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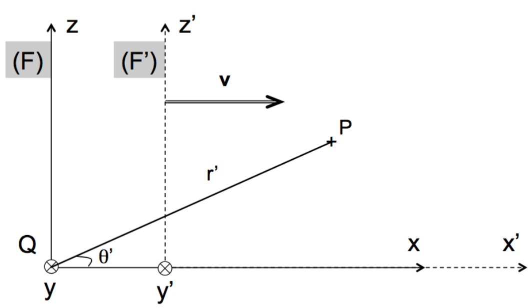
Problem set 4

Special relativity, Radiation, Transmission lines

Electromagnetism and special relativity

Problem 1: Electric field of a point charge moving with constant velocity

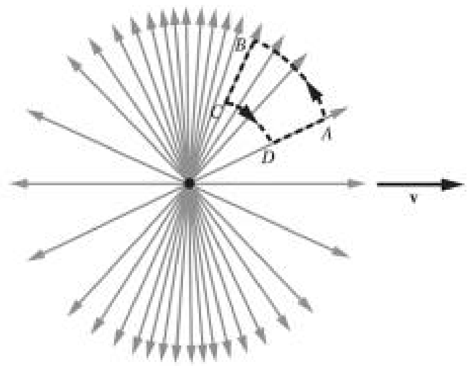
A point charge Q is at rest at the origin in an inertial frame (F). At a point P with cartesian coordinates $(x, 0, z)$ the field measured in (F) is $\mathbf{E}$. We consider another inertial frame (F') which moves in the positive x–direction with speed $\mathbf{v}$ with respect to (F).



a) Find the electric field $\mathbf{E}'$ produced at P by this charge as measured in (F') and in terms of the polar coordinates (r', θ') in (F') centered on the charge. Show that the field is parallel to the unit vector $\hat{\mathbf{r}}'$. Find $\mathbf{E}'$ in the limit $v \ll c$.

b) We are going to show that Gauss's law is satisfied in (F'). Choose as Gauss's surface the surface of a sphere at rest in (F') and centered on the charge at some particular time t'. Use spherical coordinates such that the polar angle θ' is measured from the x'–axis, so that there is azimuthal symmetry. Calculate the flux of $\mathbf{E}'$ through the surface of this sphere and show that it is equal to Q/ϵ_0. We give:

$$\int \frac{du}{(a^2 + u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2 + u^2}}.$$

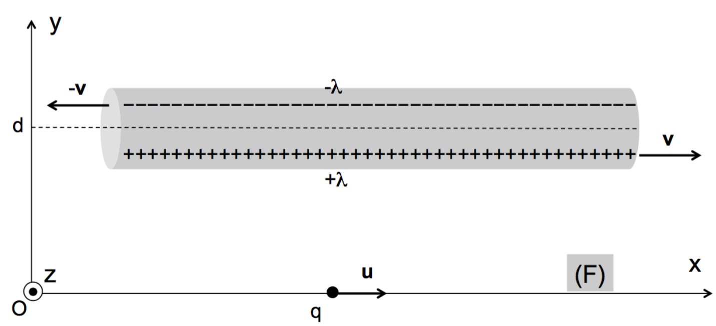


c) The figure (from E. Purcell & D. Morin, *Electricity and Magnetism*) shows a representation of the electric field in (F') (the dot in the centre represents the charge, and the density of field lines indicates the intensity of the field). Justify that the circulation of the electric field along the closed path $ABCDA$ is non zero. Comment.

d) Calculate the magnetic field $\mathbf{B}'$ of the charge Q as measured in (F'). [*Hint:* Use the fact that the magnetic field is zero in (F).] Find $\mathbf{B}'$ in the limit $v \ll c$. What is the magnetic field of a charge moving at constant velocity $\mathbf{v}$?

Problem 2: Interaction between a moving charge and other moving charges

In an inertial frame (F), we consider a line of positive charges all moving to the right with constant speed v. There is also a line of negative charges all moving to the left with the same speed v. The total charges of the positive and negative lines are equal and opposite.



We view the lines as continuous distributions with linear charge densities $+\lambda$ and $-\lambda$, as measured in (F). This is an ideal representation of a wire containing both ions and electrons. At a distance d from the axis of the wire, there is a charge q which moves to the right with speed $u < v$.

a) Calculate the net charge and the net current in the wire. Find the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ at the position of the charge, and the force $\mathbf{f} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$ on q, as measured in (F).

b) We now consider the inertial frame (F') which moves to the right at speed u. The charge q is at rest in (F'). Find the velocities v'_+ and v'_- of the positive and negative lines in (F') and show that $\gamma_\pm \equiv 1/\sqrt{1 - v'^2_\pm/c^2} = \gamma\left(1 \mp uv/c^2\right)/\sqrt{1 - u^2/c^2}$, where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Calculate the charge densities $\lambda'_\pm$ of the lines as measured in (F') and the net charge $\lambda' = \lambda'_+ + \lambda'_-$ in terms of λ, u, v and c. Find the electric field $\mathbf{E}'$ at the position of the charge, and the force $\mathbf{f}' = q\mathbf{E}'$ on q, as measured in (F'). Compare $\mathbf{f}$ and $\mathbf{f}'$.

Radiation

Problem 3: Radial oscillations of charges

An electric charge Q is distributed within a sphere of radius R in such a way that the distribution is spherically symmetric for all times. The distribution undergoes radial oscillations. Find the electric and magnetic fields outside the sphere. Does the charge distribution radiate? Comment.

Problem 4: Atom emitting light

Larmor's formula can be used to obtain a simple estimate of the lifetime for free decay of an excited atom. We consider a simple classical model of an atom, consisting of an electron with charge $q = -e$ and mass m which is bound to a heavy nucleus by a spring with spring constant $k = m\omega_0^2$. If the atom is given an excitation energy E_0 at time $t = 0$, it oscillates non–relativistically with weakly damped harmonic motion at frequency ω_0. We neglect the small change of frequency due to damping. Therefore, if we call x the direction along the electron's motion, the position of the electron relative to its equilibrium position can be written as $x(t) = A\mathrm{e}^{-t/\tau}\cos(\omega_0 t)$, where τ is a damping timescale.

a) Calculate the energy $E(t)$ of the electron at time t in terms of A, ω_0 and τ by assuming $\omega_0\tau \gg 1$.

b) Given that energy is lost through radiation, write dE/dt at time t in terms of A, ω_0, τ and other constants. Does E vary significantly during a period of oscillations? Justify that when calculating dE/dt we can take the time–average over one period of oscillation.

c) By comparing E and dE/dt obtained in questions (a) and (b), calculate the damping timescale τ. For an atom emitting visible light, we can take the wavelength to be $\lambda_0 = 5000$ $\mathring{A}$. Give the numerical value of τ, and compare with the lifetime of a typical excited state in a freely–decaying atom, which is $\sim 10^{-8}$ s. Justify the assumption $\omega_0\tau \gg 1$ used in the problem.

Please turn over...

Problem 5: Classical lifetime of a Bohr atom

In Bohr's theory of hydrogen, the electron in its ground state moves in a circular orbit of radius $a_0 = 0.53 \times 10^{-10}$ m around the (fixed) proton. Since the electron is accelerating, a classical analysis suggests that it will continuously radiate energy, and therefore the radius of the orbit would shrink with time. In this problem, we calculate the time τ it would take the electron to spiral in to the nucleus. We assume that τ is large compared to the orbital period of the electron, so that the orbit remains nearly circular at all times (this is called the *adiabatic* approximation).

a) Calculate the acceleration of the electron in terms of the radius $r(t)$ of the orbit at time t.

b) Calculate the energy $E(t)$ of the electron in terms of $r(t)$.

c) Given that energy is lost through radiation, calculate dE/dt in terms of $r(t)$. Then, using the result of question (b), write dr/dt in terms of $r(t)$. Solve for r and calculate τ. Check that τ is large compared to the orbital period of the electron, as we have assumed in this problem. Comment on the value of τ.

Problem 6: Why is the sky blue?

We consider an electron in an atom of air in the atmosphere driven at steady state by the electric field of the traveling electromagnetic wave produced by the Sun. The light emitted by the Sun covers the visible range, but here we consider a single color, that is to say a single Fourier component with frequency ω. We assume that the wave is polarized in the x–direction, so that the electric field at the location of the atom is $\mathbf{E} = E_0 \hat{\mathbf{x}} \cos(\omega t)$. We suppose that the electron is bound to the nucleus of the atom with spring constant $m\omega_0^2$, and we neglect damping (which means we assume that the driving frequency ω is not too close to the resonant frequency ω_0).

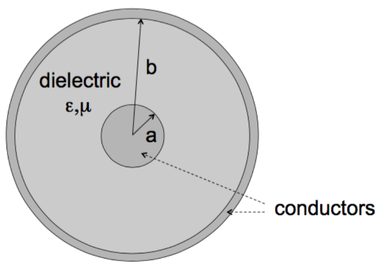
a) We call x the position of the electron relative to its equilibrium position. Write the equation of motion for the electron and calculate $x(t)$ in steady state.

b) For visible light, we have $\omega_0 \gg \omega$. Calculate the total power P_{rad} radiated by the electron. The wavelength of red and blue light is 6500 and 4500 $\mathring{A}$, respectively. Explain why the sky is blue, sunsets are red and the moon is red during total lunar eclipse.

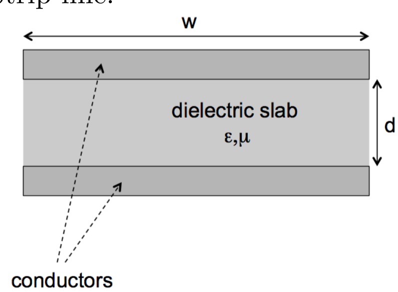
Transmission lines

Problem 7: Practical types of transmission lines
Calculate the capacitance per unit length, the inductance per unit length and the speed at which signals propagate for the following transmission lines. In each case, the conductors are separated by a material of electric permittivity ϵ and magnetic permeability μ. The figures show a cross section of the lines, which extend in the direction perpendicular to the figure.

a) Coaxial cable:



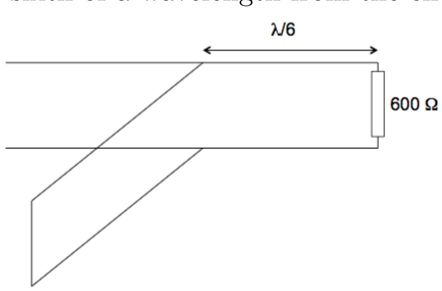
b) Strip line:



Problem 8: Short and open circuited transmission lines
Calculate the input impedance Z_1 of a length l of a loss-free transmission line of characteristic impedance Z_0 terminated by an open circuit. Calculate the input impedance Z_2 of the same transmission line terminated by a short circuit. Show that the lines can be used to create an equivalent inductor or capacitor. Show that $Z_1 Z_2 = Z_0^2$. Discuss the applications of such transmission lines (also called *stubs*).

Problem 9: Power transmitted into a load
A wave travels along a loss–free transmission line of characteristic impedance Z_1 which is terminated by a load of impedance Z_2. Show that the fraction of the incident power time–averaged over a wave period transmitted into the load is $t = 4Z_1 \,\mathrm{Re}(Z_2)/\left|Z_1 + Z_2\right|^2$.

Problem 10: Transmission lines in parallel
A transmission line with a characteristic impedance $Z_0 = 200\ \Omega$ is terminated by a resistor $R = 600\ \Omega$. Calculate the input admittance Y (inverse of the impedance) of the line at one–sixth of a wavelength from the end.



A short–circuited stub of line (with the same characteristic impedance) is added in parallel at this point. What should be its length in order to cancel the reflected signal travelling back towards the source? [*Hint:* Match the admittances.]