

Problem set 2**Electric and magnetic fields in matter**

Some of the problems below are taken from:

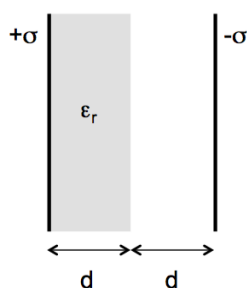
Introduction to Electrodynamics, David J. Griffiths, 4th Edition

Electric fields in matter**Problem 1: Capacitance and dielectrics**

We consider a parallel plate capacitor consisting of two metal surfaces of area A separated by a distance d . We assume $d \ll A^{1/2}$, so that edge effects can be neglected and the electric field can be considered to be uniform between the plates.

- a) The capacitor is connected to a battery so that a charge $+Q_0$ is brought from one plate to the other. Using Gauss's law, calculate the electric field E_0 between the plates. Find the potential difference ΔV_0 between the plates and the capacitance C_0 . Calculate the potential energy U_0 stored in the capacitor.
- b) We now insert a linear dielectric material between the plates while the capacitor remains connected to the battery, which supplies the potential difference ΔV_0 . Experimentally, it is found that the charge Q on the plates is increased. Explain why. By representing the system as the superposition of a vacuum capacitor and a polarized dielectric slab, calculate the total electric field E in the capacitor in terms of Q and the polarization P . By substituting $P = \epsilon_0 \chi_e E$, find the relation between the displacement vector \mathbf{D} and Q . Calculate Q in terms of Q_0 and the capacitance C in terms of C_0 . Calculate the change in the stored potential energy of the capacitor in terms of U_0 .
- c) We now come back to question (a) and disconnect the battery before inserting a dielectric material between the capacitor plates. Experimentally, it is found that the potential difference ΔV between the plates decreases. Explain why. Calculate the capacitance C in terms of C_0 and the change in the stored potential energy of the capacitor in terms of U_0 .

Problem 2: Capacitor half-filled with a dielectric

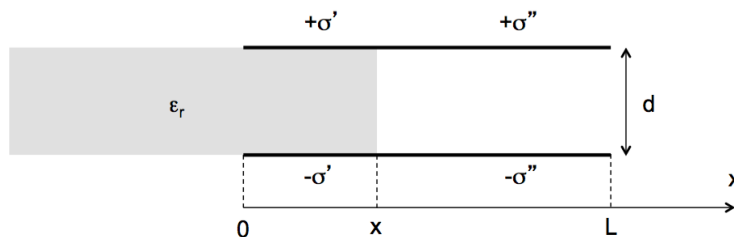


We insert a slab of linear dielectric material of dielectric constant ϵ_r and thickness d next to the positive plate in a parallel plate capacitor. The distance between the plates is $2d$, their area is A and they carry a (free) charge density $\pm\sigma$.

- Find the electric displacement vector \mathbf{D} , the electric field \mathbf{E} and the polarization \mathbf{P} in each region.
- Find the potential difference between the plates and the capacitance C . Show that the system can be regarded as two capacitors connected in series.
- Find the location and amount of all bound charge. Using the distribution of free and bound charges, recalculate the electric field \mathbf{E} in each region.

Problem 3: Force on a dielectric

We insert a portion of a slab of linear dielectric material of dielectric constant ϵ_r and thickness d on the left hand side of a parallel plate capacitor consisting of two conducting plates of length L , width w and thickness d .



- We connect the capacitor to a battery to charge the plates, and then disconnect the battery. The total charge on each plate then remains constant equal to $\pm Q$, corresponding to surface charge densities $\pm\sigma'$ and $\pm\sigma''$ on the left and right hand sides, respectively. Using Gauss's law, write the electric fields E' and E'' on the left and right hand sides in terms of σ' and σ'' . Find the relation between σ' and σ'' . Using the fact that Q is constant, write σ'' in terms of x and the different constants. Find the potential difference between the plates, and then the stored potential energy U of the system, in terms of x . Write the relation between the change in energy, dU , when the dielectric is pulled out a distance dx , and the electric force F exerted by the plates on the dielectric. Calculate F in terms of x . Does this force pull the dielectric into the capacitor or push it out?

- b) We now repeat the experiment but leave the capacitor connected to the battery, which supplies the potential difference ΔV . Calculate U in terms of x and the different constants. Write the relation between dU and F when the dielectric is pulled out a distance dx . Make sure to include *all* contributions to the work done in the system. Calculate F in terms of x and show that it is the same as in question (a).
- c) The result obtained above can be verified experimentally. However, we have assumed in our calculation that the electric field was uniform between the plates and zero outside. If that were the case in reality, would there be a force on the dielectric? Where does this force come from, and how is it possible that we obtain the right answer given the simplified model we have used?

Problem 4: Sphere with a frozen-in polarization

A sphere of radius R made of a dielectric material carries a frozen-in polarization $\mathbf{P}(\mathbf{r}) = k\mathbf{r}$, where k is a constant and \mathbf{r} is the position vector measured from the centre of the sphere. There are no free charges anywhere.

- a) Calculate the bound charges.
- b) Using Gauss's law, find the electric field \mathbf{E} inside and outside the sphere.
- c) Using the expression of \mathbf{E} and \mathbf{P} , find the electric displacement vector \mathbf{D} inside and outside the sphere. Check that it satisfies Gauss's law. Is the dielectric linear?

Problem 5: Electric field within a cavity inside a dielectric

The electric field inside a large piece of dielectric is \mathbf{E}_0 and the polarization is \mathbf{P} , so that the displacement vector is $\mathbf{D} = \epsilon_0\mathbf{E}_0 + \mathbf{P}$. A cavity is hollowed out of the material. It is small enough that the field and the polarization can be taken as uniform within it. We also assume that the polarization in the dielectric is frozen-in, so that it does not change when the cavity is hollowed out. Calculate, in the cavity, the field in terms of \mathbf{E}_0 and \mathbf{P} and the displacement in terms of \mathbf{D}_0 and \mathbf{P} in the following cases:

- a) The cavity is a small sphere [Hint: Use the superposition principle and, to calculate the electric field inside a uniformly polarized sphere, use the results of Problem 5 in Problem set 1],
- b) The cavity is shaped like a long thin needle parallel to \mathbf{P} ,
- c) The cavity is a thin, circular wafer perpendicular to the polarization.

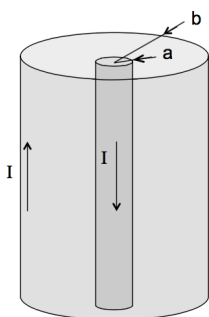
Magnetic fields in matter

Problem 6: Cylinder with a frozen-in magnetization

An infinitely long cylinder of radius R made of a magnetic material carries a frozen-in magnetization $\mathbf{M} = kr\hat{\mathbf{z}}$, where r is the distance from the axis of the cylinder, $\hat{\mathbf{z}}$ is the unit vector along the axis and k is a constant. There is no free current anywhere. Find the magnetic field \mathbf{B} inside and outside the cylinder by two different methods:

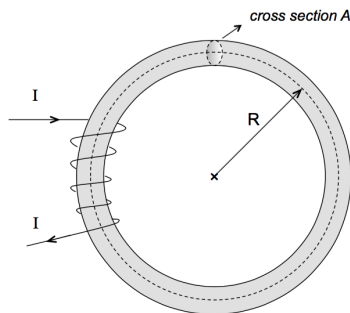
- Locate all the bound currents and calculate the field they produce,
- Use Ampère's law for \mathbf{H} and the relationship between \mathbf{B} , \mathbf{H} and \mathbf{M} to derive \mathbf{B} .

Problem 7: Magnetic field in a coaxial cable



A long coaxial cable of inner radius a and outer radius b is filled with an insulating material of magnetic susceptibility χ_m . A current I flows down the inner conductor and returns along the outer one, uniformly distributed over the surfaces. Find the magnetic field \mathbf{B} in the magnetic material between the conductors. As a check, calculate the magnetization \mathbf{M} and the bound currents, and confirm that, together with the free currents, they generate the correct field.

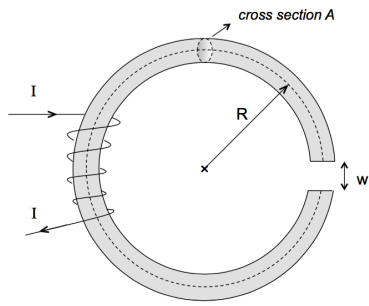
Problem 8: Air gap in an inductor



We consider a toroidal core made of an iron alloy with cross sectional area A and radius $R \gg A^{1/2}$. We make an inductor from the core by wrapping around it N turns of a wire carrying a current I , to use it as an energy storage device.

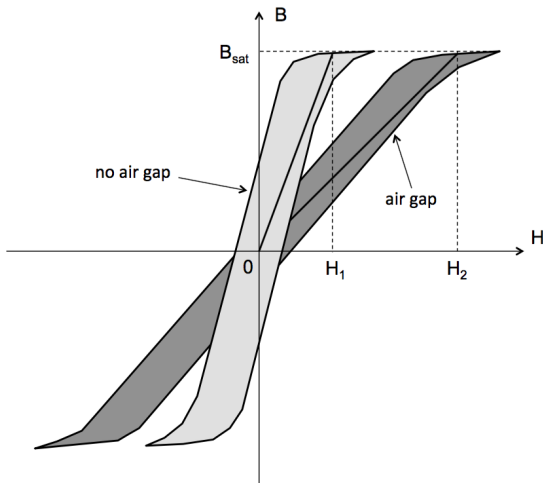
We increase the current from 0 until the material saturates, which happens when the magnetic field B inside it reaches the value B_{sat} . Before saturation is reached, the material can be regarded as being linear with relative permeability μ_r .

- Calculate the magnetic field B in the core and the inductance L . Find the maximum current I_1 that the inductor can carry before the core saturates. Calculate the magnetic energy W_1 that is stored in the inductor when the current reaches the value I_1 . If I is increased beyond I_1 , how does the inductance vary?



- b) To increase the energy that can be stored in the inductor, we have to find a way of increasing the saturation current while keeping L constant. This can be done by introducing an air gap of width w into the core. We take $w \ll R$, so that we can neglect magnetic fringing, that is to say the bulging out of the magnetic field lines as they enter air from the magnetic material.

Calculate the magnetic field B in the core and the number of turns of wire we now have to wrap around the core to keep L constant. Find the maximum current I_2 this second inductor can carry before the core saturates and compare it with I_1 . Calculate the energy W_2 stored in the magnetic field at the point of saturation and compare it with W_1 . Compare the energy stored in the magnetic material with that stored in the air gap. For numerical applications, use $R = 10$ cm, $w = 3$ mm and $\mu_r = 1500$.



- c) Some of the energy stored in the inductor can be recovered by decreasing the current back to 0. The figure shows the $B - H$ curves (hysteresis loops) measured on a toroidal core without and with gap (the straight lines inside the loops are the magnetization curves).

Show on these curves how much energy is released per unit volume by the magnetic material in the core after the current is returned to 0. Comment.

- d) Explain what would be the advantages of using iron, and especially *soft* iron, to make magnetic cores. Iron, however, has a major disadvantage when the inductor is used with alternating currents. Can you think of what it is?