# Problem set 3

## Electromagnetic waves

### Problem 1: Poynting vector and resistance heating

*This problem is not about waves but is useful to understand how energy is transferred in a circuit.*

A cylindrical resistor with radius $a$, length $L$ and resistance $R$ is connected to a battery which supplies a potential difference $V$. The resulting steady current $I$ which flows along the resistor is uniform through its cross-section.

  a) Find the electric field **E** and magnetic field **B** on the surface of the resistor.

  b) Calculate the Poynting vector **S** on the surface of the resistor. Find the power flowing through the surface of the resistor. Compare with the rate at which work is done on the charges in the resistor by the electromagnetic field.

  c) We have found above that there is a flow of energy going into the resistor. This energy has to come from outside the resistor, and therefore there has to be an electric field outside giving an inward radial Poynting vector. Justify that this is the case.

  d) The energy entering the resistor comes from the battery: energy flows *along* the wires connecting the battery to the resistor. This requires a component of the electric field *perpendicular* to the wires. Such a field is produced by charges that have to be present on the surface of the wires for a steady current to flow. Assuming the charge distribution sketched on the figure, draw the Poynting vector in the different parts of the circuit to show how energy flows from the battery to the resistor. Comment on *where* the energy flows. Is there any energy returning to the battery?
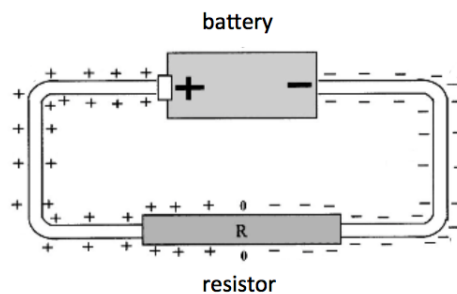


(adapted from Galili & Goihbarg, 2005, *Energy transfer in electrical circuits: a qualitative account*, American Journal of Physics, 73, 141)

## Problem 2: Standing wave and radiation pressure
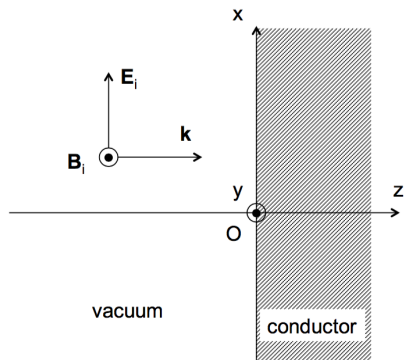


We consider an electromagnetic monochromatic plane wave which travels in vacuum and hits at normal incidence the plane surface of a *perfect* conductor which fills the half–space $z > 0$. The wave is linearly polarized in the $x$–direction and travels in the $z$–direction.

**a)** Find the transmitted and reflected waves in terms of the amplitude of the incident wave.

**b)** Write the total electric field $E$ and magnetic field $B$ in the half–space $z < 0$. Find the position (expressed in terms of the wavelength $\lambda$ of the wave) of the *nodal* planes, where $E$ and $B$ are zero, and justify why this is called a *standing* wave. Draw $E$ as a function of $z$ at different times (expressed in terms of the period $T$ of the wave).

**c)** Calculate the Poynting vector associated with the wave. Find its value in the nodal planes and describe how the energy flows in space. Calculate the time–average of the Poynting vector over a period at position $z$.

**d)** Calculate the time–average of the energy density over a period at position $z$.

**e)** Calculate the charge and current densities at the surface of the conductor. The magnetic field due to an infinite current sheet with surface current $\mathbf{K}$ is $\mu_o K/2$. Find the magnetic field due to the current flowing at the surface of the conductor, and compare with the reflected magnetic field calculated in question (a). Comment. What is the source of the reflected electric field?

**f)** Calculate the time–average over a period of the force exerted by the electromagnetic wave on the current at the surface of the conductor. Show that it is equivalent to a *radiation pressure* $P = \epsilon_0 E^2$. [Remember that the field generated by the surface current itself does not exert a force on the surface.]

**g)** The result from the previous question indicates that the radiation pressure is equal to the energy density of the electromagnetic wave. We are now going to show why this is the case. We suppose that the conductor contracts by an average distance $dz$ under the action of the radiation pressure. Find the resulting increase of the electromagnetic energy. Calculate the work done by the electromagnetic field on the conductor to make it contract. What can you conclude? [Here we only consider the time–average of the different quantities over a period.]

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## Problem 3: Fresnel's formulas
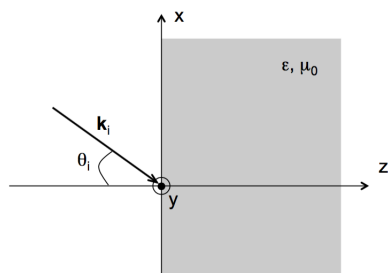


A monochromatic plane wave:

$$\tilde{\mathbf{E}}_i = \mathbf{E}_{0i}\, e^{i(\mathbf{k}_i\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_i = \frac{1}{c}\hat{\mathbf{k}}_i\times\tilde{\mathbf{E}}_i,$$

is incident on the flat surface of a linear medium with dielectric constant $\epsilon_r$ (and $\mu = \mu_0$) at an angle $\theta_i$ to the normal.

Here $\hat{\mathbf{k}}_i$ is the unit vector in the direction of propagation, and the *tilde* denotes complex quantities. We choose the origin of time so that the amplitude of the wave, $E_{0i}$, is real. This gives rise to reflected and transmitted waves:

$$\tilde{\mathbf{E}}_r = \tilde{\mathbf{E}}_{0r}\, e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_r = \frac{1}{c}\hat{\mathbf{k}}_r\times\tilde{\mathbf{E}}_r, \quad \text{and} \quad \tilde{\mathbf{E}}_t = \tilde{\mathbf{E}}_{0t}\, e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega t)}, \quad \tilde{\mathbf{B}}_t = \frac{1}{v}\hat{\mathbf{k}}_t\times\tilde{\mathbf{E}}_t,$$

where $v$ is the velocity of the wave in the dielectric.

**a)** Use the dispersion relation $\omega(k)$ and the boundary conditions at the interface to find the angle of reflection $\theta_r$ (between $\mathbf{k}_r$ and the $z$–axis) and the angle of transmission (or refraction) $\theta_t$ (between $\mathbf{k}_t$ and the $z$–axis) in terms of the angle of incidence $\theta_i$ (Snell's law). If the index of refraction $n < 1$, show that there is a critical angle $\theta_{ic}$ above which there is no transmitted wave (this is called *total internal reflection*).

**b)** We assume that the incident wave is linearly polarized with polarization normal to the plane of incidence. We assume that the reflected and transmitted waves have the same polarization (this can be shown by assuming a more general polarization and using the boundary conditions). Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of $E_{0i}$ and $\theta_i$ (Fresnel's formulas). Plot schematically the reflection coefficient $R = \left|\tilde{E}_{0r}/E_{0i}\right|^2$ as a function of $\theta_i$ for the cases $n > 1$ and $n < 1$.

**c)** Suppose now that the incident wave is linearly polarized with polarization parallel to the plane of incidence. Again we assume that the reflected and transmitted waves also have a linear polarization parallel to the plane of incidence. Use the boundary conditions to find $\tilde{E}_{0r}$ and $\tilde{E}_{0t}$ as a function of $E_{0i}$ and $\theta_i$ (Fresnel's formulas). Show that there is no reflection at a certain value of $\theta_i$, called *Brewster's angle*. Plot schematically $R$ as a function of $\theta_i$ for the cases $n > 1$ and $n < 1$. What happens to an unpolarized beam incident at Brewster's angle?

**d)** At Brewster's angle, show that $\theta_r + \theta_t = 90°$. The incident wave induces oscillations of the bound electrons at the surface of the dielectric. These electrons then act as oscillating dipoles which are the source of the reflected and transmitted waves. Given that dipoles do not radiate energy in the direction of their dipole moment, can you explain why there is no reflection at Brewster's angle?

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**Problem 4: Waves in the ionosphere**

The ionosphere is a region of the upper atmosphere which is ionized by solar (UV) radiation. It may be simply described as a dilute gas of charged particles, composed of electrons and ionized air ($N_2$ and $O_2$) molecules. The number density $n_e$ of free electrons is maximum at about 200 to 400 km above the surface of the Earth. At higher altitudes, $n_e$ decreases because the density of air molecules to be ionized decreases. At lower altitudes, $n_e$ decreases because UV radiation from the sun has been absorbed in layers above.

a) We assume that the ionosphere can be modelled as a collisionless plasma. Write the equation of motion for an electron assuming an external electric field and neglecting the local field. Show that the dielectric constant is given by $\epsilon_r = 1 - \omega_p^2/\omega^2$, with $\omega_p^2 = n_e e^2/(\epsilon_0 m)$, where $-e$ and $m$ are the electron's charge and mass, respectively. Write the dispersion relation $\omega(k)$. Show that for $\omega < \omega_p$, waves cannot propagate in the plasma.

b) Suppose that waves are emitted at the surface of the Earth toward the ionosphere. For waves with polarization both perpendicular and parallel to the plane of incidence, show that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is a total reflection back toward the Earth. [Use the results of Problem 3].

c) A radio amateur emits a wave with a wavelength of 21 m (corresponding to high frequency or shortwave broadcast) in the early evening. This wave is received by another radio amateur located 1000 km away, but not by other amateurs located closer. Assuming that the wave is being reflected by the so–called F layer of the ionosphere at a height of 300 km (the altitude at which reflection occurs can be determined from the travel time of the signal), calculate the electron density $n_e$. Compare with the known maximum and minimum F layer densities of $2 \times 10^6$ cm$^{-3}$ in the daytime and $(2\text{–}4) \times 10^5$ cm$^{-3}$ at night.

d) The cutoff frequency $\omega_p$ corresponding to $n_e \sim 10^5\text{–}10^6$ cm$^{-3}$ is of a few $10^7$ rad s$^{-1}$. FM radio and television signals have frequencies much larger than this value. What happens to these signals when they hit the ionosphere? Why are AM signals, with frequencies between 3 and 30 MHz, used for long distance (including intercontinental) communication?

e) From the BBC website: *AM reception can vary a great deal from day to night because of differences in the atmosphere. You may get good, clear reception during the day, but after sunset the signal may fade or become distorted. Signals travel further at night, so you may get interference from other transmitters, and you may even hear stations from outside the UK*. Why do radio signals travel farther at night than in the day?

## Problem 5: Anti–reflection coating
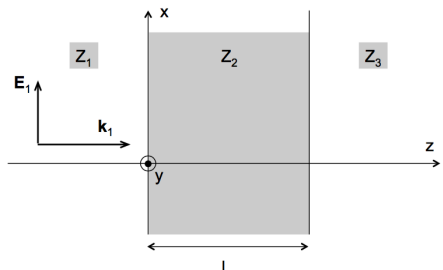


We consider three linear dielectric media with flat and parallel boundaries with impedances $Z_1$, $Z_2$ and $Z_3$. A linearly polarized electromagnetic monochromatic plane wave is normally incident from medium (1), through a layer of medium (2) of thickness $L$, into medium (3).

We consider $L = \lambda_2/4$, where $\lambda_2$ is the wavelength of the wave in medium (2).

a) Calculate the reflection coefficients $R_{12}$ between media (1) and (2) and $R_{23}$ between media (2) and (3).

b) Calculate the total reflection coefficient $R$ (that is to say the total intensity reflected into medium (1)).

c) Show that it is possible to choose $Z_1$, $Z_2$ and $Z_3$ such that there is no reflection ($R = 0$) into medium (1). Compare $R_{12}$ and $R_{23}$ in that case. By assuming that the total reflected wave in medium (1) is the superposition of the first wave reflected at the first interface and the first wave reflected at the second interface and transmitted back into medium (1), which is a good approximation when $R_{12}$ and $R_{23}$ are small compared to unity, explain why we obtain $R = 0$.

## Problem 6: Waves in a good conductor

We consider an ohmic conductor with conductivity $\sigma$, permittivity $\epsilon_0$ and permeability $\mu_0$. We assume that the conductor is excellent, that is to say $\sigma \gg \omega\epsilon_0$.

a) Show that the conductor can be described as a dielectric with complex dielectric constant $\tilde{\epsilon}_r = 1 + i\sigma/(\omega\epsilon_0)$. An electromagnetic monochromatic plane wave propagates through the conductor. Write the dispersion relation and find the wavenumber in terms of the penetration depth $\delta = \sqrt{2/(\omega\mu_0\sigma)}$. Compare the group and phase velocities. Is dispersion normal or anomalous? Show that the impedance is $Z = (1 - i)/(\delta\sigma)$.

b) Calculate the time–average over a period of the Poynting vector at a location $z$ along the direction of propagation of the wave. Calculate the time–average over a period of the rate at which work is done on the charges contained in the volume delimited by the surfaces at $z = z_1$ and $z = z_2$. Verify that it is equal to the time–average over a period of the flux of the Poynting vector into the volume.

c) Calculate the time–average over a period of the energy density. Is the energy carried mainly by the electric field or by the magnetic field? Calculate the velocity at which energy is transported. Compare with the group velocity. Comment.

*Please turn over...*

**Problem 7: Dielectric losses**

In general, the response of dielectric materials to a varying electric field is described by a *complex* dielectric constant, which is associated with energy losses. In this problem, we are going to derive an energy equation for the wave and quantify the losses.

a) Write Maxwell's equations for a dielectric with a complex dielectric constant and no conduction current nor free charges. First write the equations using complex notations, and then separate all the quantities into real and imaginary parts. Remember that the *physical* fields are the real part of the complex quantities. By taking the dot product of $\mathbf{E}$ with the equation giving $\mathbf{\nabla}\times\mathbf{B}$, derive an energy equation for the wave (follow the same procedure as in sections 4.3.1 and 4.3.2 of the Lecture Notes). Give the expression for the rate at which energy is dissipated per unit volume in the dielectric.

b) The time–average over a period of the Poynting vector in a dielectric with complex dielectric constant is:

$$\langle \mathbf{S} \rangle = \left\langle \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} \right\rangle = \hat{\mathbf{z}} c n' \frac{\epsilon_0 E_0^2}{2} e^{-2k''z},$$

where $\hat{\mathbf{z}}$ is the direction of propagation, $E_0 e^{-k''z}$ is the amplitude of the electric field, $n'$ is the real part of the complex index of refraction and $k''$ is the imaginary part of the complex wavenumber (see section 5.6.4 of the Lecture Notes). Verify that the flux of $\langle \mathbf{S} \rangle$ into a volume delimited by the surfaces at $z = z_1$ and $z = z_2$ is equal to the time–average over a period of the rate at which energy is dissipated in the volume.

c) In Maxwell's equations (in complex notations), we now add a conduction current that obeys Ohm's law. Show that the imaginary part of the dielectric constant can be incorporated into an *effective conductivity* and recover the rate of energy loss in the dielectric by drawing an analogy with ohmic dissipation.