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Problem set 1

Potentials

Problem 1: Magnetic vector potential

We consider a finite segment of straight wire of length $2L$ carrying a steady current I.

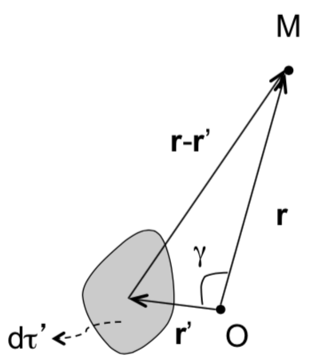
a) Calculate the magnetic vector potential $\mathbf{A}$ at a point P a distance r from the wire along the perpendicular bisector. We give:

$$\int \frac{dz}{\sqrt{z^2+a^2}} = \ln\left(z+\sqrt{z^2+a^2}\right).$$

b) Using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and assuming $L \gg r$, calculate the magnetic field at P. Check that your answer is consistent with what is expected from Ampère's law.

Problem 2: Expansion in Legendre polynomials

This problem is useful to understand that a multipole expansion is nothing more than a development in Taylor series of $1/|\mathbf{r}' - \mathbf{r}|$.



The scalar potential V created at a point M by a localised charge distribution is:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}d\tau',$$

where the integration is over the volume $\mathcal{V}$ of the distribution and the position vectors $\mathbf{r}$ and $\mathbf{r}'$ are measured from an origin O chosen arbitrarily.

Assuming $r' \ll r$, expand $1/|\mathbf{r}' - \mathbf{r}|$ in Taylor series up to 3rd order in r'/r. Use the result to give an expression for V in term of Legendre polynomials including the terms up to $l = 3$.

We give: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

Problem 3: Dipole

a) Show that the energy of a physical dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ (not necessarily uniform) is given by $U = -\mathbf{p} \cdot \mathbf{E}$.

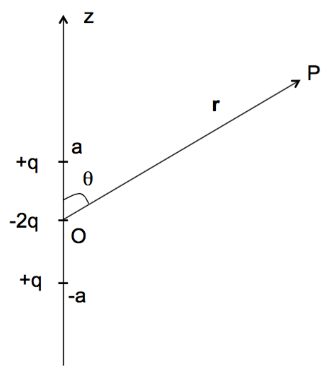
b) We consider two dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ and note $\mathbf{r}$ the position vector of $\mathbf{p}_2$ measured from $\mathbf{p}_1$. Show that the interaction energy of the two dipoles is:

$$U_{\text{int}} = \frac{1}{4\pi\epsilon_0 r^3} \left[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3\left(\mathbf{p}_1 \cdot \hat{\mathbf{r}}\right)\left(\mathbf{p}_2 \cdot \hat{\mathbf{r}}\right)\right],$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$.

c) Draw graphs showing how U_{int} depends upon the relative orientation of the dipoles in the following cases: (i) $\mathbf{p}_1$ is parallel to $\mathbf{r}$, (ii) $\mathbf{p}_1$ is perpendicular to $\mathbf{r}$.

Problem 4: Quadrupole



A system of three charges, aligned along the z–axis, consists of a charge $-2q$ at the origin O and two $+q$ charges at $z = -a$ and $z = a$. We note (r, θ, φ) the spherical coordinates.

a) Find the potential at $r \gg a$ using the multipole expansion with origin at O. Justify why this system of charges is called a quadrupole.
We give: $P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$.

b) Show that there is not translational force or couple on the quadrupole in a uniform electric field.

c) We consider a point charge Q placed at a point P a distance $r \gg a$ away from the quadrupole. Show that the torque about P on the quadrupole due to the charge is $\Gamma = 3Qqa^2 \sin 2\theta/(4\pi\epsilon_0 r^3)$ and find its direction.

d) Find the energy of the quadrupole in the potential of the charge Q. In which direction does the quadrupole rotate under the effect of the torque?

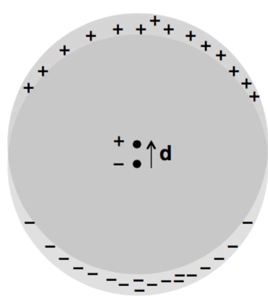
Problem 5: Electric field due to a sphere with surface charge density $\propto \cos\theta$
We consider a sphere of radius R and a spherical coordinate system (r, θ, φ) with origin at the centre of the sphere. The surface of the sphere carries a charge density $\sigma = k \cos\theta$.

a) Calculate the potential V using the fact that it satisfies Laplace's equation both inside and outside the sphere. *Hint*: write the solution of the equation in term of Legendre polynomials and choose $V = 0$ at infinity.

b) Find the electric field $\mathbf{E}$ produced by the sphere.

c) Calculate the electric dipole moment $\mathbf{p}$ of the charge distribution. Check that the electric field outside the sphere calculated in the previous question is that due to the dipole moment $\mathbf{p}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *polarization* of the sphere, which is the dipole moment per unit volume.

d) A surface charge density $\propto \cos\theta$ can be obtained by superimposing two solid spheres with same radius R, opposite charges Q and $-Q$ uniformly distributed over their volume, and centres slightly apart (separation $d \ll R$), as shown on the figure.



From a point outside the spheres, the situation is the same as if the charges were at the centre of the spheres. Find the relation between Q, $\mathbf{d}$ and $\mathbf{p}$ for the electric field outside the spheres to be the same as in the previous question. Using Gauss's theorem to obtain the contribution from each sphere, calculate the field in the region where the spheres overlap and show that it is the same as in the previous question.

Problem 6: Potential around a metal sphere in a uniform electric field

A metal sphere of radius R with no net charge is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector.

a) Explain qualitatively how the field modifies the charge distribution in the sphere and how this, in turn, affects the electric field.

b) Calculate the electric potential V outside the sphere using the fact that it satisfies Laplace's equation. *Hint*: write the solution of the equation in term of Legendre polynomials and choose $V = 0$ at the surface of the sphere.

c) Using the boundary condition on the electric field, calculate the charge density σ at the surface of the sphere.

d) Using the results from Problem 5, question b, give an expression for the electric field produced by σ. Find the total field inside the sphere. Comment.

Problem 7: Magnetic field due to a spinning sphere

We consider a sphere of radius R which carries a uniform surface charge density σ and spins with angular velocity ω around a diameter. We use spherical coordinates (r, θ, φ) with origin at the centre of the sphere and the z–axis along the rotation axis.

a) Find the surface current density $\mathbf{K}(\mathbf{r})$ as a function of θ. *Hint:* Consider a ring with a small thickness at the surface of the sphere and perpendicular to the rotation axis.

b) Justify that the magnetic field produced by the surface current can be written under the form $\mathbf{B} = \boldsymbol{\nabla}\Psi$, where Ψ is a scalar function. Show that Ψ satisfies Laplace's equation inside and outside the sphere. Write the boundary conditions on Ψ.

c) Write Ψ in term of Legendre polynomials. Show that the $l = 1$ term in the expansion, with appropriate coefficients, satisfies Laplace's equation inside and outside the sphere with the boundary conditions given above. Justify that this is **the** solution.

d) Find the magnetic field $\mathbf{B}$ produced by the sphere.

e) Consider a ring with a small thickness at the surface of the sphere and perpendicular to the rotation axis. Write the magnetic dipole moment of this ring. Find the total magnetic dipole moment $\mathbf{m}$ of the sphere.

f) Check that the magnetic field outside the sphere calculated in question d is that due to the dipole moment $\mathbf{m}$. What can you conclude about the higher multipoles? Express the field inside the sphere as a function of the *magnetization* of the sphere, which is the dipole moment per unit volume.

Problem 8: Separation of variables in cylindrical coordinates

a) Solve Laplace's equation by separation of variables in cylindrical coordinates, assuming there is no dependence on z.

b) Consider an infinitely long straight wire along the z–axis which carries a uniform line charge λ. Calculate the electric field using Gauss's theorem. Find the potential and check that it is a solution of Laplace's equation found in question a.

c) Consider an infinitely long metal pipe of radius R placed at right angle to a uniform electric field $\mathbf{E}_0$. Using the result from question a and appropriate boundary conditions, find the potential outside the pipe. Calculate the charge density induced on the pipe.