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Mathematics 5 Ziyan Li

Q. 1) (a) An equation is linear if when $y_1(x)$ and $y_2(x)$ are both solutions to the Complementary Function, then any linear combination $Ay_1 + By_2$ is also a solution to the CF

~~Linear equation can be written in the form.~~

(b) homogeneous equation is such that there is no term in the equation that is a function of only the independent variable.

Example ① => $y'' + 2y' + 3y = 4x$

② => $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$

The linearities are exact in both cases.

Properties :

Homogeneous & Linear : Only Complementary function.

Linear : Complementary Function plus Particular Integral

2. $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad ①$$

Introduce $\quad V(r,t) = r\,u(r,t)$

$\Rightarrow \quad u(r,t) = \frac{1}{r}V(r,t)$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left[\frac{1}{r}V(r,t)\right]$$

$$= \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}$$

$$r^2\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial r} - V$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = r\frac{\partial^2 v}{\partial r^2} + \cancel{\frac{\partial v}{\partial r}} - \cancel{\frac{\partial v}{\partial r}} = r\frac{\partial^2 v}{\partial r^2}$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial^2 v}{\partial r^2}$$

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{1}{r}v(r,t)\right) = \frac{1}{c^2}\frac{1}{r}\frac{\partial^2 v}{\partial t^2}$$

$$\therefore ① \Rightarrow \cancel{\frac{1}{r}}\frac{\partial^2 v}{\partial r^2} = \cancel{\frac{1}{r}}\frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \Rightarrow \frac{\partial^2 v}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}$$

$v(r,t) = V(\xi,\eta) \qquad \xi = r + ct \;, \; \eta = r - ct$

$$\Rightarrow \frac{\partial \xi}{\partial r} = 1 \quad \frac{\partial \eta}{\partial r} = 1 \quad \frac{\partial \xi}{\partial t} = c \quad \frac{\partial \eta}{\partial t} = -c$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \xi}\right) + \frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \xi}\right)\frac{\partial \xi}{\partial r} + \frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \xi}\right)\frac{\partial \eta}{\partial r}$$

$$+ \frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \eta}\right)\frac{\partial \xi}{\partial r} + \frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \eta}\right)\frac{\partial \eta}{\partial r} = \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + 2\frac{\partial^2 v}{\partial \xi \partial \eta}$$

$$\frac{\partial v}{\partial t} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = c\frac{\partial v}{\partial \xi} - c\frac{\partial v}{\partial \eta}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial v}{\partial \xi} - c\frac{\partial v}{\partial \eta}\right).$$

$$= c\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \xi}\right)\frac{\partial \xi}{\partial t} + c\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \xi}\right)\frac{\partial \eta}{\partial t} - c\frac{\partial}{\partial \xi}\left(\frac{\partial v}{\partial \eta}\right)\frac{\partial \xi}{\partial t}$$

$$- c\frac{\partial}{\partial \eta}\left(\frac{\partial v}{\partial \eta}\right)\frac{\partial \eta}{\partial t}.$$

$$= c^2\frac{\partial^2 v}{\partial \xi^2} + c^2\frac{\partial^2 v}{\partial \eta^2} - 2c^2\frac{\partial^2 v}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \Rightarrow 4c^2\frac{\partial^2 v}{\partial \xi \partial \eta} = 0$$

$$\Rightarrow \frac{\partial^2 v}{\partial \xi \partial \eta} = 0 \quad (2)$$

Integrate (2) w.r.t η

$$\Rightarrow \frac{\partial v}{\partial \xi} = f'(\xi) \quad (3)$$

Integrate (3) w.r.t. ξ gives

$$v(\xi, \eta) = f(\xi) + g(\eta)$$

$$\therefore v(r,t) = f(r+ct) + g(r-ct).$$

$\therefore$ ~~$u(r,t) = \frac{1}{r}[f(r+g$~~

$$u(r,t) = \frac{1}{r}[f(r+ct) + g(r-ct)]$$

The solution represents a spherical wave that propagates spherically symmetrically along the radial direction.

3. $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Separation of Variables

try $y = X(x)\,T(t)$ we have

$$T\frac{\partial^2 X}{\partial x^2} = \frac{1}{c^2} X \cancel{\frac{\partial^2 T}{\partial x^2}} \frac{\partial^2 T}{\partial t^2}$$

$$\Rightarrow \quad \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \lambda$$

L.H.S is a function only of x and R.H.S is a function only of t so if they always equal then they must both be a constant (let it be λ)

$\therefore$ if $\lambda = 0$

$$\frac{d^2X}{dx^2} = 0 \Rightarrow X(x) = \cancel{A_0 x}\ A_0 x + B_0$$

$$\frac{d^2T}{dt^2} = 0 \Rightarrow T(t) = C_0 t + D_0$$

If $\lambda > 0 \Rightarrow \lambda = k^2$

then $$\frac{d^2X}{dx^2} - k^2 X = 0 \Rightarrow X = A_k^+ e^{kx} + B_k^+ e^{-kx}$$

$$\frac{d^2T}{dt^2} - c^2k^2 T = 0 \Rightarrow T = C_k^+ e^{kct} + \cancel{B_k} D_k^+ e^{-kct}.$$

if $\lambda < 0 \Rightarrow \lambda = -k^2$

$$\therefore \frac{d^2X}{dx^2} + k^2 X = 0 \Rightarrow X = A_k^- \cos(kx) + B_k^- \sin(kx)$$

$$\frac{d^2T}{dt^2} + k^2c^2 T = 0 \Rightarrow T = C_k^- \cos(kct) + D_k^- \sin(kct)$$

For displacement on a string we want the sinusoidal solutions

$$\therefore \; y(x,t) = \sum_{k} \left(A_k \cos(kx) + B_k \sin(kx)\right)\left(C_k \cos(kct) + D_k \sin(kct)\right)$$

Boundary Conditions:

$y(0,t) = y(a,t) = 0$ since string is stretched between these ~~two~~ points. $x=0$ and $x=a$.

$y(0,t) = 0 \Rightarrow$

$$\sum_{k} A_k \left(C_k \cos(kct) + D_k \sin(kct)\right) = 0 \quad \Rightarrow \quad A_k = 0$$

$y(a,t) = 0 \Rightarrow$

$$\sum_{k} B_k \sin(ka) \left(C_k \cos(kct) + D_k \sin(kct)\right) = 0$$

For B_k to be non-zero, $\sin(ka) = 0$

$\therefore \; ka = n\pi \quad (n = 1, 2, 3, \ldots)$

$(\sin(-x) = -\sin(x)$ so negative n's can be incoporated$)$

$\therefore \; k = \frac{n\pi}{a}$

$\therefore$ General solution is

$$y(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left[C_n \cos\left(\frac{n\pi c t}{a}\right) + D_n \sin\left(\frac{n\pi c t}{a}\right)\right]$$

(ii)

$$y(x,0) = L\sin\left(\frac{\pi x}{a}\right) \qquad \frac{\partial y}{\partial t}(x,0) = 0$$

① ②

② ⟹

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{a} \sin\left(\frac{n\pi x}{a}\right)\left[-C_n \sin\left(\frac{n\pi c(0)}{a}\right) + D_n \cos\left(\frac{n\pi c(0)}{a}\right)\right].$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c}{a} \sin\left(\frac{n\pi x}{a}\right)\left[D_k\right].$$

$$\Rightarrow \quad D_n = 0$$

$$\cancel{\therefore y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi c t}{a}\right)}$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi c t}{a}\right).$$

① ⟹ $$L\sin\left(\frac{\pi x}{a}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right)$$

Easily see that $C_n = L$ if $n = 1$
$C_n = 0$ if $n \neq 1$

$$\therefore \boxed{y(x,t) = L\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi c t}{a}\right)}$$

✓

(ii) $y(x,0) = 0$ ①, $\frac{\partial y}{\partial t}(x,0) = V \sin(\frac{2\pi x}{a})$

① $\Rightarrow$ $0 = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{a}) [C_n]$

$\Rightarrow$ $C_n = 0$

$\cancel{②\Rightarrow}$ $y = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi x}{a}) \sin(\frac{n\pi c t}{a})$

② $\Rightarrow$ $V \sin(\frac{2\pi x}{a}) = \sum_{n=1}^{\infty} \frac{n\pi c}{a} D_n \sin(\frac{n\pi x}{a})$

Easily see that $D_n = 0$ if $n \neq 2$

$V = \frac{n\pi c}{a} D_n$ if $n = 2$

$\therefore D_2 = \frac{Va}{2\pi c}$

$$\boxed{y(x,t) = \frac{Va}{2\pi c} \sin(\frac{2\pi x}{a}) \sin(\frac{2\pi c t}{a})}$$

(iii) ~~$y(x,0) = L \sin$~~

Let $y_1 = L \sin(\frac{\pi x}{a}) \cos(\frac{\pi c t}{a})$ then $y_1(x,0) = L \sin(\frac{\pi x}{a})$

$\frac{\partial y_1}{\partial t}(x,0) = 0.$

$y_2 = \frac{Va}{2\pi c} \sin(\frac{2\pi x}{a}) \sin(\frac{2\pi c t}{a})$ then $y_2(x,0) = 0$

$\frac{\partial y_2}{\partial t}(x,0) = V \sin(2\pi x / a)$

Consider $y = y_1 + y_2$ then

$y(x,0) = y_1(x,0) + y_2(x,0) = L \sin(\frac{\pi x}{a})$

$\frac{\partial y}{\partial t}(x,0) = \frac{\partial y_1}{\partial t}(x,0) + \frac{\partial y_2}{\partial t}(x,0) = V \sin(\frac{2\pi x}{a})$

$\therefore$ $y = y_1 + y_2$ satisfies the boundary conditions.

and it is the solution.

$$\therefore \boxed{y(x,t) = L\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi c t}{a}\right) + \frac{Va}{2\pi c}\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi c t}{a}\right)}$$

(iv) $y(x,0) = 2L\sin\left(\frac{3\pi x}{2a}\right)\cos\left(\frac{\pi x}{2a}\right)$

$$= 2L\left[\frac{1}{2}\sin\left(\frac{3\pi x}{2a} + \frac{\pi x}{2a}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{2a} - \frac{\pi x}{2a}\right)\right]$$

$$= L\sin\left(\frac{2\pi x}{a}\right) + L\sin\left(\frac{\pi x}{a}\right)$$

$\because$ $\frac{\partial y}{\partial t}(x,0) = 0$

$\therefore$ By exactly the same argument in (i) we find that the solution.

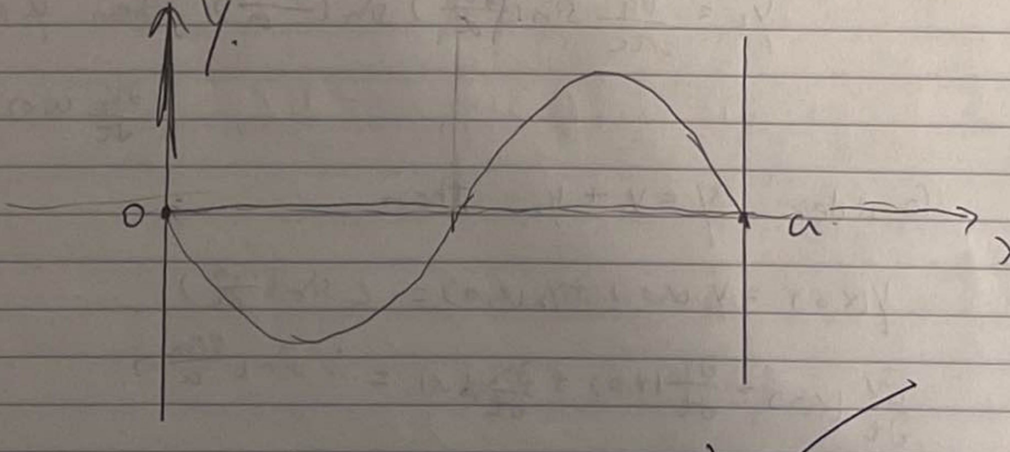
$$\boxed{y(x,t) = L\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi c t}{a}\right) + L\sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi c t}{a}\right)}$$

At $t = \frac{a}{2c}$

$\cancel{y(x,t)}$

$$y\left(x, \frac{a}{2c}\right) = L\sin\left(\frac{\pi x}{a}\right)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} + L\sin\left(\frac{2\pi x}{a}\right)\underbrace{\cos(\pi)}_{-1}$$

$$= -L\sin\left(\frac{2\pi x}{a}\right)$$



4 $$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2} \quad ① \qquad T = T(x, t)$$

try $T = F(t)\,X(x)$

$$\cancel{X}\frac{dF}{dt} = K F \frac{d^2X}{dx^2}$$

$$\Rightarrow \quad \frac{1}{F}\frac{dF}{dt} = \frac{K}{X}\frac{d^2X}{dx^2} = \lambda = \text{constant}$$

If λ is real, $\frac{dF}{dt} = \lambda F$ generates exponential solutions in t, which contradicts the boundary condition $T(x, 0) = T_0\cos\omega t$. $\therefore$ λ has to be imaginary to give oscillatory solution

$\therefore$ let $\lambda = -i\Omega \Rightarrow \frac{dF}{dt} = -i\Omega F \Rightarrow F = e^{-i\Omega t}$

$$\frac{K}{X}\frac{d^2X}{dx^2} = -i\Omega \Rightarrow \frac{d^2X}{dx^2} + \frac{i\Omega}{K}X = 0$$

$$\Rightarrow X = Ae^{\sqrt{-i\Omega/K}\,x} + Be^{-\sqrt{-i\Omega/K}\,x}$$

$$\cancel{\sqrt{\frac{i\Omega}{K}}} \quad \sqrt{\frac{-i\Omega}{K}} = \sqrt{-i}\sqrt{\frac{\Omega}{K}} = \frac{1-i}{\sqrt{2}}\sqrt{\frac{\Omega}{K}} = (1-i)\sqrt{\frac{\Omega}{2K}}$$

let $\cancel{\alpha = \sqrt{\frac{\Omega}{2K}}}$ $\quad \frac{1}{\alpha} = \sqrt{\frac{\Omega}{2K}}$ then

$$X = Ae^{(1-i)x/\alpha} + Be^{-(1-i)x/\alpha}$$

As $x \to \infty$ we want finite solution

so $A = 0$

$$X = Be^{-(1-i)x/\alpha} = Be^{-x/\alpha}e^{ix/\alpha}$$

$$\therefore \tilde{T} = FX = Be^{-x/\alpha} e^{ix/\alpha} e^{-i\Omega t}$$

$$= Be^{-x/\alpha} e^{i(x/\alpha - \Omega t)}$$

$\tilde{T}$ is a solution to ①

If $\tilde{T} = T_R + iT_I$ ~~wh~~ $(T_R, T_I \in \mathbb{R})$.

then $\frac{\partial}{\partial t}(T_R + iT_I) = K\frac{\partial^2}{\partial x^2}(T_R + iT_I)$

$$\Rightarrow \left(\frac{\partial T_R}{\partial t} - K\frac{\partial^2 T_R}{\partial x^2}\right) = -i\left(\frac{\partial T_I}{\partial t} - K\frac{\partial^2 T_I}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial T_R}{\partial t} = K\frac{\partial^2 T_R}{\partial x^2} \text{ and } \frac{\partial T_I}{\partial t} = K\frac{\partial^2 T_I}{\partial x^2}$$

$\Rightarrow T_R$ and T_I both solve ①

To match boundary condition $T(0,t) = T_0 \cos \omega t$

we take ~~Re~~ $T = T_R$.

$$\therefore T = Re(\tilde{T}) = Be^{-x/\alpha}\cos\left(\frac{x}{\alpha} - \Omega t\right)$$

$$T(0,t) = T_0 \cos \omega t = B\cos(-\Omega t) = B\cos(\Omega t).$$

$\therefore$ ~~$T_0 = +$~~ $B = T_0$, $\Omega = \pm\omega$

Note: try $\Omega = -\omega$ we find T does not satisfy ①

So $\Omega = +\omega$, then $\frac{1}{\alpha^2} = \frac{\omega}{2K} = \frac{1}{a^2} \Rightarrow \alpha = a$

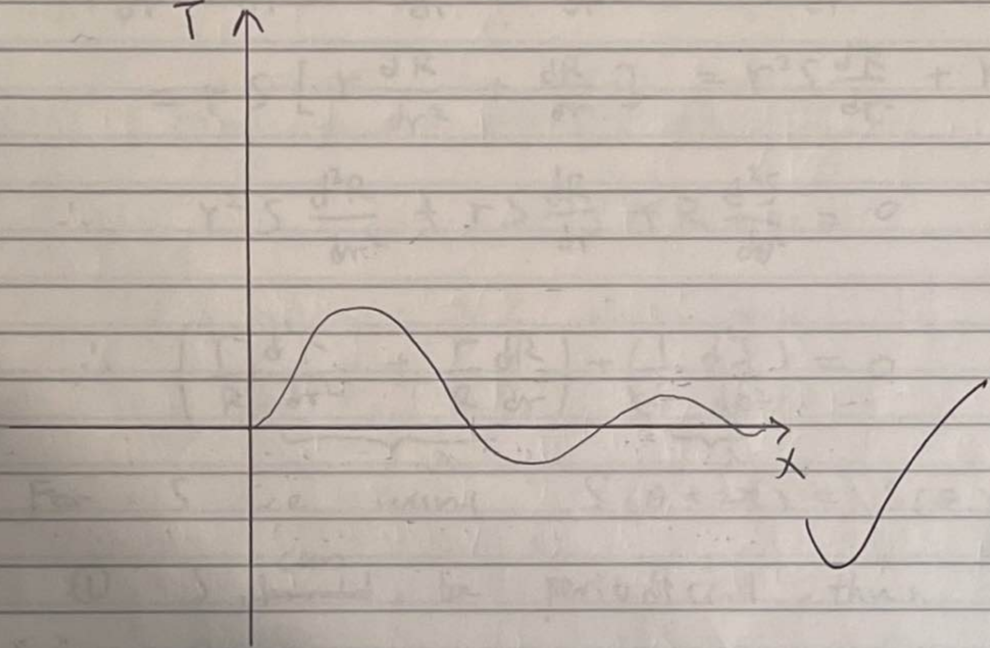
$$\therefore \boxed{T(x,t) = T_0 e^{-x/a}\cos\left(\frac{x}{a} - \omega t\right)}$$

when $\omega t = \pi/2$

$$T(x, \tfrac{\pi}{2\omega}) = T_0 e^{-x/a} \cos\left(\frac{x}{a} - \frac{\pi}{2}\right)$$

$$\cancel{= T_0 e^{-x/a} \cos}$$

$$= T_0 e^{-x/a} \sin\left(\frac{x}{a}\right)$$



5

$$r\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial \theta^2}=0$$

try $V=R(r)\,S(\theta)$

$$\frac{\partial^2 V}{\partial\theta^2}=R\frac{d^2S}{d\theta^2}$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)=r\frac{\partial}{\partial r}\left(rS\frac{dR}{dr}\right)=rS\frac{d}{dr}\left(r\frac{dR}{dr}\right)$$

$$=rS\left[r\frac{d^2R}{dr^2}+\frac{dR}{dr}\right]=r^2S\frac{d^2R}{dr^2}+rS\frac{dR}{dr}$$

$$\therefore\quad r^2S\frac{d^2R}{dr^2}+rS\frac{dR}{dr}+R\frac{d^2S}{d\theta^2}=0$$

$$\therefore\quad \underbrace{\left(\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}\right)}_{-\lambda}+\underbrace{\left(\frac{1}{S}\frac{d^2S}{d\theta^2}\right)}_{\lambda}=0$$

For S we want $S(\theta+2\pi)=S(\theta)$

① S ~~should~~ can be periodical, thus $\lambda=-k^2$

$$\frac{1}{S}\frac{d^2S}{d\theta^2}=-k^2\implies\frac{d^2S}{d\theta^2}+k^2S=0$$

$$\implies S=C_k\cos(k\theta)+D_k\sin(k\theta)$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}=k^2\implies r^2R''+rR'-k^2R=0$$

~~try $R(r)=\sum_{k=0}^{\infty}A_kr^k$ then.~~

$$\cancel{r^2\sum_{k=0}^{\infty}A_kk(k-1)r^{k-2}+r\sum_{k=0}^{\infty}A_kk(r^{k-1})-k^2\sum_{k=0}^{\infty}A_kr^k}$$

$$\cancel{=\sum_{k=0}^{\infty}A_k\left[k(k-1)+k-k^2\right]}$$

try $R(r)=\sum_{n=0}^{\infty}A_nr^n$

then $r^2\sum_{n=0}^{\infty}A_nn(n-1)r^{n-2}+r\sum_{n=0}^{\infty}A_nnr^{n-1}-k^2\sum_{n=0}^{\infty}A_nr^n=0$

$$\therefore \sum_{n=0}^{\infty} A_n[n(n-1) + n - k^2] r^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n [n^2 - k^2] r^n = 0$$

$\therefore n^2 = k^2 \quad \therefore n = k$ or $-k$

$$\therefore R(r) = A_k r^k + B_k r^{-k} = A_k r^k + \frac{B_k}{r^k}$$

(2) S can also be constant.

$\therefore$ ~~$T(r,\theta)$ $V(r,\theta)$~~

~~$V(r,\theta) = \sum_{k=}$~~

If ~~$k=0$, then~~ $\lambda = 0$, then

$$S'' = 0 \Rightarrow S = C_0\theta + D_0$$

$$S(\theta + 2\pi) = S(\theta) \Rightarrow C_0 = 0 \Rightarrow S = D_0$$

~~$R'' + rR' =$~~ $rR'' + R' = 0$

$$\therefore r\frac{dR'}{dr} = -R' \Rightarrow \frac{dR'}{R'} = -\frac{dr}{r}$$

$$\therefore \ln(R') = -\ln r + A_0''$$

$$\therefore R' = \frac{A_0'}{r}$$

$$\frac{dR}{dr} = \frac{A_0'}{r} \Rightarrow R = A_0' \ln(r) + B_0'$$

$\therefore \quad A_0 = A_0' D_0 \quad , \quad B_0 = B_0' D_0$

then $S(\theta)R(r) = A_0 \ln(r) + B_0$

(3) if ~~S~~ $\lambda = +k^2$.

then $S = C' e^{kt} + D' e^{-kt}$ can not be periodical, so we ignore this situation

$$\therefore \boxed{V(r,\theta) = A_0 \ln r + B_0 + \sum_{k=1}^{\infty} \left(A_k r^k + \frac{B_k}{r^k}\right)\left(C_k \cos(k\theta) + D_k \sin(k\theta)\right).}$$

Any linear combination of solutions is always a solution because the Laplace equation is linear and homogeneous.

(i) ~~$V(a,\theta) = V_0(1+\cos\theta)$~~

V is finite ~~at~~ as $r \to 0$

$$\Rightarrow \quad B_k = 0 \;, \quad A_0 = 0$$

$$\therefore V = B_0 + \sum_{k=1}^{\infty} r^k \left(C_k \cos(k\theta) + D_k \sin(k\theta)\right)$$

$$V(a,\theta) = V_0 + V_0 \cos\theta = \left(\sum_{k=1}^{\infty} a^k \left(C_k \cos(k\theta) + D_k \sin(k\theta)\right).\right) + B_0.$$

$$\Rightarrow \quad B_0 = V_0 \;, \quad D_k = 0 \;, \quad C_k = 0 \text{ if } k \neq 1$$

$$a C_1 = V_0 \quad \therefore \; C_1 = \frac{V_0}{a}$$

$$\therefore \boxed{V(r,\theta) = V_0 + \frac{V_0}{a} r \cos(\theta)} \quad \text{for } r \leq a.$$

(ii) V is finite as $r \to \infty$

$$\therefore A_k = 0 \;, \quad A_0 = 0.$$

$$V(b,\theta) = 2V_0 \sin^2\theta = 2V_0\left(\frac{1-\cos 2\theta}{2}\right) = V_0 - V_0 \cos 2\theta$$

$$\therefore V_0 - V_0 \cos 2\theta = B_0 + \sum_{k=1}^{\infty} \frac{1}{b^k}\left(C_k \cos(k\theta) + D_k \sin(k\theta)\right)$$

$$\therefore \quad B_0 = V_0 \;, \quad D_k = 0 \;, \quad C_k = 0 \text{ if } k \neq 2.$$

$$\frac{C_2}{b^2} = -V_0 \quad \Rightarrow \quad C_2 = -b^2 V_0$$

$$\therefore \boxed{V(r,\theta) = V_0 - \frac{b^2 V_0}{r^2}\cos(2\theta)} \quad \text{for } r \geq b$$

6. $u = u(x, y, t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

try $u = X(x)\, Y(y)\, T(t)$

$$XY \frac{d^2 T}{dt^2} = c^2 YT \frac{d^2 X}{dx^2} + c^2 XT \frac{d^2 Y}{dy^2}.$$

$$\therefore \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{\cancel{c^2}\, 1}{X} \frac{d^2 X}{dx^2} + \frac{\cancel{c^2}\, 1}{Y} \frac{d^2 Y}{dy^2}.$$

For oscillatory solutions

$\cancel{try}$ $\frac{1}{X} \frac{d^2 X}{dx^2} = \cancel{-k^2}\ -k_1^2.$

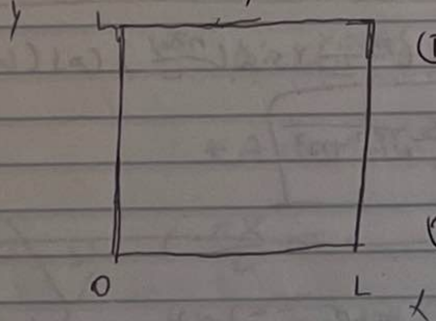
$$\Rightarrow \frac{d^2 X}{dx^2} + k_1^2 X = 0 \Rightarrow \cancel{X = A_m \cos(mx)}$$

$$X = A_{k_1} \cos(k_1 x) + B_{k_1} \sin(k_1 x).$$

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_2^2 \Rightarrow \frac{d^2 Y}{dy^2} + k_2^2 Y = 0.$$

$$Y = C_{k_2} \cos(k_2 y) + D_{k_2} \sin(k_2 y)$$

Boundary Conditions for x and y:



① $u(0, y, t) = 0$

$\Rightarrow X(0) = 0 \Rightarrow A_{k_1} = 0.$

② $u(L, y, t) = 0$

$\Rightarrow X(L) = 0 \Rightarrow B_{k_1} \sin(k_1 L) = 0$

$= k_1 L = m\pi$ $(m = 1, 2, 3, \cdots)$

$\Rightarrow k_1 = \frac{m\pi}{L}$

③ $u(x, 0, t) = 0 \Rightarrow Y(0) = 0 \Rightarrow C_{k_2} = 0.$

④ $u(x, L, t) = 0 \Rightarrow Y(L) = 0 \Rightarrow D_{k_2} \sin(k_2 L) = 0$

$\Rightarrow k_2 L = n\pi \Rightarrow k_2 = \frac{n\pi}{L}$ $(n = 1, 2, 3, \cdots)$

For $T(t)$ we have

$$\cancel{\text{再}} \quad \frac{1}{c^2 T}\frac{d^2T}{dt^2} = -k_1^2 - k_2^2$$

$$\therefore \; T'' + (k_1^2 + k_2^2)c^2 T = 0$$

$$\cancel{T(t) = E_{m,n} \cos}$$

let $\sqrt{(k_1^2+k_2^2)c^2} = \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]^{\frac{1}{2}} c$

$$= \left[(m^2+n^2)\left(\frac{\pi}{L}\right)^2\right]^{\frac{1}{2}} c = \boxed{\frac{\pi c}{L}(m^2+n^2)^{1/2}}$$

$$= \cancel{\Omega_{m,n}} \; \omega_{m,n}.$$

then $\cancel{T'' + \Omega_{m,n}^2 T = 0}$ $\; T'' + \omega_{m,n}^2 T = 0.$

$$\therefore T(t) = E_{m,n}\cos(\omega_{m,n}t) + F_{m,n}\sin(\omega_{m,n}t).$$

$$= E_{m,n}\cos(\omega_{m,n}t + \phi) = \cancel{E'_{m,n}} \; \cancel{E_{m,n}\cos(\omega_{m,n}t + \phi)}$$

we are always free to choose ϕ $t=0$ to make ϕ vanish

~~we can~~ are always ~~free to~~ choose ~~t=0 the point t=0 such that $\phi = 0$~~

$$\boxed{\frac{\partial u}{\partial t} = 0 \text{ at } t=0 \Rightarrow F_{m,n} = 0 \quad \text{OR}}$$

$$\therefore \; T(t) = E_{m,n}\cos(\omega_{m,n}t)$$

$\cancel{\text{x}}$ let $A_{m,n} = B_{k_1} D_{k_2} E_{m,n}$ then

$$\boxed{\cancel{\text{再}} \; u(x,y,t) = A_{m,n}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi y}{L}\right)\cos(\omega_{m,n}t) \quad \text{where} \; \boxed{\omega_{m,n} = \frac{\pi c}{L}\sqrt{m^2+n^2}}}$$

m and n are positive integers

The lowest mode

$$\omega_{1,1} = \sqrt{2}\frac{\pi c}{L}$$

Second lowest mode $\omega_{2,1} = \omega_{1,2} = \sqrt{5}\frac{\pi c}{L}$.

$\therefore$ the required ratio is $\boxed{\sqrt{\frac{5}{2}}}$

For the combinations of $\omega_{2,1}$ and $\omega_{1,2}$

If we take $A_{1,2} = -A_{2,1} = -A$, then

~~$u(x,y,t) =$~~

$$u(x, y=x, t) = A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right)$$
$$- A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right)$$
$$= 0$$

$\therefore$ The $y=x$ diagonal is the node.

If we take $A_{1,2} = A_{2,1} = A$ then

$$u = (x, y=L-x, t) = A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi(L-x)}{L}\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right).$$
$$+ A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi(L-x)}{L}\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right)$$
$$= A\sin\left(\frac{2\pi x}{L}\right)\sin\left(-\frac{\pi x}{L}+\pi\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right)$$
$$+ A\sin\left(\frac{\pi x}{L}\right)\sin\left(-\frac{2\pi x}{L}+2\pi\right)\cos\left(\sqrt{5}\frac{\pi c t}{L}\right)$$

$\rightarrow \sin\left(-\frac{\pi x}{L}+\pi\right) = \sin\left(\frac{\pi x}{L}\right)$

$\sin\left(-\frac{2\pi x}{L}+2\pi\right) = -\sin\left(\frac{2\pi x}{L}\right)$

$\therefore u(x, y=L-x, t) = 0$

$\therefore$ The $y=L-x$ diagonal is the node.

Triangular membrane requires ~~A_{n}~~, $A_{m,n} = \pm A_{n,m}$

If $m = n$ then ~~...~~ $w = 0$ constantly

which makes no sense physically. So.

it requires $m \neq n$.

Hence the lowest frequency is

when $m = 2$ $n = 1$ or $m = 1$, $n = 2$

~~we~~ we have

$$\boxed{\omega_{1,2} = \omega_{2,1} = \sqrt{5}\,\frac{\pi c}{L}}$$

✓

7. $R = R(\rho) = J_m(x) \qquad x = \gamma\rho \quad , \gamma^2 \equiv \frac{\omega^2}{c^2} - p^2$

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \left[\underbrace{\left(\frac{\omega^2}{c^2} - p^2\right)}_{\gamma^2}\rho^2 - m^2\right] R = 0$$

即

$$\rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + [\underbrace{\gamma^2\rho^2}_{x^2} - m^2] R = 0$$

$$\therefore \rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + [x^2 - m^2] R = 0$$

$\because x = \gamma\rho \quad \therefore \quad \rho = \frac{x}{\gamma}$. ~~$\frac{d}{d\rho} = \frac{dx}{d\rho}\frac{d}{dx}$~~

$$\frac{d}{d\rho} = \frac{dx}{d\rho}\frac{d}{dx} = \gamma \frac{d}{dx}$$

$$\therefore \rho \frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) = \cancel{\frac{x}{\gamma}\frac{d}{}} \frac{x}{\cancel{\gamma}}(\cancel{\gamma}) \frac{d}{dx}\left[\frac{x}{\cancel{\gamma}}(\cancel{\gamma}) \frac{dR(x)}{dx}\right]$$

$$= x \frac{d}{dx}\left(x \frac{dR(x)}{dx}\right) \cancel{+ (x^2 - m^2) J_m(x) = 0}$$

$$\therefore \quad \boxed{x \frac{d}{dx}\left(x \frac{dR(x)}{dx}\right) + (x^2 - m^2) R(x) = 0}$$

The solutions to this kind of equation is the Bassel functions. , so we have

$$\boxed{x \frac{d}{dx}\left(x \frac{dJ_m(x)}{dx}\right) + (x^2 - m^2) J_m(x) = 0}$$

where $J_m(x)$ is the m^{th} Bassel function.

$$\frac{d^2Z}{dz^2} = -p^2 Z \implies \frac{d^2Z}{dz^2} + p^2 Z = 0$$

$$\implies \cancel{Z = A\cos(px) - Z =}$$

$$Z(z) = A_p \cos(pz) + B_p \sin(pz)$$

At $z = 0, h$ $\quad Z \sim \cos\left(\frac{n\pi z}{h}\right)$

$$\implies p = \frac{n\pi}{h}$$

$$R(\rho = a) = 0 \implies J_m(x = \gamma a) = 0$$

$$\implies \gamma a = \alpha_{mr} \implies \gamma = \frac{\alpha_{mn}}{a}$$

$$\therefore \gamma = \frac{\omega^2}{c^2} - p^2$$

$$\therefore \frac{\omega^2}{c^2} - \frac{n^2\pi^2}{h^2} = \frac{\alpha_{mr}^2}{a^2}$$

$$\implies \boxed{\frac{\omega^2}{c^2} = \frac{\alpha_{mr}^2}{a^2} + \frac{n^2\pi^2}{h^2}}$$

8 (i) $\frac{d^2P}{d\phi^2} = -m^2P \Rightarrow \frac{d^2P}{d\phi^2} + m^2P = 0$

$\Rightarrow$ $m=0$ $P(\phi) = A_0\phi + B_0$

we require $P(\phi+2\pi) = P(\phi)$ $\therefore A_0 = 0$

$\therefore P = B_0$, a constant solution.

$\Rightarrow$ $m \neq 0$ $P(\phi) = A_m\cos(m\phi) + B_m\sin(m\phi)$

we require $P(\phi+2\pi) = P(\phi)$

$\therefore$ m has to be integers

$\therefore$ m is an integer or zero.

(ii) $m=0 \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \lambda T = 0$

Sub $T = \cos\theta \Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(-\sin^2\theta\right) + \lambda\cos\theta = 0$

$\Rightarrow \frac{1}{\sin\theta}\left(-2\sin\theta\cos\theta\right) + \lambda\cos\theta = 0$

$\Rightarrow (-2+\lambda)\cos\theta = 0$

$\Rightarrow \lambda = 2$

$\therefore$ $T = \cos\theta$ solves the $m=0$ equation for T with $\lambda = 2$

(iii) $\nabla^2 V = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] = 0$

Sub in $V = RTP$ and multiply by $\frac{r^2\sin\theta}{RTP}$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] + \frac{\sin\theta}{T}\frac{d}{d\theta}\left[\sin\theta\frac{dT}{d\theta}\right] + \underbrace{\frac{1}{P}\frac{d^2P}{d\phi^2}}_{-m^2} = 0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left[r^2\frac{dR}{dr}\right] + \underbrace{\frac{1}{T}\frac{d}{d\theta}\left[\sin\theta\frac{dT}{d\theta}\right] - \frac{m^2}{\sin^2\theta}}_{-\lambda} = 0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda$$

$$\Rightarrow \boxed{\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda R.}$$

For $\lambda = 2$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = 2R = (1+1)(1)R$$

try r^k and solve the quadratic for k.

$\therefore$ $\cancel{r^2R'' + 2R}$

$$\therefore \boxed{R(r) = Ar + \frac{B}{r^2}} \checkmark$$

(iv) General solution for $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \lambda R$

(let $\lambda = l(l+1)$) is $R(r) = C_{lm} r^{l} + D_{lm} r^{-(l+1)}$

General solution for $\frac{d^2P}{d\phi^2} = -m^2 P$ is

$$P(\phi) = A_{lm}\cos(m\phi) + B_{lm}\sin(m\phi)$$

General solutions for $-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + \frac{m^2}{\sin^2\theta}T = \lambda T$;

is let $\lambda = l(l+1)$ and $x = \cos\theta$

$$\frac{d}{dx} = \cancel{\frac{dx}{d\theta}}\frac{d\theta}{dx}\frac{d}{d\theta} = \frac{-1}{\sin\theta}\frac{d}{d\theta}.$$

$$\Rightarrow \frac{d}{d\theta} = -\sin\theta\frac{d}{dx}$$

$$\Rightarrow \left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right) - \frac{m^2}{1-x^2}\right]T = -l(l+1)T.$$

$$\therefore T = P_l^m(x) = P_l^m(\cos\theta)$$

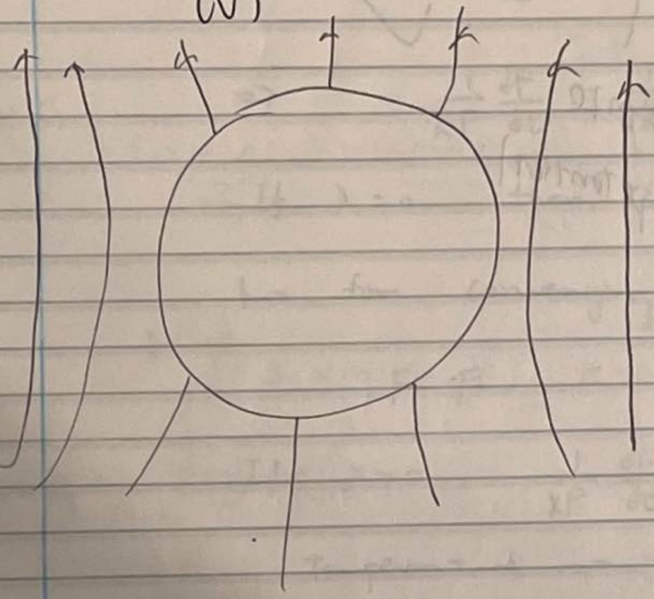
where $P_m^l(\cos\theta)$ is the associated Legendre functions.

∴ Solution is

$$\boxed{V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} [C_{lm} r^{l} + D_{lm} r^{-(l+1)}] P_l^m(\cos\theta)[A_{lm}\cos(m\phi) + B_{lm}\sin(m\phi)]}$$

For $\lambda = 2$ $m=0$ $T = \cos\theta$, $\boxed{V(r,\theta,\phi) = \cos\theta\,(Ar + \frac{B}{r^2})}$.

(IV)



~~In the~~ This is a situation with z-axis symmetry

$\therefore V(r,\theta,\phi) = V(r,\theta)$

$$\therefore V = \sum_{l=0}^{\infty} (C_l r^l + D_l r^{-(l+1)}) P_l(\cos\theta)$$

where $P_l(\cos\theta)$ is the l^{th} order Legendre polynomial for $\cos\theta$.

$$V(a,\theta) = 0 \Rightarrow \sum_{l=0}^{\infty} [C_l a^l + D_l \frac{1}{a^{l+1}}] P_l(\cos\theta) = 0$$

$$\Rightarrow \cancel{C_l = 0} \quad C_l a^l = -D_l \frac{1}{a^{l+1}}$$

$$\cancel{\therefore V = \sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}} P_l(\cos\theta)}. \quad \Rightarrow D_l = -C_l a^{2l+1}$$

$$V(\infty,\theta) = -Er\cos\theta = -E r^1 P_1(\cos\theta)$$

$$\cancel{\sum_{l=0}^{\infty} \frac{D_l}{r^{l+1}}}$$

$$V = \sum_{l=0}^{\infty} C_l [r^l - a^{2l+1} \frac{1}{r^{l+1}}] P_l(\cos\theta).$$

when $r \to \infty$ $\frac{1}{r^{l+1}} \to 0$.

$$\therefore V = \sum_{l=0}^{\infty} C_l r^l P_l(\cos\theta) = -E r P_1(\cos\theta)$$

Orthogonality of Legendre polynomial $\Rightarrow$

$\therefore \cancel{C_l = Ex}$, $C_1 = -E$, $C_l = 0$ if $l \neq 1$

$\therefore D_1 = a^3 E$, $D_l = 0$ if $l \neq 1$.

$\Rightarrow$ ~~[illegible]~~

$$\boxed{V = \left[-Er + \frac{a^3 E}{r^2} \right] \cos\theta}$$

uniform field

dipole potential

9. $$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \qquad T = T(x,t)$$

try $T = X(x) F(t)$

$$\Rightarrow \quad X \frac{dF}{dt} = \kappa F \frac{d^2 X}{dx^2}$$

$$\Rightarrow \quad \frac{1}{\kappa F}\frac{dF}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \text{const} = \lambda.$$

If $\lambda = 0$ then $X(x) = A_0 x + B_0$

but for converging temperature at $x = L$, $A_0 = 0 \therefore X = B_0$

$F = F_0 \quad \therefore XF = F_0 B_0 = T_0$

If $\lambda > 0$ $\frac{1}{\kappa F}\frac{dF}{dt} = \lambda$ would result in diverging temperature at $x = L$ so λ cannot be greater than 0

For $\lambda < 0$, let $\lambda = -m^2$

then $\frac{1}{\kappa F}\frac{dF}{dt} = -m^2$

$$\frac{dF}{dt} = -m^2 \kappa F \Rightarrow F = F_m e^{-\kappa m^2 t}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -m^2 \Rightarrow \frac{d^2X}{dx^2} + m^2 X = 0.$$

$$\therefore X = A_m \cos(mx) + B_m \sin(mx)$$

$\therefore$ General solution :

$\cancel{T(x,t) = \sum_m A_m \cos(m}$

$$T(x,t) = T_0 + \sum_m [A_m \cos(mx) + B_m \sin(mx)] e^{-\kappa m^2 t}$$

~~The end at $x = L$ is perfectly insulated~~

$$\cancel{\Rightarrow \frac{\partial T}{\partial x}(L,t) = 0}$$

Boundary Conditions:

① $T(x, \infty) \to \cancel{0}\ 100°C$.

$\therefore \quad 100°C = T_0$

② $T(x=0, t) = \cancel{T_0}\ 100°C$

$$\therefore \quad 100°C = 100°C + \sum_m [A_m \underbrace{\cos(mx)}_{1} + B_m \underbrace{\sin(mx)}_{0}] e^{-km^2 t}.$$

$\Rightarrow \quad A_m = 0$

$$\therefore \quad T = 100 + \sum_m B_m \sin(mx) e^{-km^2 t}.$$

③ At $x=L$ the bar is perfectly insulated

$$\therefore \quad \frac{\partial T}{\partial x}(L, t) = 0$$

$$\therefore \quad \sum_m m B_m \cos(mL) e^{-km^2 t} = 0.$$

for $B_m \neq 0$, $\cos(mL) = 0$

$$\therefore \quad mL = (n+\tfrac{1}{2})\pi$$

$$\therefore \quad m = \frac{(2n+1)\pi}{2L}$$

$$\therefore T = 100 + \sum_{n=0}^{\infty} B_n \sin\left(\frac{(2n+1)\pi x}{2L}\right) \exp\left(-k\left(\frac{(2n+1)\pi}{2L}\right)^2 t\right).$$

④ $T(x, 0) = 0$

$$\Rightarrow \quad -100 = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2L}\pi x\right)$$

$$\therefore \quad -100 \int_0^{2L} \sin\left(\frac{2m+1}{2L}\pi x\right) dx = \sum_{n=0}^{\infty} B_n \underbrace{\int_0^{2L} \sin\left(\frac{2m+1}{2L}\pi x\right) \sin\left(\frac{2n+1}{2L}\pi x\right) dx}_{L\delta_{mn}.}$$

$$\Rightarrow \quad -100 \times \frac{4L}{(2m+1)\pi} = \sum_{n=0}^{\infty} B_n L \delta_{mn} = L B_m$$

$$\therefore \quad B_m = -100 \times \frac{4}{(2m+1)\pi} \Rightarrow B_n = -100 \times \frac{4}{(2n+1)\pi}.$$

$$\therefore\ T = 100 - 100 \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{2n+1}{2L}\pi x\right) \exp\left(-k\left(\frac{2n+1}{2L}\pi\right)^2 t\right).$$

$$\boxed{\frac{T}{100} = 1 - \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin\left(\frac{2n+1}{2L}\pi x\right) \exp\left(-k\left(\frac{2n+1}{2L}\pi\right)^2 t\right).}$$

At large t, we ignore n=1, 2, 3, ...

$$T \simeq 100 - \frac{400}{\pi} \sin\left(\frac{\pi x}{2L}\right) \exp\left(-\frac{k\pi^2}{4L^2} t\right).$$

$$\langle T \rangle = \frac{1}{L} \int_0^L T(x,t)\, dx.$$

$$= 100 - \frac{400}{\pi L} \exp\left(-\frac{k\pi^2}{4L^2} t\right) \underbrace{\int_0^L \sin\left(\frac{\pi x}{2L}\right) dx}_{\frac{2L}{\pi}}.$$

$$= 100 - \frac{800}{\pi^2} \exp\left(-\frac{k\pi^2}{4L} t\right).$$

for $\langle T \rangle = 90$, $\frac{800}{\pi^2} \exp\left(-\frac{k\pi^2}{4L} t\right) = 10.$

$$\therefore\ \exp\left(-\frac{k\pi^2}{4L} t\right) = \frac{\pi^2}{80}$$

$$\therefore\ \frac{k\pi^2}{4L} t = \ln\left(\frac{80}{\pi^2}\right)$$

$$\therefore\ t = \ln\left(\frac{80}{\pi^2}\right) \frac{4 \times 1}{0.001 \times \pi^2} = \boxed{848\ s}$$

10. $$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0$$

try $V = R(r)\,T(\theta)$

$$T\frac{\partial}{\partial r}\left(r^2\frac{dR}{dr}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0$$

$\therefore$ $\cancel{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{dR}{}\right)}$ $\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{fn only of } r} + \underbrace{\frac{1}{\sin\theta\,T}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)}_{\text{fn only of } \theta} = 0.$

$\therefore$ let $\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1)$

$\Rightarrow$ ~~$R(r) = \alpha_n r^n$~~ $R(r) = \frac{\alpha_n}{r^{n+1}} + \beta_n r^n$

then $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = -n(n+1)\,T$

~~try~~ let $x = \cos\theta$ then $\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} = -\sin\theta\frac{d}{dx}$.

$\therefore$ $\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right] + n(n+1)\,T = 0.$

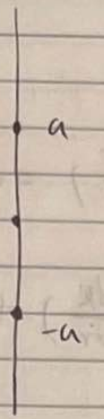
Solutions are $T(\theta) = P_n(\cos\theta)$ where $P_n(\cos\theta)$ is the n^{th} order Legendre Polynomial

$\therefore$ General solution:

$$\boxed{V(r,\theta) = \sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{\alpha_n}{r^{n+1}} + \beta_n r^n\right).}$$

(ii) Along z-axis $P_n(\cos(\theta=0)) = P_n(1) = 1$ $\therefore$ we can solely work on the function $R(r)$.

(a)



Along the z-axis

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r-a} - \frac{1}{r+a}\right].$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[\frac{1}{1-\frac{a}{r}} - \frac{1}{1+\frac{a}{r}}\right].$$

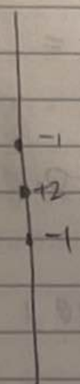
$$= \frac{1}{4\pi\epsilon_0 r}\left[\left(1+\frac{a}{r}+\left(\frac{a}{r}\right)^2+\cdots\right) - \left(1-\left(\frac{a}{r}\right)+\left(\frac{a}{r}\right)^2-\left(\frac{a}{r}\right)^3+\cdots\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[\frac{2a}{r} + \frac{2a^3}{r^3} + \frac{2a^5}{r^5} + \cdots\right).$$

$$\therefore \boxed{V(r,\theta) = \frac{1}{2\pi\epsilon_0 r}\sum_{n=0}^{\infty}\frac{a^{2n+1}}{r^{2n+1}}P_{2n+1}(\cos\theta)}$$

✓

(b)



Along the z-axis

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{-1}{r-a} + \frac{-1}{r+a} + \frac{2}{r}\right].$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[2 - \frac{1}{1-\frac{a}{r}} - \frac{1}{1+\frac{a}{r}}\right].$$

$$= \frac{1}{4\pi\epsilon_0 r}\left[2 - \left(1+\left(\frac{a}{r}\right)+\left(\frac{a}{r}\right)^2+\cdots\right) - \left(1-\left(\frac{a}{r}\right)+\left(\frac{a}{r}\right)^2-\cdots\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0 r}\left(-2\left(\frac{a}{r}\right)^2 - 2\left(\frac{a}{r}\right)^4 - \cdots\right).$$

$$= \frac{-1}{2\pi\epsilon_0 r}\left(\left(\frac{a}{r}\right)^2 + \left(\frac{a}{r}\right)^4 + \cdots\right)$$

$$\therefore \boxed{V(r,\theta) = \frac{-1}{2\pi\epsilon_0 r}\sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta)}$$

✓

$$\frac{1}{a^2} = \frac{\omega}{2k}.$$

$$\frac{dy}{dt} + \alpha y = f(t)$$

non-homog...
but linear

9.

$$\cancel{\frac{d\theta}{dt}}\ \cancel{\frac{\partial\theta}{\partial t}}\ \frac{\partial}{\partial t}\oint a\hat{n}\cdot ds.$$

$=$

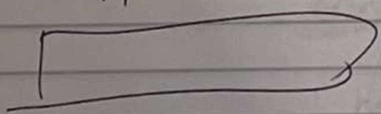
① $T(x=0, t>0) = 100°C.$

perfectly insulated. ⟹ no heat flow

⟹ no temperature difference

between two points.

$T_1 > T_2.$



$\longrightarrow$

will be heat flow

② $\frac{\partial T}{\partial x}(L, t) = 0.$

③ $T(x, 0) = 0°C \quad 0 < x < L.$

④ $T(x, t \to \infty) = 100°C.$

10.

Compute the potential where you can compute easily and use it as a boundary conditions

ϕ

$x + \frac{x}{2} + \frac{x}{\vdots}$

$$\frac{a_{n+2}}{a_n} \to \frac{2}{k}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$$

set $x = 1$.

$f(1) = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\therefore$ For Hermite Polynomials.

~~Series~~ Recurrence Series

must terminate somewhere.