To: Caroline Terquem

B1 Problem set 1 Ziyan Li

① (a) (A) $\underline{u} = (\alpha x, -\alpha y) \rightarrow u_x = \alpha x, \quad u_y = -\alpha y$

Streamline $\dfrac{dx}{u_x} = \dfrac{dy}{u_y}$

$\therefore \dfrac{dx}{\alpha x} = \dfrac{dy}{-\alpha y} \quad \therefore \quad \dfrac{dx}{x} = \dfrac{dy}{-y}$

$\rightarrow \dfrac{dx}{x} + \dfrac{dy}{y} = 0 \Rightarrow \ln x + \ln y = C$

$\rightarrow \ln(xy) = C \Rightarrow \boxed{xy = \underset{\text{const}}{C}}$

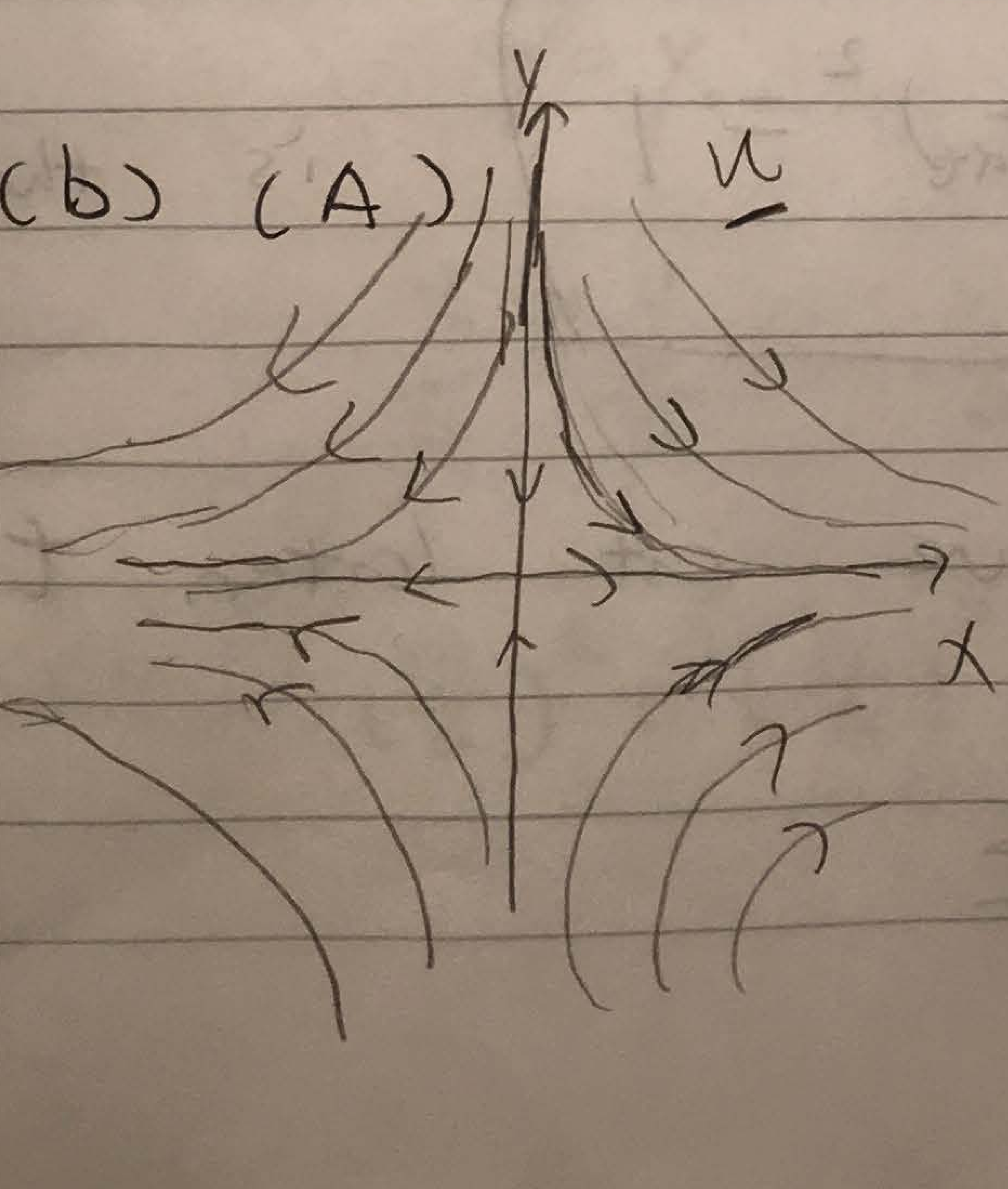
(B) $\underline{u} = (\gamma y, 0) \rightarrow u_x = \gamma y, \quad u_y = 0$

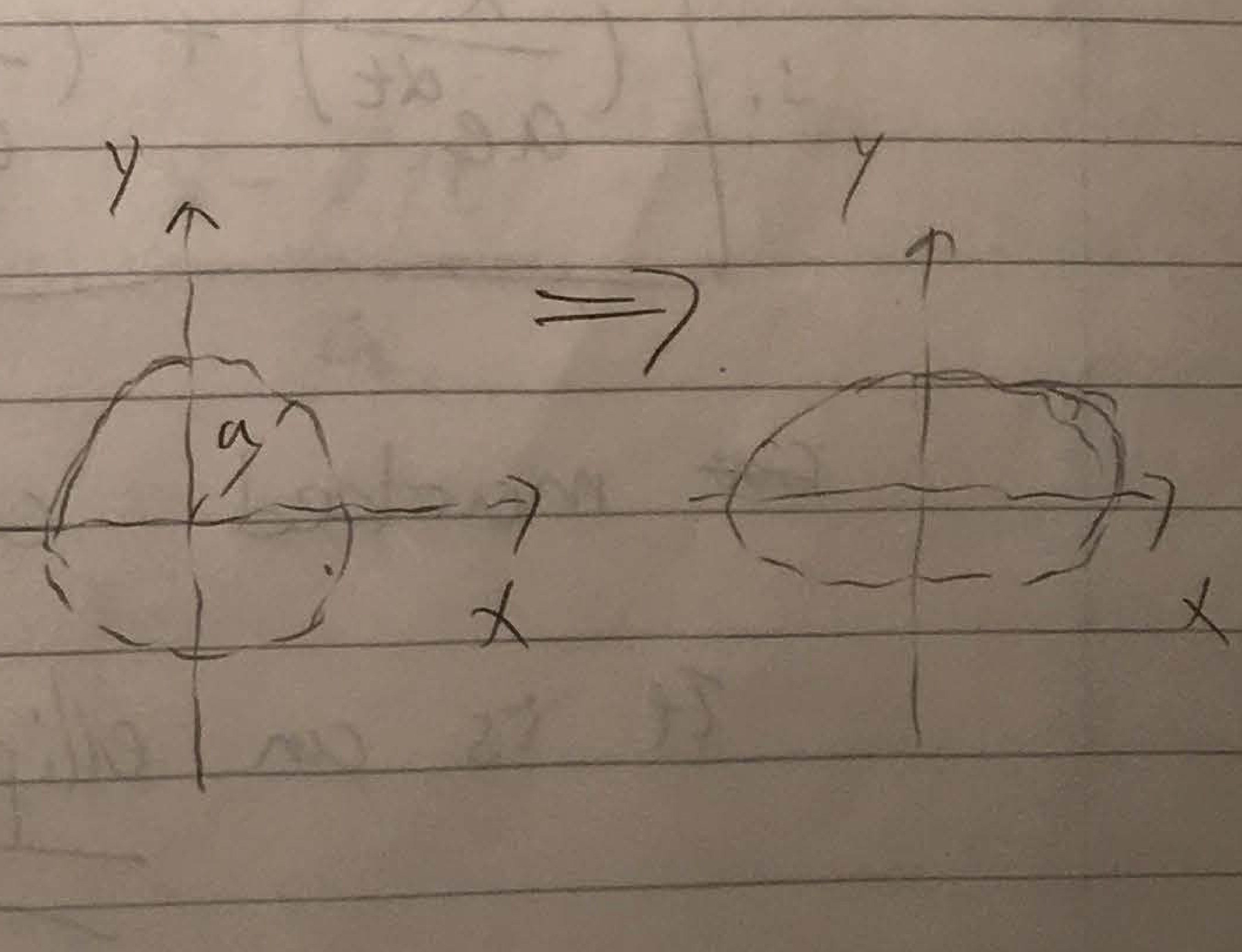
$\therefore \dfrac{dx}{\gamma y} = \dfrac{dy}{0}$ for dx to be finite

we need $dy = 0 \rightarrow \boxed{y = \text{Const}}$

(straight horizontal lines)

(b) (A)





At $t=0$ $x^2+y^2=a^2$ is marked

Assume point (x_0, y_0) is on the circle at $t=0$, then $x_0^2+y_0^2=a^2$

$\because$ $\underline{u} = (u_x, u_y) = \alpha(x, -y)$

$\therefore$ $\frac{dx}{dt} = \alpha x$ $\frac{dy}{dt} = -\alpha y$

track a point :

$\because$ $x(t=0) = x_0$, $y(t=0) = y_0$

is the initial position for the point ~~initially~~

$\therefore$ $x(t) = x_0 e^{\alpha t}$ $y(t) = y_0 e^{-\alpha t}$

at time t

$\therefore$ $\frac{x_0}{a} = \frac{x}{ae^{\alpha t}}$ $\frac{y_0}{a} = \frac{y}{ae^{-\alpha t}}$

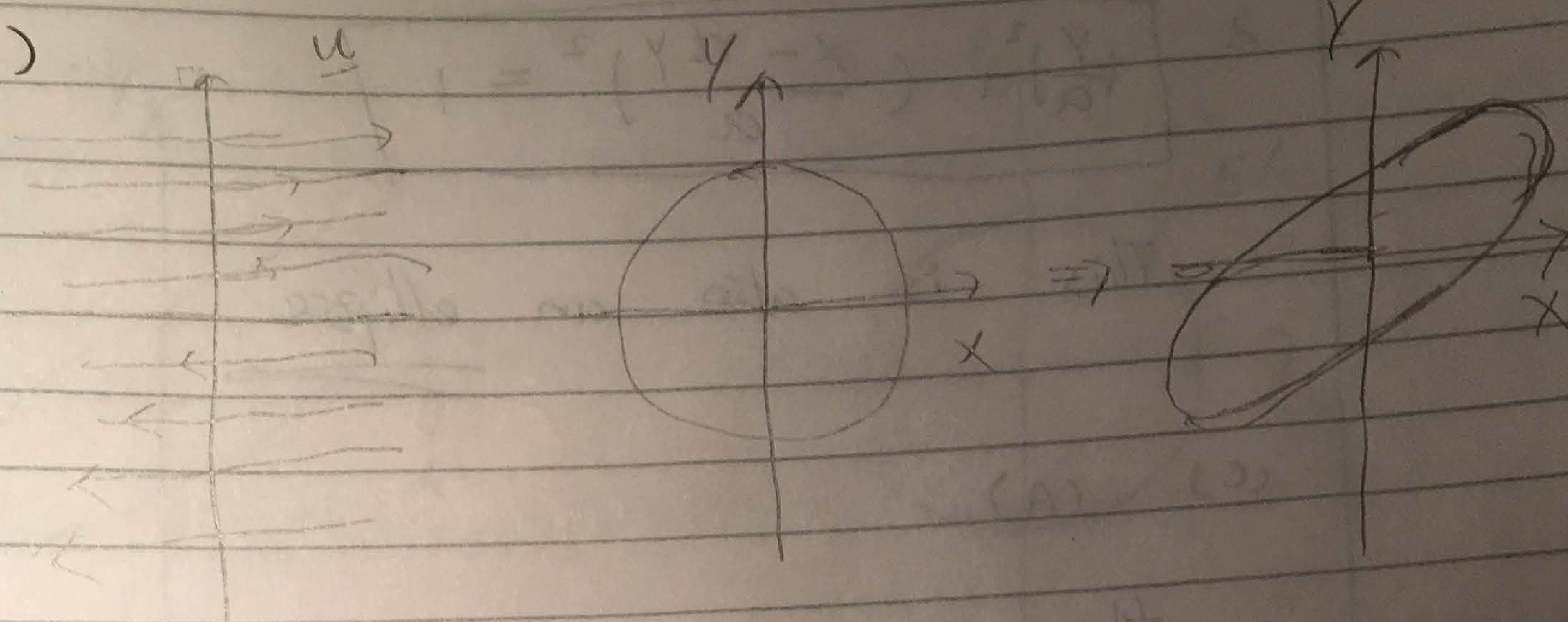
$\because$ $x_0^2+y_0^2=a^2$ $\therefore$ $\left(\frac{x_0}{a}\right)^2 + \left(\frac{y_0}{a}\right)^2 = 1$

$\therefore$ $\boxed{\left(\frac{x}{ae^{\alpha t}}\right)^2 + \left(\frac{y}{ae^{-\alpha t}}\right)^2 = 1}$ is the

marked curve at later $t>0$

It is an <u>ellipse</u>

(B)



Still tracking the point at (x_0, y_0) at time $t=0$. $x_0^2 + y_0^2 = a^2$

$\rightarrow \quad \underline{u} = (u_x, u_y) = (\gamma y, 0)$

$\therefore \quad \cancel{\frac{dx}{dt} = \alpha x} \quad \frac{dx}{dt} = \gamma y \quad \frac{dy}{dt} = 0$

$\therefore \quad y = y_0 \rightarrow \frac{dx}{dt} = \gamma y_0 \rightarrow \cancel{x = \gamma y_0 x + c}$

$x = \gamma y_0 t + c$

at $t=0$, $x = x_0 \Rightarrow c = x_0$

$\therefore \quad x = \gamma y_0 t + x_0$

$y = y_0$

$\therefore \quad \frac{y_0}{a} = \frac{y}{a}, \quad \frac{x_0}{a} = \frac{x - \gamma y t}{a}$

$\because \quad \left(\frac{x_0}{a}\right)^2 + \left(\frac{y_0}{a}\right)^2 = 1$

$\therefore$ $$\boxed{\left(\frac{y}{a}\right)^2 + \left(\frac{x - \gamma t y}{a}\right)^2 = 1}$$

This is also an ellipse

(C) (A)

the area ~~S~~ for ellipse is

$$S = \pi (a e^{\alpha t})(a e^{-\alpha t})$$

$$= \pi a^2 = \text{Area of circle } x^2 + y^2 = a^2$$

$\therefore$ Area unchanged

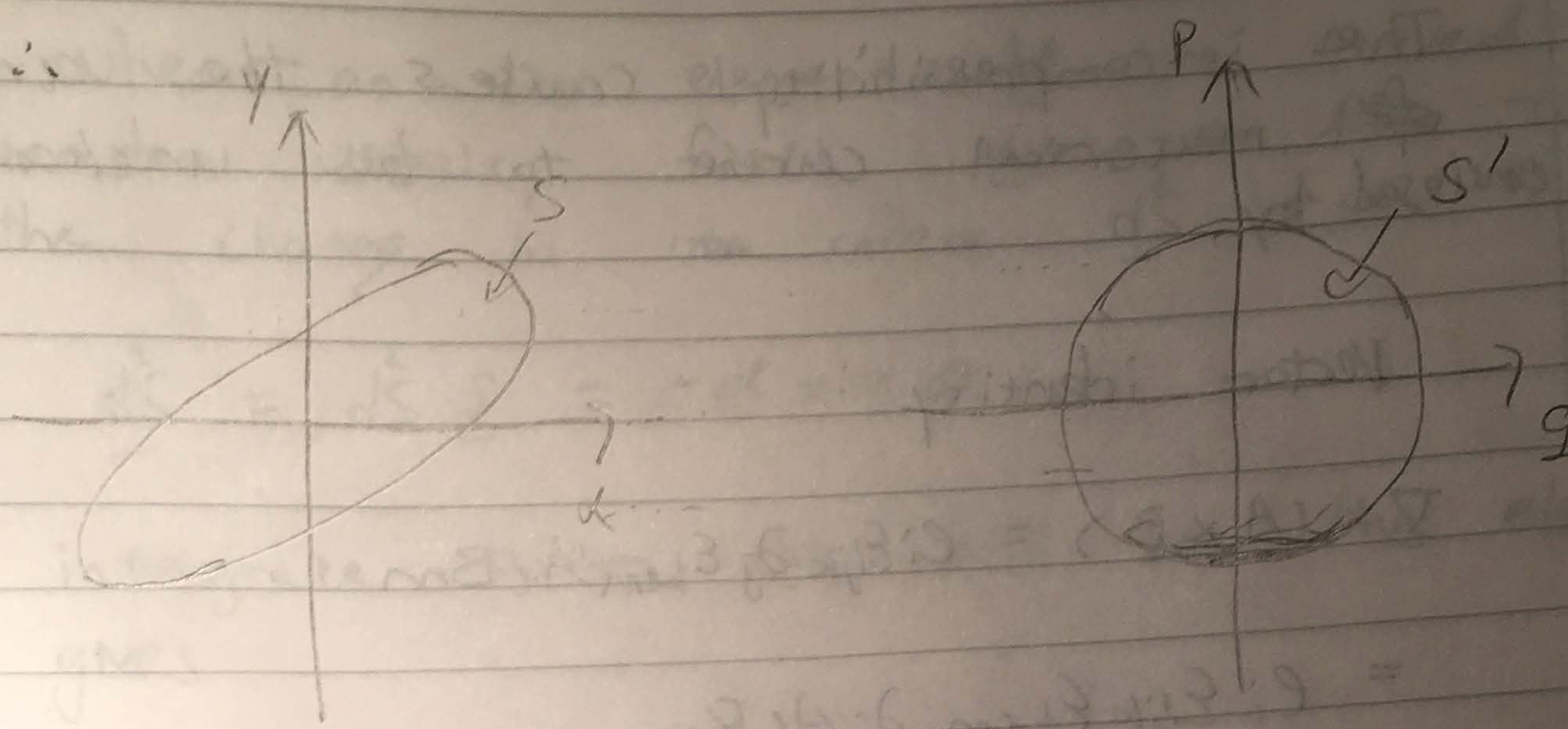
(B) change of variable

$$p = y \qquad q = x - \gamma t y$$

then Jacobian $\left|\frac{\partial(p, q)}{\partial(x, y)}\right| = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{vmatrix}$.

$$= \left| \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x} \right| = |-1| = 1$$

$\therefore$ $dp\,dq = dx\,dy$



In x-y space the ~~as~~ region S:

$\frac{y^2}{a^2} + \frac{(x-\gamma y t)^2}{a^2} = 1$ is ~~mat~~ mapped

to the p, q space as the region S':

$p^2 + q^2 = a^2$

$\because$ Jacobine $= 1$ $\quad dp\,dq = dx\,dy$

$\therefore$ Area $S = \int_S dx\,dy = \int_{S'} dp\,dq$

$=$ area of circle $p^2 + q^2 = a^2$

$= \boxed{\pi a^2}$

$\rightarrow$ Area unchanged

The incompressibility causes the area ~~of~~ enclosed by material curve to be unchanged.

Vector identity:

$$\underline{\nabla} \times (\underline{A} \times \underline{B}) = e_i \varepsilon_{ijk} \partial_j \varepsilon_{klm} A_l B_m$$

$$= e_i \varepsilon_{ijk} \varepsilon_{klm} \partial_j A_l B_m$$

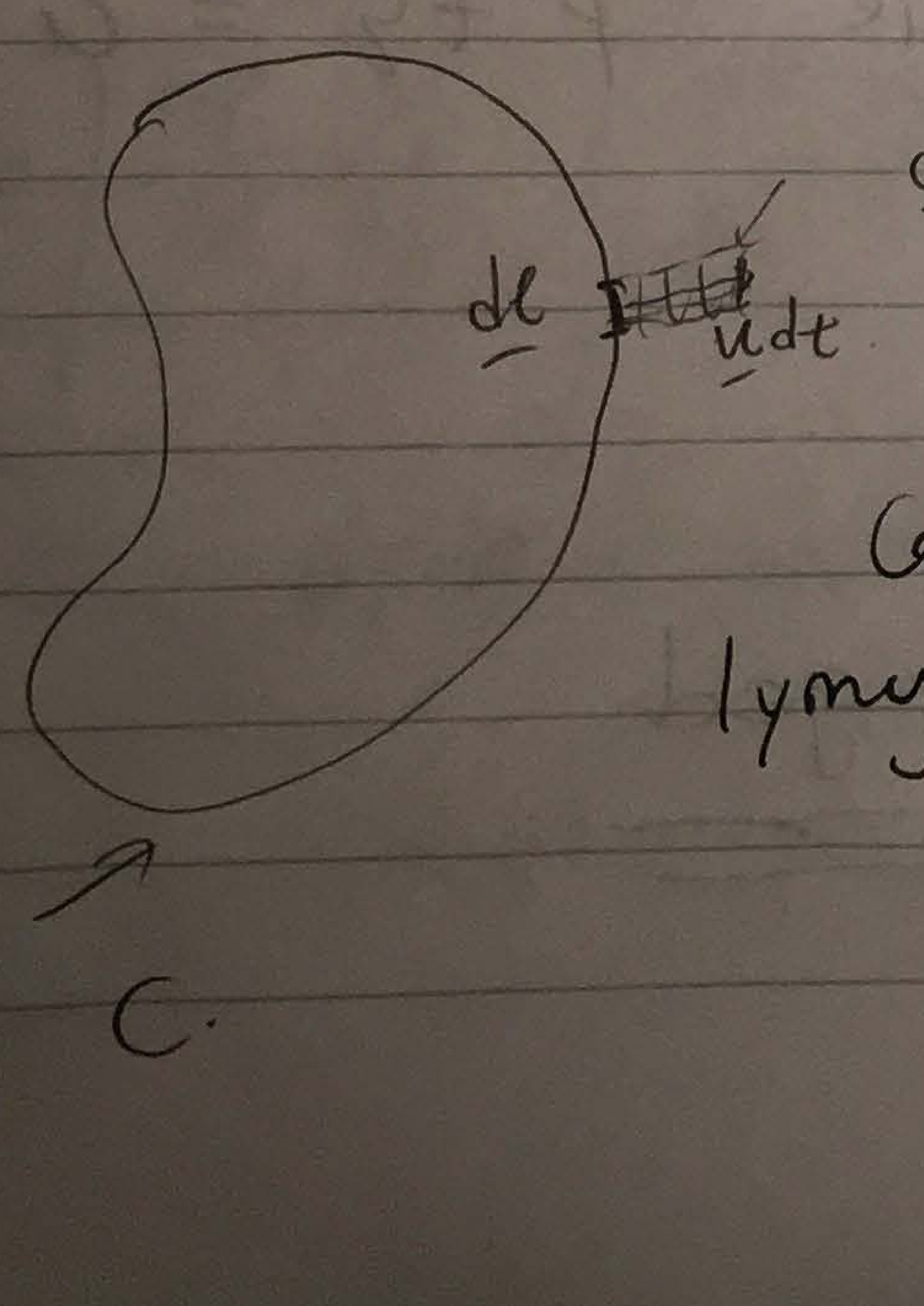
$$= e_i (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) \cancel{(B_m \partial_j A_l + A_l \partial_j A} (A_l \partial_j B_m + (\partial_j A_l) B_m)$$

$$= e_i [A_i \partial_j B_j - A_j \partial_j B_i + B_j \partial_j A_i - B_i \partial_j A_j].$$

$$= \underline{A}(\underline{\nabla} \cdot \underline{B}) - (\underline{A} \cdot \underline{\nabla})\underline{B} + (\underline{B} \cdot \underline{\nabla})\underline{A} - \underline{B}(\underline{\nabla} \cdot \underline{A}). \quad (1)$$

If $\underline{B}$ is a constant vector, then

$$\underline{\nabla} \times (\underline{A} \times \underline{B}) = (\underline{B} \cdot \underline{\nabla})\underline{A} - \underline{B}(\underline{\nabla} \cdot \underline{A}) \quad \cancel{\underline{A}(\underline{\nabla} \cdot \underline{B}) - (\underline{A} \cdot \underline{\nabla})\underline{B}} \quad (2)$$



$$d\underline{s} = d\underline{l} \times \underline{u}\, dt.$$

Consider a surface area lying on the x-y plane when an line element $\underline{dl}$ is moved under velocity field $\underline{u}$, for time dt, the change in area $\underline{d^2S}$ is

$$\underline{d^2S} = d^2S\,\underline{\hat{z}} = -\underline{dl} \times \underline{u}\,dt$$

integrate over all the line elements gives

$$\underline{dS} = dS\,\underline{\hat{z}} = dt \oint_C \underline{u} \times \underline{dl}$$

$$\therefore \frac{\underline{dS}}{dt} = \oint_C \underline{u} \times \underline{dl} = \frac{dS}{dt}\,\underline{\hat{z}}$$

let $\underline{b}$ be an arbitrary constant vector

$$\underline{b} \cdot (\underline{u} \times \underline{dl}) = -\underline{dl} \cdot (\underline{u} \times \underline{b})$$

$$\therefore \oint_C \underline{b} \cdot (\underline{u} \times \underline{dl}) = -\oint_C \underline{dl} \cdot (\underline{u} \times \underline{b})$$

$$\Rightarrow \underline{b} \cdot \oint_C \underline{u} \times \underline{dl} = -\oint_C \underline{dl} \cdot (\underline{u} \times \underline{b})$$

$$= -\int_S \underline{dS} \cdot (\underline{\nabla} \times (\underline{u} \times \underline{b}))$$

Stoke's Theorem

$$= -\int_S \underline{dS} \cdot [(\underline{b} \cdot \underline{\nabla})\underline{u} - \underline{b}(\underline{\nabla} \cdot \underline{u})]$$

using (2)

$$= \underline{b} \cdot \int_S (\underline{\nabla} \cdot \underline{u}) \, d\underline{S} - \int ((\underline{b} \cdot \underline{\nabla}) \underline{u}) \cdot d\underline{S}$$

Consider $((\underline{b} \cdot \underline{\nabla}) \underline{u}) \cdot d\underline{S}$

$$\cancel{= b_j \partial_j \quad b_j \, dS_i \, \partial_j u}$$

$$= dS_i (b_j \partial_j u_i)$$

$$= b_j \partial_j (u_i dS_i) - u_i (b_j \partial_j dS_i)$$

$d\underline{S}$ is treated as a constant vector inside the integral (when computing the integrand)

$$\therefore \ \underline{\partial_j dS_i = 0}$$

$$\therefore ((\underline{b} \cdot \underline{\nabla}) \underline{u}) \cdot d\underline{S} = dS_i \, b_j \partial_j u_i$$

$$= b_j \partial_j (u_i dS_i)$$

$$= \underline{b} \cdot \underline{\nabla} (\underline{u} \cdot d\underline{S})$$

$$\therefore \cancel{\frac{d\underline{S}}{dt} \hat{\underline{z}} = \frac{d\underline{S}}{dt} \quad \underline{b} \cdot \oint_C \underline{u} \times d\underline{t}}$$

$$\cancel{=} \quad \cancel{= \underline{b} \cdot \int_S d\underline{S} (\underline{\nabla} \cdot \underline{u}) - \nabla}$$

$\therefore \quad \underline{b} \cdot \oint_C \underline{u} \times d\underline{l} = \underline{b} \cdot \int_S [(\nabla \cdot \underline{u}) d\underline{S} - \nabla(\underline{u} \cdot d\underline{S})]$

$\because$ $\underline{b}$ is arbitrary

$$\therefore \quad \oint_C \underline{u} \times d\underline{l} = \int_S [(\nabla \cdot \underline{u}) d\underline{S} - \nabla(\underline{u} \cdot d\underline{S})]$$

For (A) and (B) Fluids are incompressible $\Leftrightarrow$ $\nabla \cdot \underline{u} = 0$ ③

$\because$ $d\underline{S} = dS\,\hat{\underline{z}}$, $\underline{u}$ is along x-y plane

$\rightarrow$ $\therefore$ $\underline{u} \cdot d\underline{S} = 0$ ④

$\because$ ③ , ④

$$\therefore \quad \frac{dS}{dt}\hat{\underline{z}} = \frac{d\underline{S}}{dt} = \oint_C \underline{u} \times d\underline{l} = 0$$

$$\rightarrow \quad \frac{dS}{dt} = 0$$

$\rightarrow$ Area S is unchanged

(d) For (A), ~~the major semi-axis~~ the ellipse is

~~$a(t) =$~~ $\therefore$

$$\left(\frac{x}{ae^{\alpha t}}\right)^2 + \left(\frac{y}{ae^{-\alpha t}}\right)^2 = 0$$

$\therefore$ major semi-axis is

$$a_A(t) = ae^{\alpha t}$$

For (B) the ellipse is

$$\left(\frac{y}{a}\right)^2 + \left(\frac{x - \gamma y t}{a}\right)^2 = 1$$

$$\therefore \quad y^2 + x^2 - 2\gamma t xy + (\gamma^2 t^2) y^2 = a^2$$

$$\therefore \quad x^2 - 2\gamma t\, xy + (\gamma^2 t^2 + 1) y^2 = a^2$$

put in quadratic form

$$\underbrace{(x, y)}_{\underline{x}^T} \underbrace{\begin{pmatrix} 1 & -\gamma t \\ -\gamma t & \gamma^2 t^2 + 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\underline{x}} = a^2$$

matrix

$\therefore \quad \underline{x}^T Q \underline{x} = a^2$

introducing new coordinates $\underline{y} = P\underline{x}$

$\therefore \ \underline{y}^T = \underline{x}^T P^T$, where P is the diagonalising basis transformation matrix of Q such that $\hat{Q} = PQP^T$ is diagonal

$\cancel{\hat{Q} = \text{diag}(\lambda_1, \ldots)} \quad \hat{Q} = (\lambda_1, \lambda_2)$

$P^T = P^{-1} \Rightarrow$ ~~P is rotation.~~

then $a^2 = \underline{x}^T Q \underline{x} = \underline{y}^T \underbrace{PQP^T}_{\hat{Q}} \underline{y} = \underline{y}^T \hat{Q} \underline{y}$

$\therefore$ $\cancel{a^2 = \lambda_1 y}$ if $\underline{y} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, then

$$a^2 = (x')^2 \lambda_1 + (y')^2 \lambda_2$$

Now finding the eigenvalues of Q

$$\det \begin{pmatrix} 1-\lambda & -\gamma t \\ -\gamma t & \gamma^2 t^2 + 1 - \lambda \end{pmatrix} = 0.$$

$\therefore \ (1-\lambda)(\gamma^2 t^2 + 1 - \lambda) - \gamma^2 t^2 = 0$

$\therefore \ \gamma^2 t^2 - \lambda \gamma^2 t^2 + 1 - \lambda - \lambda + \lambda^2 \cancel{- \gamma^2 t^2} = 0$

$\therefore \ \lambda^2 - (\gamma^2 t^2 + 2)\lambda + 1 = 0$

$y = Px$ is an unscaled coordinate transfer matrix

Jacobian $= \det(P) = 1$

$\det(PP^T) = \det(I) = 1$

$|\det(P)| = 1$

$\Downarrow$

$\det P = \pm 1$

$\therefore \lambda = \frac{(\gamma^2t^2+2) \pm \sqrt{(\gamma^2t^2+2)^2 - 4}}{2}$

$\therefore \lambda = \frac{1}{2}\left[(\gamma^2t^2+2)\left(1 \pm \sqrt{1 - \frac{4}{(\gamma^2t^2+2)^2}}\right)\right]$

As $t \to \infty$

(let $\sqrt{1+x} \simeq 1 + \frac{1}{2}x$ for $x \ll 1$)

$\lambda \simeq \frac{1}{2}\left[(\gamma^2t^2+2)\left(1 \pm \left(1 - \frac{2}{(\gamma^2t^2+2)^2}\right)\right)\right]$.

$\therefore \lambda_{min} = \lambda_- = \frac{1}{\cancel{2}}(\gamma^2t^2+2)\left(\frac{\cancel{2}}{(\gamma^2t^2+2)^2}\right)$

$= \frac{1}{\gamma^2t^2+2} \underset{t \to \infty}{\simeq} \frac{1}{\gamma^2t^2}$

$\therefore$ Semi major axis $a_B(t) \simeq \sqrt{\frac{a^2}{\lambda_{min}}} = \cancel{a}\ a\gamma t$.

As $t \to \infty$

$\frac{\infty}{\infty}$ L'hopital

$\lim_{t \to \infty} \frac{a_A(t)}{a_B(t)} = \frac{ae^{\alpha t}}{a\gamma t} \Rightarrow$

$= \lim_{t \to \infty} \frac{\alpha e^{\alpha t}}{\gamma} \to \infty$

$\therefore$ $a_A(t)$ grows faster than $a_B(t)$

(A) grows faster than (B)

2 (a) For 2D incompressible fluid

~~$\nabla \times (\phi \hat{z}) = 0$~~ if velocity $\underline{u} = (u_x, u_y)$

~~then let~~ Then $\underline{\nabla} \cdot \underline{u} = 0$

ϕ = velocity potential

irrotational fluid $\underline{u} = \underline{\nabla}\phi \Rightarrow u_x = \partial_x \phi,\ u_y = \partial_y \phi$

$\therefore\ \underline{\nabla} \cdot \underline{u} = 0 \Leftrightarrow \partial_{xx}\phi + \partial_{yy}\phi = 0 \Leftrightarrow \nabla^2 \phi = 0$

(i) $\phi = C(x^2+y^2)$

$\nabla^2\phi = C(2+2) = 4C \neq 0 \rightarrow$ Not incompressible

(ii) $\phi = C(x^2-y^2)$

$\nabla^2\phi = C(2-2) = 0 \rightarrow$ incompressible

(b) For 2D irrotational fluid $\underline{u} = (u_x, u_y)$

$\rightarrow \partial_x u_y - \partial_y u_x = 0$

ψ = Stream function

incompressible fluid ~~$u_x = \frac{\partial \psi}{\partial y}$~~ $u_x = \partial_y \psi$

$u_y = -\partial_x \psi$

$\therefore\ -\partial_{xx}\psi - \partial_{yy}\psi = 0 \Rightarrow \nabla^2\psi = 0$

(iii) $\psi = C(x^2+y^2)$ $\nabla^2\psi = 4C \neq 0$

$\rightarrow$ Not irrotational

(iv) $\psi = C(x^2 - y^2)$ $\quad \nabla^2 \psi = 0 \rightarrow$ irrotational

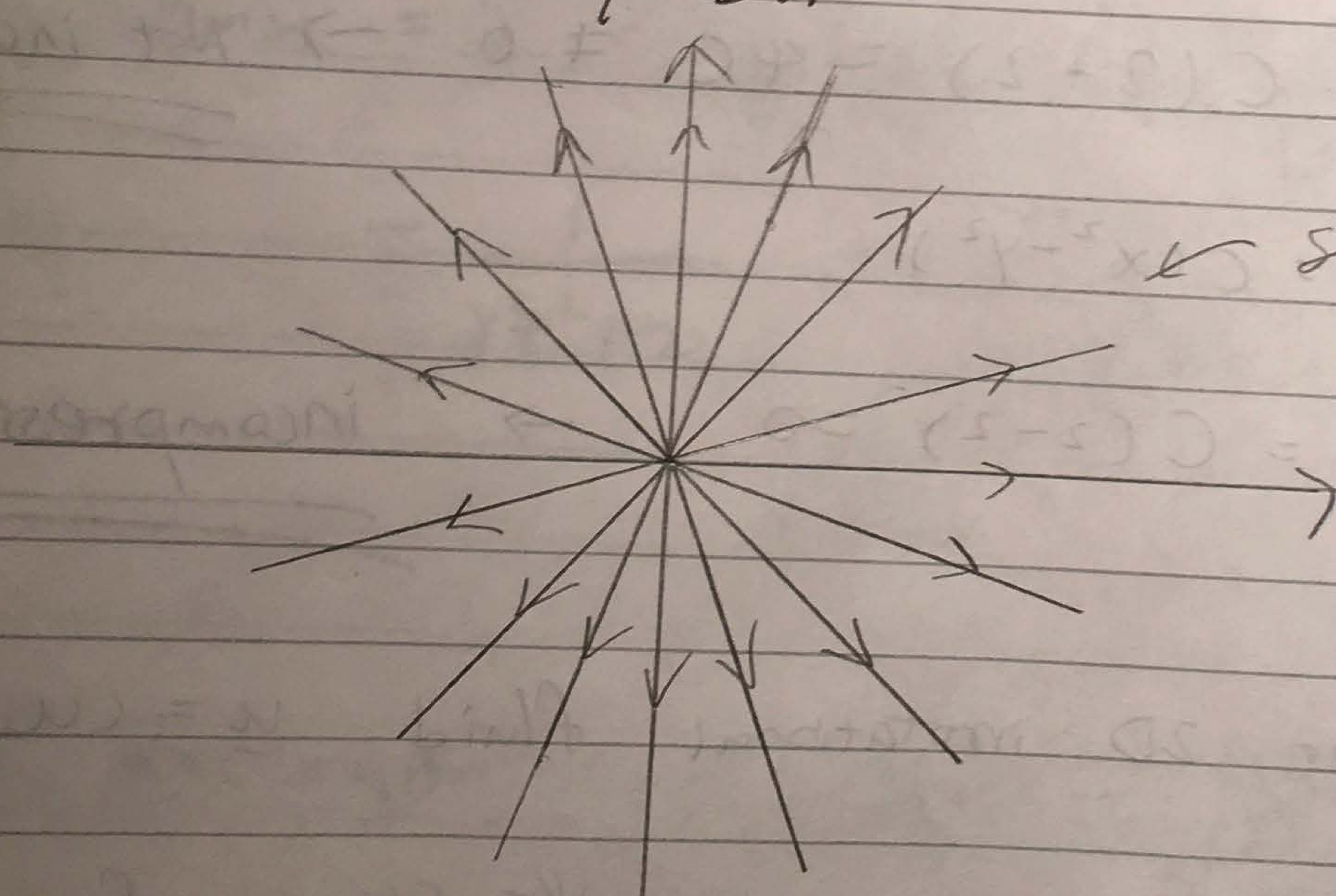
(c) Streamlines

(i) $\phi = C(x^2 + y^2)$

$u_x = \partial_x \phi = 2Cx$ $\quad u_y = \partial_y \phi = 2Cy$

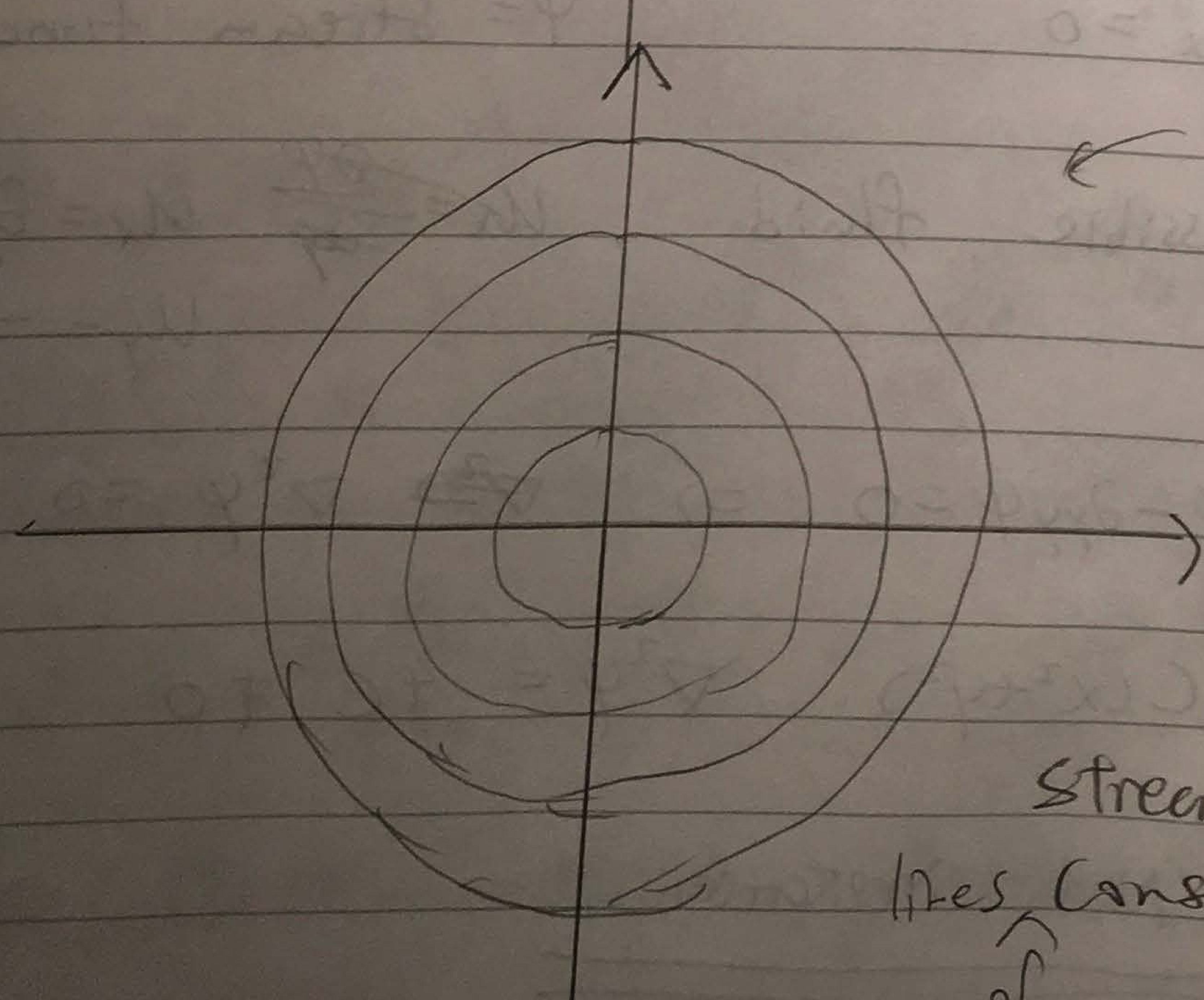
Streamline $\quad \frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln x = \ln y + \underset{\substack{\uparrow \\ \text{constant}}}{D'}$

$\Rightarrow \quad y = Dx$



Streamline

$y = Dx$



constant ϕ

$x^2 + y^2 = a^2$

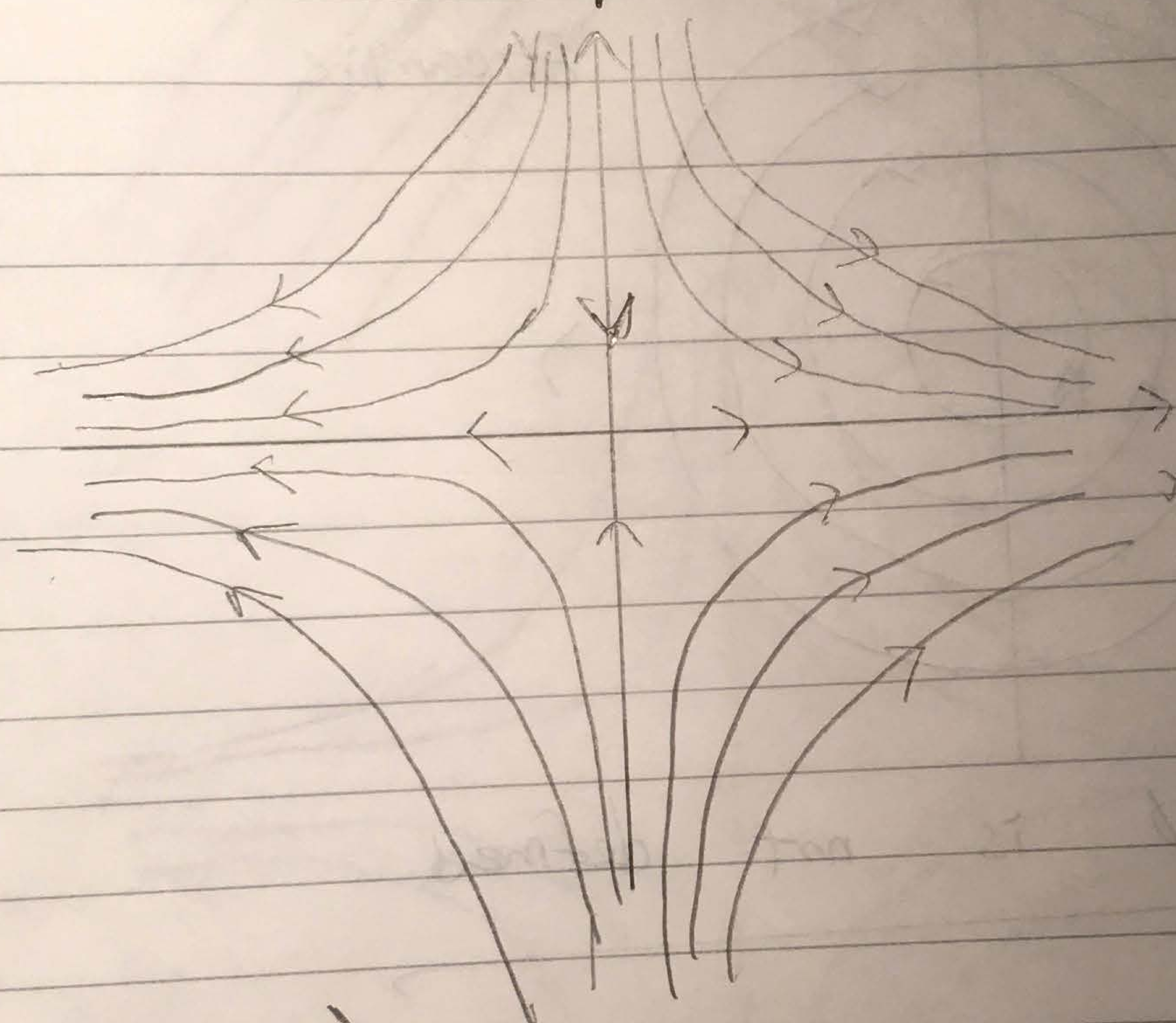
Streamlines $\perp$ lines of constant ϕ

(ii) $\phi = C(x^2 - y^2)$

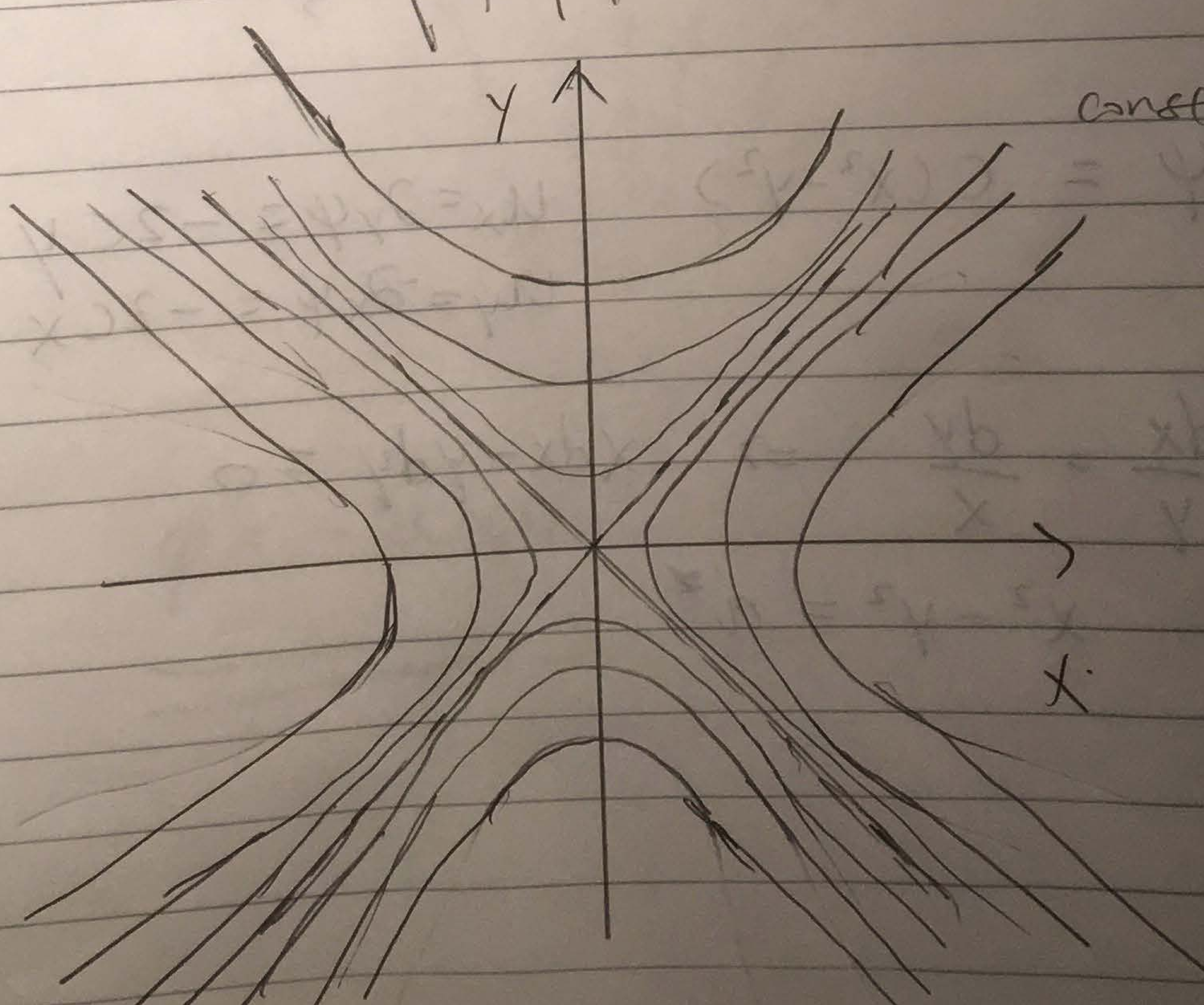
$$u_x = \partial_x \phi = 2Cx \qquad u_y = \partial_y \phi = -2Cy$$

Stream lines $\dfrac{dx}{x} = -\dfrac{dy}{y}$

$$\therefore \ln x = -\ln y + D' \Rightarrow xy = D$$



Stream line



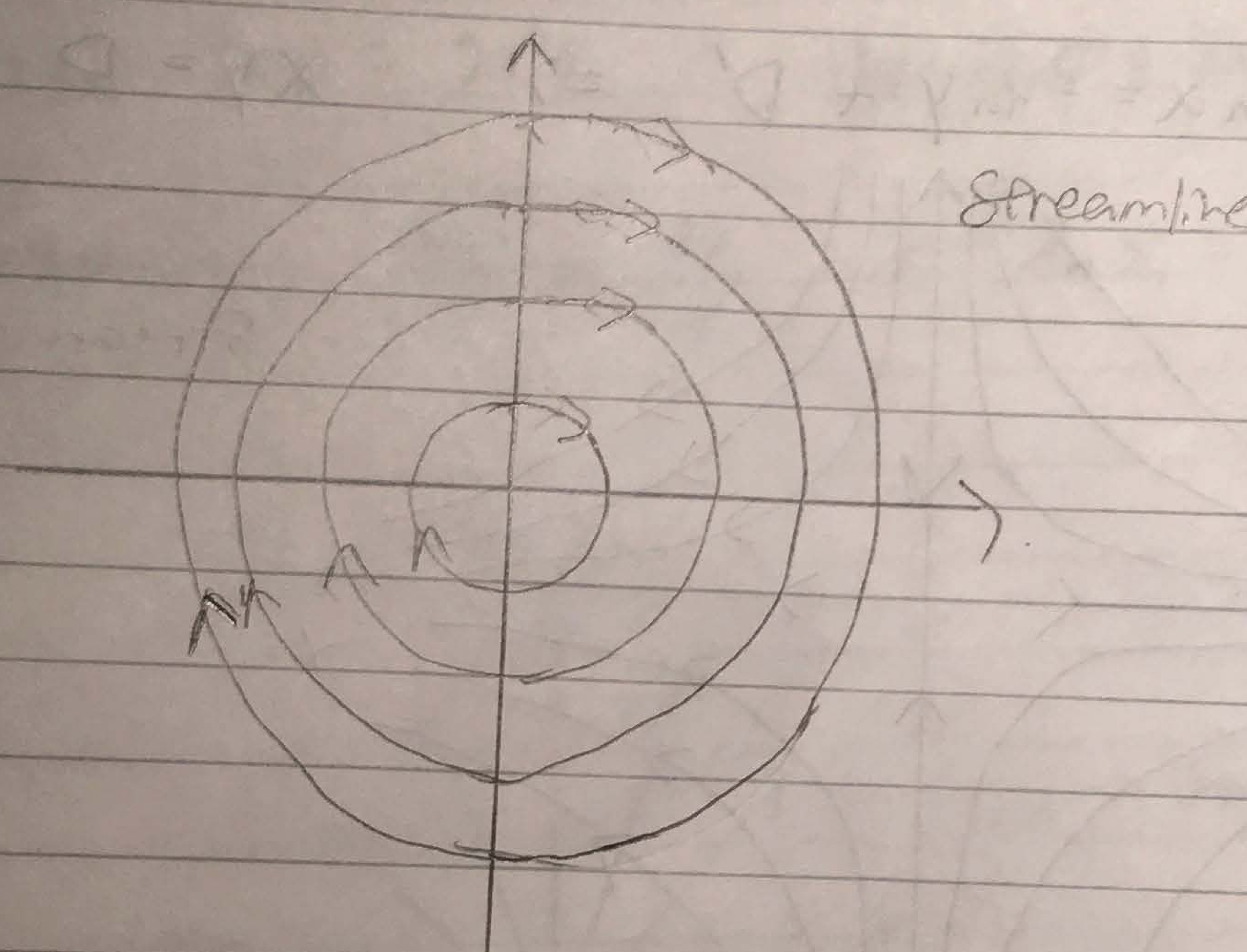
Constant ϕ

(iii) $\psi = C(x^2+y^2)$ $\quad u_x = \partial_y \psi = 2Cy$

$u_y = -\partial_x \psi = -2Cx$

Streamline $\quad \dfrac{dx}{y} = -\dfrac{dy}{x}$

$\Rightarrow \quad x\,dx + y\,dy = 0 \quad \Rightarrow \quad x^2+y^2 = a^2$



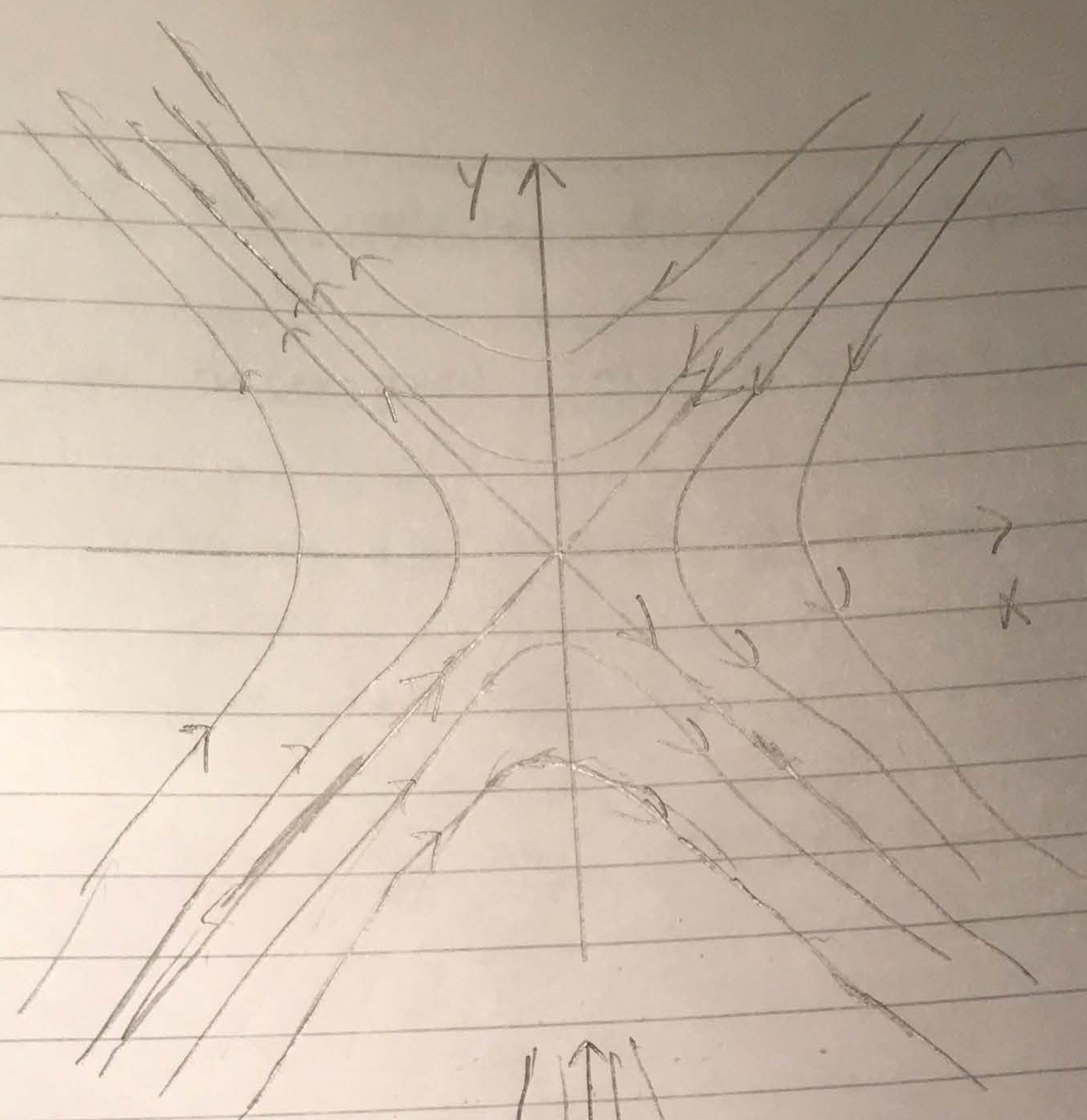
ϕ is not defined

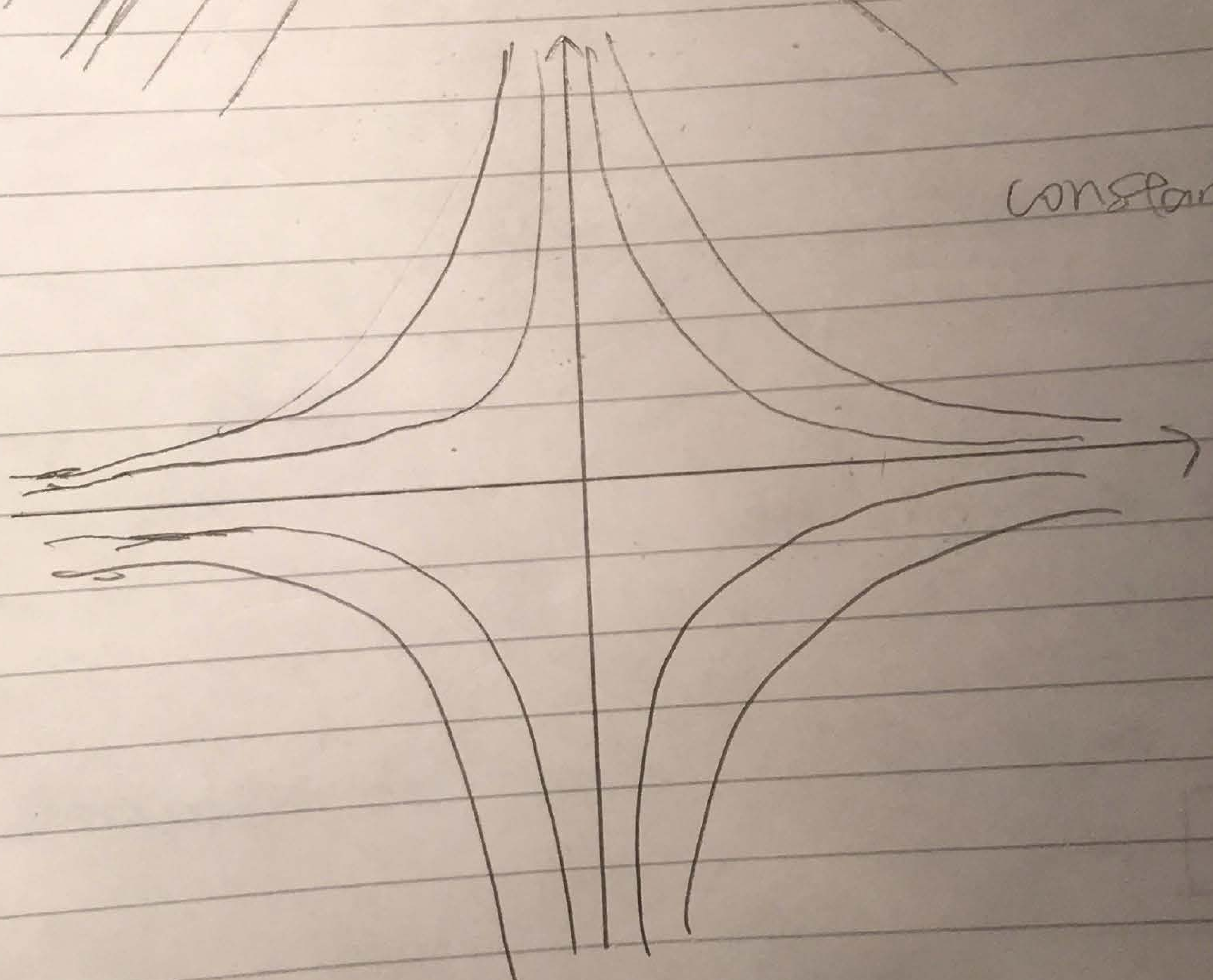
(iv) $\psi = C(x^2-y^2)$ $\quad u_x = \partial_y \psi = -2Cy$

$u_y = -\partial_x \psi = -2Cx$

$\therefore \quad \dfrac{dx}{y} = \dfrac{dy}{x} \quad \Rightarrow \quad x\,dx - y\,dy = 0$

$\Rightarrow \quad x^2 - y^2 = a^2$





$\phi = -2Cxy$

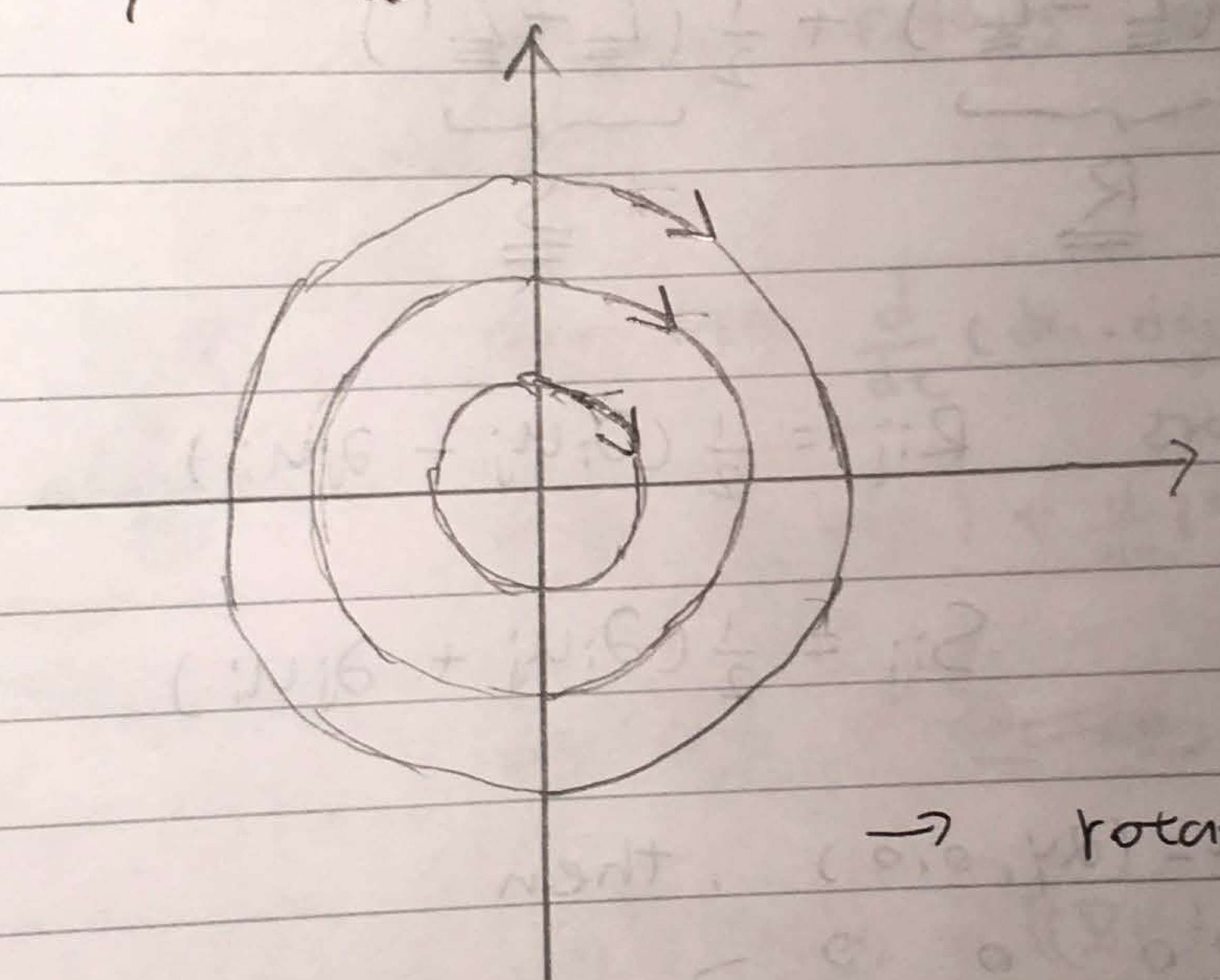
③ (i) the velocity field $\underline{u} = (\alpha y, 0, 0)$

can be decomposed into $\underline{u} = \underline{u_r} + \underline{u_s}$, where

$\underline{u_r} = \frac{1}{2}\alpha(y, -x, 0)$ and $\underline{u_s} = \frac{1}{2}\alpha(y, x, 0)$

streamline for $\underline{u_r}$:

$$\frac{dx}{y} = -\frac{dy}{x} \Rightarrow x\,dx + y\,dy = 0 \Rightarrow \boxed{x^2 + y^2 = C}$$

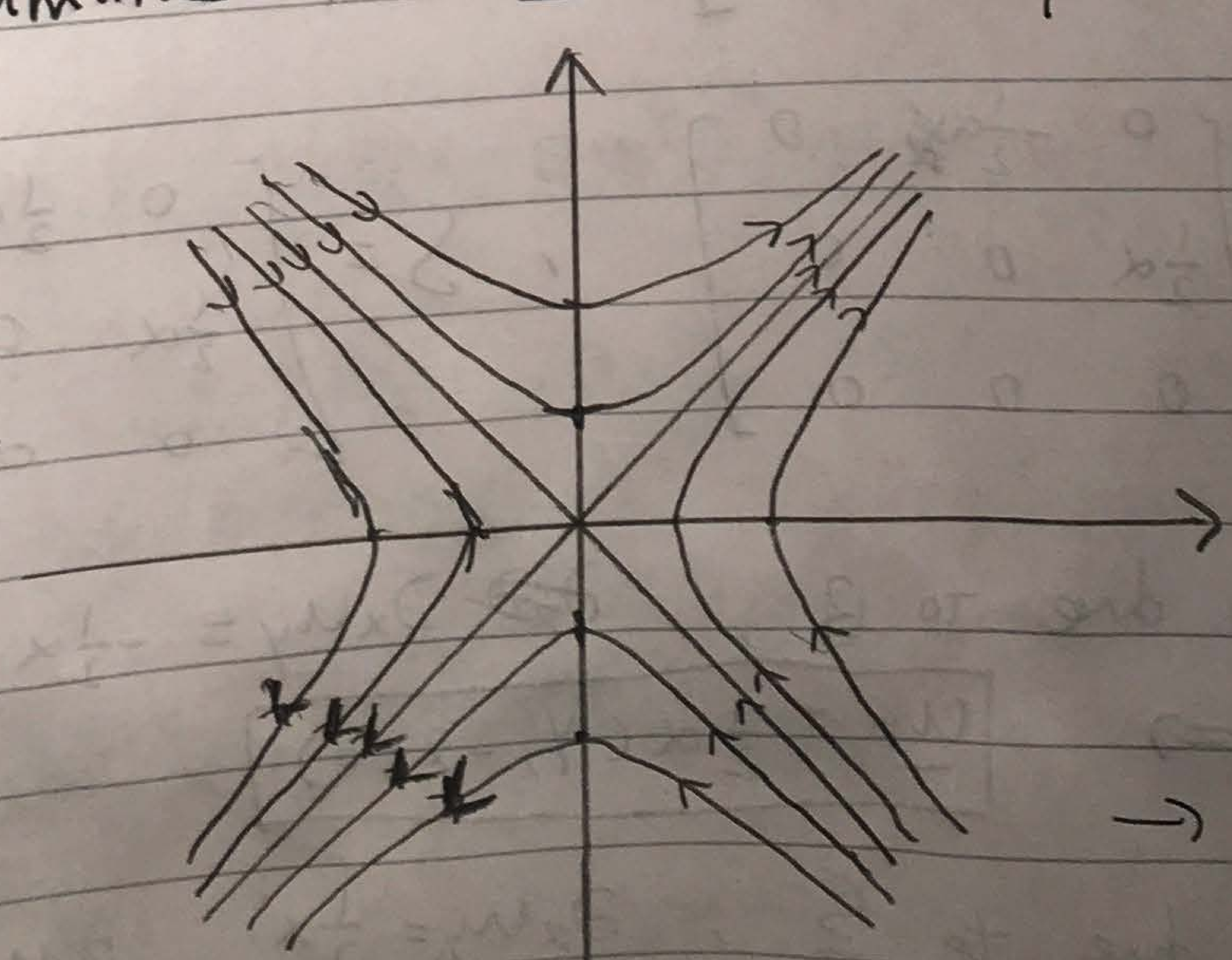


→ rotation flow

streamline for $\underline{u_s}$:

$$\frac{dx}{y} = \frac{dy}{x} \Rightarrow x\,dx - y\,dy = 0$$

$$\Rightarrow \boxed{x^2 - y^2 = C}$$



→ straining flow

(ii) The velocity gradient tensor is

$$\underline{\underline{L}} = \nabla \underline{u} = \frac{\partial \underline{u}}{\partial \underline{x}} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

OR $\underline{L_{ij} = \partial_i u_j}$

$$\underline{\underline{L}} = \underbrace{\tfrac{1}{2}(\underline{\underline{L}} - \underline{\underline{L}}^T)}_{\underline{\underline{R}}} + \underbrace{\tfrac{1}{2}(\underline{\underline{L}} + \underline{\underline{L}}^T)}_{\underline{\underline{S}}}$$

Define tensors $R_{ij} = \frac{1}{2}(\partial_i u_j - \partial_j u_i)$

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

If ~~R~~ $\underline{u} = (\alpha y, 0, 0)$, then

$$\underline{\underline{L}} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow R = \begin{bmatrix} 0 & -\frac{1}{2}\alpha & 0 \\ \frac{1}{2}\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & \frac{1}{2}\alpha & 0 \\ \frac{1}{2}\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For velocity due to $\underline{\underline{R}}$, $\partial_x u_y = -\frac{1}{2}\alpha$, $\partial_y u_x = \frac{1}{2}\alpha$

$$\Rightarrow \boxed{\underline{u_r} = \tfrac{1}{2}\alpha(y, -x, 0)}$$

For velocity due to $\underline{\underline{S}}$, $\partial_x u_y = \frac{1}{2}\alpha$, $\partial_y u_x = \frac{1}{2}\alpha$

$$\Rightarrow \boxed{\underline{u_s} = \tfrac{1}{2}\alpha(y, x, 0)}$$

In fact any velocity field can be ~~dom~~ decomposed into ~~the~~ rotation and strain parts.

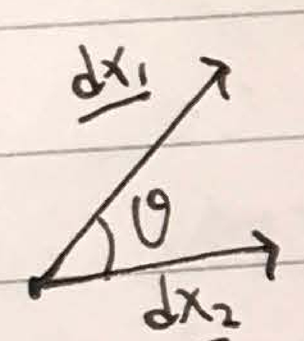
$$\underline{u}(\underline{r}+d\underline{r},t) = \underline{u}(\underline{r},t) + \frac{\partial \underline{u}}{\partial x}dx + \frac{\partial \underline{u}}{\partial y}dy + \frac{\partial \underline{u}}{\partial z}dz$$

$$= \underline{u}(\underline{r},t) + (\nabla \underline{u})^{T}\cdot d\underline{r} \qquad \text{If } (\nabla \underline{u})_{ij} = \frac{\partial u_i}{\partial x_j}$$

$$\Rightarrow d\underline{u} = \nabla \underline{u} \cdot d\underline{x}$$

$$\cancel{\left((\nabla \underline{u} \cdot d\underline{r})_j = \partial_i u_j \, dr_i\right)} \qquad \cancel{\left((\nabla \underline{u})^T \cdot d\underline{r}\right)_i = \partial_j u_i \, dr_j}$$

Consider 2 arbitrary vectors at point P, namely $d\underline{x}_1$ and $d\underline{x}_2$



then ~~the~~ $\frac{d}{dt}(d\underline{x}_1 \cdot d\underline{x}_2)$

$$= d\underline{x}_1 \cdot \frac{d}{dt}(d\underline{x}_2) + \frac{d}{dt}(d\underline{x}_1)\cdot d\underline{x}_2$$

$$= d\underline{x}_1^T \cdot d\underline{u}_2 + \cancel{d\underline{x}_2 \, d\underline{u}_1} \; d\underline{u}_1^T \cdot d\underline{x}_2$$

$$= d\underline{x}_1^T \cdot (\nabla \underline{u}) d\underline{x}_2 + \left((\nabla \underline{u}) d\underline{x}_1\right)^T \cdot d\underline{x}_2$$

$$= d\underline{x}_1^T (\nabla \underline{u}) d\underline{x}_2 + d\underline{x}_1^T (\nabla \underline{u})^T d\underline{x}_2$$

$$= 2\, d\underline{x}_1^T \left[\frac{1}{2}\left((\nabla \underline{u}) + (\nabla \underline{u})^T\right)\right] d\underline{x}_2 = 2\, d\underline{x}_1^T \underline{\underline{S}}\, d\underline{x}_2$$

If $d\underline{x}_1 = d\underline{x}_2$, then $d\underline{x}_1 \cdot d\underline{x}_1 = |d\underline{x}_1|^2 =$ length of a line element in the fluid

$$\frac{d}{dt}(|d\underline{x}_1|^2) = 2\, d\underline{x}_1^T \underline{\underline{S}}\, d\underline{x}_1$$

If $\nabla \underline{u}$ is anti-symmetric $\Rightarrow \underline{\underline{S}} = \underline{0} \Rightarrow |d\underline{x}_1|^2$ remains constant

$\Rightarrow$ no deformation for velocity field $\underline{u}_r$ with anti-symmetric velocity gradient.

The rate of deformation (strain) of $|dx_1|^2$ ~~is~~ only depends on the symmetric part of velocity gradient.

Consider $\underline{u_s}$ with symmetric gradient

$$\underline{\nabla}\underline{u_s} = \begin{bmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_y \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{bmatrix} \qquad (\partial_i u_j = \partial_j u_i)$$

$$\underline{\nabla} \times \underline{u_s} = \hat{\underline{x}}(\underbrace{\partial_y u_z - \partial_z u_y}_{0}) + \hat{\underline{y}}(\underbrace{\partial_z u_x - \partial_x u_z}_{0}) + \hat{\underline{z}}(\underbrace{\partial_x u_y - \partial_y u_x}_{0})$$

$\Rightarrow$ Velocity field that has ~~zero~~ symmetric gradient tensor has zero curl, so is irrotational

$\Rightarrow$ we can always express any velocity field as

$$\underline{u} = \underline{u_r} + \underline{u_s}$$

$\underline{\nabla}\underline{u_r} = \frac{1}{2}(\underline{\nabla}\underline{u} - (\underline{\nabla}\underline{u})^T)$

gives no deformation

$\underline{\nabla} \times \underline{u} = \underline{\nabla} \times \underline{u_r}$

~~gives~~ gives all rotation

$\underline{\nabla}\underline{u_s} = \frac{1}{2}(\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T)$

gives no rotation

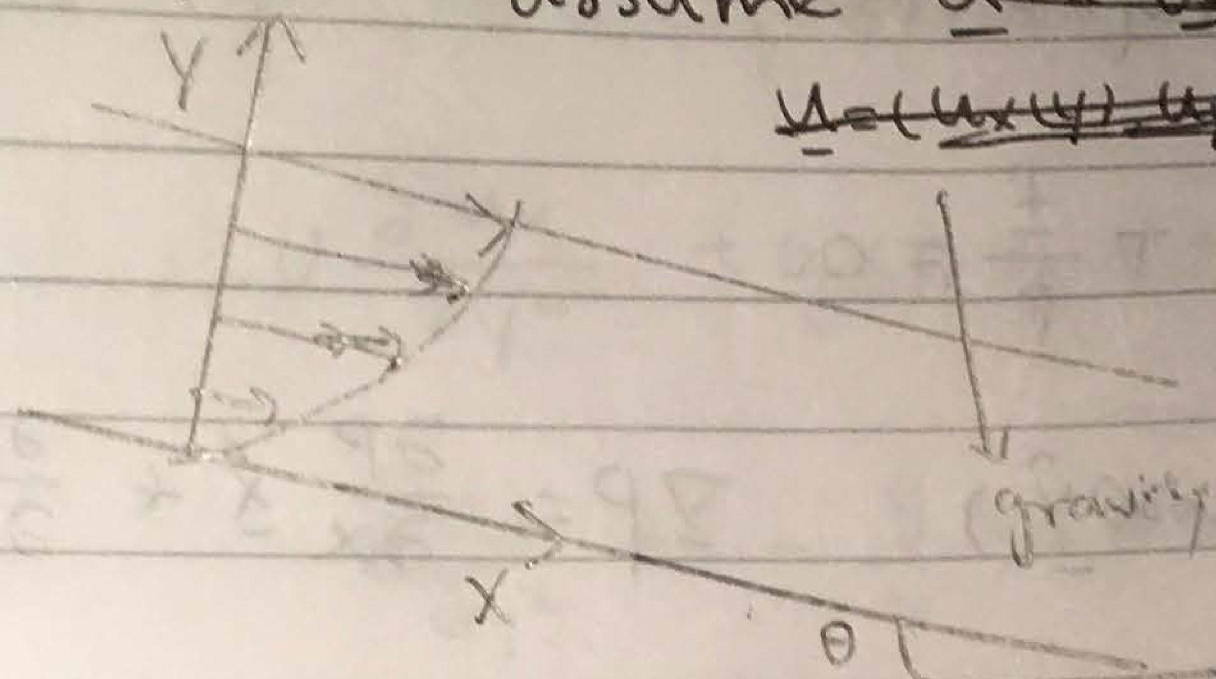
$\underline{\nabla} \times \underline{u_s} = 0$

gives all deformation.

④ (a) assume $\underline{u} = (u_x, u_y)^T$ ~~$\underline{u} = \underline{u}(y)$~~ ~~$\underline{u} = (u_x, u_y, 0)^T$~~

~~$\underline{u} = (u_x(y), u_y(y))$~~ and $u_x = u_x(y)$

$u_y = u_y(y)$



~~By symmetry $u_z = 0$~~

No slip boundary condition $\Rightarrow u_x(y=0) = 0$

incompressibility $\Rightarrow \underline{\nabla} \cdot \underline{u} = 0 \Rightarrow \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0$

$\because u_x = u_x(y) \quad \therefore \frac{\partial u_x}{\partial x} = 0 \Rightarrow \frac{\partial u_y}{\partial y} = 0$

$\because u_y = u_y(y)$ and $\frac{\partial u_y}{\partial y} = 0 \quad \therefore \boxed{u_y \text{ is constant}}$

Applying boundary condition $u_y(0) = 0$

$\Rightarrow \boxed{u_y = 0}$

(b) the Navier-Stokes equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla}) \underline{u} = -\frac{1}{\rho} \underline{\nabla} P + \nu \nabla^2 \underline{u} + \frac{\underline{f}}{\rho}$$

Steady flow $\Rightarrow \frac{\partial \underline{u}}{\partial t} = 0$

$\underline{u} = (u_x(y), 0)^T \Rightarrow (\underline{u} \cdot \underline{\nabla}) \underline{u} = \begin{pmatrix} u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} \\ u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} \end{pmatrix}$

$u_y = 0, \ \frac{\partial u_x}{\partial x} = 0 \Rightarrow$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ~~$\Rightarrow 0$~~

$\Rightarrow (\underline{u} \cdot \underline{\nabla}) \underline{u} = \underline{0}$

$\Rightarrow$ the N-S equation reduces to

$$-\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u} + \frac{\underline{f}}{\rho} = 0$$

$$\underline{f} = \rho g(\sin\theta\,\hat{\underline{x}} - \cos\theta\,\hat{\underline{y}}) \qquad \underline{\nabla} p = \frac{\partial p}{\partial x}\hat{\underline{x}} + \frac{\partial p}{\partial y}\hat{\underline{y}}$$

$$\nabla^2 \underline{u} = \frac{\partial^2 u_x}{\partial y^2}\hat{\underline{x}}$$

$$\Rightarrow \quad \hat{\underline{x}}: \quad \boxed{-\frac{1}{\cancel{\rho}}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u_x}{\partial y^2} + \cancel{\rho} g\sin\theta = 0}$$

$$\hat{\underline{y}}: \quad \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y} - \cancel{\rho} g\cos\theta = 0}$$

(c) $\quad \hat{\underline{y}} \Rightarrow \quad \frac{\partial p}{\partial y} = -\rho g\cos\theta \quad ①$

$\because$ flow is under the influence of gravity
No applied pressure gradient across the x-direction.

$\Rightarrow$ expect

$$\frac{\partial p}{\partial x} = 0 \qquad \Rightarrow \qquad p = p(y)$$

$$① \Rightarrow \quad p = -\rho g y\cos\theta + C(x)$$

Boundary condition: at $y=d$, $p=p_0$ for all x

$$\Rightarrow \quad p_0 = -\rho g d\cos\theta + C(x)$$

$$\Rightarrow \quad C = p_0 + \rho g d\cos\theta \qquad \boxed{\frac{\partial p}{\partial x} = \frac{\partial C}{\partial x} = 0} \text{ indeed}$$

$$\Rightarrow \quad \boxed{p = p_0 + \rho g(d-y)\cos\theta}$$

(d) the $\hat{x}$ equation with $\frac{\partial P}{\partial x} = 0$ is

$$\nu \frac{\partial^2 u_x}{\partial y^2} + g \sin\theta = 0 \quad \Rightarrow \quad \cancel{\nu \frac{d^2 u_x}{dy^2}}$$

$$\frac{d^2 u_x}{dy^2} = -\frac{g}{\nu} \sin\theta$$

$$\Rightarrow \quad u_x(y) = -\frac{g}{2\nu} \sin\theta \, y^2 + C_1 y + C_2$$

At $y = 0$, $u_x = 0 \Rightarrow C_2 = 0$

At $y = d$, $\frac{du_x}{dy} = 0 \Rightarrow$

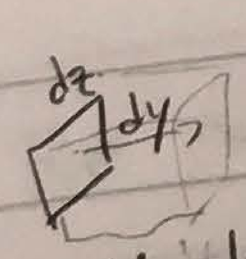
at $y = d$, $-\frac{g}{\nu} \sin\theta \cdot d + C_1 = 0$

$$\Rightarrow C_1 = \frac{g}{2\nu} \sin\theta \cdot \cancel{d}\, 2d$$

$$\Rightarrow u_x(y) = -\frac{g}{2\nu} \sin\theta \, y^2 + \frac{g}{2\nu} \sin\theta (2dy)$$

$$\Rightarrow \boxed{u_x(y) = \frac{g}{2\nu} y(2d - y) \sin\theta}$$

(e) Volume ~~flux~~ flow through $dydz$ in time dt is.



$dV = (dz dy)(u_x dt)$

$\Rightarrow$ volume flux is

$$\frac{dV}{dt} = u_x dy dz$$

Volume flux per unit ~~distance~~ distance along $\hat{z}$ is $\frac{dV}{dz\,dt} = u_x dy$ for length dy along $\hat{y}$

Hence total volume flux per dz is

$$\int_0^d u_x \, dy = \frac{g}{2\nu}\sin\theta \int_0^d 2dy - y^2 \, dy$$

$$= \frac{g}{2\nu}\sin\theta \left[dy^2 - \frac{1}{3}y^2 \right]_0^d$$

$$= \frac{g}{2\nu}\sin\theta \left[\frac{2}{3}d^3 \right] = \boxed{\frac{gd^3\sin\theta}{3\nu}}$$

(5) the Reynolds number is

$$Re = \frac{UL}{\nu}$$, where U = velocity scale
L = length scale
ν = kinematic viscosity.

(a) for a jumbo ~~jet~~ jet

$U \sim 1000\ km/h \sim 278\ m/s$

$L \sim 10\ m$

$\nu \sim 1.5 \times 10^{-5}\ m^2/s.$

$\boxed{Re \sim 1.85 \times 10^{8}}$

(b) ~~for~~ a human swimmer

$U \sim 2 m/s$

$L \sim 2m$ $\qquad \nu \sim 10^{-6} m^2/s.$

$\Rightarrow \boxed{Re \sim 4 \times 10^{6}}$

(c) for a thick layer of treacle draining off a spoon.

$U \sim 10^{-2} m/s$ $\qquad \cancel{L \sim 10^{-2} m/s}$ $\quad L \sim 10^{-1} m/s$

$\nu \sim 10^{-1} m^2 s^{-1}$

$\Rightarrow \boxed{Re \sim 10^{-2}}$

(d) ~~at~~ for a bacterium swim in water

$U \sim 100\ \mu m/s = 1 \times 10^{-4}\ m/s.$

$L \sim 5\ \mu m = 5 \times 10^{-6}\ m$

$\nu \sim 10^{-6}\ m^2/s$

$\Rightarrow \boxed{Re \sim 5 \times 10^{-4}}$

(6) $(\underline{u}\cdot\underline{\nabla})\underline{u} = -\frac{1}{\rho}\underline{\nabla}p + \frac{1}{\rho\mu}(\underline{\nabla}\times\underline{B})\times\underline{B} + \nu\nabla^2\underline{u}$ $\boxed{1}$

$(\underline{u}\cdot\underline{\nabla})\underline{B} = (\underline{B}\cdot\underline{\nabla})\underline{u} + \frac{1}{\sigma\mu}\nabla^2\underline{B}$ $\boxed{2}$

Define length scale L, velocity scale U and magnetic scale B_0

U = typical velocity of system
B_0 = typical magnetic field of system
L = typical distance across which the system's dynamics change significantly.

Define dimensionless variables and operators.

$\tilde{\underline{u}} = \frac{\underline{u}}{U}$ $\quad \tilde{\underline{\nabla}} = L\underline{\nabla}$

$\tilde{\underline{B}} = \frac{\underline{B}}{B_0}$ $\quad \left(\frac{\tilde{p}}{\rho}\right) = \frac{1}{U^2}\frac{p}{\rho}$ $\quad \tilde{t} = \frac{U}{L}t.$

$\boxed{1} \Rightarrow \frac{U^2}{L}(\tilde{\underline{u}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{u}} = -\frac{U^2}{L}\tilde{\underline{\nabla}}\left(\frac{\tilde{p}}{\rho}\right) + \frac{B_0^2}{\rho\mu L}(\tilde{\underline{\nabla}}\times\tilde{\underline{B}})\times\tilde{\underline{B}}$
$+ \frac{\nu U}{L^2}\tilde{\underline{\nabla}}^2\tilde{\underline{u}}$

$\Rightarrow (\tilde{\underline{u}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{u}} = -\tilde{\underline{\nabla}}\left(\frac{\tilde{p}}{\rho}\right) + \frac{B_0^2}{\rho\mu U^2}(\tilde{\underline{\nabla}}\times\tilde{\underline{B}})\times\tilde{\underline{B}}$
$+ \frac{\nu}{UL}\tilde{\underline{\nabla}}^2\tilde{\underline{u}}$

$\boxed{2} \Rightarrow \frac{UB_0}{L}(\tilde{\underline{u}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{B}} = \frac{UB_0}{L}(\tilde{\underline{B}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{u}} + \frac{1}{\sigma\mu}B_0\frac{1}{L^2}\tilde{\underline{\nabla}}^2\tilde{\underline{B}}$

$\Rightarrow (\tilde{\underline{u}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{B}} = (\tilde{\underline{B}}\cdot\tilde{\underline{\nabla}})\tilde{\underline{u}} + \frac{1}{\sigma\mu UL}\tilde{\underline{\nabla}}^2\tilde{\underline{B}}$

So we have 3 dimensionless control parametres

$$\boxed{\begin{aligned} R_1 &= Re = \frac{UL}{\nu} \\ R_2 &= \frac{\rho\nu U^2}{B_0^2} \\ R_3 &= \sigma\nu UL \end{aligned}}$$

→ ~~RE~~ R_1 is the usual Reynold's number describing the viscosity of fluid

→ R_2 is the ratio between magnetic energy and kinetic energy. It describes the strength of magnetic field.

→ R_3 describe the ~~CO~~ strength of electric current generated by the flow of fluid. as a response to an external field

→ Condition for Dynamical ~~Sy~~ Similarity is

① ~~RA~~ R_1, R_2, R_3 are equal for 2 systems. ~~and the 2 systems have~~

② Boundary conditions of 2 systems are scaled accordingly.

7 (a) the dimensionless Navier-Stoke's equation:

$$\frac{\partial \tilde{\underline{u}}}{\partial t} + (\tilde{\underline{u}} \cdot \tilde{\underline{\nabla}})\tilde{\underline{u}} = -\tilde{\underline{\nabla}}\left(\frac{\tilde{\Delta P}}{\rho}\right) + \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{u}}$$

inertial term $\quad (Re = \frac{UL}{\nu}) \quad$ viscous term

(i) if $Re \ll 1$, inertial term can be neglected

$$\therefore \quad 0 = -\tilde{\underline{\nabla}}\left(\frac{\tilde{\Delta P}}{\rho}\right) + \frac{\nu}{UL}\tilde{\nabla}^2\tilde{\underline{u}} \qquad \left(\begin{matrix}\nabla \sim \frac{1}{L} \\ u \sim U\end{matrix}\right)$$

$$\Rightarrow \quad -\frac{1}{\rho}\nabla(\Delta P) + \nu\nabla^2\underline{u} = 0$$

$$\Rightarrow \quad \frac{\Delta P}{\rho L} \sim \frac{\nu U}{L^2} \quad \Rightarrow \quad \boxed{\Delta P \sim \frac{\rho \nu U}{L}}$$

(ii) if $Re \gg 1$, viscous term can be neglected

$$\frac{\partial \tilde{\underline{u}}}{\partial t} + (\tilde{\underline{u}} \cdot \tilde{\underline{\nabla}})\tilde{\underline{u}} = -\tilde{\underline{\nabla}}\left(\frac{\tilde{\Delta P}}{\rho}\right) \qquad \left(\begin{matrix}\nabla \sim \frac{1}{L} \\ u \sim U\end{matrix}\right)$$

$$\Rightarrow \quad \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \underline{\nabla})\underline{u} = -\underline{\nabla}\left(\frac{\Delta P}{\rho}\right)$$

$$\frac{U^2}{L} \sim \frac{\Delta P}{\rho L} \quad \Rightarrow \quad \boxed{\Delta P \sim \rho U^2}$$

(b) the viscosities $\quad \nu_{water} \approx 1 \times 10^{-6}\ m^2/s$

$\nu_{air} \approx \cancel{?}\ 15 \times 10^{-6}\ m^2/s$

For Dynamical Similarity

$\Rightarrow Re(air) = Re(water)$

$$\frac{U_{air} L_{air}}{\nu_{air}} = \frac{U_{water} L_{water}}{\nu_{water}}$$

∵ same ~~wind tunnel~~ size of plate ∴ $L_{water} = L_{air}$

∴ $$\frac{U_{water}}{U_{air}} = \frac{\nu_{water}}{\nu_{air}} = \cancel{\frac{1}{4}} \boxed{\frac{1}{15}}$$

~~U_{water}~~

typical frequency scale $= \frac{1}{T} \sim \frac{U}{L}$

∴ ~~$f_{water} = (f_{air}) \frac{U_{air}}{...}$~~

$$f_{water} = f_{air} \frac{U_{water}}{U_{air}} = \frac{1}{15}(0.5\ s^{-1})$$

$$= \boxed{0.033\ s^{-1}}$$

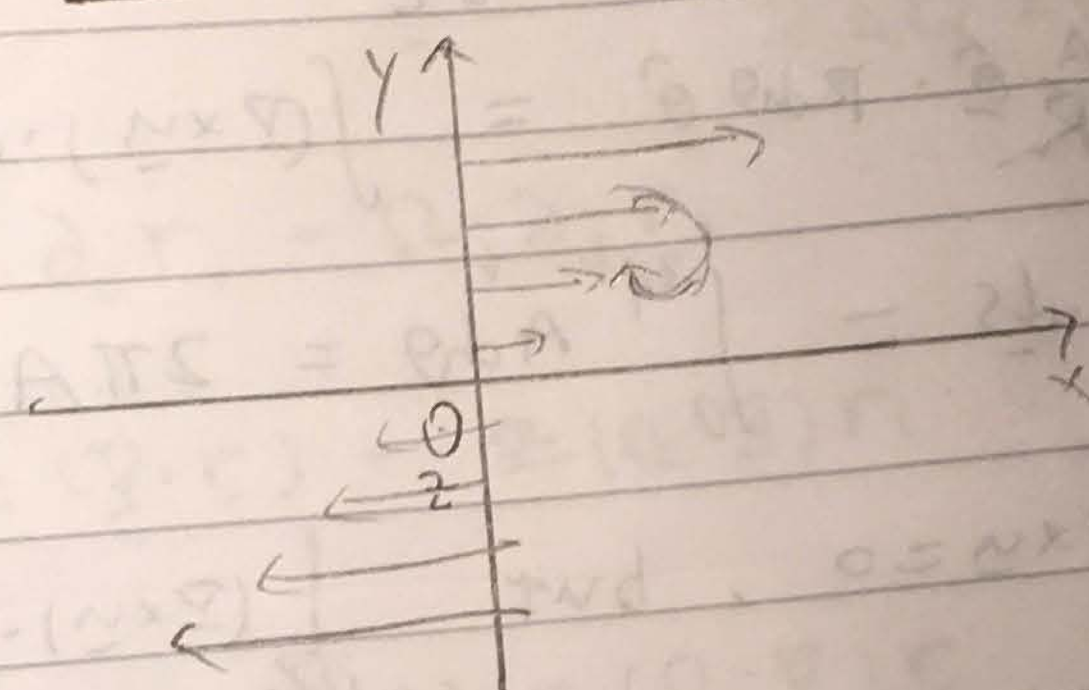
8 (a) Vorticity describes the local rotation of the fluid's velocity field.

$$\underline{\omega} = \underline{\nabla} \times \underline{u}$$

(i) $\underline{u} = (\alpha y, 0, 0)^T$

$$\underline{\omega} = \underline{\nabla} \times \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \alpha y & 0 & 0 \end{vmatrix} = (0, 0, -\alpha)$$

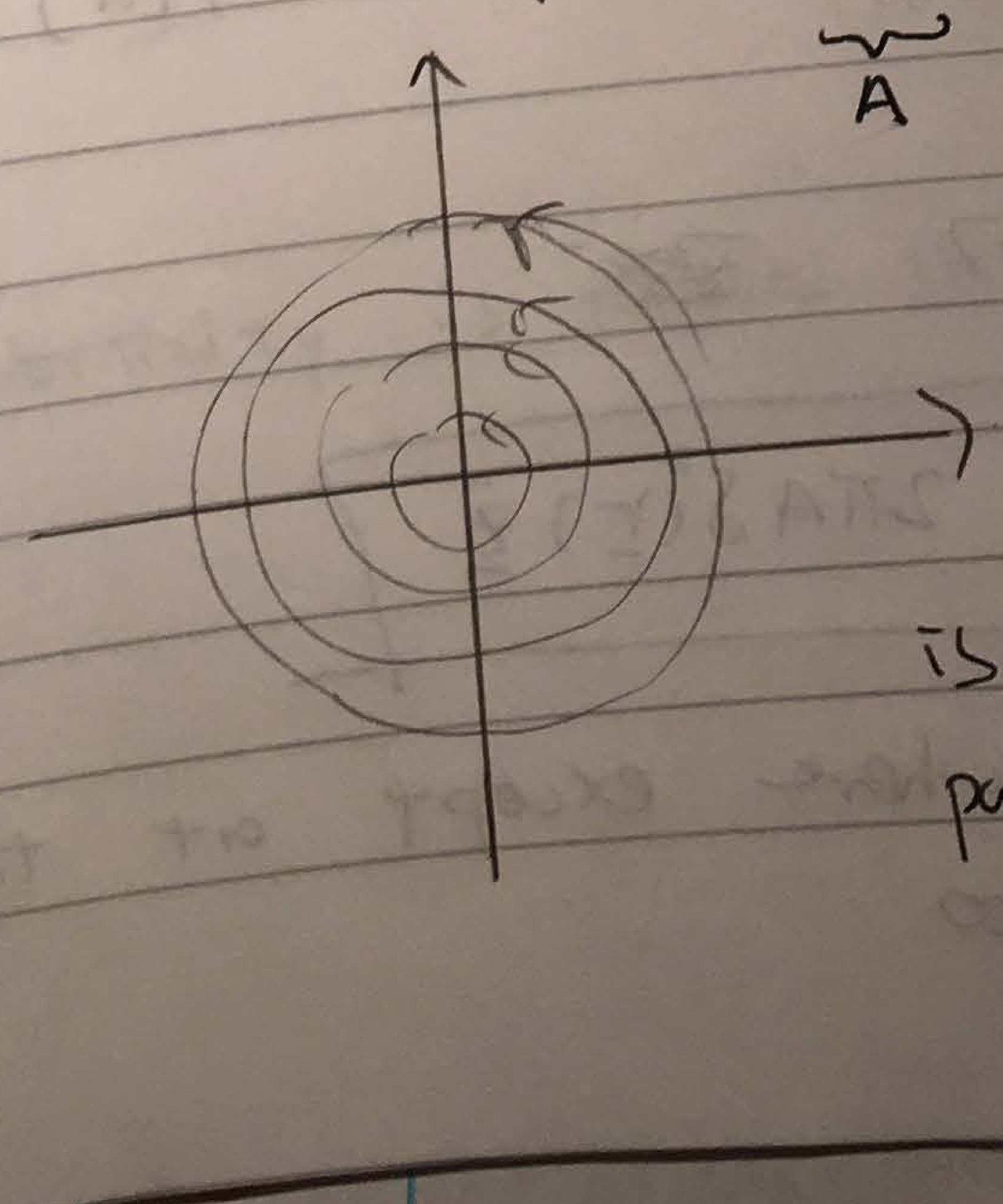
$$= \boxed{-\alpha \hat{\underline{z}}}$$



Vorticity meter ~~counterclock~~ has a clockwise local rotation going into the page

(ii) $\underline{u} = \frac{A}{r}\hat{\underline{\theta}}$

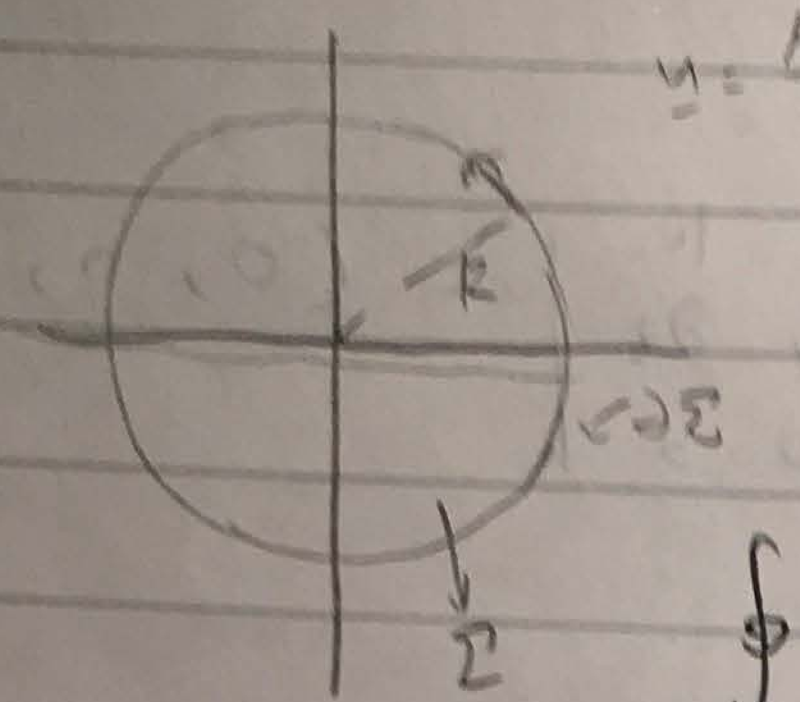
$$\underline{\nabla} \times \underline{u} = \frac{1}{r} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & \hat{\underline{z}} \\ \partial_r & \partial_\theta & \partial_z \\ 0 & \underbrace{\frac{A}{r} \cdot r}_{A} & 0 \end{vmatrix} = (0, 0, 0) = \underline{0}$$



Although the $\underline{u}$-field looks rotating, the vorticity meter will not rotate because there is no local spin for this particular $\underline{u}$-field

Except at the origin, where the cancel[illegible]
$\frac{A}{r}\cdot r$ is not well-defined.

$\therefore$ At $r=0$, $\underline{\omega} \neq 0$



~~Stoke's formula~~.

Stoke's theorem

$$\oint_{\partial\Sigma} \underline{u}\cdot d\underline{l} = \int_{\Sigma} (\nabla\times\underline{u})\cdot d\underline{S}$$

$$\Rightarrow \oint \frac{A}{R}\hat{\theta}\cdot R\,d\theta\,\hat{\theta} = \int (\nabla\times\underline{u})\cdot d\underline{S}$$

$$\Rightarrow \int_{\Sigma} (\nabla\times\underline{u})\cdot d\underline{S} = \int_0^{2\pi} A\,d\theta = 2\pi A$$

for $r\neq 0$ $\nabla\times\underline{u} = 0$, but $\int_{\Sigma} (\nabla\times\underline{u})\cdot d\underline{S} = 2\pi A \neq 0$

$\therefore$ we must have at $r=0$, $(\nabla\times\underline{u}) \neq 0$

the Dirac delta $\delta^2(\underline{r}) = \frac{1}{r}\delta(r)\delta(\theta)$

let $(\nabla\times\underline{u}) = C\,\delta^2(\underline{r})\,\hat{\underline{z}}$, then $\int (\nabla\times\underline{u})\cdot d\underline{S}$

$$= \int_0^{2\pi}\int_0^{R} \frac{1}{r} C\,\delta(r)\,\delta(\theta)\,r\,dr\,d\theta = C$$

$$\left(\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a) \right)$$

$\Rightarrow$ $C = 2\pi A$ $\Rightarrow$

$$\boxed{\underline{\omega} = \nabla\times\underline{u} = 2\pi A\,\delta^2(\underline{r})\,\hat{\underline{z}}}$$

vorticity is 0 everywhere except at the origin, where vorticity $\to \infty$

(b) angular velocity Ω = constant

rigid body $\underline{u} = \underline{\Omega} \times \underline{r}$

$\therefore$ vorticity $\underline{\omega} = \underline{\nabla} \times \underline{u} = \underline{\nabla} \times (\underline{\Omega} \times \underline{r})$

$\Rightarrow \omega_i = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \Omega_l r_m$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_j \Omega_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \underbrace{\Omega_l \partial_j}_{} r_m$ ∂_i

$\partial_i \Omega_j = 0$, Ω is constant spatially

$= \Omega_i \partial_j r_j - \Omega_j \partial_j r_i$

$= \Omega_i (\underline{\nabla} \cdot \underline{r}) - (\underline{\Omega} \cdot \underline{\nabla}) r_i$

$\Rightarrow \underline{\omega} = \underline{\Omega} (\underline{\nabla} \cdot \underline{r}) - (\underline{\Omega} \cdot \underline{\nabla}) \underline{r}$

$\underline{\nabla} \cdot \underline{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$

$$(\underline{\Omega} \cdot \underline{\nabla}) \underline{r} = \begin{pmatrix} \Omega_x \frac{\partial x}{\partial x} + \Omega_y \frac{\partial y}{\partial x} + \Omega_z \frac{\partial z}{\partial x} \\ \Omega_x \frac{\partial y}{\partial x} + \Omega_y \frac{\partial y}{\partial y} + \Omega_z \frac{\partial y}{\partial z} \\ \Omega_x \frac{\partial z}{\partial x} + \Omega_y \frac{\partial z}{\partial y} + \Omega_z \frac{\partial z}{\partial z} \end{pmatrix} = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} = \underline{\Omega}$$

$\Rightarrow \underline{\omega} = 3\underline{\Omega} - \underline{\Omega} = \boxed{2\underline{\Omega}}$

vorticity is twice the angular velocity.