Mathematical Methods MT2011: Problems 1

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(mostly recycled from Fabian Essler's MT2009 problems)

Linear independence, dimension, bases

1. Show that the space of 2×2 matrices is a linear vector space. What is its dimension? Give a basis for this space.

2. What is the dimension of the space of $n \times n$ matrices? Give a basis for this space.

3. What is the dimension of the space of $n \times n$ matrices all of whose components are zero except possibly for the diagonal components?

4. What is the dimension of the space of symmetric 2×2 matrices, i.e., 2×2 matrices such that $A = A^{\mathrm{T}}$? (Recall that the transpose A^{T} of a matrix A is defined by $(A^{\mathrm{T}})_{ij} = A_{ji}$.) Exhibit a basis for this space.

5. Consider the vector space of all functions of a variable t. Show that the following pairs of functions are linearly independent. (a) 1, t (b) t, t^2 (c) e^t, t (d) $\sin t$, $\cos t$.

6. What are the coordinates of the function $f(t) = 3\sin t + 5\cos t$ with respect to the basis $\{\sin t, \cos t\}$?

7. What are the dimensions of the vector spaces spanned by the following sets of vectors (they are given in Cartesian form)?
(a) $\{(1,1)^{\mathrm{T}}, (1,2)^{\mathrm{T}}\}$
(b) $\{(1,0)^{\mathrm{T}}, (1,0)^{\mathrm{T}}\}$
(c) $\{(1,1,2)^{\mathrm{T}}, (-2,0,1)^{\mathrm{T}}, (-1,1,3)^{\mathrm{T}}\}$
(d) $\{(1,1,1,1)^{\mathrm{T}}, (1,-1,1,-1)^{\mathrm{T}}, (1,1,-1,-1)^{\mathrm{T}}, (1,-1,-1,1)^{\mathrm{T}}\}$
(e) $\{(1,2,3)^{\mathrm{T}}, (1,-2,1)^{\mathrm{T}}, (4,1,4)^{\mathrm{T}}, (4,5,6)^{\mathrm{T}}\}$

If the number of vectors is greater than the dimension, choose some of them to form a set of basis vectors and express the remaining vectors as linear combinations of them. Which of the bases are orthogonal?

*8. $|v_1\rangle, \ldots, |v_m\rangle$ is a list of linearly dependent vectors with $|v_1\rangle \neq 0$. Show that:
(a) there exists at least one j between 2 and m such that $|v_j\rangle \in \mathrm{span}\,(|v_1\rangle, \ldots, |v_{j-1}\rangle)$;
(b) if we choose k to be the *smallest* index such that $|v_k\rangle \in \mathrm{span}\,(|v_1\rangle, \ldots, |v_{k-1}\rangle)$ then the vectors $|v_1\rangle$, ..., $|v_{k-1}\rangle$ are linearly independent.

***9.** Recall that $|v\rangle$ is an eigenvector of the operator A with eigenvalue λ if $A\,|v\rangle = \lambda\,|v\rangle$. Suppose that λ_1, ..., λ_m are distinct eigenvalues of A and $|v_1\rangle$, ..., $|v_m\rangle$ are the corresponding eigenvectors. Show that $|v_1\rangle$, ..., $|v_m\rangle$ are linearly independent.

Inner product, orthogonality

10. Let $\vec{a}_1, \ldots, \vec{a}_n$ be vectors in $\mathbb{R}^n$ and assume that they are mutually perpendicular (i.e., any two of them are orthogonal) and none of them is equal to 0. Prove that they are linearly independent.

11. Find real values α and β such that the complex vectors $\mathbf{u} = \alpha \begin{pmatrix} 1+\mathrm{i} \\ 1-\mathrm{i} \end{pmatrix}$ and $\mathbf{v} = \beta \begin{pmatrix} 1-\mathrm{i} \\ 1+\mathrm{i} \end{pmatrix}$ are normalised. What is the value of the scalar product $\mathbf{u}^\dagger \cdot \mathbf{v}$? Prove that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Are there any further linearly independent two-dimensional complex vectors? If so, find the necessary vectors to make an orthogonal basis. Express the vector $\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}$ as a linear combination of the basis vectors.

12. Prove the **Triangle Inequality**: given a norm $\||\mathbf{a}\rangle\| = \sqrt{\langle \mathbf{a}|\mathbf{a}\rangle}$ defined through the inner product $\langle\cdot|\cdot\rangle$ we have for any two vectors $|\mathbf{v}\rangle$ and $|\mathbf{w}\rangle$ in a linear vector space

$$\||\mathbf{v}\rangle + |\mathbf{w}\rangle\| \leq \||\mathbf{v}\rangle\| + \||\mathbf{w}\rangle\| \,. \tag{Q12.1}$$

13. Construct a third vector which is orthogonal to the following pairs and normalise all three vectors.
(a) $(1, 2, 3)^T$, $(-1, -1, 1)^T$
(b) $(1 + \mathrm{i}\sqrt{3}, 2, 1 - \mathrm{i}\sqrt{3})^T$, $(1, -1, 1)^\mathrm{T}$
*(c) $(1 - \mathrm{i}, 1, 3\mathrm{i})^\mathrm{T}$, $(1 + 2\mathrm{i}, 2, 1)^\mathrm{T}$.

14. Using the Gram–Schmidt procedure construct an orthonormal set of vectors from the following:
$\vec{x}_1 = (0, 0, 1, 1)^\mathrm{T}$, $\vec{x}_2 = (1, 0, -1, 0)^\mathrm{T}$, $\vec{x}_3 = (1, 2, 0, 2)^\mathrm{T}$, $\vec{x}_4 = (2, 1, 1, 1)^\mathrm{T}$.

15. Consider the vector space of continuous, complex-valued functions on the interval $[-\pi, \pi]$. Show that

$$\langle \mathbf{f}|\mathbf{g}\rangle = \int_{-\pi}^{\pi} \mathrm{d}t\, f^\star(t) g(t) \tag{Q15.1}$$

defines a scalar product on this space. Are the following functions orthogonal with respect to this scalar product? (a) $\sin t$, $\cos t$ (b) $\exp(\mathrm{i}nt)$, $\exp(\mathrm{i}kt)$, n, k integers (c) t^2, t^4.

16. Let $S(\mathbf{a}, \mathbf{b}) = \mathbf{a}^\dagger M \mathbf{b}$, where M is a matrix. What conditions do we need to impose on M if $S(\mathbf{a}, \mathbf{b})$ is to define a scalar product between the vectors $\mathbf{a}$ and $\mathbf{b}$?

17. Let $\mathcal{V}$ be the real vector space of all real symmetric $n \times n$ matrices and define the scalar product of two matrices A, B by

$$\langle A|B\rangle = \mathrm{tr}(AB), \tag{Q17.1}$$

where $\mathrm{tr}\, A$ denotes the trace of A (i.e., the sum of the diagonal elements). Show that this indeed fulfills the requirements of a scalar product.