Mathematical Methods MT2011: Problems 3

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(mostly recycled from Fabian Essler's MT2009 problems)

Fourier series

1. Sketch graphs of the following functions in the range $-2\pi < x < 2\pi$ given that all the functions are periodic with period 2π:

(a) $f(x) = |x|, \quad -\pi < x < \pi;$
(b) $f(x) = x, \quad -\pi < x < \pi;$
(c) $f(x) = \begin{cases} 0, & -\pi < x < 0, \\ x, & 0 < x < \pi. \end{cases}$

2. State whether the following functions are even, odd or neither even nor odd:

(a) $\sin x$ (b) $\cos x$ (c) $\sin^2 x$ (d) $\cos^2 x$ (e) $\sin 2x$

(f) $\cos 2x$ (g) $x \cos x$ (h) $\sin x + \cos x$ (i) $x + \sin x$ (j) e^x.

3. Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \sin x, & 0 < x < \pi. \end{cases} \tag{Q3.1}$$

Fourier sine and cosine series

Non-periodic functions defined on an interval $[a, b]$ can be represented as Fourier series by *continuing* them outside $[a, b]$ in order to make them periodic. For example, Figure 1 below shows a function $f(x)$ defined on the interval $[0, a]$, while Figure 2 shows two different continuations of $f(x)$. The first is even in x, while the second is odd, but both coincide with $f(x)$ in the interval $[0, a]$.

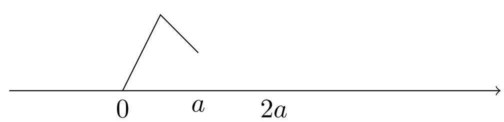


Figure 1: A function $f(x)$ deinfed on the interval $[0, a]$.

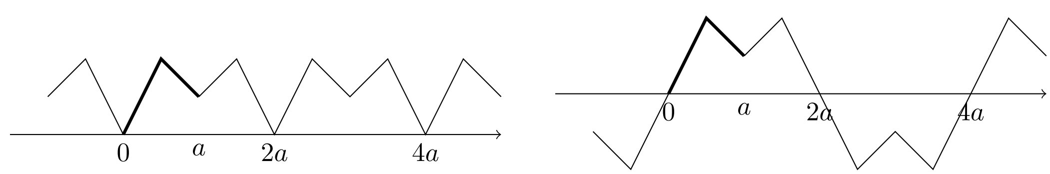


Figure 2: Two different periodic continuations of $f(x)$ with periods $2a$ and $4a$ respectively.

4. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function

$$f(x) = x \sin x, \quad 0 < x < \pi. \tag{Q4.1}$$

In order to determine the Fourier sine series, you need to extend the function to the interval $[-\pi, 0]$ by requiring that $f(-x) = -f(x)$ so that the function becomes odd. Similarly, to obtain the cosine series, you need to extend the function so that $f(-x) = f(x)$.

5. Find (a) the Fourier sine series and (b) the Fourier cosine series for the function $f(x) = x^2$ on the interval $0 < x < 2$.

Generalised Fourier series

6. By applying the the Gram–Schmidt procedure to the list of monomials 1, x, x^2, ..., show that the first three elements of an orthonormal basis for the space $L^2_w(-\infty, \infty)$ with weight function $w(x) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}$ are $e_0(x) = 1$, $e_1(x) = \sqrt{2}x$, $e_2(x) = \frac{1}{\sqrt{2}}(2x^2 - 1)$.

Explain why any well-behaved function $f : \mathbb{R} \to \mathbb{C}$ may be expressed as a series of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n e_n(x) \mathrm{e}^{-\frac{1}{2}x^2}, \qquad \text{(Q6.1)}$$

and obtain an expression for the coefficient a_k in terms of $f(x)$ and the $e_i(x)$.

The series on the RHS of (Q6.1) is a **Gauss–Hermite** series; the polynomials $e_i(x)$ turn out to be (normalised) Hermite polynomials. Later in the course we *define* Hermite polynomials to be eigenfunctions of a particular differential operator that satisfy a particular normalisation convention.

Fourier transforms and generalized functions

Recall that the Fourier transform of a function $f(x)$ is defined to be

$$\tilde{f}(k) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x.$$

7. Evaluate $\tilde{f}(k)$ for

$$f(x) = \begin{cases} (a-x)/a^2, & \text{for } 0 \le x \le a, \\ (a+x)/a^2, & \text{for } -a \le x \le 0, \\ 0 & \text{otherwise.} \end{cases} \tag{Q7.1}$$

[You may find it helpful to sketch $f(x)$.]

Sketch $\tilde{f}(k)$ as a function of k, marking in particular (a) the value of $\tilde{f}(k)$ at $k = 0$ (b) the position of the first zeros on either side of $k = 0$.

Explain the reason for the value of $\tilde{f}(0)$ in geometrical terms.

8. This problem is for students who have already taken the complex analysis short option (and they are strongly advised to do it).

Evalue the Fourier transforms of the following functions for $a > 0$ (using the residue theorem where appropriate):

(a) $f(x) = \frac{2a}{a^2+x^2}$, (b) $f(x) = \exp(-a|x|)$, (c) $f(x) = \frac{1}{\cosh x}$.

9. Let $\tilde{f}(k)$ be the Fourier transform of $f(x)$. Show that

(a) the Fourier transform of $f(ax)$ is $\frac{1}{a}\tilde{f}(k/a)$

(b) the Fourier transform of $f(a+x)$ is $\mathrm{e}^{\mathrm{i}ka}\tilde{f}(k)$

(c) the Fourier transform of $\mathrm{e}^{\mathrm{i}qx}f(x)$ is $\tilde{f}(k-q)$

(d) the Fourier transform of $\frac{\mathrm{d}f}{\mathrm{d}x}$ is $\mathrm{i}k\tilde{f}(k)$

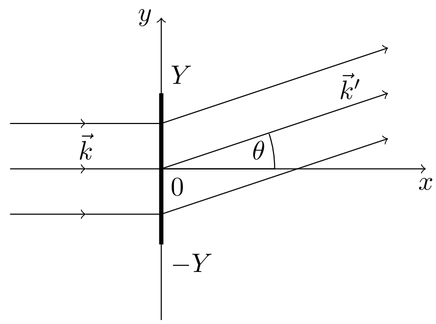
(e) the Fourier transform of $xf(x)$ is $\mathrm{i}\frac{\mathrm{d}\tilde{f}(k)}{\mathrm{d}k}$

10. By taking the Fourier transform of the differential equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2} - K^2\Phi = f(x), \tag{Q10.1}$$

show that its solution can be written as

$$\Phi(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{\mathrm{i}kx} \frac{\tilde{f}(k)}{k^2 + K^2}. \tag{Q10.2}$$



11. A direct application of Fourier transform techniques is *Fraunhofer diffraction*. See the discussion of p.443 of Riley, Hobson & Bence. Consider a beam of monochromatic light of wavelength λ incident on a two-dimensional screen of width $2Y$. The direction of the incident beam is specified by the wavevector $\vec{k}$. The magnitude of $\vec{k}$ is given by wavenumber $k = |\vec{k}| = \frac{2\pi}{\lambda}$.

The essential quantity in the Fraunhofer diffraction pattern is the dependence of the observed amplitude (and hence intensity) on the angle θ between the viewing direction (which we specify through a vector $\vec{k}'$ and the direction of the incident beam $\vec{k}$. We suppose that at the position $(0, y)$ the amplitude of the transmitted light is $f(y)$ per unit length in the y direction (note that $f(y)$ may be complex). The function $f(y)$ is called an *aperture function*. Both the screen and the beam are assumed infinite in the z direction. The total light amplitude at a position

$$\vec{r}_0 = x_0\vec{e}_x + y_0\vec{e}_y, \tag{Q11.1}$$

with $x_0 > 0$ will be the superposition of all Huygens' wavelets originating from the various parts of the screen. For very large $|\vec{r}_0|$ these can be treated as plane waves to give

$$A(\vec{r}_0) = \int_{-Y}^{Y} \mathrm{d}y\, f(y) \frac{\exp\left[\mathrm{i}\vec{k}' \cdot (\vec{r}_0 - y\vec{e}_y)\right]}{|\vec{r}_0 - y\vec{e}_y|}. \tag{Q11.2}$$

Here the factor $\exp\left[\mathrm{i}\vec{k}' \cdot (\vec{r}_0 - y\vec{e}_y)\right]$ represents the phase change undergone by the light in travelling from the point $y\vec{e}_y$ on the screen to the point $\vec{r}_0$. The denominator represents the reduction in amplitude with distance. Writing $\vec{k}' = k\cos\theta\vec{e}_x + k\sin\theta\vec{e}_y$ and assuming that $|\vec{r}_0| \gg Y$ we can approximate $|\vec{r}_0 - y\vec{e}_y| \approx |\vec{r}_0|$ and hence approximate (Q11.2) as

$$A(\vec{r}_0) = \frac{\exp(\mathrm{i}\vec{k}' \cdot \vec{r}_0)}{|\vec{r}_0|} \int_{-Y}^{Y} \mathrm{d}y\, f(y) \exp\left[-\mathrm{i}ky\sin\theta\right]. \tag{Q11.3}$$

Now using that $f(y) = 0$ for $|y| > Y$ (remember, the slit has half-width Y) we can extend the integration limits to $\pm\infty$. Then the intensity in direction θ is given by

$$I(\theta) = |A|^2 = \frac{2\pi}{|\vec{r}_0|^2} \left|\tilde{f}(k\sin\theta)\right|^2 \equiv \frac{2\pi}{|\vec{r}_0|^2} \hat{I}(\theta). \tag{Q11.4}$$

Use this to calculate $\hat{I}(\theta)$ for

(a) the "single finite slit" diffraction pattern, given by the aperture function

$$f(x) = \begin{cases} 1, & \text{for } |x| < b/2, \\ 0, & \text{otherwise;} \end{cases} \tag{Q11.5}$$

(b) The "two narrow slits" diffraction pattern, given by the aperture function

$$f(x) = \delta(x + a/2) + \delta(x - a/2); \tag{Q11.6}$$

(c) The "two finite slits" diffraction pattern, given by

$$f(x) = \begin{cases} 1, & \text{for } \frac{a-b}{2} \le x \le \frac{a+b}{2}, \\ 1, & \text{for } -\frac{a+b}{2} \le x \le -\frac{a-b}{2}, \\ 0, & \text{otherwise,} \end{cases} \tag{Q11.7}$$

where $b < a$. How is the result of (c) related to those for (a) and (b)? Can you explain this in terms of the convolution theorem?