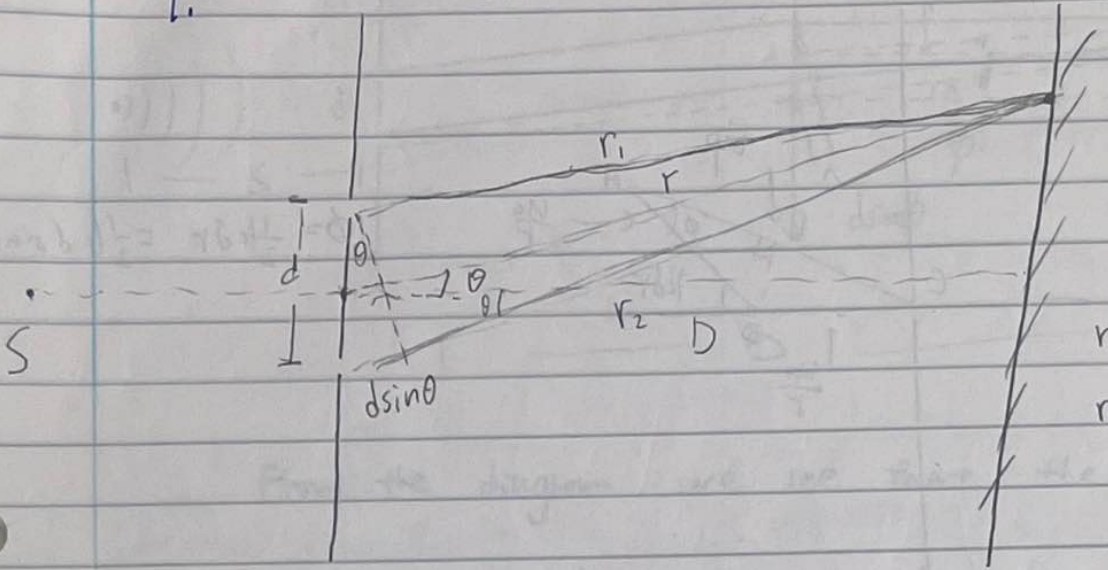
To: Caroline Terquem

Excellent!

1 Optics (1 - 10) Ziyan Li

1.




$\delta r = r_2 - r_1 = d\sin\theta$

$r = \frac{D}{\cos\theta}$

$r_1 = r - \frac{\delta r}{2}$

$r_2 = r + \frac{\delta r}{2}$

(a) $$u_P = \frac{u_0}{r_1} e^{i(kr_1 - \omega t)} + \frac{u_0}{r_2} e^{i(kr_2 - \omega t)}$$

$$\approx \frac{u_0}{r} e^{i(kr - \omega t)} [\underbrace{e^{i(k\delta r/2)} + e^{-i(k\delta r/2)}}_{2\cos(k\delta r/2)}]$$

$$= \frac{2u_0}{r} e^{i(kr - \omega t)} \cos(\tfrac{1}{2} kd\sin\theta)$$

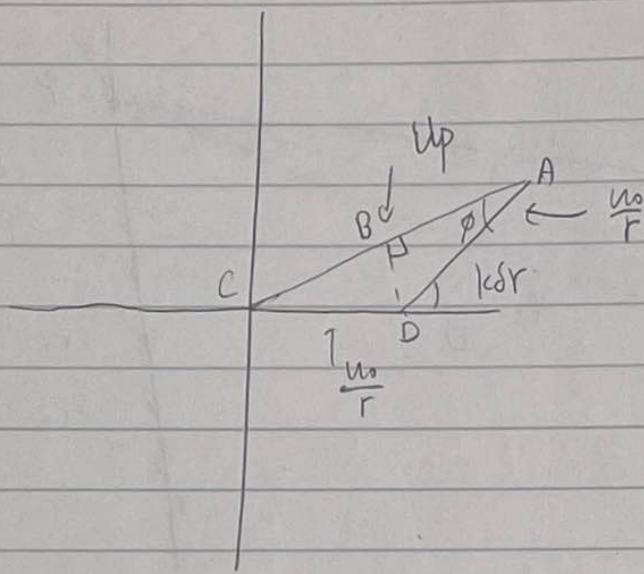
$$I_p = |u_p|^2 = 4\left(\frac{u_0 \cos\theta}{D}\right)^2 \cos^2(\tfrac{1}{2} kd\sin\theta)$$

$\theta$ is small $\cos\theta \approx 1$ $\sin\theta \approx \theta \approx \tan\theta \approx \frac{y}{D}$

$$I(0) = \frac{4u_0^2}{D^2} \approx 4\left(\frac{u_0 \cos\theta}{D}\right)^2, \quad I_p = I(\theta)$$

$$\therefore \boxed{I(\theta) = I(0) \cos^2(\tfrac{1}{2} kd\sin\theta) \quad (k = \tfrac{2\pi}{\lambda})}$$ ✓

(b)




$\phi = \frac{1}{2}k\delta r = \frac{1}{2}kd\sin\theta$

By Geometry : $\frac{AB}{AD} = \cos\phi = \cos(\frac{1}{2}k\delta r)$

$|u_p| = 2AB + CD = 2AB = 2AD\cos\phi$

$= \frac{2u_0}{r}\cos(\frac{1}{2}kd\sin\theta)$

$I(\theta) = |u_p|^2 = \boxed{I(0)\cos^2(\frac{1}{2}kd\sin\theta)}$ ✓ $k = \frac{2\pi}{\lambda}$

(c) Central maximum : $\theta_0 = 0$

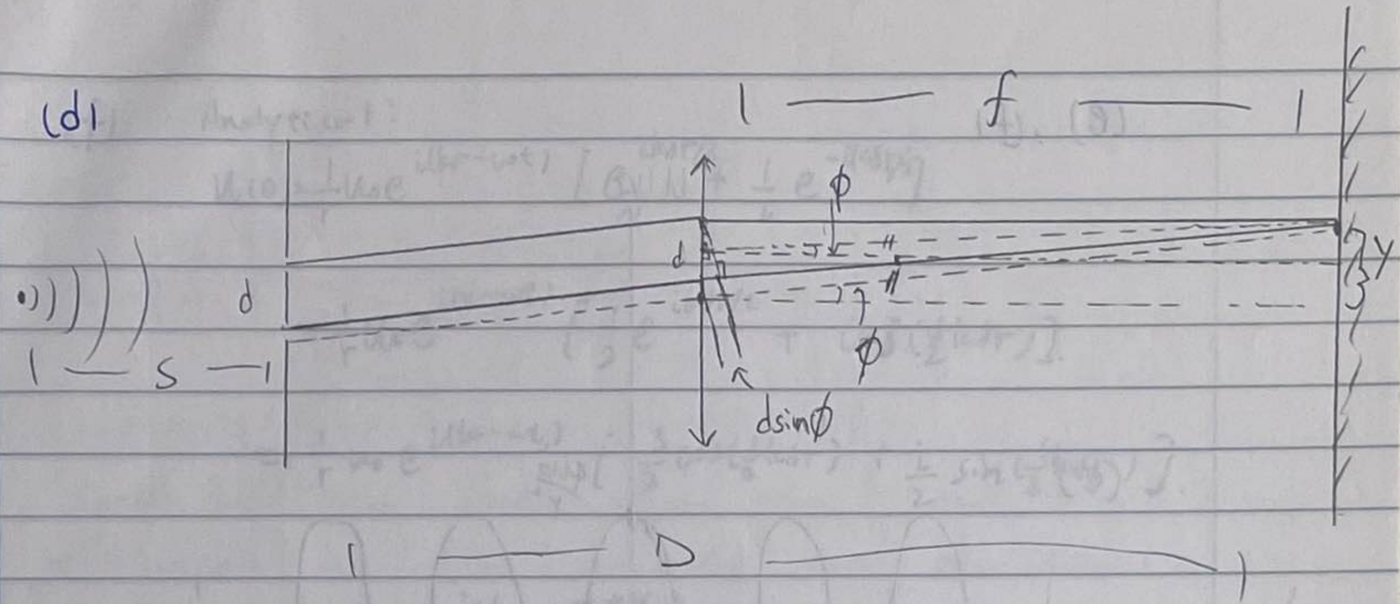
First maximum : $\frac{1}{2}kd\sin\theta_1 = \pi$

$\because \sin\theta \approx \theta$ $\therefore \frac{1}{2}\frac{2\pi}{\lambda}d\theta_1 = \pi$ $\Rightarrow$ $\theta_1 = \frac{\lambda}{d}$

~~$\therefore \boxed{\theta = \frac{\lambda}{d}}$~~

$\because \Delta\theta = \theta_1 - \theta_0 \Rightarrow \boxed{\Delta\theta = \frac{\lambda}{d}}$ ✓

(d)




From the diagram we see that the path difference is ~~still~~ ~~$\delta r$ = $kd\sin\theta$ $d\sin\theta$~~ $\delta r = d\sin\phi$, with the angle $\phi$ being $\phi \simeq \frac{y}{f}$

~~$\therefore$ $\delta r$ remains the same the intensity distribution~~

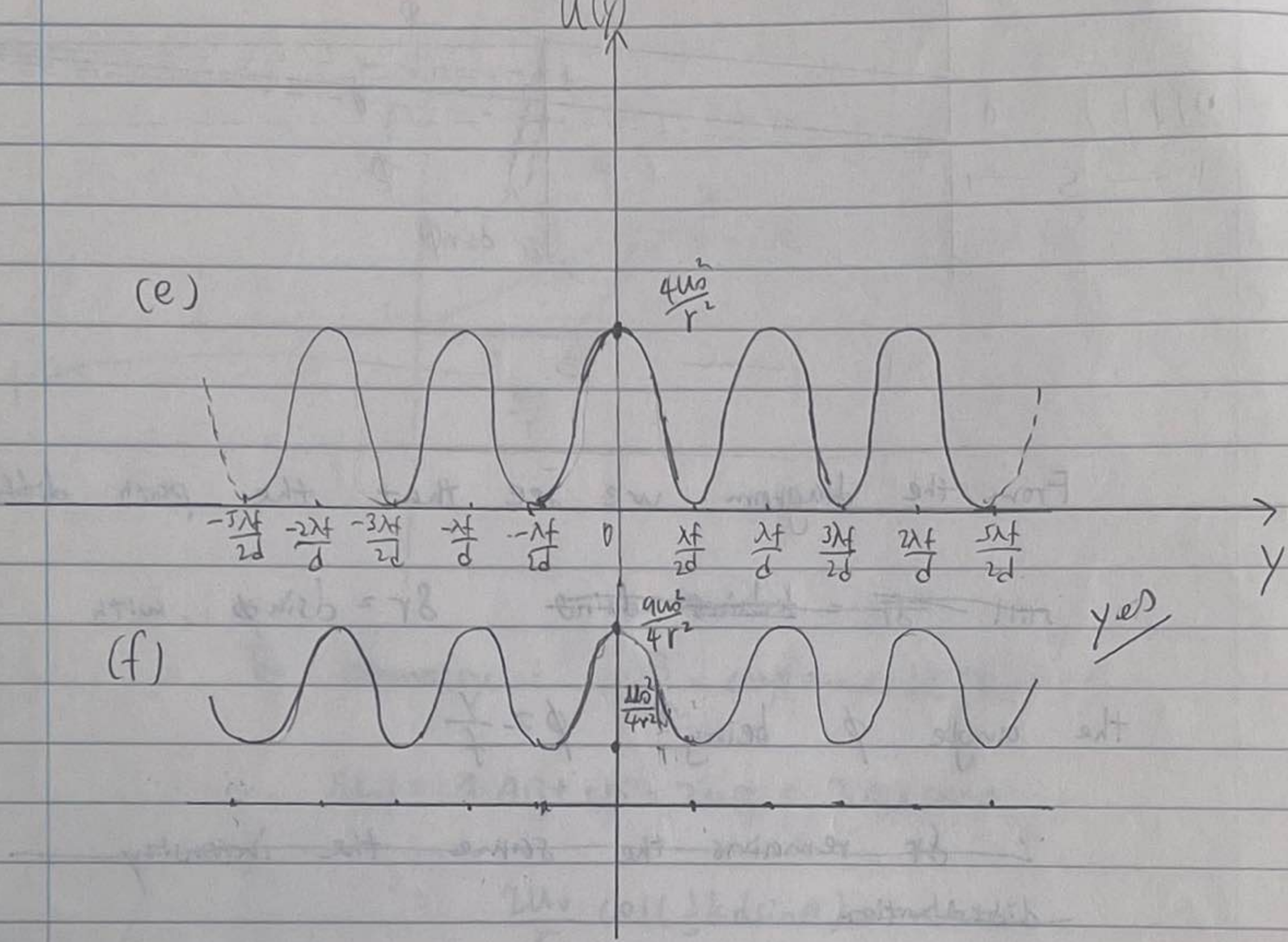
Following the same derivation as (a) we see that ~~$I(\theta)$~~ $I(\phi) = I(0)\cos^2\left(\frac{1}{2}k\delta r\right)$

Now $\delta r = d\sin\phi \simeq d\phi \simeq \frac{dy}{f}$, $k = \frac{2\pi}{\lambda}$

$$\therefore I(\phi) = I(0)\cos^2\left(\frac{\pi d y}{\lambda f}\right)$$

First maximum $\frac{\pi d y}{\lambda f} = \pi \Rightarrow \boxed{y = \frac{\lambda f}{d}}$ ✓

(e), (f)




~~In (f)~~ In (f), the interference is never completely destructive.

(f) Analytical:

$$u(\theta) = \frac{1}{r} u_0 e^{i(kr-\omega t)} \left[ e^{ik\delta r/2} + \frac{1}{2} e^{-ik\delta r/2} \right]$$

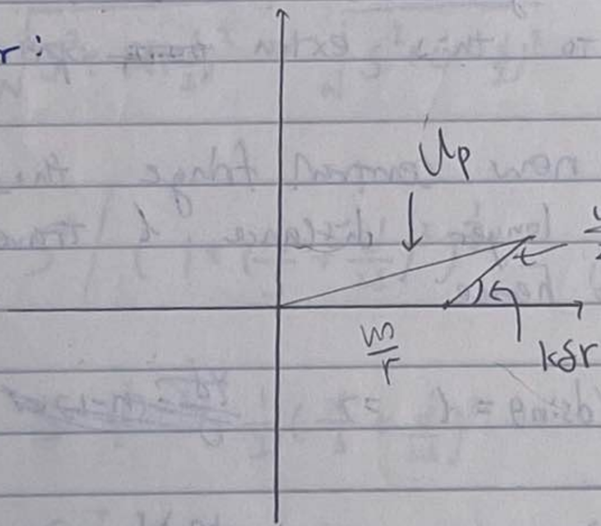
$$= \frac{1}{r} u_0 e^{i(kr-\omega t)} \left[ \frac{1}{2} e^{ik\delta r/2} + \cos\left(\tfrac{1}{2}k\delta r\right) \right].$$

$$= \frac{1}{r} u_0 e^{i(kr-\omega t)} \left[ \frac{3}{2}\cos\left(\tfrac{1}{2}k\delta r\right) + \frac{i}{2}\sin\left(\tfrac{1}{2}k\delta r\right) \right].$$

$$= \frac{1}{r} u_0 e^{i(kr-\omega t+\alpha)} \left[ \frac{9}{4}\cos^2\left(\tfrac{1}{2}k\delta r\right) + \frac{1}{4}\sin^2\left(\tfrac{1}{2}k\delta r\right) \right]^{\frac{1}{2}}$$

$\left[\alpha = \arctan\left(\tfrac{1}{3}\tan\left(\tfrac{1}{2}k\delta r\right)\right)\right]$ $$= \frac{1}{r} u_0 e^{i(kr-\omega t+\alpha)} \left( \cos(k\delta r) + \frac{5}{4} \right)^{\frac{1}{2}}$$

Phasor:




$\delta r = d\sin\theta$

$k\delta r = \frac{2\pi d \sin\theta}{\lambda}$

$\sin\theta \simeq \frac{y}{f}$

$k\delta r = \frac{2\pi d y}{\lambda f}$

$$|u_p| = \frac{u_0}{r}\left[ 1^2 + \left(\tfrac{1}{2}\right)^2 + 2(1)\left(\tfrac{1}{2}\right)\cos(k\delta r) \right]^{\frac{1}{2}}$$

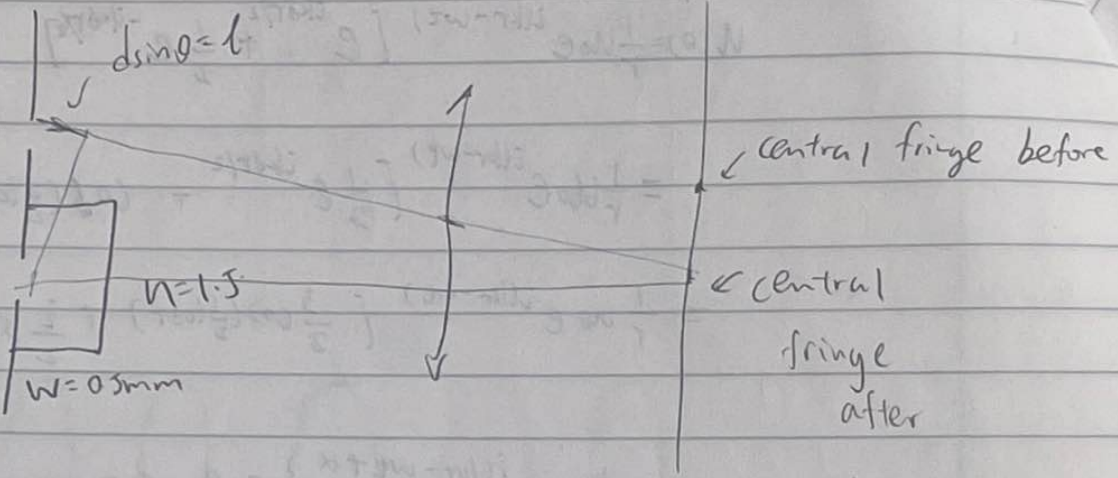
$$= \frac{u_0}{r}\left[ \frac{5}{4} + \cos(k\delta r) \right]^{\frac{1}{2}}.$$

$$\therefore I_p = \left(\frac{u_0}{r}\right)^2 \left[ \frac{5}{4} + \cos(k\delta r) \right] \quad \cancel{I(\theta) = \frac{9}{4}\left(\frac{u_0}{r}\right)^2}$$

$$\therefore \cancel{I(\theta)} \quad I(\theta) \propto \left[ \frac{5}{4} + \cos(kd\sin\theta) \right]$$

$\sin\theta \simeq \theta \approx \frac{y}{f}$ $\Rightarrow$ $$\boxed{I(\theta) \propto \left[ \frac{5}{4} + \cos\left(\frac{2\pi d y}{\lambda f}\right) \right].}$$ ✓

(g)




Extra optical path length $l = (n-1)w$

* Central Fringe shifts towards the slit with the glass due to this extra ~~path~~ optical path length.

* For the new central fringe this $l$ is compensated by the longer distance $l$ traveled by the upper beam, hence

$$d\sin\theta = l \implies \cancel{\frac{yd}{D} = (n-1)w} \quad \frac{yd}{f} = (n-1)w$$

m'th fringe is at $y = \frac{m\lambda f}{d}$ according to (d)

$\therefore$ If central fringe has shifted $m$ fringes, then

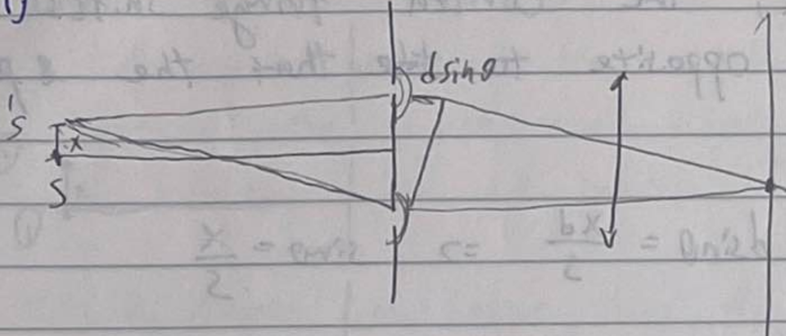
$$\left(\frac{m\lambda f}{d}\right)\left(\frac{d}{f}\right) = (n-1)w$$

$$\implies \boxed{m = \frac{(n-1)w}{\lambda}} \checkmark$$

$h = 0.5$ mm, $n = 1.5$ $\therefore$ $m = \frac{0.25\text{ mm}}{\lambda}$

$\therefore$ Central fringe has shifted ~~to~~ towards the slit with the glass plate by $\frac{0.25\text{mm}}{\lambda}$ fringes approximately. Spacing remains the same

(h)




In this case when the light rays arrives at the slits the two ~~bee~~ beams emerge from the slits are not in phase

This extra ~~phase~~ path length is

$$\Delta l = \sqrt{S^2 + (x+\tfrac{d}{2})^2} - \sqrt{S^2 + (x-\tfrac{d}{2})^2}$$

$$= S\left[\left(1+\left(\frac{x}{S}+\frac{d}{2S}\right)^2\right)^{\frac{1}{2}} - \left(1+\left(\frac{x}{S}-\frac{d}{2S}\right)^2\right)^{\frac{1}{2}}\right]$$

$$= S\left[\cancel{1} + \frac{1}{2}\left(\frac{x}{S}+\frac{d}{2S}\right)^2 - \cancel{1} - \frac{1}{2}\left(\frac{x}{S}-\frac{d}{2S}\right)^2\right]$$

$$= \frac{1}{2S}\left[\left(x+\frac{d}{2}\right)^2 - \left(x-\frac{d}{2}\right)^2\right]$$

$$= \frac{1}{2S}(2x)(d) = \frac{xd}{S}$$

∴ The central fringe will shift. The ~~upper~~ lower beam in the diagram travels longer by $\frac{xd}{S}$ before the slits, so it must travel $\frac{xd}{S}$ less after the slits. ∴ Central fringe shifts down if point source moves up

* In fact, the central fringe shifts to the direction opposite to ~~the~~ that the point source moves.

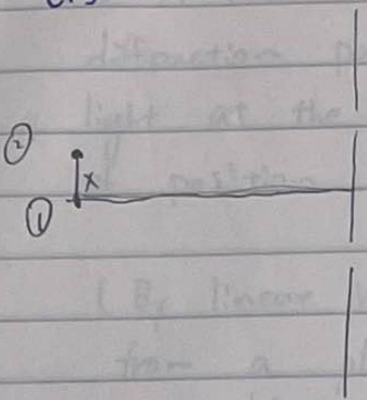
$\therefore \quad d\sin\theta = \frac{xd}{s} \Rightarrow \sin\theta = \frac{x}{s}$

$\Rightarrow \frac{y}{f} = \frac{x}{s} \Rightarrow$ if $y = \frac{m\lambda f}{d}$, then

$\frac{m\lambda \cancel{f}}{d\cancel{f}} = \frac{x}{s} \Rightarrow m = \boxed{\frac{xd}{\lambda s}}$

$\therefore$ shifts by $\frac{xd}{\lambda s}$ fringes.

(i)




From (g) we know central fringe of ② shifts down: $d\sin\theta = \frac{xd}{s} \Rightarrow d\frac{y}{f} = \frac{xd}{s} \Rightarrow y = \frac{x}{s}f$

If the central fringe of source ② coincide with the first ~~maximum~~ minimum of source ① than the ~~net~~ overlap of interference patterns make the fringes invisible.

In this ~~cas~~ case: $y = \frac{\lambda f}{2d}$

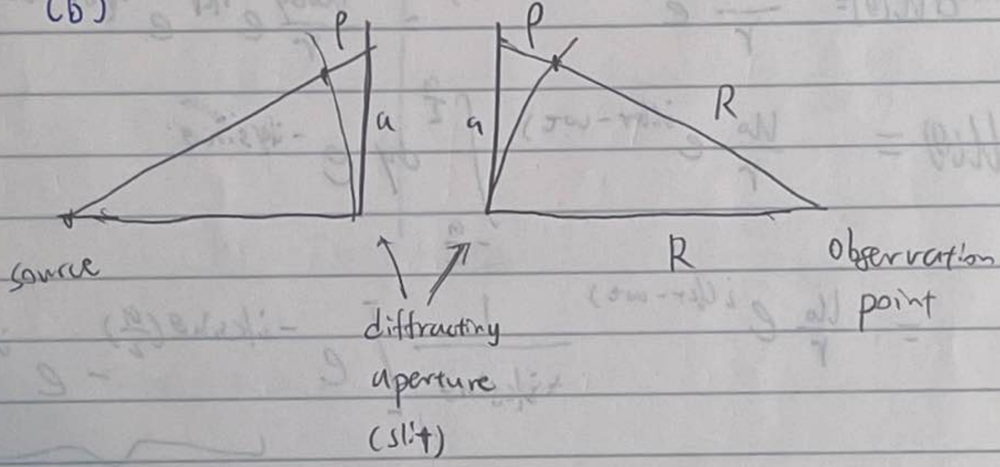
$\therefore \frac{\lambda f}{2d} = \frac{x}{s}f \Rightarrow x = \frac{\lambda s}{2d} = \frac{(500\times10^{-9})(20\times10^{-3})}{2\times(0.5\times10^{-3})} = \boxed{1\times10^{-5}\,m}$

$\therefore$ maximum size for a singe source of incoherent emitter is of ~~raidus~~ radius $\boxed{1\times10^{-5}m}$

2. (a) A Fraunhofer diffraction is ~~pattern~~ a ~~diffr~~ diffraction pattern for which the phase of the light at the observation point is a linear function of position for all points in the diffracting aperture.

(By linear we mean that the wave front deviates from a plane wave by less than $\lambda/20$ across the diffracting aperture)

(b)




$$(R+\rho)^2 = R^2 + a^2$$

$$\Rightarrow \cancel{R^2} + 2R\rho + \rho^2 = \cancel{R^2} + a^2$$

For $\rho \leq \frac{\lambda}{20}$, $\cancel{2R\rho + \rho^2 \leq a^2}$

$\cancel{\therefore \rho^2 \ll 2R\rho \quad \therefore 2R\rho \leq a^2}$

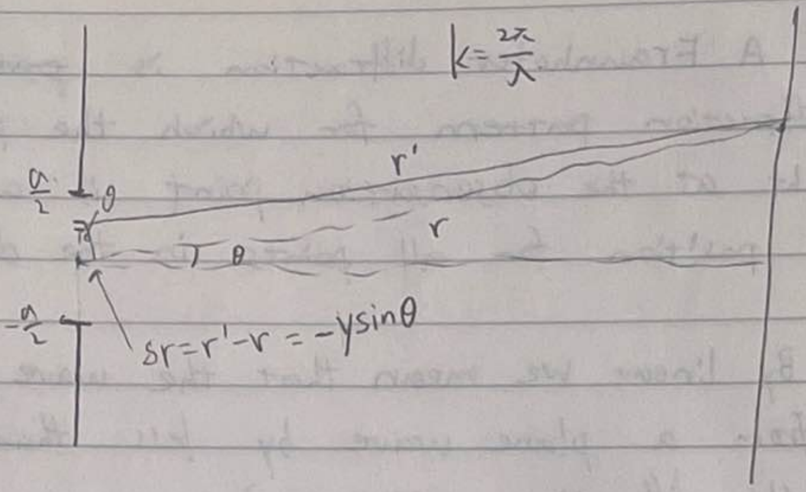
$\therefore 2R\rho \gg \rho^2$

$\therefore 2R\rho = a^2 \Rightarrow R = \frac{a^2}{2\rho}$

$\because \rho \leq \frac{\lambda}{20} \quad \therefore R \geq \frac{10a^2}{\lambda}$

$\Rightarrow$ minimum distance $\boxed{R = \frac{10a^2}{\lambda}}$ ✓

(c)




Amplitude in plane of aperture : $U_0$ per unit length

$$dU(\theta) = \frac{U_0 dy}{r} e^{i(kr'(\theta)-\omega t)} = \frac{U_0 dy}{r} e^{ikr} e^{i(k\delta r - \omega t)}$$

$$\Rightarrow U(\theta) = \frac{U_0}{r} e^{i(kr-\omega t)} \int_{-\frac{a}{2}}^{\frac{a}{2}} dy\, e^{-iky\sin\theta}$$

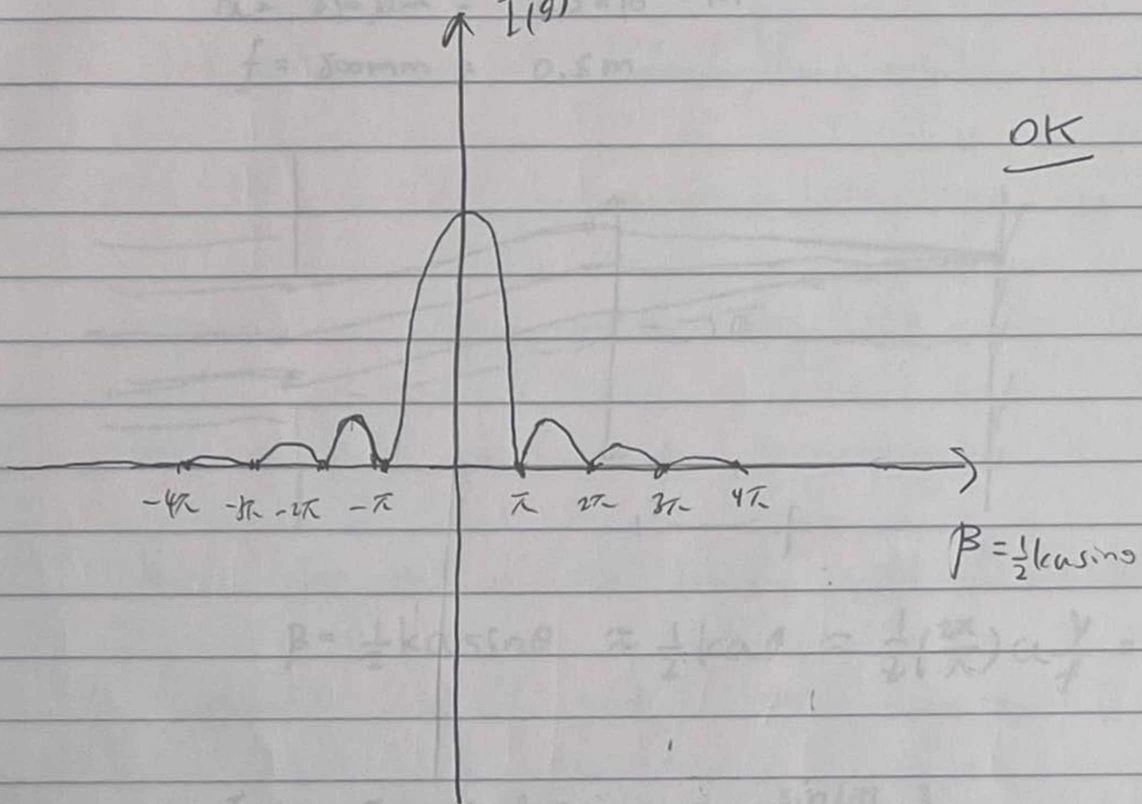
$$= \frac{U_0}{r} e^{i(kr-\omega t)} \frac{1}{+ik\sin\theta} \Big[ \underbrace{e^{-ik\sin\theta(\frac{a}{2})} - e^{ik\sin\theta(\frac{a}{2})}}_{+2i\sin(\frac{1}{2}ka\sin\theta)} \Big]$$

$$= \frac{U_0 a}{r} e^{i(kr-\omega t)} \frac{\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}$$

let $\boxed{\beta = \frac{1}{2}ka\sin\theta \quad , k = \frac{2\pi}{\lambda} \quad , I_0 = \left(\frac{U_0 a}{r}\right)^2}$

$$I(\theta) = |U(\theta)|^2 = I_0\left(\frac{\sin(\beta)}{\beta}\right)^2 = \boxed{I \operatorname{sinc}^2(\beta)} \checkmark$$

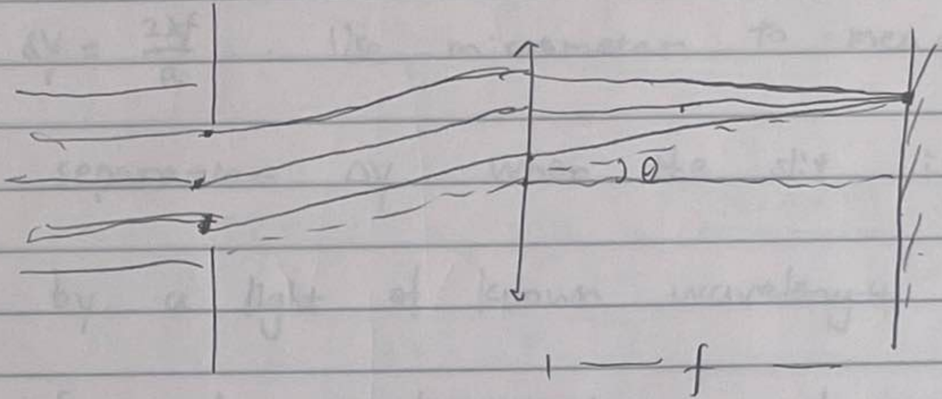
OK



3. $\lambda = 600\,nm = 6\times10^{-7}\,m$

$a = 250\,\mu m = 2.5\times10^{-4}\,m$

$f = 500\,mm = 0.5\,m$



$$\beta = \frac{1}{2}ka\sin\theta \approx \frac{1}{2}ka\theta \approx \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)a\frac{y}{f} = \frac{\pi a y}{\lambda f}$$

$$I(\theta) = I(0)\,\mathrm{sinc}^2(\beta) = I(0)\left(\frac{\sin(\beta)}{\beta}\right)^2$$

First minima $\beta = \pm\pi$ :

$\beta = \pi \Rightarrow \quad \pi = \frac{\pi a y}{\lambda f} \Rightarrow \quad y_+ = \frac{\lambda f}{a}$

$\beta = -\pi \Rightarrow \quad -\pi = \frac{\pi a y}{\lambda f} \Rightarrow \quad y_- = -\frac{\lambda f}{a}$

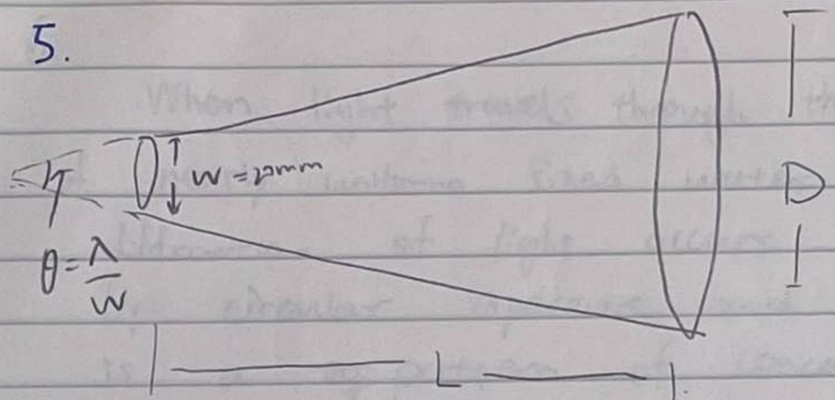
$\therefore \quad \Delta y = y_+ - y_- = \frac{2\lambda f}{a} = 0.0024\,m$

$= \boxed{2.4\times10^{-3}\,m}$ ✓

4. $a \approx 100\,\mu m = 1 \times 10^{-4}\,m$

The separation of two first ~~successive~~ minima is $\Delta y = \frac{2\lambda f}{a}$. Use micrometer to measure this separation $\Delta y$ when the slit is illuminated by a light of known wavelength $\lambda$. Vary the focus length of the lens used. ~~According to~~ ~~$\Delta y$~~ and plot $\Delta y$ against the focal length $f$. Then since $\Delta y = (\frac{2\lambda}{a}) f$, slope $= \frac{2\lambda}{a}$ $\Rightarrow$ $\boxed{a = \frac{2\lambda}{slope}}$

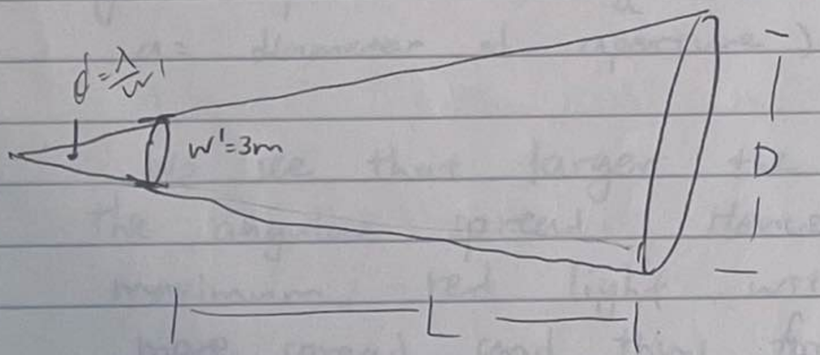
5.




$$D \approx L\theta = \frac{\lambda L}{w} = \frac{(700\times10^{-9})(390{,}000\times10^{3})}{(20\times10^{-3})}$$

$$= \boxed{1.37\times10^{4}\ m}$$

$(D \gg 20mm \therefore ok)$




$$D' \approx L\theta' = \frac{\lambda L}{w'} = 91m$$ ~~91m~~

( we cannot say $91m \gg 3m$ $\therefore$ we must include the w in D)

$$\therefore \quad D = 3m + D' = \boxed{94\ m}$$

6.

When light travels through thin clouds made up of nearly uniform sized water droplets ~~or so~~, diffraction of light occurs. This is a diffraction by circular aperture and the resultant pattern is a ~~of~~ pattern of concentric rings, with the central circular maximum significantly brighter ~~that~~ than other local maxima.

The angular size of this central maximum is given by $\delta\theta = 1.22\frac{\lambda}{a}$ ~~and~~ ($\lambda$ = wavelength, $a$ = diameter of aperture )

We see that larger the $\lambda$ gives larger the angular spread. Hence for ~~the~~ the ~~ce~~ central maximum, red light, with larger $\lambda$, has more spread and thus forms the ~~perif~~ periphery.

$$\text{angular diameter} = 2\delta\theta = 2^\circ = \pi/90 \approx 0.0349$$

$$\Rightarrow \delta\theta = 0.0175$$

$$a = 1.22\frac{\lambda}{\delta\theta} \qquad \lambda = \lambda_{red} = 700\,nm$$

$$\therefore a = 1.22\frac{700\times10^{-9}}{0.0175} = \boxed{4.88\times10^{-5}\,m}$$ ✓

7.

$$u_P = -\frac{i}{\lambda}\int_S \frac{u_0\, dS}{r}\, \eta(\underline{n},\underline{r})\, \exp(ikr)$$

The factor $\frac{1}{r}$ indicates that for a point source, which produces spherical wave, the amplitude ~~of~~ falls of with distance travelled inversely with the relationship $A \propto \frac{1}{r}$. And therefore Intensity $\propto \frac{1}{r^2}$

$\eta(\underline{n},\underline{r})$ is the obliquity factor that accounts for the fact that the wave propagates only in the forward direction.

Assumptions: $\eta(\underline{n},\underline{r}) = 1$ (ignore obliquity factor)

$dS \to dx$ (restrict to 1-D)

ignore $\frac{1}{r}$ term as $r$ varies small

~~Frau~~ Fraunhofer diffraction condition satisfied

$$e^{ikr} = e^{ikr'} e^{ik\sin\theta x} \quad (\beta = k\sin\theta)$$

Absorbing $e^{ikr'}$ into constant of proportionality gives (set $u_0 = u(x)$)

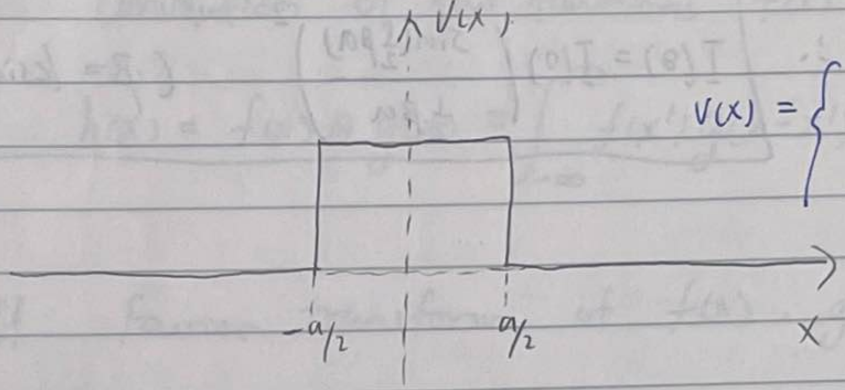
$$u_P \Rightarrow A(\beta) = \alpha\int_{-\infty}^{\infty} u(x)\, e^{i\beta x}\, dx$$

$\therefore$ The angular distribution of the Fraunhofer diffraction pattern can be represented (is proportional to) the Fourier transform of amplitude distribution function $u(x)$.

For a single slit of width $a$,

$$A(\beta) = \alpha \int_{-a/2}^{a/2} u(x) e^{i\beta x} dx$$ ✓

8.




$$V(x) = \begin{cases} 1 & |x| < \frac{a}{2} \\ 0 & |x| > \frac{a}{2} \end{cases}$$

$$A(\beta) = \alpha \int_{-\infty}^{\infty} V(x) e^{i\beta x} \, dx$$

$$= \alpha \int_{-a/2}^{a/2} e^{i\beta x} \, dx = \alpha \left(\frac{1}{i\beta}\right) \left[ e^{i\beta x} \right]_{-a/2}^{a/2}$$

$$= \frac{\alpha}{i\beta} \left[ e^{\frac{i}{2}\beta a} - e^{-\frac{i}{2}\beta a} \right] = \frac{\alpha}{i} \frac{2i \sin(\frac{1}{2}\beta a)}{\beta}$$

$$= (\alpha a) \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a}$$

when $\beta \to 0$, $\frac{1}{2}\beta a \to 0$ $\quad \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \to 1$

$$\therefore A(\beta) \to A(0) \Rightarrow A(0) \to \alpha a.$$

$$\therefore \boxed{A(\beta) = A(0) \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \qquad (\beta = k \sin\theta)}$$

Intensity pattern $\frac{I(\beta)}{I(0)} = \left(\frac{A(\beta)}{A(0)}\right)^2$ ✓

$$\therefore \boxed{I(\theta) = I(0)\left(\frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a}\right)^2 \quad (\beta = k\sin\theta)}$$ ✓

9. Convolution of two functions $f(x)$ and $g(x)$ is

$$h(x) = f(x) \otimes g(x) = \int_{-\infty}^{\infty} f(x')\, g(x-x')\, dx'$$

If Fourier transform of $f(x)$, $g(x)$, $h(x)$ are ~~the~~ $F(\beta)$, $G(\beta)$, $H(\beta)$ respectively, then

Convolution Theorem states that

$$\underline{H(\beta) = F(\beta)\, G(\beta) \times 2\pi}$$

$$u(x) = \begin{cases} 1, & -\frac{d}{2} - \frac{a}{2} < x < -\frac{d}{2} + \frac{a}{2} \\ & \text{or } \frac{d}{2} - \frac{a}{2} < x < \frac{d}{2} + \frac{a}{2} \\ 0, & \text{otherwise} \end{cases}$$

Consider $V(x) = \begin{cases} 1 & |x| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$ ✓

and $W(x) = \delta(x + \frac{d}{2}) + \delta(x - \frac{d}{2})$ ✓

then the convolution of $V$ and $W$ is

$$h(x) = V(x) \otimes W(x) = \int_{-\infty}^{\infty} V(x')\, W(x-x')\, dx'$$

$$= \int_{-\infty}^{\infty} V(x')\,\delta(x-x'-\tfrac{d}{2})\,dx' + \int_{-\infty}^{\infty} V(x')\,\delta(x-x'+\tfrac{d}{2})\,dx'$$

$$= V(x-\tfrac{d}{2}) + V(x+\tfrac{d}{2}) \checkmark$$

$$V(x-\tfrac{d}{2}) = \begin{cases} 1, & |x-\tfrac{d}{2}| < \tfrac{a}{2} \\ 0, & \text{otherwise} \end{cases}$$

$\checkmark$

$$V(x+\tfrac{d}{2}) = \begin{cases} 1, & |x+\tfrac{d}{2}| < \tfrac{a}{2} \\ 0, & \text{otherwise} \end{cases}$$

Hence $h(x) = V(x-\frac{d}{2}) + V(x+\frac{d}{2}) = U(x)$

$$\therefore \boxed{U(x) = V(x) \otimes W(x)}$$

$U$ can be represented by the convolution of two simpler functions $V$ and $W$

Fraunhofer diffraction pattern :

$$U(\beta)\ \cancel{A(\beta)} = \propto \int_{-\infty}^{\infty} u(x)\, e^{i\beta x}\, dx$$

$$\because u(x) = V(x) \otimes W(x) \qquad \therefore U(\beta) = 2\pi V(\beta) W(\beta)$$

$$V(\beta) \cancel{=} \propto \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \qquad \text{(from Q8 )}$$

$$W(\beta) = \propto \int_{-\infty}^{\infty} w(x)\, e^{i\beta x}\, dx$$

$$= \propto \int_{-\infty}^{\infty} \delta(x - \tfrac{d}{2})\, e^{i\beta x}\, dx + \propto \int_{-\infty}^{\infty} \delta(x + \tfrac{d}{2})\, e^{i\beta x}\, dx$$

$$= \propto \left[ e^{\frac{1}{2}\beta d} + e^{-\frac{1}{2}\beta d} \right] \propto \cos(\tfrac{1}{2}\beta d)$$

$$\therefore U(\beta) \cancel{=} \propto V(\beta) W(\beta) \propto \cos(\tfrac{1}{2}\beta d) \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a}$$

$$\Rightarrow \boxed{U(\beta) = U(0) \cos(\tfrac{1}{2}\beta d) \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \qquad (\beta = \tfrac{2\pi}{\lambda} \sin\theta)}$$

yes

10.

$$u_p = -\frac{i}{\lambda}\int \frac{u_0\, dS}{r} e^{ikr} \eta(\underline{n}, \underline{r})$$

$\eta(\underline{n}, \underline{r}) = 1$ . $\frac{1}{r} \approx 1$

$u_0$ = ~~so~~ some function only of $y \propto T(y)$

$dS = dx\, dy$

Fraunhofer condition $e^{ikr} = e^{ikr'} e^{-iksin\theta y} e^{-iksin\phi x}$

$\therefore u_p \propto \int_{x,y} T(y) e^{-ikysin\theta} e^{-ikxsin\phi} dx\, dy$ .

$\theta$ is angle spread in y-direction
$\phi$ is angular spread in x direction

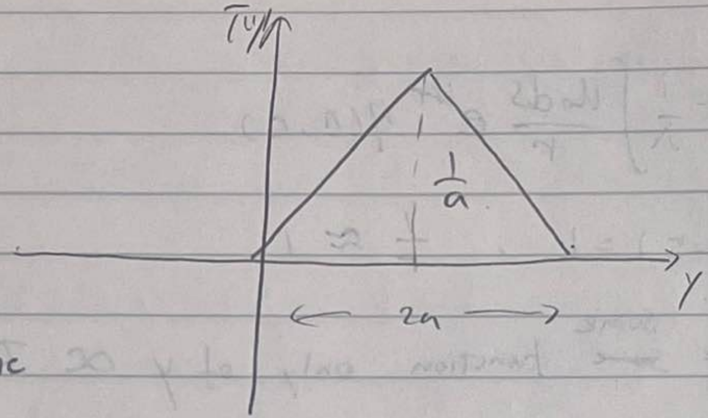
$$= \int e^{-ikxsin\phi} dx \int T(y) \exp(-ikysin\theta)\, dy$$

$$\propto \underbrace{\int_{-\infty}^{\infty} T(y) \exp(-ikysin\theta)\, dy}_{A(\theta)} \quad ✓$$

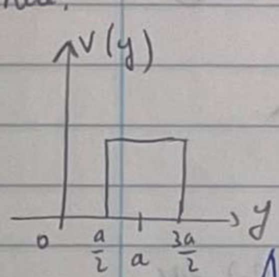
$\because I_p \propto |u_p|^2 \propto |A(\theta)|^2$

$$\therefore A(\theta) = \int_{-\infty}^{\infty} T(y) \exp(-ikysin\theta)\, dy. \quad \text{works}$$

as expected




you can also use the fact that $T(y)$ is a self-convolution of $V(y)$, where:




$$T(y) = \cancel{\begin{cases} \frac{x}{a^2} & 0<x<a \\ \frac{(2a-x)}{a^2} & a<x<2a \\ 0 & \text{otherwise.} \end{cases}} \quad \begin{cases} \frac{y}{a^2} & 0<y<a \\ \frac{(2a-y)}{a^2} & a<y<2a \\ 0 & \text{otherwise} \end{cases}$$

$$A(\theta) = \int_{-\infty}^{\infty} T(y) \exp(-iky\sin\theta)\, dy$$

$$= \int_{\cancel{-\infty}\,0}^{\cancel{\infty}\,a} \frac{y}{a^2} \exp(-iky\sin\theta)\, dy + \int_{\cancel{-\infty}\,a}^{\cancel{\infty}\,2a} \frac{(2a-y)}{a^2} \exp(-iky\sin\theta)\, dy$$

(let $\beta = k\sin\theta$)

$$= \frac{1}{a^2}\left[ \int_0^a y e^{-i\beta y} dy + \int_a^{2a} (2a-y) e^{-i\beta y} dy \right]$$

$$= \frac{1}{a^2}\left[ -\frac{1}{i\beta} y e^{-i\beta y} + \frac{1}{\beta^2} e^{-i\beta y} \right]_0^a$$

$$+ \frac{1}{a^2}\left[ -\frac{2a}{i\beta} e^{-i\beta y} + \frac{1}{i\beta} y e^{-i\beta y} - \frac{1}{\beta^2} e^{-i\beta y} \right]_a^{2a}$$

$$= \frac{1}{a^2}\Big[ -\frac{1}{i\beta} a e^{-i\beta a} + \frac{1}{\beta^2} e^{-i\beta a} - \frac{1}{\beta^2}$$

$$- \frac{2a}{i\beta} e^{-2i\beta a} + \frac{1}{i\beta} 2a e^{-2i\beta a} - \frac{1}{\beta^2} e^{-2i\beta a}$$

$$+ \frac{2a}{i\beta} e^{-i\beta a} - \frac{1}{i\beta} a e^{-i\beta a} + \frac{1}{\beta^2} e^{-i\beta a} \Big]$$

$$= \frac{1}{a^2}\Big[ \cancel{-\frac{2a}{i\beta} e^{-i\beta a}} + \cancel{\frac{2a}{i\beta} e^{-i\beta a}} + \cancel{\frac{2a}{i\beta} e^{-2i\beta a}}$$

$$\cancel{- \frac{2a}{i\beta} e^{-2i\beta a}} + \frac{2}{\beta^2} e^{-i\beta a} - \frac{1}{\beta^2} e^{-2i\beta a} - \frac{1}{\beta^2} \Big)$$

$$= \frac{1}{a^2\beta^2}\Big[ (e^{-i\beta a} - e^{-2i\beta a}) + (e^{-i\beta a} - 1) \Big]$$

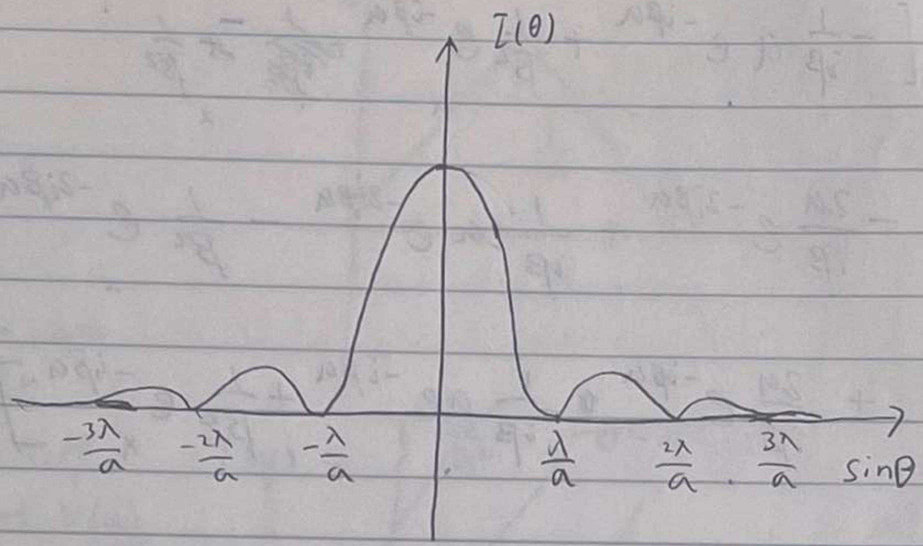
$$= \frac{1}{a^2\beta^2}\Big[ e^{-i\beta a} \Big]\Big[ 2 - (e^{i\beta a} + e^{-i\beta a}) \Big]$$

$$= \frac{2}{a^2\beta^2} e^{-i\beta a} (1 - \cos(\beta a))$$

$$\therefore \; A(\theta) = \frac{2e^{-i\beta a}}{a^2} \frac{(1-\cos(\beta a))}{\beta^2} \qquad (\beta = k\sin\theta)$$

$$\cancel{= \frac{e^{-i\beta a}}{a} \left( \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \right)^2}$$

$$\boxed{= \frac{1}{2} e^{-i\beta a} \left( \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \right)^2}$$



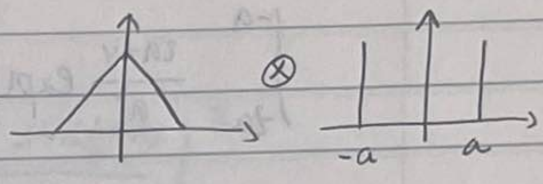

Hence $\boxed{I(\theta) \propto \left(\frac{\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}\right)^4}$ yes

Where ~~we when~~ we choose $\theta$ to be measured from $y=a$ to the screen.

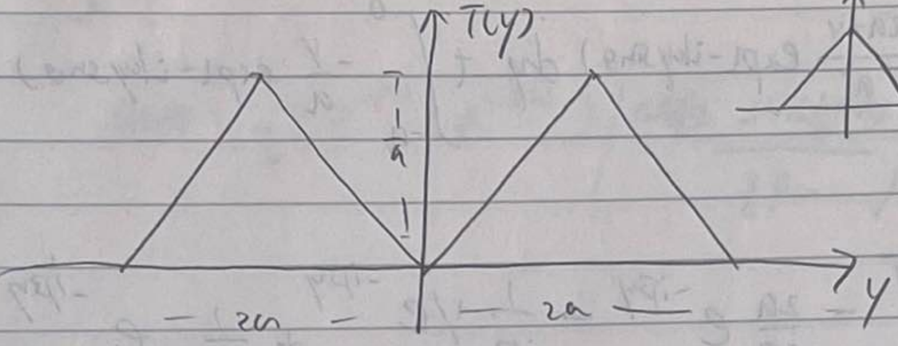
At minima $\frac{1}{2}ka\sin\theta = n\pi$

$\Rightarrow \frac{\pi}{\lambda} a\sin\theta = n\pi \Rightarrow \sin\theta = \frac{n\lambda}{a}$

You can also use the fact that $T(y) =$




(δ functions)




$$T(y) = \begin{cases} \dfrac{2a+y}{a^2} & -2a<y<a \\[2ex] \dfrac{-y}{a^2} & a<y<0 \\[2ex] \dfrac{y}{a^2} & 0<y<a \\[2ex] \dfrac{2a-y}{a^2} & a<y<2a \\[2ex] 0, & \text{otherwise} \end{cases}$$

We know from previous part that

$$\int_0^a \frac{y}{a^2} \exp(-iky\sin\theta)\, dy + \int_a^{2a} \frac{(2a-y)}{a^2} \exp(-iky\sin\theta)\, dy$$

$$= \frac{1}{2} e^{-i\beta a} \left( \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \right)^2$$

~~make the substitution $z = -y$, $\beta' = -\beta$ we get~~

$$\int_{-2a}^{-a} \frac{2a+y}{a^2} \exp(-iky\sin\theta)\,dy + \int_{-a}^{0} -\frac{y}{a} \exp(-iky\sin\theta)\,dy.$$

$$= \frac{1}{a^2}\left[ -\frac{2a}{i\beta} e^{-i\beta y} - \frac{1}{i\beta} y e^{-i\beta y} + \frac{1}{\beta^2} e^{-i\beta y} \right]_{-2a}^{-a}.$$

$$+ \frac{1}{a^2}\left[ \frac{1}{i\beta} y e^{-i\beta y} \cancel{+} - \frac{1}{\beta^2} e^{-i\beta y} \right]_{-a}^{0}.$$

$$= \frac{1}{a^2}\Big[ \cancel{-\frac{2a}{i\beta} e^{i\beta a}} \cancel{- \frac{1}{i\beta}(-a)} e^{i\beta a} + \frac{1}{\beta^2} e^{i\beta a}$$

$$\cancel{+ \frac{2a}{i\beta} e^{2i\beta a}} \cancel{+ \frac{1}{i\beta}(-2a)} e^{2i\beta a} - \frac{1}{\beta^2} e^{2i\beta a}$$

$$- \frac{1}{\beta^2} \cancel{- \frac{1}{i\beta}(-a)} e^{i\beta a} + \frac{1}{\beta^2} e^{i\beta a} \Big].$$

$$= \frac{1}{a^2\beta^2}\left[e^{i\beta a}\right]\left[2 - (e^{i\beta a} + e^{-i\beta a})\right]$$

$$= \frac{1}{2} e^{i\beta a} \left( \frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a} \right)^2$$

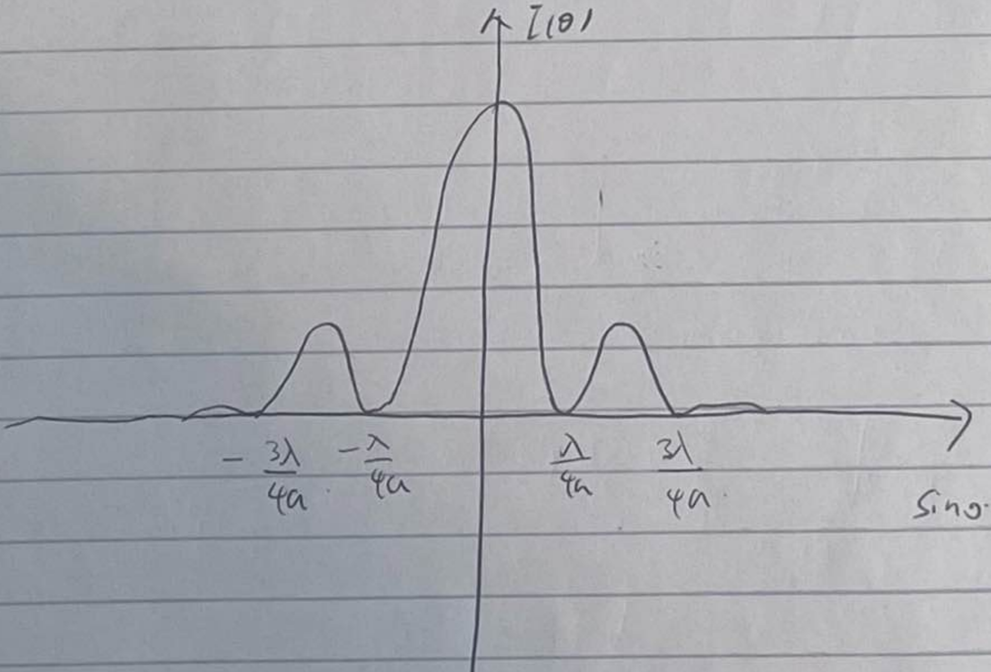
∴

$$A(\theta) = \frac{1}{2}\left(e^{i\beta a} + e^{-i\beta a}\right)\left(\frac{\cancel{\sin\frac{1}{2}}\ \sin(\frac{1}{2}\beta a)}{\frac{1}{2}\beta a}\right)^2$$

$$\Rightarrow \quad A(\theta) = \cos(\beta a)\left(\frac{\sin(\frac{1}{2}\beta a)}{\frac{1}{2}(\beta a)}\right)^2$$

$$\therefore \quad \boxed{I(\theta) \propto \cos^2(ka\sin\theta)\left(\frac{\sin(\frac{1}{2}ka\sin\theta)}{\frac{1}{2}ka\sin\theta}\right)^4}$$

Yes




minima: $\frac{1}{2}ka\sin\theta = n\pi \Rightarrow \sin\theta = \frac{n\lambda}{a}$

and $ka\sin\theta = (n+\frac{1}{2})\pi$.

$\therefore \quad \frac{2\pi}{\lambda}a\sin\theta = (n+\frac{1}{2})\pi \Rightarrow \sin\theta = \frac{1}{2}(n+\frac{1}{2})\frac{\lambda}{a}$