Second–Year Electromagnetism **Summer 2015**
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Vacation work: Problem set 0

Revisions

At the start of the second year, you will receive the second part of the Electromagnetism course. This vacation work contains a set of problems that will enable you to revise the material covered in the first year Electromagnetism course.

Some of the problems below are taken from:
Introduction to Electrodynamics, David J. Griffiths, 4th Edition
Electricity and Magnetism, Edward M. Purcell and David J. Morin, 3rd Edition.

Electrostatics

Problem 1: Field and potential from charged ring
A thin ring of radius a carries a charge q uniformly distributed. Consider the ring to lie in the x–y plane with its centre at the origin.

a) Find the electric field $\mathbf{E}$ at a point P on the z–axis.

b) Find the electric potential V at P.

c) A charge $-q$ with mass m is released from rest far away along the axis. Calculate its speed when it passes through the centre of the ring. (Assume that the ring is fixed in place).

Problem 2: Field from charged disc
The ring in the previous problem is replaced by a thin disc of radius a carrying a charge q uniformly distributed. Consider the disc to lie in the x–y plane with its centre at the origin.

a) Find the electric field $\mathbf{E}$ at a point P on the z–axis.

b) Check that the values of $\mathbf{E}$ at $z = 0$ and in the limit $z \gg a$ are consistent with expectations.

 Please turn over...

Problem 3: Hydrogen atom

According to quantum mechanics, the hydrogen atom in its ground state can be described by a point charge $+q$ (charge of the proton) surrounded by an electron cloud with a charge density $\rho(r) = -C\mathrm{e}^{-2r/a_0}$. Here a_0 is the Bohr radius, 0.53×10^{-10} m, and C is a constant.

a) Given that the total charge of the atom is zero, calculate C.

b) Calculate the electric field at a distance r from the nucleus.

c) Calculate the electric potential, $V(r)$, at a distance r from the nucleus. We give:

$$\int \left(\frac{1}{\alpha r'} + 1\right) \frac{\mathrm{e}^{-\alpha r'}}{r'} dr' = -\frac{\mathrm{e}^{-\alpha r}}{\alpha r}.$$

Problem 4: Energy of a charged sphere

We consider a solid sphere of radius a and charge Q uniformly distributed.

a) Calculate the electric field $E(r)$ and the electric potential $V(r)$ at a distance r from the centre of the sphere.

Find the energy U stored in the sphere three different ways:

b) Use the potential energy of the charge distribution due to the potential $V(r)$:

$$U = \frac{1}{2} \int_{\mathcal{V}} \rho V \, d\tau,$$

where ρ is the charge density and the integral is over the volume $\mathcal{V}$ of the sphere.

c) Use the energy stored in the field produced by the charge distribution:

$$U = \int_{\text{space}} \frac{\epsilon_0 E^2}{2} d\tau,$$

where the integral is over *all space*.

d) Calculate the work necessary to assemble the sphere by bringing successively thin charged layers at the surface.

Problem 5: Conductors

A metal sphere of radius R_1, carrying charge q, is surrounded by a thick concentric metal shell of inner and outer radii R_2 and R_3. The shell carries no net charge.

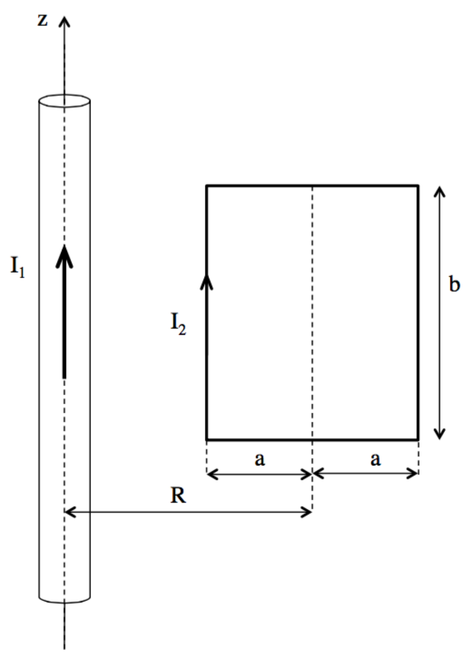
a) Find the surface charge densities at R_1, R_2 and R_3.

b) Find the potential at the centre, choosing $V = 0$ at infinity.

c) Now the outer surface is grounded. Explain how that modifies the charge distribution. How do the answers to questions (a) and (b) change?

Magnetostatics

Problem 6: Force on a loop



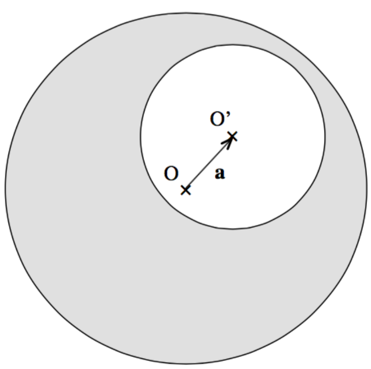
A long thin wire carries a current I_1 in the positive z–direction along the axis of a cylindrical co-ordinate system. A thin, rectangular loop of wire lies in a plane containing the axis, as represented on the figure. The loop carries a current I_2.

a) Find the magnetic field due to the long thin wire as a function of distance r from the axis.

b) Find the vector force on each side of the loop which results from this magnetic field.

c) Find the resultant force on the loop.

Problem 7: Magnetic field in off–centre hole



A cylindrical rod carries a uniform current density J. A cylindrical cavity with an arbitrary radius is hollowed out from the rod at an arbitrary location. The axes of the rod and cavity are parallel. A cross section is shown on the figure. The points O and O' are on the axes of the rod and cavity, respectively, and we note $\mathbf{a} = \mathbf{OO'}$.

a) Show that the field inside a solid cylinder can be written as $\mathbf{B} = (\mu_0 J/2)\hat{\mathbf{z}}\times\mathbf{r}$, where $\hat{\mathbf{z}}$ is the unit vector along the axis and $\mathbf{r}$ is the position vector measured perpendicularly to the axis.

b) Show that the magnetic field inside the cylindrical cavity is uniform (in both magnitude and direction).

Problem 8: Magnetic field at the centre of a sphere

A spherical shell with radius a and uniform surface charge density σ spins with angular frequency ω around a diameter. Find the magnetic field at the centre.

Problem 9: Motion of a charged particle in a magnetic field

A long thin wire carries a current I in the positive z–direction along the axis of a cylindrical co-ordinate system. A particle of charge q and mass m moves in the magnetic field produced by this wire. We will neglect the gravitational force acting on the particle as it is very small compared to the magnetic force.

a) Is the kinetic energy of the particle a constant of motion?

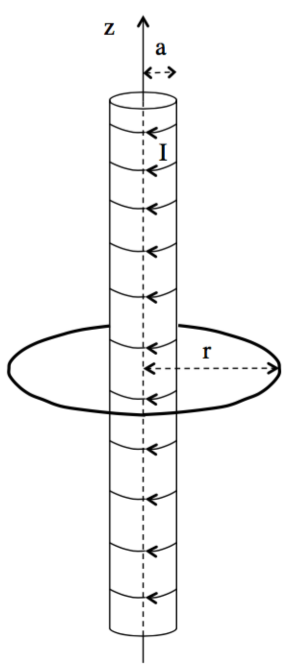
b) Find the force $\mathbf{F}$ on the particle, in cylindrical coordinates.

c) Obtain the equation of motion, $\mathbf{F} = md\mathbf{v}/dt$, in cylindrical coordinates for the particle.

d) Suppose the velocity in the z–direction is constant. Describe the motion.

Electromagnetic induction

Problem 10: Growing current in a solenoid



An infinite solenoid has radius a and n turns per unit length. The current grows linearly with time, according to $I(t) = kt$, $k > 0$. The solenoid is looped by a circular wire of radius r, coaxial with it. We recall that the magnetic field due to the current in the solenoid is $B = \mu_0 nI$ inside the solenoid and zero outside.

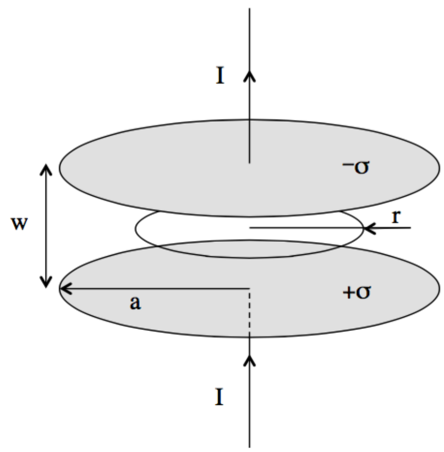
a) Without doing any calculation, explain which way the current induced in the loop flows.

b) Use the integral form of Faraday's law, which is $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt$, to find the electric field in the loop for both $r < a$ and $r > a$. Check that the orientation of $\mathbf{E}$ agrees with the answer to question (a).

c) Verify that your result satisfies the local form of the law, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.

Maxwell's equations

Problem 11: Energy flow into a capacitor



A capacitor has circular plates with radius a and is being charged by a constant current I. The separation of the plates is $w \ll a$. Assume that the current flows out over the plates through thin wires that connect to the centre of the plates, and in such a way that the surface charge density σ is uniform, at any given time, and is zero at $t = 0$.

a) Find the electric field between the plates as a function of t.

b) Consider the circle of radius $r < a$ shown on the figure (and centered on the axis of the capacitor). Using the integral form of Maxwell's equation $\boldsymbol{\nabla}\times\mathbf{B} = \epsilon_0\mu_0\partial\mathbf{E}/\partial t$ over the surface delimited by the circle, find the magnetic field at a distance r from the axis of the capacitor.

c) Find the energy density u and the Poynting vector $\mathbf{S}$ in the gap. Check that the relation:

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{S},$$

is satisfied.

d) Consider a cylinder of radius $b < a$ and length w inside the gap. Determine the total energy in the cylinder, as a function of time. Calculate the total power flowing into the cylinder, by integrating the Poynting vector $\mathbf{S}$ over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the cylinder.

e) When $b = a$, and assuming that we can still neglect edge effects in that case, check that the total power flowing into the capacitor is:

$$\frac{d}{dt}\left(\frac{1}{2}QV\right),$$

where V is the voltage across the capacitor (since $QV/2$ is the energy stored in the electric field in the capacitor).