\theta \, d\theta}_{dx} = 0$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2\theta) \, d\theta$$

$$= \frac{1}{2} \times 2\pi = \pi$$

$$\Rightarrow \boxed{\mathcal{L} = 2\pi\rho U A}$$

The circulation

$$\Gamma_c = \oint_{r=a} \underline{u} \cdot \underline{dl} = \oint_{r=a} (\cancel{u_r \hat{\underline{r}}} + u_\theta \hat{\underline{\theta}}) \cdot a\, d\theta \, \hat{\underline{\theta}}$$

$$= \int_0^{2\pi} a u_\theta \Big|_{r=a} d\theta = \int_0^{2\pi} -2aU\sin\theta - A \, d\theta$$

$$= -\int_0^{2\pi} A \, d\theta = \boxed{-2\pi A}$$

$$\Rightarrow \boxed{\mathcal{L} = -\rho U \Gamma_c}$$

4. (a)

Suppose fluid motion is irrotational

$\rightarrow$ $\cancel{u = \frac{\partial\phi}{}}$ $\underline{u} = \nabla\phi$, $\underline{u} = (u, v)$

$\rightarrow$ $u = \frac{\partial\phi}{\partial x}$, $v = \frac{\partial\phi}{\partial y}$

Fluid particles on the surface must remain on the surface.

$\eta(x,t)$ is the ~~equation~~ height of free surface

$\rightarrow$ equation of free surface is $y = \eta(x, t)$

~~Co~~ Consider $F(x,y,t) = y - \eta(x,t) = 0$

It is constant for any fluid particle on the free surface, so

$\frac{DF}{Dt} = 0$ on free surface

$\Rightarrow$ $\frac{\partial F}{\partial t} + (\underline{u}\cdot\underline{\nabla})F = 0$ on $y = \eta(x,t)$

$\Rightarrow$ $\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$

$\because F = y - \eta(x,t)$ $\therefore$ $\frac{\partial F}{\partial t} = -\frac{\partial \eta}{\partial t}$

$\frac{\partial F}{\partial x} = -\frac{\partial \eta}{\partial x}$ $\frac{\partial F}{\partial y} = 1$

$$\Rightarrow \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v \quad \text{on} \quad y = \eta(x,t) \quad ①$$

For inviscid flow $\nu = 0$

$\Rightarrow$ Euler equation with body ~~forn~~ force is

$$\frac{D\underline{u}}{Dt} = -\frac{\nabla P}{\rho} - g\hat{\underline{y}}, \quad \frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla\left(\frac{P}{\rho} + gy\right)$$

~~The corresponding B~~

Use $(\underline{u}\cdot\nabla)\underline{u} = (\nabla\times\underline{u})\times\underline{u} + \nabla(\frac{1}{2}u^2)$

$\because \underline{u} = \nabla\phi \quad \therefore \nabla\times\underline{u} = 0$

$$\therefore \frac{\partial \underline{u}}{\partial t} = -\nabla\left(\frac{P}{\rho} + \frac{1}{2}\underline{u}^2 + gy\right)$$

$$\because \underline{u} = \nabla\phi \quad \therefore \nabla\left(\frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\underline{u}^2 + gy\right) = 0$$

$$\Rightarrow \frac{\partial\phi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}\underline{u}^2 + gy = G(t)$$

(G is an arbitrary function of t)

$G(t)$ can be chosen freely because

$\underline{u} = \nabla\phi$ so ~~the $\frac{\partial\phi}{\partial t}$ does not~~ adding a function of t to ϕ doesn't change $\underline{u}$

~~Choose G(t)=~~

Assume $P = P_0 = 1\,\text{atm}$ everywhere on $y=\eta(x,t)$

choose $G(t) = \frac{P_0}{\rho}$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \cancel{\frac{P_0}{\rho}} + \frac{1}{2}\underline{u}^2 + g\eta = \cancel{\frac{P_0}{\rho}}$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta = 0 \quad \text{on } y=\eta(x,t) \quad ②$$

Now we make small amplitude approximations:
assume $(\eta-h)$, u, v are small, we neglect all quadratic terms of η, u, and v.

so ① $\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v$

quadratic

$$\Rightarrow \quad \frac{\partial \eta}{\partial t} = v = \cancel{\frac{\partial \phi}{\partial y}} \quad \cancel{\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}}$$

at $y=\eta(x,y)$

$$\therefore \quad \frac{\partial \eta}{\partial t} = v(x,\eta,t) = v(x,h,t) + (\eta-h)\frac{\partial v}{\partial y}$$

quadratic.

$$= v(x,h,t)$$

~~$\frac{\partial \phi}{\partial y}(x,h,t)=$~~

$$\Rightarrow \quad \frac{\partial \eta}{\partial t} = v \quad \text{at } y=h$$

$$\Rightarrow \quad \boxed{\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} \quad \text{at } y=h}$$

~~rota~~ $\Rightarrow$ rate of change of ~~heig~~ height of wave is the y-component of velocity

② $\Rightarrow$ $$\frac{\partial \phi}{\partial t} + \underbrace{\frac{1}{2}(u^2+v^2)}_{\text{quadratic}} + g\eta = 0 \quad \text{at } y=\eta(x,t)$$

$$\Rightarrow \frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } y=\eta(x,t)$$

$$\cancel{\eta(x,t) = \eta(h,t) + \frac{\partial \eta}{\partial x}}$$

$$\frac{\partial \phi}{\partial t}(x,\eta,t) = \frac{\partial \phi}{\partial t}(x,h,t) + \frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial y}(\eta-h)\right]$$

$$= \frac{\partial \phi}{\partial t}(x,h,t) + \frac{\partial}{\partial t}\underbrace{((\eta-h))(V)}_{\text{quadratic}}$$

$$\simeq \frac{\partial \phi}{\partial t}(x,h,t)$$

$$\Rightarrow \boxed{\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } y=h}$$

→ (this correspond to the equal pressure at free surface)

→ At the bottom of the channel ($y=0$)
No slip boundary condition applies for only the y-direction (not the x-direction because the viscosity is 0)

$$\Rightarrow V = 0 \quad \text{at } y=0$$

$$\Leftrightarrow \boxed{\frac{\partial \phi}{\partial y} = 0 \quad \text{at } y=0}$$

(b) the ~~im~~ incompressibility condition

$$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (\nabla^2 \phi = 0)$$

try $\begin{cases} \eta(x,t) = A\cos(kx - \omega t) \\ \phi(x,y,t) = f(y)\sin(kx - \omega t) \end{cases}$ (wave solutions)

$$-k^2 f \sin(kx-\omega t) + \frac{d^2 f}{dy^2}\sin(kx-\omega t) = 0$$

$$\Rightarrow \frac{d^2 f}{dy^2} - k^2 f = 0$$

$$\Rightarrow f(y) = Ce^{ky} + De^{-ky}$$

$\because \frac{\partial \phi}{\partial y} = 0$ at $y=0$ $\therefore \frac{df}{dy} = 0$ at $y=0$

$$\Rightarrow ke^{ky}(C-D) = 0 \Rightarrow C = D$$

$$\Rightarrow f(y) = C(e^{ky} + e^{-ky}) = \underbrace{C\cosh(ky)}_{\text{absorb the factor 2}}$$

$\therefore \phi = C\cosh(ky)\sin(kx - \omega t)$

$\eta = A\cos(kx - \omega t)$.

at $y=h$, $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$

$$\Rightarrow \omega A \sin(\cancel{kh} - \omega t) = kC\sinh(kh)\sin(\cancel{kh} - \omega t) \quad ③$$

$$\frac{\partial \phi}{\partial t} = -g\eta$$

$$\Rightarrow \quad -\omega C \cosh(kh) \cos(\cancel{kx} - \omega t)$$

$$= -gA \cos(\cancel{kx} - \omega t) \quad (4)$$

$$(3) \Rightarrow \quad \frac{\omega}{k} = \frac{C}{A} \sinh(kh)$$

$$(4) \Rightarrow \quad \frac{C}{A} = \frac{g}{\omega \cosh(kh)}$$

$$\Rightarrow \quad \frac{\omega}{k} = \frac{g \sinh(kh)}{\omega \cosh(kh)} \quad \Rightarrow \quad \boxed{\omega^2 = gk \tanh(kh)} \quad (1)$$

(c) we treat $x = kh$ as a small quantity $(kh \ll 1)$ and expand $\tanh(kh) = \tanh(x)$

$$\omega^2 = gk \tanh(kh)$$

$$\rightarrow \quad \omega = \sqrt{gk}\,(\tanh(kh))^{\frac{1}{2}}$$

Maclaurin series for ~~tanh(kh)~~ $(\tanh(x))$ at $x = kh$

~~tanh~~

Maclaurin series

$$f(x) = f(0) + \frac{1}{1!} f'(0) x + \frac{1}{2!} f''(0) x^2 + \cdots$$

$$\tanh(0) = 0$$

$$\cancel{\frac{d}{dx}\tanh^{\frac{1}{2}}(x) = \frac{1}{2}\tanh^{-\frac{1}{2}}(x)}.$$

$$\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x) = \frac{1}{\cosh^2(x)}$$

$$\Rightarrow \frac{d}{dx}\tanh(x)\Big|_{x=0} = \frac{1}{\frac{1+1}{2}} = 1$$

$$\frac{d^2}{dx^2}\tanh(x)\Big|_{x=0} = -2\operatorname{sech}^2(x)\tanh(x)\Big|_{x=0} = 0$$

$$\frac{d^3}{dx^3}\tanh(x)\Big|_{x=0} = 4\operatorname{sech}^2(x)\tanh^2(x)\Big|_{x=0}$$

$$-2\operatorname{sech}^4(x)\Big|_{x=0}$$

$$= 0 - 2 = -2$$

$$\Rightarrow \cancel{\tanh(x)}\ \tanh(x) \simeq \frac{1}{1!}x - \frac{2}{3!}x^3$$

$$= x - \frac{x^3}{3}$$

$$\therefore (\tanh(x))^{\frac{1}{2}} \simeq \left(x - \frac{x^3}{3}\right)^{\frac{1}{2}}$$

$$= \sqrt{x}\cdot\sqrt{1 - \frac{x^2}{3}}$$

$$\simeq \sqrt{x}\left(1 - \frac{x^2}{6}\right) \sim x^{\frac{1}{2}} - \frac{x^{5/2}}{6}$$

$$\cancel{\rightarrow \sqrt{g}\sqrt{\tanh(kh)} \simeq \sqrt{kh}\left(\sqrt{kh} - \frac{(kh)^{5/2}}{6}\right)}$$

$$\cancel{= kh - \frac{k^3h^3}{6}}$$

$$\cancel{\Rightarrow \omega = \sqrt{\;}}$$

$$\omega = \sqrt{gk}\sqrt{\tanh(kh)} \simeq \sqrt{g}\cdot\sqrt{k}\left(\sqrt{kh} - \frac{(kh)^{5/2}}{6}\right),$$ or

$$= \underbrace{\sqrt{gh}}_{c_0}\left(k - \frac{k^3h^2}{6}\right)$$

$$= \boxed{c_0\left(k - \frac{k^3h^2}{6}\right)}$$

(d) The Korteweg-de Vries equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \left(\frac{3c_0}{2h}\right)\eta\frac{\partial \eta}{\partial x} + \left(\frac{c_0 h^2}{6}\right)\frac{\partial^3 \eta}{\partial x^3} = 0$$

→ The $\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x}$ is the non-dispersive, linear, traveling wave term that originates from the wave propagating at constant speed. It corresponds to the "k" term in the dispersion relation $\omega = c_0 \left(k - \frac{k^3 h^2}{6}\right)$. It ensures that the ~~shape of~~ wave ~~is unchanged as it~~ propagates. as time passes.

→ The $\frac{3c_0}{2h}\eta\frac{\partial \eta}{\partial x}$ term is the non-linear term

Its origin is the small but non-negligible amplitude of wave compare to the depth of water i.e. $0 \neq \frac{\eta}{h} \ll 1$. It causes the shape of wave to be non-periodic and changing. ~~It doesn't contribute to~~ (if without dispersion)

→ The $\frac{c_0 h^2}{6}\frac{\partial^3 \eta}{\partial x^3}$ term is the dispersive term.

Its origin is the dispersion of wave, i.e. the "k^3" term in $\omega = c_0 \left(k^2 - \frac{k^3 h^2}{6}\right)$. For this term to exist we need $\frac{h^2}{L^2}$, where L is the horizontal scale ($\frac{\partial}{\partial x} \sim L$), is small but non-negligible. It causes the ~~dispersion of wave~~ weak dispersion that balances the non-linear term

The balance between the dispersive term and the non-linear term makes the shape of the wave permanant

For non-linear and dispersive effects to balance, we need

$$0 < \frac{\eta}{h} \sim \frac{h^2}{L^2} \ll 1$$

→ If $\frac{\eta}{h} \ll \frac{h^2}{L^2}$, wave is linear and dispersive
(small amplitude linear approximation)

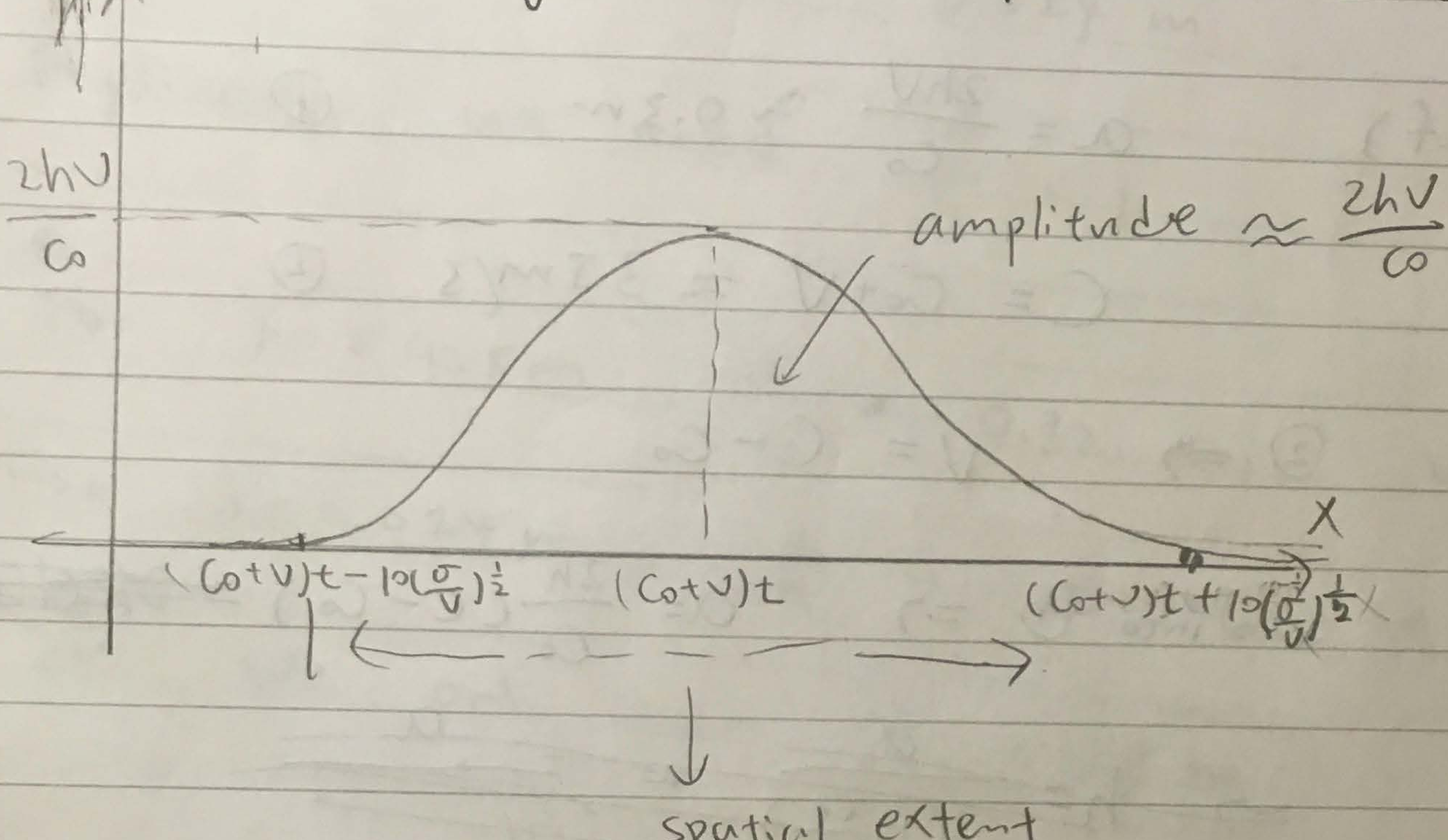
If $\frac{\eta}{h} \gg \frac{h^2}{L^2}$, wave is ~~not~~ nonlinear with changing shape. (shallow water approximation)

Only if $\frac{\eta}{h} \sim \frac{h^2}{L^2}$ we can ~~write~~ get soliton solutions from the Korteweg-de Vries equation.

(e) A possible solution is

$$\eta(x,t) = \frac{2hV}{C_0}\,\mathrm{sech}^2\left[\frac{1}{2}\left(\frac{V}{\sigma}\right)^{\frac{1}{2}}\{x-(C_0+V)t\}\right]$$

where $\sigma = \frac{C_0h^2}{6}$, V is a parameter, $C_0 = \sqrt{gh}$



→ Amplitude $= \frac{2hV}{C_0}$

$\mathrm{sech}^2(x)$ goes to 0 as $\Delta x \to \pm 5$ $(\Delta x = x-(C_0+V)t)$

$\therefore$ spatial extent $\simeq 2\times\frac{5}{\frac{1}{2}\left(\frac{V}{\sigma}\right)^{\frac{1}{2}}} = 20\left(\frac{\sigma}{V}\right)^{\frac{1}{2}}$

wave speed $= C_0+V$

→ Different from the linear wave that satisfies (1), This soliton wave has only one pulse rather than the oscillatory sinusoidal shape. Also, the speed of soliton wave depends on its amplitude, rather than the wavelength, as the linear wave does.

→ Also, if ~~we~~ one soliton is superposed on another soliton, the result is not a solution of the Korteweg-de Vries equation because the equation is non-linear.

(f) $$a = \frac{2hV}{C_0} \approx 0.3\,m \quad (1)$$

$$C = C_0 + V \approx 3.5\,m/s \quad (2)$$

(2) ⇒ $$V = C - C_0$$

sub into (1) ⇒ $$a = \frac{2h}{C_0}(C - C_0) = \cancel{2h\left(\frac{C_0}{C}\right)}$$

$$\cancel{\Rightarrow h = \frac{a}{2\left(1-\frac{C_0}{C}\right)} = \frac{a}{2C}}$$

$\because C_0 = \sqrt{gh}$

$$\therefore \quad a = 2h\left(\frac{C}{C_0} - 1\right) = 2h\left(\frac{C}{\sqrt{gh}} - 1\right)$$

$$\Rightarrow \left(\frac{2C}{\sqrt{g}}\right)(\sqrt{h}) - 2h - a = 0$$

let $J = \sqrt{h}$, then

$$2J^2 - \left(\frac{2C}{\sqrt{g}}\right)J + a = 0 \qquad (J > 0)$$

$$\Rightarrow \quad J = \frac{\frac{2C}{\sqrt{g}} \pm \sqrt{\left(\frac{2C}{\sqrt{g}}\right)^2 - 8a}}{4}$$

$\because$ $g = 9.8$, $c = 3.5$, $a = 0.3$

$\therefore$ $J = 0.962\ \sqrt{m}$ or $0.156\ \sqrt{m}$

$\Rightarrow$ $h = 0.925\ m$ or $0.024\ m$

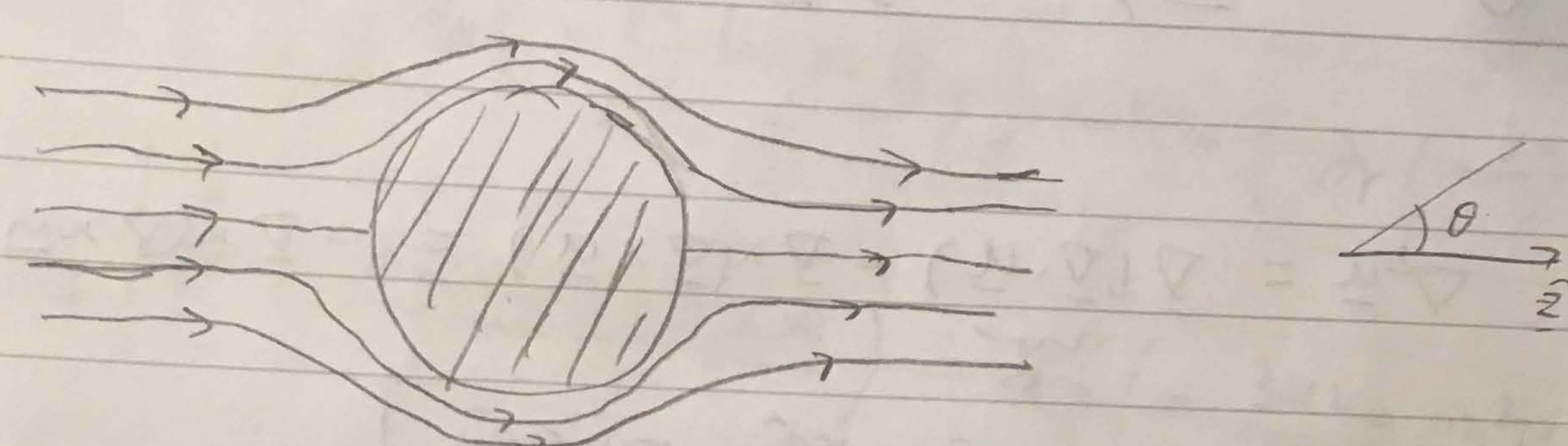
Physically we need $\frac{\eta}{h} < 1$

For $h = 0.925\ m$, $\frac{a}{h} \sim 0.32 < 1$ ✓

For $h = 0.024\ m$, $\frac{a}{h} \sim 12.5 > 1$ ✗

$\Rightarrow$ we only have $\boxed{h = 0.925\ m}$

5. (a)



(b) (i) Stoke's equations

$$\underline{\nabla} \cdot \underline{u} = 0 \quad , \quad \underline{\nabla} P = \eta \nabla^2 \underline{u}$$

Given $\quad u_r = u_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)$

$$u_\theta = -u_0 \sin\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)$$

$$P = P_0 - \frac{3\eta u_0 a}{2r^2} \cos\theta \quad , \quad u_\phi = 0 \quad \left(\frac{\partial}{\partial\phi} = 0\right)$$

$\longrightarrow \underline{\nabla} \cdot \underline{u} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} (u_\theta \sin\theta)$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} \left[u_0 \cos\theta \left(r^2 - \frac{3ar}{2} + \frac{a^3}{2r} \right) \right]$$

$$+ \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta} \left[-u_0 \sin^2\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) \right]$$

$$= \frac{1}{r^2} u_0 \cos\theta \left(2r - \frac{3a}{2} - \frac{a^3}{2r^2} \right)$$

$$+ \frac{1}{r\sin\theta} \left(-2u_0 \sin\theta \cos\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) \right)$$

$$= U_0 \cos\theta \left(\cancel{\frac{2}{r}} - \cancel{\frac{3a}{2r^2}} - \cancel{\frac{a^3}{2r^4}} - \cancel{\frac{2}{r}} + \cancel{\frac{3a}{2r^2}} + \cancel{\frac{a^3}{2r^4}} \right)$$

$$= 0 \quad \Rightarrow \quad \underline{\nabla} \cdot \underline{u} = 0$$ ✓ <u>verified</u>

$$\rightarrow \quad \nabla^2 \underline{u} = \underline{\nabla}(\underbrace{\underline{\nabla} \cdot \underline{u}}_{0}) - \underline{\nabla} \times (\underline{\nabla} \times \underline{u}) = -\underline{\nabla} \times (\underline{\nabla} \times \underline{u})$$

$$\underline{\nabla} \times \underline{u} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta\, \hat{e_\phi} \\ \partial_r & \partial_\theta & \partial_\phi \\ U_r & rU_\theta & 0 \end{vmatrix}$$

$$= \left(\hat{e_r}(0) - r\hat{e_\theta}(0) + r\sin\theta\, \hat{e_\phi} \left(\partial_r (rU_\theta) - \partial_\theta (U_r) \right) \right) \times \frac{1}{r^2 \sin\theta}$$

$$\partial_r (rU_\theta) = -U_0 \sin\theta\, \partial_r \left(r - \frac{3a}{4} - \frac{a^3}{4r^2} \right)$$

$$= -U_0 \sin\theta \left(1 + \frac{a^3}{2r^3} \right)$$

$$\partial_\theta (U_r) =$$

$$= U_0 \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3} \right) \partial_\theta \cos\theta$$

$$= -U_0 \sin\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3} \right)$$

$$\Rightarrow \quad \underline{\nabla} \times \underline{u} = r\sin\theta\, \hat{e_\phi} [U_0 \sin\theta] \left[-\frac{3a}{2r} \right] \cdot \frac{1}{r^2 \sin\theta}$$

$$= -\frac{3a}{2} U_0 \frac{\sin\theta}{r^2} \hat{e_\phi}$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{u}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{e_r}} & r\hat{\underline{e_\theta}} & r\sin\theta\, \hat{\underline{e_\phi}} \\ \partial r & \partial\theta & \partial\phi \\ 0 & 0 & -\frac{3a}{2} u_0 \frac{\sin\theta}{r^2} \times r\sin\theta \end{vmatrix}.$$

$$= \frac{1}{r^2 \sin\theta} \left[\hat{\underline{e_r}} \left(\partial_\theta \left(-\frac{3a}{2} u_0 \frac{\sin^2\theta}{r} \right) \right) - r\hat{\underline{e_\theta}} \left(\partial r \left(-\frac{3a}{2} u_0 \frac{\sin^2\theta}{r} \right) \right) \right]$$

$$= \frac{-1}{r^2 \sin\theta} \left[\hat{\underline{e_r}} \left(\frac{3au_0 \sin\theta\cos\theta}{r} \right) - \hat{\underline{e_\theta}} \left(- \frac{3au_0 \sin^2\theta}{2r} \right) \right]$$

$$= - \left[\hat{\underline{e_r}} \left(\frac{3u_0 a\cos\theta}{r^3} \right) + \hat{\underline{e_\theta}} \left(\frac{3u_0 a\sin\theta}{2r^3} \right) \right]$$

$$\Rightarrow \eta \nabla^2 \underline{u} = - \underline{\nabla} \times (\underline{\nabla} \times \underline{u})\, \eta$$

$$= \underline{\frac{3u_0 \eta a\cos\theta}{r^3} \hat{\underline{e_r}} + \frac{3u_0 a\eta\sin\theta}{2r^3} \hat{\underline{e_\theta}}}$$

$$\because\ p = p_0 - \frac{3\eta u_0 a\cos\theta}{2r^2}$$

$$\therefore\ \underline{\nabla} p = \frac{\partial p}{\partial r} \hat{\underline{e_r}} + \frac{1}{r}\frac{\partial p}{\partial\theta} \hat{\underline{e_\theta}}$$

$$= \underline{\frac{3\eta u_0 a\cos\theta}{r^3} \hat{\underline{e_r}} + \frac{3\eta u_0 a\sin\theta}{2r^3} \hat{\underline{e_\theta}}}$$

$\Rightarrow \quad \underline{\nabla} p = \eta \nabla^2 \underline{u}$ <u>verified</u>

(ii) The boundary conditions.

→ No slip boundary conditions:

$u_r = 0$, $u_\theta = 0$ at surface of sphere $r = a$

Sub $r=a$ into expressions for u_r, u_θ gives

$$u_r(a,\theta) = U_0 \cos\theta \left(1 - \frac{3a}{2a} + \frac{a^3}{2a^3}\right) = 0 \quad \text{verified}$$

$$u_\theta(a,\theta) = -U_0 \sin\theta \left(1 - \frac{3a}{4a} - \frac{a^3}{4a^3}\right) = 0 \quad \text{verified}$$

→ Boundary condition at infinity $r \to \infty$:

The flow should become uniform as $r \to \infty$

i.e. $\underline{u} = \text{const} \times \underline{\hat{e_z}}$

As $r \to \infty$, $u_r \to U_0 \cos\theta$

$u_\theta \to -U_0 \sin\theta$

$\therefore \underline{u} \to U_0(\cos\theta \, \underline{\hat{e_r}} - \sin\theta \, \underline{\hat{e_\theta}})$

$= U_0 \underline{\hat{e_z}}$ verified

(c) $\sigma_{rr} = -p + 2\eta \frac{\partial u_r}{\partial r}$

$$= -p_o + \frac{3\eta u_o a}{2r^2}\cos\theta + 2\eta u_o \cos\theta \left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right)$$

$$\rightarrow \sigma_{rr}\Big|_{r=a} = -p_o + \frac{3\eta u_o a}{2a^2}\cos\theta + \cancel{\frac{3\eta u_o a}{2a^2}\cos\theta - \frac{3\eta u_o a^3}{2a^4}\cos\theta}$$

$$= \boxed{-p_o + \frac{3\eta u_o}{2a}\cos\theta}$$

$$\sigma_{\theta\theta} = -p_o + 2\eta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)$$

$$= -p_o + \frac{3\eta u_o a}{2r^2}\cos\theta + 2\eta\left(\frac{1}{r}(-u_o\cos\theta)\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right) + \frac{1}{r}u_o\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\right)$$

$$\rightarrow \sigma_{\theta\theta}\Big|_{r=a} = -p_o + \frac{3\eta u_o}{2a}\cos\theta + 2\eta(0+0)$$

$$= \boxed{-p_o + \frac{3\eta u_o}{2a}\cos\theta}$$

$$\sigma_{\phi\phi} = -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right)$$

$$= -p_o + \frac{3\eta u_o a}{2r^2}\cos\theta + \left(0 + \frac{1}{r}\left(u_o\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) - u_o\sin\theta\frac{\cos\theta}{\sin\theta}\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\right)\right)2\eta$$

$$\rightarrow \sigma_{\phi\phi}\Big|_{r=a} = -p_o + \frac{3\eta u_o}{2a}\cos\theta + 2\eta\left(\frac{1}{r}(0+0)\right)$$

$$= \boxed{-p_o + \frac{3\eta u_o}{2a}\cos\theta}$$

$$\sigma_{r\theta} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$= \eta\left(r\frac{\partial}{\partial r}\left[-u_0\sin\theta\left(\frac{1}{r} - \frac{3a}{4r^2} - \frac{a^3}{4r^4}\right)\right]\right.$$

$$\left. + \frac{1}{r}\frac{\partial}{\partial\theta}\left[u_0\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\right]\right)$$

$$= \eta\left(r(-u_0\sin\theta)\left(-\frac{1}{r^2} + \frac{3a}{2r^3} + \frac{a^3}{r^5}\right)\right.$$

$$\left. - \frac{1}{r}u_0\sin\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\right)$$

$$\Rightarrow \sigma_{r\theta}\Big|_{r=a} = \sigma_{\theta r}\Big|_{r=a} = -\eta a u_0 \sin\theta\left(\frac{1}{a} + \frac{1}{2a}\right)$$

$$= \boxed{-\frac{3}{2}\frac{\eta u_0}{a}\sin\theta}$$

$$\rightarrow \sigma_{r\phi} = \eta\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right)\right) = \boxed{0}$$

$$= \sigma_{\phi r}$$

$$\rightarrow \sigma_{\theta\phi} = \sigma_{\phi\theta} = \eta\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right)$$

$$= \boxed{0}$$

~~Hence the viscous stress tensor :~~

~~$\underline{\underline{\sigma}} =$~~

Hence the viscous stress tensor :

$$\underline{\underline{\sigma}} = \begin{pmatrix} -P_0 + \frac{3\eta u_0}{2a}\cos\theta & -\frac{3\eta u_0}{2a}\sin\theta & 0 \\ -\frac{3\eta u_0}{2a}\sin\theta & -P_0 + \frac{3\eta u_0}{2a}\cos\theta & 0 \\ 0 & 0 & -P_0 + \frac{3\eta u_0}{2a}\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix}$$

The stress on sphere is

$\underline{t} = \underline{\underline{\sigma}}\,\underline{\hat{n}}$ ($\underline{\hat{n}}$ is the unit normal of sphere)

$\rightarrow$ $t_i = \sigma_{ij} n_j$, for a sphere , $\underline{\hat{n}} = 1\underline{\hat{e_r}} + 0\underline{\hat{e_\theta}} + 0\underline{\hat{e_\phi}}$

$= \underline{\hat{e_r}}$

$\therefore n_r = 1$, $n_\theta = 0$, $n_\phi = 0$

$$\Rightarrow \quad t_r = \sigma_{rr} n_r + \sigma_{r\theta} n_\theta + \sigma_{r\phi} n_\phi$$
$$= \sigma_{rr}$$
$$= \underline{-P_0 + \frac{3\eta u_0}{2a}\cos\theta}$$

$$\Rightarrow \quad t_\theta = \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta + \sigma_{\theta\phi} n_\phi$$
$$= \sigma_{\theta r}$$
$$= \underline{-\frac{3\eta u_0}{2a}\sin\theta}$$

$\cancel{t}$ stress $\underline{t} = t_r \hat{\underline{e_r}} + t_\theta \hat{\underline{e_\theta}}$ on the sphere

(d) the drag D acting on the sphere is, by symmetry, along $\hat{\underline{e_z}}$

So the ~~stress~~ stress along z-direction is

$$t_z = t_r \hat{\underline{e_r}} \cdot \hat{\underline{e_z}} + t_\theta \hat{\underline{e_\theta}} \cdot \hat{\underline{e_z}}$$

$$= t_r (\sin\theta\cos\phi \hat{\underline{e_x}} + \sin\theta\sin\phi \hat{\underline{e_y}} + \cos\theta \hat{\underline{e_z}}) \cdot \hat{\underline{e_z}}$$

$$t_\theta (\cos\theta\cos\phi \hat{\underline{e_x}} + \cos\theta\sin\phi \hat{\underline{e_y}} - \sin\theta \hat{\underline{e_z}}) \cdot \hat{\underline{e_z}}$$

$$= t_r \cos\theta - t_\theta \sin\theta$$

$$= \cancel{P_0} - P_0\cos\theta + \frac{3}{2}\frac{\eta u_0}{a}\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

$$= -P_0\cos\theta + \frac{3}{2}\frac{\eta u_0}{a}$$

$$\cancel{D = \int_0^{2\pi}\int_0^{\pi}}$$

$$D = \int t_z \, dS = \int_0^{2\pi}\int_0^{\pi} -P_0\cos\theta + \frac{3}{2}\frac{\eta u_0}{a} \, a^2 \sin\theta \, d\theta \, d\phi$$

$$\Rightarrow \int_0^{2\pi} d\phi = 2\pi$$

$$\rightarrow \int_0^{\pi} \cos\theta\sin\theta \, d\theta$$

$$= \int_{\sin\theta = 0}^{\sin\theta = 0} \sin\theta \, d(\sin\theta) = 0$$

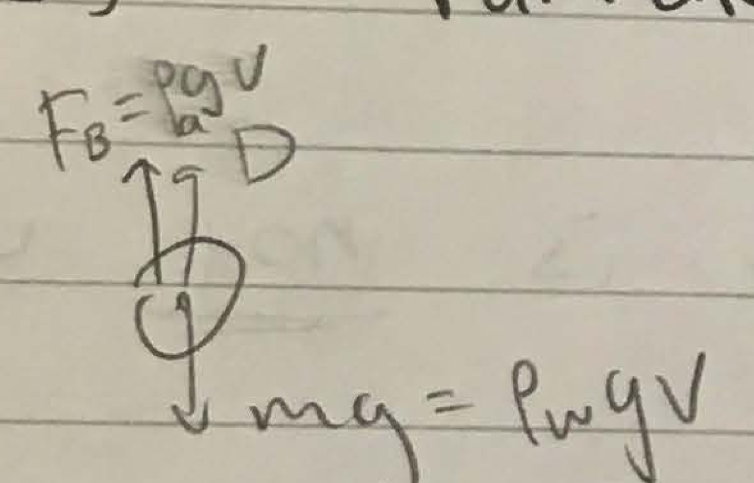
$\rightarrow \quad \therefore \cancel{D = \int_0}$

$$D = 2\pi \int_0^{\pi} \frac{3\eta u_0}{2\cancel{a}} \cdot a^{\cancel{2}} \sin\theta \, d\theta$$

$$= 2\pi \cdot \frac{3\eta u_0}{2} a \cdot \Big(\underbrace{-\cos\theta \Big|_0^{\pi}}_{2} \Big)$$

$$= \boxed{6\pi\eta a u_0}$$

(e) rain drop radius $= a$



density of water $= \rho_w$

density of air $= \rho_a$

$\Rightarrow$ Balance of forces:
(mg, F_B - bouyancy, D).

$\cancel{\rho_a g V = \rho}$

$$mg = F_B + D$$

$$\rightarrow \rho_w g V = \rho_a g V + D.$$

$$\rightarrow \rho_w g \frac{4}{3}\cancel{\pi} a^{\cancel{3}\,2} = \rho_a g \frac{4}{3}\cancel{\pi} a^{\cancel{3}\,2} + 6\cancel{\pi}\eta \cancel{a} u_T \quad (\text{at } u_0 = u_T),$$

$$6\cancel{\pi}\eta u_T = \frac{4}{3}(\rho_w - \rho_a) g a^2$$

$$\Rightarrow u_T = \frac{2}{9\eta_a}(\rho_w - \rho_a) g a^2$$

η_a = dynamic viscosity of air =

$$= \cancel{1.983\times10^{-5}\ Pa\,s}\quad \overbrace{1.5\times10^{-5}\,m^2/s}^{\nu} \times \cancel{1.29\,kg/m^3}\ 1.23\,kg/m^3$$

$$= 1.85\times10^{-5}\ kg\,m^{-1}s^{-1}$$

$\rho_w = 1\times10^3\ kg/m^3$ $\qquad a = 1\times10^{-3}\,m$

$\rho_a = 1.29\ kg/m^3$ $\qquad g = 9.8\,m/s^2$

$\Rightarrow$ $\cancel{U_T = 117.5\,m/s}$

$$\boxed{U_T = 117.6\,m/s}$$

The use of Stoke's Law is not valid

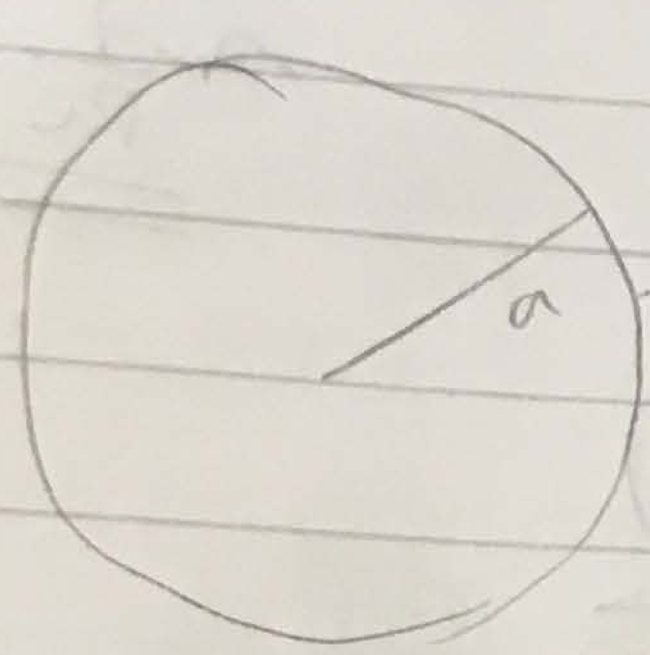
here because the Reynolds number

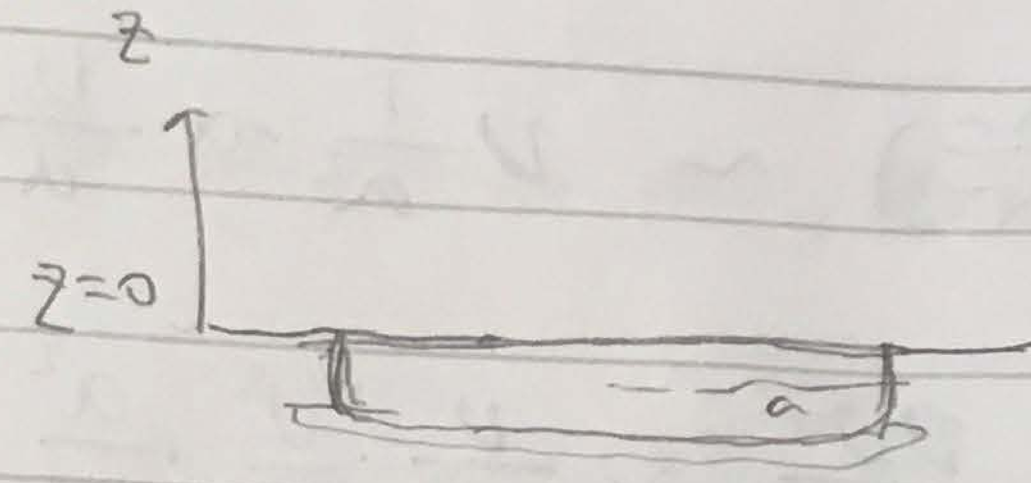
$$Re = \frac{LU}{\nu} \sim \frac{1\times10^{-3}\,m\times 10\,m/s}{1\times10^{-5}\,m^2/s}$$

~ 1000 is not low at all

$\therefore$ we cannot use Stoke's Law which needs $Re \ll 1$

6. (a)





Navier Stokes ~~in~~ equation ~~(steady flow)~~

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla}P + \nu\nabla^2\underline{u} \quad , \quad \underline{\nabla}\cdot\underline{u} = 0$$

System is symmetrical in θ $\Rightarrow$ $u_\theta = 0$, $\frac{\partial}{\partial\theta} = 0$

$\Rightarrow$ N-S equation in $\hat{e_r}$:

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \nu\frac{\partial^2 u_r}{\partial z^2} - \nu\frac{u_r}{r^2}$$

the radial length scale $\sim a$

velocity scale $u_r \sim U$.

(time scale $t \sim \frac{U}{a}$)

$\rightarrow$ $u_r\frac{\partial u_r}{\partial r} \sim \frac{U^2}{a}$ $\Rightarrow$ $\frac{\partial u_r}{\partial t} \sim \frac{U^2}{a}$

vertical length scale $\sim h$

$\rightarrow$ $\frac{\partial u_r}{\partial z} \sim \frac{U}{h}$

$\because \underline{\nabla}\cdot\underline{u} = 0$ $\therefore$ $\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$

$\Rightarrow$ $\frac{1}{a^2}\cdot a\cdot U \sim \frac{U_z}{\cancel{a}h}$

$\Rightarrow$ $u_z \sim \frac{hU}{a}$

$\rightarrow$ $u_z\frac{\partial u_r}{\partial z} \sim \frac{U}{h}\cdot\frac{hU}{a} \sim \frac{U^2}{a}$

$\rightarrow \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) \sim \nu \frac{1}{a^2}\cdot a\cdot\frac{U}{a} \sim \frac{\nu U}{a^2} \sim \frac{\nu}{Ua}\cdot\frac{U^2}{a}$

$\sim \underline{\frac{1}{Re}\frac{U^2}{a}}$

$\rightarrow \nu\frac{\partial^2 u_r}{\partial z^2} \sim \frac{\nu U}{h^2} \sim \frac{\nu}{Ua}\cdot\frac{U^2}{a}\cdot\frac{a^2}{h^2}$

$\sim \underline{\frac{1}{Re}\frac{U^2}{a}\left(\frac{a^2}{h^2}\right)}$

$\nu\frac{u_r}{r^2} \sim \frac{\nu U}{a^2} \sim \frac{\nu}{Ua}\frac{U^2}{a} \sim \underline{\frac{1}{Re}\frac{U^2}{a}}$

ie For low Re, and $h \ll a$, we have

$$\frac{1}{Re}\frac{U^2}{a}\left(\frac{a^2}{h^2}\right) \gg \frac{1}{Re}\frac{U^2}{a} \gg \frac{U^2}{a}$$

∴ The only non-pressure term that remains is

$\nu\frac{\partial^2 u_r}{\partial z^2}$

$\therefore\quad -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\frac{\partial^2 u_r}{\partial z^2} = 0 \qquad \because \eta = \rho\nu$

$\therefore\quad \underline{\frac{\partial P}{\partial r} = \eta\frac{\partial^2 u_r}{\partial z^2}}$

(b) The N-S equation in $\hat{e_z}$ is

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right)$$

$$\frac{\partial u_z}{\partial t} \sim \frac{h}{a}\frac{U}{t} \sim \frac{h}{a}\cdot\frac{U^2}{a}$$

$$u_r \frac{\partial u_z}{\partial r} \sim \frac{h}{a}\frac{U^2}{a}$$

$$u_z \frac{\partial u_z}{\partial z} \sim \frac{Uh}{ah}\cdot U\frac{h}{a} \sim \frac{h}{a}\frac{U^2}{a}$$

$$\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) \sim \nu\frac{h}{a}\frac{U}{a^2} \sim \frac{1}{Re}\frac{U^2}{a}\frac{h}{a}$$

$$\nu\frac{\partial^2 u_z}{\partial z^2} \sim \frac{1}{Re}\frac{U}{a}\left(\frac{a^2}{h^2}\right)\left(\frac{h}{a}\right)$$

Basically every term is scaled down by a factor of $\frac{h}{a} \ll 1$

$\Rightarrow$ ignore every term in this equation except the pressure term because they are all small compare to $\frac{1}{\rho}\frac{\partial P}{\partial r} \sim \frac{1}{Re}\frac{U}{a}\left(\frac{a^2}{h^2}\right)$.

$\Rightarrow$ so they are even smaller than

$$\frac{1}{\rho}\frac{\partial P}{\partial z} \sim \frac{1}{Re}\frac{U}{a}\left(\frac{a^3}{h^3}\right)$$

$\Rightarrow$ $\frac{1}{\rho}\frac{\partial P}{\partial z} = 0$ $\Rightarrow$ P independant of z

$\because$ P is also independent of θ due to symmetry

$\therefore$ $\underline{\underline{P = P(r,t)}}$

So equations are

$$\frac{\partial P}{\partial r} = \eta \frac{\partial^2 u_r}{\partial z^2} \quad , \quad \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

and $\frac{\partial P}{\partial r}$ independent of z

$$\Rightarrow \frac{\partial u_r}{\partial z} = \frac{1}{\eta}\frac{\partial P}{\partial r} z + C_1$$

$$\Rightarrow u_r = \frac{1}{2\eta}\frac{\partial P}{\partial r} z^2 + C_1 z + C_2$$

No Slip Boundary conditions : $u_r(z=0) = 0 \Rightarrow C_2 = 0$

$$u_r(z=-h) = 0 \Rightarrow \frac{1}{2\eta}\frac{\partial P}{\partial r} h^{\cancel{2}} - C_1 \cancel{h} = 0$$

$$\Rightarrow C_1 = \frac{1}{2\eta}\frac{\partial P}{\partial r} h$$

$$\Rightarrow u_r = \frac{1}{2\eta}\frac{\partial P}{\partial r} z^2 + \frac{1}{2\eta}\frac{\partial P}{\partial r} h z$$

$$\Rightarrow \boxed{u_r = \frac{1}{2\eta}\frac{\partial P}{\partial r} z(z+h)}$$

(c) The incompressibility condition is

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r)$$

$$= -\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{r}{2\eta}\frac{\partial P}{\partial r} z(z+h)\right]$$

$$= -\frac{1}{2\eta} z(z+h) \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)$$

integrate w.r.t z

$$\Rightarrow \quad u_z = -\frac{1}{2\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)\left(\frac{z^3}{3} + \frac{hz^2}{2} + C_3\right)$$

No Slip Boundary condition

$u_z = 0$ at $z = 0$ $\Rightarrow$ $C_3 = 0$

$$\Rightarrow \quad u_z = -\frac{1}{2\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)\left(\frac{z^3}{3} + \frac{hz^2}{2}\right)$$

At $z = -h$, $u_z = W$

$$\Rightarrow \quad W = -\frac{1}{2\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)\underbrace{\left(-\frac{h^3}{3} + \frac{h^3}{2}\right)}_{h^3/6}$$

$$= -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)$$

(e) Consider the stress tensor

$$\sigma_{ij} = -p\delta_{ij} + \eta(\partial_i u_j + \partial_j u_i)$$

$$\because \frac{\partial p}{\partial r} = \eta \frac{\partial^2 u_r}{\partial z^2} \quad \therefore \frac{p}{a} \sim \eta \frac{U}{h^2}$$

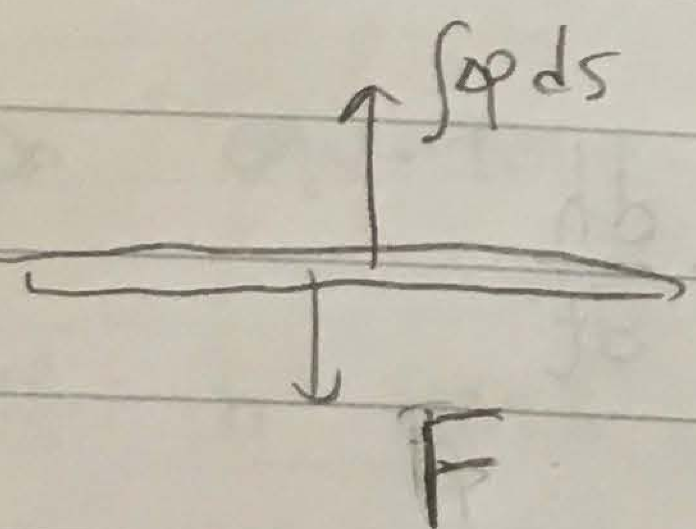
$$\therefore p \sim \frac{\eta U a}{h^2} \sim \eta \frac{U}{h}\left(\frac{a}{h}\right)$$

all $\eta \partial_i u_j$'s have order $\sim \eta \frac{U}{h}$ or less

(e.g. $\eta \frac{U}{a}$)

$$\therefore p \gg \partial_i u_j$$

$\Rightarrow$ in thin ~~film~~ film approximation the stress is ~~dominant~~ dominantly due to pressure alone.



pressure inside film is less than pressure outside because

$$\Delta p = p - p_0 = \frac{3\eta W}{h^3}(a^2 - r^2) \quad \text{and} \quad W < 0, \quad r < a$$

$\therefore$ to balance forces, we need downward force F given by:

$$F = \int \Delta p \, ds = \int_0^{2\pi}\int_0^a \frac{3\eta W}{h^3}(a^2 - r^2)\, r\, dr\, d\theta$$

$$= \frac{3\eta W}{h^3} \cdot 2\pi \cdot \int_0^a a^2 r - r^3 \, dr$$

$$= \frac{3\eta W}{h^3} \cdot 2\pi \underbrace{\left(a^2 \cdot \frac{a^2}{2} - \frac{a^4}{4}\right)}_{\frac{a^4}{4}} = \underline{\underline{\frac{3\pi}{2}\frac{\eta a^4 W}{h^3}}}$$

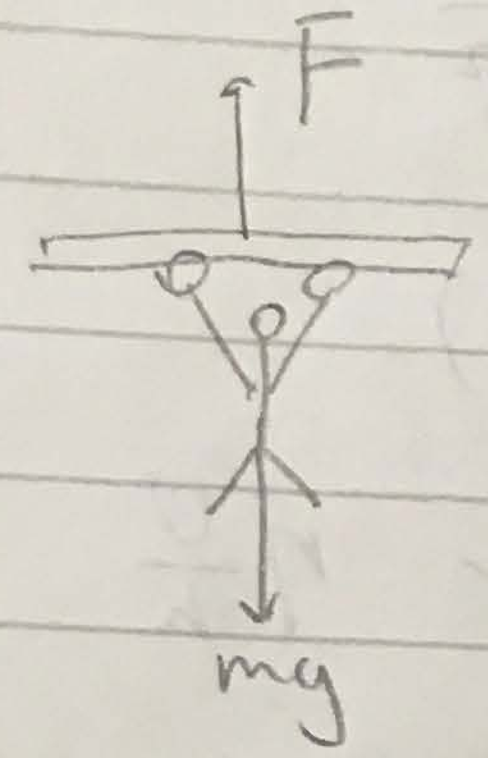
(f)

$a_0 = 10\,cm = 0.1\,m$

$h_0 = 1\,mm = \cancel{0.001}\ 1\times10^{-3}\,m$

$\eta = 300\ kg\,m^{-1}s^{-1}$

$m = 50\ kg$



(choose h, w to be positive, i.e. down is positive)

$\cancel{m\frac{d^2h}{dt^2}}$

$w = \frac{dh}{dt}$

$$m\frac{d^2h}{dt^2} = mg - F$$

$$\Rightarrow \quad m\frac{d^2h}{dt^2} = mg - \frac{3\pi\eta}{2}\frac{a^4}{h^3}\frac{dh}{dt}$$

$\because\ V = \pi a_0^2 h_0 = \pi a^2 h \quad \therefore\ a^4 = \frac{V^2}{\pi^2 h^2}$

$$\Rightarrow \quad m\frac{d^2h}{dt^2} = mg - \frac{3\cancel{\pi}\eta}{2}\frac{V^2}{\pi^{\cancel{2}}}\frac{1}{h^5}\frac{dh}{dt}$$

$$\Rightarrow \quad m\frac{d^2h}{dt^2} = mg - \frac{3\eta V^2}{2\pi}\underbrace{\frac{1}{h^5}\frac{dh}{dt}}_{-\frac{1}{4}\left(\frac{d}{dt}\frac{1}{h^4}\right)}$$

$$\Rightarrow \quad \frac{d^2h}{dt^2} = g + \frac{3\eta V^2}{8\pi m}\frac{d}{dt}\left(\frac{1}{h^4}\right)$$

integrate w.r.t t $\Rightarrow$

$$\frac{dh}{dt} = gt + \frac{3\eta V^2}{8\pi m}\frac{1}{h^4} + C$$

At $t=0$, $\frac{dh}{dt}=0$, $h=h_0$

$$\therefore \quad C = -\frac{3\eta V^2}{8\pi m}\frac{1}{h_0^4} = -\frac{3\eta \pi a_0^4 h_0^2}{8\pi m}\frac{1}{h_0^4}$$

$$= -\frac{3\eta a_0^4 h_0^2}{8m}$$

$$\Rightarrow \quad \frac{dh}{dt} = gt - \frac{3\eta a_0^4 h_0^2}{8m}\left(\frac{1}{h_0^4} - \frac{1}{h^4}\right) \quad ①$$

Thin film approximation only works for

$h \sim h_0$, not $h \gg h_0$ ($h \sim a_0$ for example)

So equation ① should only work for $h \sim h_0$

If h gets too large, then approximation breaks down and Spiderman drops.

$\because \frac{dh}{dt}$ is small, so ignore it ~~[illegible]~~

At time when approximation no longer works,

$h \gg h_0 \quad \rightarrow \quad \frac{1}{h^4} \ll \frac{1}{h_0}$

$$\Rightarrow \quad gt - \frac{3\eta a_0^4}{8mh_0^2} = 0$$

$$\Rightarrow \quad t = \frac{3\eta a_0^4}{8mgh_0^2} = \frac{3(300)(0.1)^4}{8(50)(9.8)(1\times10^{-3})^2}$$

$$\approx \boxed{23\ s}$$

→ For real fluid the viscosity would be much lower, so the time of attachment, which $\propto \eta$, should be much shorter.

(d)

$$w = -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = -\frac{12\eta w}{h^3} r$$

integrate $\Rightarrow$ $$r\frac{\partial P}{\partial r} = -\frac{6\eta w}{h^3} r^2 + C_4$$

$$\Rightarrow \frac{\partial P}{\partial r} = -\frac{6\eta w}{h^3} r + \frac{C_4}{r}$$

For finite pressure gradient at $r=0$, need $C_4 = 0$

$$\Rightarrow \frac{\partial P}{\partial r} = -\frac{6\eta w}{h^3} r$$

$\Rightarrow$ integrate $\Rightarrow$ $\cancel{P = -\frac{6\eta w}{h^3}}$

$$P = -\frac{3\eta w}{h^3} r^2 + C_5$$

Boundary condition : At $r=a$, $P = P_0 = 1\text{atm}$

$$\therefore P_0 = -\frac{3\eta w}{h^3} a^2 + C_5 \Rightarrow C_5 = P_0 + \frac{3\eta w}{h^3} a^2$$

$$\Rightarrow \underline{\underline{P - P_0 = \frac{3\eta w}{h^3}(a^2 - r^2)}}$$