Mathematical Methods MT2011: Problems 2

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(mostly recycled from Fabian Essler's MT2009 problems)

Matrix addition and multiplication

1. The matrix equation $\mathbf{c} = A\mathbf{a} + \mathbf{b}$ can be written using index notation together with the convention that repeated indices are summed over as $c_i = A_{ij}a_j + b_i$. Write the following expressions in matrix form: (a) $b_j = A_{ji}a_i$ (b) $b_l = A^\star_{kl}a_k$ (c) $b_jA_{jk}b_k$ (d) $C_{ij} = A_{ki}B_{kj}$ (e) $F_{ij} = B_{jk}A_{ki}$ (f) $A_{ij}\delta_{jk}B_{ki}$.

2. By considering the matrices $A = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ show that $AB = 0$ does not imply that either A or B is the zero matrix. Allowing A and B to be any square matrices, show that $AB = 0$ implies that at least one of them is singular, i.e., has zero determinant.

3. Show that if $[A, B] = 0$ then

$$(A + B)^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i}, \qquad \text{(Q3.1)}$$

identifying clearly where you make use of the fact that $[A, B] = 0$. Use this result to show that if $[A, B] = 0$ then $\exp A \exp B = \exp(A + B)$. Give an example of A,B for which $[A, B] \neq 0$ and $\exp A \exp B \neq \exp(A + B)$.

4. Consider the matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the matrices are symmetric? Which are Hermitian? By calculating the commutators of these matrices, show that they can be written as $[\sigma^a, \sigma^b] = 2\mathrm{i}\epsilon_{abc}\sigma^c$, where ϵ_{abc} is the alternating tensor and on the right-hand side the summation convention is employed (i.e., the index c is summed over). Write $\exp(\mathrm{i}\alpha\sigma^y)$ (α is a real number) as a 2×2 matrix. What does it represent? Show that $\exp(\mathrm{i}\alpha\sigma^y)$ is unitary without writing it explicitly as a 2×2 matrix.

5. Let A be an operator on a complex vector space $\mathcal{V}$ and let $|v_1\rangle$ and $|v_2\rangle$ be any two vectors from $\mathcal{V}$. Verify that

$$4\langle v_2| A |v_1\rangle = \langle v_+| A |v_+\rangle - \langle v_-| A |v_-\rangle + \langle v'_+| A |v'_+\rangle \mathrm{i} - \langle v'_-| A |v'_-\rangle \mathrm{i}, \qquad \text{(Q5.1)}$$

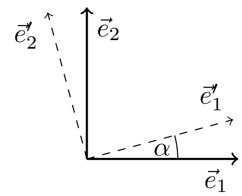
where $|v_\pm\rangle \equiv |v_1\rangle \pm |v_2\rangle$ and $|v'_\pm\rangle \equiv |v_1\rangle \pm \mathrm{i}\,|v_2\rangle$. Use this result to show that

(i) if $\langle v| A |v\rangle = 0$ for all vectors $|v\rangle \in \mathcal{V}$ then $A = 0$;

(ii) A is Hermitian if and only if $\langle v| A |v\rangle$ is real for all $|v\rangle \in \mathcal{V}$. (Hint: if $\langle v| A |v\rangle$ is real then $\langle v| A |v\rangle - \langle v| A |v\rangle^\star = 0$.)

Change of basis

6. Consider the vector space $\mathcal{V}$ of arrows in the plane. Let A be the linear operator that rotates all vectors by 45 degrees and then reflects them with respect to the horizontal. Let $B_1 = \{\vec{e}_1, \vec{e}_2\}$ be the standard Cartesian basis and let $B_2 = \{\vec{e}'_1, \vec{e}'_2\}$ be another orthonormal basis of $\mathcal{V}$ which is obtained from $\{\vec{e}_1, \vec{e}_2\}$ by a rotation by an angle α (see figure). Write down the transformation that takes $\vec{e}_{1,2}$ to $\vec{e}'_{1,2}$ in matrix form. What are the coordinate representations of A with respect to the bases B_1 and B_2 respectively? What is the matrix equation that relates these two coordinate representations?



Rank, trace, determinant

7. Find the rank of the following matrices by reducing them to upper triangular form:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 5 & -2 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix}. \tag{Q7.1}$$

8. Here is one way of proving that $\det(AB) = \det A \det B$, where A and B are $n \times n$ matrices.

(i) Verify that the matrices involved can written as

$$A = \begin{pmatrix} e_1 A \\ \vdots \\ e_n A \end{pmatrix}, \quad B = \begin{pmatrix} e_1 B \\ \vdots \\ e_n B \end{pmatrix}, \quad AB = \begin{pmatrix} \sum_{j_1} A_{1j_1} e_{j_1} B \\ \vdots \\ \sum_{j_n} A_{nj_n} e_{j_n} B \end{pmatrix}, \tag{Q8.1}$$

where e_k is a row vector having 1 in the k^{th} slot and zeros everywhere else.

(ii) Show that

$$\det \begin{pmatrix} \alpha\vec{a} + \beta\vec{b} \\ \vec{c}_2 \\ \vdots \\ \vec{c}_n \end{pmatrix} = \alpha \det \begin{pmatrix} \vec{a} \\ \vec{c}_2 \\ \vdots \\ \vec{c}_n \end{pmatrix} + \beta \det \begin{pmatrix} \vec{b} \\ \vec{c}_2 \\ \vdots \\ \vec{c}_n \end{pmatrix}, \tag{Q8.2}$$

where $\vec{a}$, $\vec{b}$, $\vec{c}_2$, ..., $\vec{c}_n$ are n-dimensional row vectors and α, β are scalars.

(iii) Explain why

$$\det(AB) = \sum_{j_1=1}^{n} \cdots \sum_{j_n=1}^{n} A_{1j_1} \cdots A_{nj_n} \det \begin{pmatrix} e_{j_1} B \\ \vdots \\ e_{j_n} B \end{pmatrix}. \tag{Q8.3}$$

(iv) We need only consider terms in which $(j_1, \ldots, j_n)$ are distinct – why? Hence show that (Q8.3) can be written as

$$\det(AB) = \sum_P A_{1P(1)} \cdots A_{nP(n)} \operatorname{sgn}(P) \det \begin{pmatrix} e_1 B \\ \vdots \\ e_n B \end{pmatrix} = \det A \det B. \tag{Q8.4}$$

9. Show by direct expansion of the determinant that

$$\det(I + \epsilon A) = 1 + \epsilon \operatorname{tr} A + O(\epsilon^2) \tag{Q9.1}$$

and hence that

$$\det(\exp A) = \exp(\operatorname{tr} A). \tag{Q9.2}$$

10. By taking the trace of both sides, prove that there are no finite-dimensional matrix representations of the momentum operator p and the position operator x which satisfy $[p, x] = -\mathrm{i}\hbar$. Why does this argument fail if the matrices are infinite dimensional (as Heisenberg's were)?

Rotations, eigenvalues, eigenvectors

11. Which of these matrices represents a rotation?

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix}. \tag{Q11.1}$$

Find the angle and axis of the rotation. What does the other matrix represent?

12. Let $A = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(a) Multiply A and B together and interpret the result in terms of 2D rotations.

(b) Evaluate the product $A^{\mathrm{T}}A$ and interpret the result. Evaluate $A^{\mathrm{T}}B$ and interpret the result.

(c) Find the eigenvalues and eigenvectors of A. (Remember that they do not have to be real. Why not?)

13.

(a) Find the eigenvalues of the Pauli matrix $\sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Normalise the two corresponding eigenvectors, $\vec{u}_1$, $\vec{u}_2$, so that $\vec{u}_1^\dagger \cdot \vec{u}_1 = \vec{u}_2^\dagger \cdot \vec{u}_2 = 1$. Check that $\vec{u}_1^\dagger \cdot \vec{u}_2 = 0$. Form the matrix $U = (\, \vec{u}_1 \quad \vec{u}_2 \,)$ and verify that $U^\dagger U = I$. Evaluate $U^\dagger \sigma^y U$. What have you learned from this calculation?

(b) A general 2-component complex vector $\vec{v} = (c_1, c_2)^{\mathrm{T}}$ is expanded as a linear combination of the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ via

$$\vec{v} = \alpha\vec{u}_1 + \beta\vec{u}_2, \tag{Q13.1}$$

where α and β are complex numbers. Determine α and β in terms of c_1 and c_2 in two ways: (i) by equating corresponding components of (Q13.1), (ii) by showing that $\alpha = \vec{u}_1^\dagger \cdot \vec{v}$, $\beta = \vec{u}_2^\dagger \cdot \vec{v}$ and evaluating these products.

14. Verify that the matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has eigenvalues -1, 1, 2 and find the associated normalised eigenvectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$. Construct the matrix $R = (\, \vec{u}_1 \quad \vec{u}_2 \quad \vec{u}_3 \,)$ and show that it is orthogonal and that it diagonalises A.

15. Construct a real symmetric matrix whose eigenvalues are 2, 1 and -2, and whose corresponding normalised eigenvectors are $\frac{1}{\sqrt{2}}(0,1,1)^{\mathrm{T}}$, $\frac{1}{\sqrt{2}}(0,1,-1)^{\mathrm{T}}$ and $(1,0,0)^{\mathrm{T}}$.

16. Find the eigenvalues and eigenvectors of the matrix $F = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Hence, proving the validity of the method you use, find the values of the elements of the matrix F^n, where n is a positive integer.

17. Write down the matrix R_1 for a three-dimensional rotation through $\pi/4$ about the z-axis and the matrix R_2 for a rotation through $\pi/4$ about the x-axis. Calculate $Q_1 = R_1 R_2$ and $Q_2 = R_2 R_1$; explain geometrically why they are different.

18. By finding the eigenvectors of the Hermitian matrix $H = \begin{pmatrix} 10 & 3\mathrm{i} \\ -3\mathrm{i} & 2 \end{pmatrix}$ construct a unitary matrix U such that $U^\dagger H U = D$, where D is a real diagonal matrix.

19. Which of the following matrices have a complete set of eigenvectors in common?

$$A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}, \quad C = \begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix}, \quad D = \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}. \qquad \text{(Q19.1)}$$

Construct the common set of eigenvectors where possible.

20. What are the eigenvalues and eigenvectors of the matrix $\sigma^+ = \frac{1}{2}(\sigma^x + \mathrm{i}\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$? Can σ^+ be diagonalised?

Quadratic forms

21.

(i) Show that the quadratic form $4x^2 + 2y^2 + 2z^2 - 2xy + 2yz - 2zx$ can be written as $\vec{x}^{\mathrm{T}} V \vec{x}$ where V is a symmetric matrix. Find the eigenvalues of V. Explain why, by rotating the axes, the quadratic form may be reduced to the simple expression $\lambda x'^2 + \mu y'^2 + \nu z'^2$; what are λ, μ, ν?

(ii) The components of the current density vector $\vec{j}$ in a conductor are proportional to the components of the applied electric field $\vec{E}$ in simple (isotropic) cases: $\vec{j} = \sigma\vec{E}$. In crystals, however, the relation may be more complicated, though still *linear*, namely of the form $j_a = \sum_{b=1}^{3} \sigma_{ab} E_b$, where σ_{ab} for the entries in a real symmetric 3×3 matrix and the index a runs from 1 to 3. In a particular case, the quantities σ_{ab} are given (in certain units) by $\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$. Explain why by a rotation of the axes the relation between $\vec{j}$ and $\vec{E}$ can be reduced to $j'_1 = \tilde{\sigma}_1 E'_1$, $j'_2 = \tilde{\sigma}_2 E'_2$, $j'_3 = \tilde{\sigma}_3 E'_3$ and find $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$.