To: Ana Lopez

Mathematical method 4 Ziyan Li

1)

Consider the equation

① $\sum_{n=1}^{N} C_n \psi_n(x) = 0$ $\because$ $\psi_n(x)$ are orthogonal

$$\langle \psi_m | \psi_n \rangle = \int_a^b dx\, w(x)\, \psi_n(x)\, \psi_m^*(x) = \delta_{mn} \int_{-\infty}^{\infty} dx\, w(x) |\psi_m(x)|^2$$

take $\langle \psi_m |$ and act on equation ① gives

$$\sum_{n=1}^{N} C_n \langle \psi_m | \psi_n \rangle = 0$$

$$\Rightarrow \sum_{n=1}^{N} C_n \delta_{mN} \int_{-\infty}^{\infty} dx\, w(x) |\psi_m(x)|^2 = 0$$

$$\Rightarrow C_m \underbrace{\int_{-\infty}^{\infty} dx\, w(x) |\psi_m(x)|^2}_{> 0 \text{ for non zero } \psi_m} = 0$$

$\Rightarrow$ $C_m = 0$ for $m = 1, \cdots, N$

$\therefore$ The $\psi_n(x)$'s are linearly independent.

2) (i)

$$f(x) = \sum_{n=1}^{\infty} a_n u_n(x) \quad ①$$

$u_n(x)$'s are orthonormal

$$\Rightarrow \quad \langle u_n | u_m \rangle = \int_a^b dx\, \underbrace{u_n^*(x)}_{} u_m(x) = \delta_{mn}$$

$= u_n(x)$ since $u_n(x)$'s are real

take $\langle u_m |$ and act on equation ① gives

$$\int_a^b dx\, u_m(x) f(x) = \sum_{n=1}^{\infty} \left[\underbrace{\int_a^b dx\, u_m(x) u_n(x)}_{\delta_{mn}} \right] a_n$$

$$= \sum_{n=1}^{\infty} \delta_{mn} a_n = a_m$$

$$\Rightarrow \quad a_m = \int_a^b dx\, u_m(x) f(x)$$

$$\Rightarrow \quad \boxed{a_n = \int_a^b u_n(x) f(x)\, dx.}$$

(ii)

$$\int_a^b [f(x)]^2 dx = \int_a^b (a_1 u_1(x) + a_2 u_2(x) + \dots)(a_1 u_1(x) + a_2 u_2(x) + \dots)\, dx$$

$$= \int_a^b [\, a_1^2 u_1^2(x) + a_2^2 u_2^2(x) + \dots + 2 a_1 a_2 u_1(x) u_2(x) + \dots \,]\, dx$$

$$\int_a^b a_i^2 u_i^2(x)\, dx = a_i^2 \int \underbrace{dx\, u_i^2(x)}_{=1} = a_i^2$$

$$\int_a^b 2 a_i a_j u_i(x) u_j(x)\, dx = 2 a_i a_j \int_a^b \underbrace{u_i(x) u_j(x)}_{\delta_{ij} \text{ for } i \neq j,\ \text{which is } 0} = 0$$

$$\therefore \quad \int_a^b [f(x)]^2 dx = a_1^2 + a_2^2 + \dots = \sum_{n=1}^{\infty} a_n^2$$

QED.

$x^2y'' + 2xy' + \frac{1}{4}$

3. $\hat{L} y = x^2 y'' + 2xy' + \frac{1}{4} y$ $\qquad \frac{d}{dx}\left(x^2 \frac{dy}{dx}\right) + \frac{1}{4} y$

$w(x) = 1$

sub $x = e^t \Rightarrow t = \ln(x) \Rightarrow \frac{dt}{dx} = \frac{1}{x} = e^{-t}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)e^{-t}$$

$$= \left[e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right]e^{-t}$$

$$= e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}$$

$$\therefore \hat{L} y = e^{2t}\left[e^{-2t}\frac{d^2y}{dt^2} - e^{-2t}\frac{dy}{dt}\right] + 2e^t e^{-t}\frac{dy}{dt} + \frac{1}{4}y$$

$$= \frac{d^2y}{dt^2} - \frac{dy}{dt} + 2\frac{dy}{dt} + \frac{1}{4}y$$

$$= \frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{1}{4}y$$

$$\hat{L} y = \lambda y \Rightarrow \frac{d^2y}{dt^2} + \frac{dy}{dt} + \left(\frac{1}{4} - \lambda\right)y = 0$$

If $\lambda \neq 0$:

~~char~~ Auxillary equation $\Rightarrow m^2 + m + \left(\frac{1}{4} - \lambda\right) = 0$

$$\Rightarrow \left(m + \frac{1}{2}\right)^2 = \lambda \qquad \therefore m = -\frac{1}{2} \pm \sqrt{\lambda}$$

$$\therefore y(t) = A\exp\left[\left(-\frac{1}{2} + \sqrt{\lambda}\right)t\right] + B\exp\left[\left(-\frac{1}{2} - \sqrt{\lambda}\right)t\right]$$

boundary conditions:

① : $y(x=1) = 0 \Rightarrow y(t=0) = 0$

$\therefore A + B = 0 \Rightarrow B = -A$

② : $y(x=e) = 0 \Rightarrow y(t=1) = 0$

$$\therefore A\left\{\exp\left[\left(-\frac{1}{2} + \sqrt{\lambda}\right)\right] - \exp\left[\left(-\frac{1}{2} - \sqrt{\lambda}\right)\right]\right\} = 0$$

For $y \neq 0$, we need

$$\exp\left[-\tfrac{1}{2} + \sqrt{\lambda}\right] = \exp\left[-\tfrac{1}{2} - \sqrt{\lambda}\right].$$

If $\lambda > 0$:

$$\cancel{\exp} \quad -\tfrac{1}{2} + \sqrt{\lambda} = -\tfrac{1}{2} - \sqrt{\lambda} \Rightarrow \sqrt{\lambda} = 0 \Rightarrow \lambda = 0$$

There is a contradiction

If $\lambda < 0$: let $\lambda = -k^2$ then $\sqrt{\lambda} = \sqrt{-k^2} = ik$

$$e^{-\frac{1}{2}+ik} = e^{-\frac{1}{2}-ik} \Rightarrow e^{ik} = e^{-ik}.$$

$$\Rightarrow e^{2ik} = 1 \Rightarrow k = n\pi \qquad (n = \pm1, \pm2, \pm3, \ldots)$$

$$\therefore \boxed{\lambda_n = -n^2\pi^2} \Rightarrow \sqrt{\lambda_n} = in\pi.$$

$$y_n = \cancel{A\{e e^{-\frac{1}{2}}} \; A e^{-\frac{1}{2}t}\left(e^{in\pi t} - e^{-in\pi t}\right)$$

$$= A e^{-\frac{1}{2}t}\left(2i\sin(n\pi t)\right).$$

$$\therefore \boxed{y_n = 2iAe^{-\frac{1}{2}t}\sin(n\pi t)}$$

Now to normalize the eigenfunctions we need to find the weight function ~~w(x)~~ $w(t)$

we put $\hat{L}$ in the Sturm-Liouville form

OR just work with x variable $\int_0^1 dx\,|y_n(x)|^2 = 1$

$$\hat{L}y = \frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{1}{4}y \quad \text{or} \iff \hat{L} = c_2\frac{d^2}{dt^2} + c_1\frac{d}{dt} + c_0$$

$$\therefore c_2 = 1 \qquad c_1 = 1 \qquad c_0 = \frac{1}{4}$$

$$p(t) = \exp\left(\int_0^t \frac{c_1}{c_2}\,dt'\right) = e^t.$$

$$w(t) = \frac{p(t)}{c_2} = \frac{e^t}{1} = e^t.$$

$$q(t) = c_0 w(t) = \frac{1}{4}e^t$$

$\therefore \hat{L} = e^{-t}\left[\frac{d}{dt}\left(e^{t}\frac{d}{dt}\right) + \frac{1}{4}e^{t}\right]$ with $w(t) = e^{t}$

$\therefore$ to normalise y_n's we need

$$\cancel{\langle y_m | y_n \rangle = \int_0^1 y_m^* y_n e^t \, dt = \delta_{mn}}$$

$$\langle y_n | y_n \rangle = \int_0^1 |y_n|^2 e^t \, dt = 1$$

~~let A be real then,~~

$$\cancel{\therefore (2iA)^2 \int_0^1 e^t e^{-\frac{1}{2}t} e^{-\frac{1}{2}t}}$$

$$|2iA|^2 \int_0^1 e^t e^{-\frac{1}{2}t} e^{-\frac{1}{2}t} \sin^2(n\pi t) \, dt = 1$$

$e^t\frac{d^2}{dt^2} + e^t\frac{d}{dt}$

$$\therefore |2iA|^2 \underbrace{\int_0^1 \sin^2(n\pi t)\, dt}_{\frac{1}{2}} = 1$$

$\cancel{\therefore 4A^2\left(\frac{1}{2}\right) = 1}$

$\cancel{\therefore 2A^2 = 1 \Rightarrow A = \pm\frac{1}{\sqrt{2}}}$

~~let A be positive $\Rightarrow A = \frac{1}{\sqrt{2}}$~~

$\therefore |2iA|^2 = 2$ let $A = iB$ (B is real)

then $(-2B)^2 = 2 \quad \therefore 4B^2 = 2 \Rightarrow B = \pm\frac{1}{\sqrt{2}}$

let $B = -\frac{1}{\sqrt{2}}$ then $A = -i\frac{1}{\sqrt{2}}$

$\therefore 2iA = 2(i)(-i)\frac{1}{\sqrt{2}} = \sqrt{2}$

$\therefore \; y_n = \sqrt{2}\, e^{-\frac{1}{2}t} \sin(n\pi t).$

finally,

If $\lambda = 0$ $\quad m^2 + m + \frac{1}{4} = 0 \implies m = -\frac{1}{2}$

$$\therefore \; y = (A + Bt)\, e^{-\frac{1}{2}t}.$$

$t = 0,\ y = 0 \implies A = 0 \quad \therefore \; y = Bt\, e^{-\frac{1}{2}t}.$

$t = 1,\ y = 0 \implies 0 = B(1)(1) \implies B = 0$

$\therefore y = 0$

$\therefore$ $\cancel{\lambda = 0}$ λ cannot be 0

~~$\therefore$ Overall:~~

~~$\lambda_n = -n^2\pi^2$ for $n = \pm 1, \pm 2, \dots$~~

~~$y_n = \sqrt{2}\, e^{-\frac{1}{2}t} \sin(n\pi t)$~~

~~$y_n = \sqrt{2}\, e^{-\frac{1}{2}\ln x}$~~

$\because t = \ln x \;\therefore e^{-\frac{1}{2}t} = e^{-\frac{1}{2}\ln(x)} = \left(e^{\ln x}\right)^{-\frac{1}{2}} = x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$

$\therefore$ Overall

$\lambda_n = -n^2\pi^2$ for $n = \pm 1, \pm 2, \dots$

$y_n = \sqrt{\frac{2}{x}} \sin(n\pi \ln(x))$

✓

4. (a)

$$\mathcal{L} = \frac{d}{dx} + x .$$

$$\langle u|\mathcal{L}|v\rangle = \int_{-\infty}^{\infty} dx\, u^* \left(\frac{dv}{dx} + xv\right).$$

$$= \int_{-\infty}^{\infty} dx\, u^* \frac{dv}{dx} + \int_{-\infty}^{\infty} dx\, u^* x v$$

$$= \underbrace{[u^* v]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} v\frac{du^*}{dx}\, dx + \int_{-\infty}^{\infty} dx\, u^* x v$$

$$= \int_{-\infty}^{\infty} v \underbrace{\left(-\frac{d}{dx} + x\right)}_{\mathcal{L}^\dagger} u^*\, dx$$

$$= \left(\int_{-\infty}^{\infty} v^* \mathcal{L}^\dagger u\, dx\right)^* = \langle v|\mathcal{L}^\dagger|u\rangle^*$$

$$\therefore\ \mathcal{L}^\dagger = -\frac{d}{dx} + x \neq \frac{d}{dx} + x = \mathcal{L}.$$

$\therefore$ $\mathcal{L}$ is not hermitian.

(b) $\mathcal{L} = -i\frac{d}{dx} + x^2$

~~$\langle u|\mathcal{L}|v\rangle$~~ $\langle u|\mathcal{L}|v\rangle = \int_{-\infty}^{\infty} u^*(-i\frac{d}{dx} + x^2)v \, dx$

$$= \int_{-\infty}^{\infty} -iu^* \frac{dv}{dx} + u^* x^2 v \, dx.$$

$$= \int_{-\infty}^{\infty} -iu^* \frac{dv}{dx} dx + \int_{-\infty}^{\infty} u^* x^2 v \, dx.$$

$$= \underbrace{-i[u^* v]_{-\infty}^{\infty}}_{0} + i\int_{-\infty}^{\infty} v \frac{du^*}{dx} dx + \int_{-\infty}^{\infty} vx^2 u^* dx$$

$$= \int_{-\infty}^{\infty} iv\frac{du^*}{dx} + vx^2 u^* \, dx.$$

$$= \left(\int_{-\infty}^{\infty} v^*(-i\frac{d}{dx} + x^2) u \, dx\right)^*$$

$$= \langle v|\mathcal{L}^\dagger|u\rangle^*$$

$\therefore \mathcal{L}^\dagger = -i\frac{d}{dx} + x^2 = \mathcal{L}$

$\therefore \mathcal{L}$ is hermitian.

(c) $\mathcal{L} = ix\frac{d}{dx}$

$$\langle u|\mathcal{L}|v\rangle = \int_{-\infty}^{\infty} u^* \left(ix\frac{d}{dx}\right) v \, dx$$

$$= \int_{-\infty}^{\infty} u^* (ixv') \, dx = \int_{-\infty}^{\infty} ixu^* v' \, dx$$

$$= [ixu^* v]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v (ixu^*)' \, dx$$

$\because \lim_{x\to\pm\infty} xf(x) = 0$ and $u, v \in \{f(x)\}$ $\therefore [ixu^* v]_{-\infty}^{\infty} = 0$

$$\therefore \langle u|\mathcal{L}|v\rangle = -i\int_{-\infty}^{\infty} vx\frac{du^*}{dx} + vu^* \, dx$$

$$= \left(\int_{-\infty}^{\infty} v^* \left(ix\frac{d}{dx} + 1\right) u \, dx\right)^*$$

$$= \langle v|\mathcal{L}^\dagger|u\rangle^*$$

$$\therefore \mathcal{L}^\dagger = ix\frac{d}{dx} + 1 \neq ix\frac{d}{dx} = \mathcal{L}$$

$\therefore$ $\mathcal{L}^\dagger \neq \mathcal{L}$ $\therefore$ $\mathcal{L}$ is not hermitian

(d) $\mathcal{L} = ix\frac{d^3}{dx^3}$

$$\langle u|\mathcal{L}|v\rangle = \int_{-\infty}^{\infty} u^* \, ix \frac{d^3 v}{dx^3} \, dx = \int_{-\infty}^{\infty} (ixu^*) v''' dx$$

$$= \underbrace{[ixu^* v'']_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} (ixu^*)' v'' \, dx$$

$$= -\underbrace{[(ixu^*)' v']_{-\infty}^{\infty}}_{0} + \int_{-\infty}^{\infty} (ixu^*)'' v' \, dx$$

$$= \underbrace{[(ixu^*)'' v]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} (ixu^*)''' v \, dx$$

$$= -i \int_{-\infty}^{\infty} (xu^*)''' v \, dx$$

$$= -i \int_{-\infty}^{\infty} v(xu^{*'''} + 3u^{*''}) \, dx.$$

$$= -i \int_{-\infty}^{\infty} v x u^{*'''} \, dx - 3i \int_{-\infty}^{\infty} v u^{*''} \, dx.$$

$$= \left(\int_{-\infty}^{\infty} v^* \left(ix\frac{d^3}{dx^3} + 3i\frac{d^2}{dx^2}\right) u \, dx \right)^*$$

$$= \langle v|\mathcal{L}^\dagger|u\rangle^*$$

$$\therefore \mathcal{L}^\dagger = ix\frac{d^3}{dx^3} + 3i\frac{d^2}{dx^2} \neq \mathcal{L}$$

$\therefore \mathcal{L}$ is not hermitian

5. By definition

$$\int_a^b u^*(x)\,[A^\dagger v(x)]\,w(x)\,dx = \int_a^b [Au(x)]^*\,v(x)\,w(x)\,dx$$

$$\Rightarrow \int_a^b u^*(x)\,[(A^\dagger)^\dagger v(x)]\,w(x)\,dx = \int_a^b [A^\dagger u(x)]^*\,v(x)\,w(x)\,dx$$

$$= \left(\int_a^b [A^\dagger u(x)]\,V^*(x)\,w(x)\,dx\right)^*$$

$$= \left(\int_a^b v^*(x)\,[A^\dagger u(x)]\,w(x)\,dx\right)^*$$

$$= \left(\int_a^b [A v(x)]^*\,u(x)\,w(x)\,dx\right)^*$$

$$= \int_a^b u^*(x)\,[A v(x)]\,w(x)\,dx$$

$\because$ This is true for any u and v

$\therefore$ $\underline{(A^\dagger)^\dagger = A}$

(a) $$\int_a^b u^*(x)\,[(A^\dagger + A)\,v(x)]\,w(x)\,dx$$

$$= \int_a^b u^*(A^\dagger v)\,w\,dx + \int_a^b u^* A v\,w\,dx$$

$$= \int_a^b ((A^\dagger)^\dagger u)^* v\,w\,dx + \int_a^b (A^\dagger u)^* v\,w\,dx$$

$$= \int_a^b (Au)^* v\,w\,dx + \int_a^b (A^\dagger u)^* v\,W\,dx$$

$$= \int_a^b ((A + A^\dagger)u)^* v\,w\,dx.$$ $\Rightarrow$ $\underline{A + A^\dagger \text{ is hermitian}}$

$$\int_a^b u^*(x)\,[i(A-A^\dagger)v(x)]\,w(x)\,dx$$

$$\cancel{= \int_a^b u^* [iAv]\, w\, dx + \int_a^b u^* (-i)\,(A^\dagger v)\, w\, dx}$$

$$\cancel{= \int_a^b}$$

$$\cancel{= \int_a^b}$$

$$= \int_a^b u^* [i(A-A^\dagger)v]\, w\, dx$$

$$= \left(\int_a^b v^* i[A^\dagger u]\, w\, dx\right)^* + \left(\int_a^b v^* (-i)[Au]\, w\, dx\right)^*.$$

$$= \int_a^b [A^\dagger u]^* (-i)\, v\, w\, dx + \int_a^b [Au]^* (i)\, v\, w\, dx$$

$$= \int_a^b [i(A-A^\dagger)w]^*\, v\, w\, \cancel{dx}$$

$\therefore$ $i(A-A^\dagger)$ is hermitian.

(b) let $B = A + A^\dagger$, $C = i(A - A^\dagger)$ then

B and C are both hermitian

$$A + A^\dagger = B$$
$$A - A^\dagger = -iC$$

$$\therefore \quad A = \frac{B - iC}{2}, \quad A^\dagger = \frac{B + iC}{2}.$$

$\because$ A is arbitrary non-hermitian operator.
this shows that non-hermitian operator can be expressed as a linear combination of hermitian operators.

6.

$$\hat{L}y_m = \lambda_n y_m \qquad \hat{L}y_n = \lambda_n y_n$$

$$(\lambda_m - \lambda_n)\int_a^b y_m y_n w\,dx = \lambda_m \int_a^b y_m y_n w\,dx - \lambda_n \int_a^b y_m y_n w\,dx.$$

$$= \int_a^b (\hat{L}y_m)\, y_n w\,dx - \int_a^b (\hat{L}y_n)\, y_m w\,dx$$

$$= \int_a^b \left(\cancel{\tfrac{1}{w}}\right)\left((Py_m')' + qy_m\right) y_n \cancel{w}\,dx - \int_a^b \left(\cancel{\tfrac{1}{w}}\right)\left((Py_n')' + qy_n\right) y_m \cancel{w}\,dx.$$

$$= \int_a^b y_n(Py_m')' + qy_m y_n\,dx - \int_a^b y_m(Py_n')' + qy_m y_n\,dx.$$

$$= \int_a^b y_n(Py_m')'\,dx - \int_a^b y_m(Py_n')'\,dx.$$

$$= \underbrace{[y_n P y_m']_a^b}_{0} - \cancel{\int_a^b y_n' P y_m'\,dx} - \underbrace{[y_m P y_n']_a^b}_{0} + \cancel{\int_a^b y_m' P y_n'\,dx}.$$

$$= 0$$

$\because$ $(\lambda_m - \lambda_n) \neq 0$ (λ_m and λ_n are distinct eigenvalues)

$$\therefore \int_a^b y_m y_n w\,dx = 0$$

QED

7. $$xy'' + (k+1-x)y' = \lambda y$$

compare with $$\frac{1}{w}(py')' = \lambda y$$

$$\Rightarrow \frac{p}{w}y'' + \frac{p'}{w}y' = \lambda y.$$

$$\therefore \frac{p}{w} = x \qquad \frac{p'}{w} = (k+1-x)$$

$$\therefore \frac{p'}{p} = \frac{k+1-x}{x} = \frac{k+1}{x} - 1$$

$$\therefore \int^{p} \frac{dp}{p} = \int^{x} \left(\frac{k+1}{x'} - 1\right) dx'$$

$$\therefore \ln p = (k+1)\ln(x) - x$$

$$p(x) = x^{k+1}e^{-x}$$

$$w(x) = \frac{p(x)}{x} = x^k e^{-x} \qquad \therefore \frac{1}{w(x)} = \frac{e^x}{x^k}$$

$\therefore$ The Sturm-Liouville form of this equation is

$$\boxed{\frac{e^x}{x^k}\frac{d}{dx}\left(x^{k+1}e^{-x}\frac{dy}{dx}\right) = \lambda y}$$

The boundary condition is

$$\left[u^* p(x) \frac{dv}{dx}\right]_a^b = 0$$

The ~~natr~~ natural limit $[a, b]$ is such that the boundary condition is true no matter ~~we~~ what ~~u~~u and v is

$\therefore P(a) = P(b) = 0$

$\therefore$ if $k > -1$ $\quad a = 0, \quad b = \infty$

if $k \le -1$ $\quad \cancel{a = -\infty \quad b = \infty}$

there is no natural limits.

8 (a) $H\psi(x) = E\psi(x)$

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 \qquad \varepsilon = \frac{E}{\hbar\omega} \Rightarrow E = \hbar\omega\varepsilon.$$

$$y = x\sqrt{\frac{m\omega}{\hbar}} \Rightarrow \frac{dx}{dy} = \sqrt{\frac{\hbar}{m\omega}}$$

$$\therefore \frac{d}{dy} = \frac{d}{dx}\left(\frac{dx}{dy}\right) = \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}$$

$$\frac{d^2}{dy^2} = \frac{\hbar}{m\omega}\frac{d}{dx} \Rightarrow \frac{d^2}{dx^2} = \frac{m\omega}{\hbar}\frac{d^2}{dy^2}.$$

$$H\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

$$\therefore \cancel{\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = \frac{\hbar^2}{m\omega^2 x^2}\frac{d^2\psi}{}}$$

$$\cancel{\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2 = E}$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(E - \frac{1}{2}m\omega^2x^2\right)\psi = 0$$

$$\therefore \frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2\psi}{dy^2} + \left(\hbar\omega\varepsilon - \frac{1}{2}m\omega^2x^2\right)\psi = 0$$

$$\therefore \hbar\omega\frac{d^2\psi}{dy^2} + \hbar\omega\Big(2\varepsilon - \underbrace{\frac{m\omega}{\hbar}x^2}_{y^2}\Big)\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dy^2} + (2\varepsilon - y^2)\psi = 0$$

(b) When $y \to \infty$

$$2\varepsilon - y^2 \to -y^2$$

$$\therefore \frac{d^2\Phi}{dy^2} = y^2\Phi \Rightarrow \frac{d^2\Phi}{dy^2} - y^2\Phi = 0$$

Try $\Phi(y) = Ay^k e^{-y^2/2}$

$$\frac{d\Phi}{dy} = Aky^{k-1}e^{-y^2/2} + Ay^k e^{-y^2/2}(-y)$$

$$= Aky^{k-1}e^{-y^2/2} - Ay^{k+1}e^{-y^2/2}.$$

$$= A(ky^{k-1} - y^{k+1})e^{-y^2/2}.$$

$$\frac{d^2\Phi}{dy^2} = A(k(k-1)y^{k-2} - (k+1)y^k)e^{-y^2/2}$$

$$+ A(ky^{k-1} - y^{k+1})(-y)e^{-y^2/2}.$$

$$= A(k(k-1)y^{k-2} - (k+1)y^k - ky^k + y^{k+2})e^{-y^2/2}.$$

$$\cancel{= (k(k-1)y \quad Ay^k e^{-y^2/2}}$$

$$= \underbrace{(k(k-1)y^{-2} - (2k+1) + y^2)}_{\to y^2 \text{ as } y \to \infty} Ay^k e^{-y^2/2}.$$

$$= y^2 Ay^k e^{-y^2/2} = y^2\Phi$$

$\therefore$ When $y \to \infty$ $\Phi \to Ay^k e^{-y^2/2}$.

(c)

$$\frac{d^2\Phi}{dy^2} + (2\varepsilon - y^2)\Phi = 0$$

Substitute $\Phi = u e^{-y^2/2}$

$$\frac{d\Phi}{dy} = u' e^{-y^2/2} - yu e^{-y^2/2} = (u' - yu) e^{-y^2/2}.$$

$$\frac{d^2\Phi}{dy^2} = (u' - yu)' e^{-y^2/2} - y(u' - yu) e^{-y^2/2}.$$

$$= e^{-y^2/2} (u'' - yu' - u - yu' + y^2 u)$$

$$= e^{-y^2/2} (u'' - 2yu' + y^2 u - u).$$

$$\therefore \quad e^{-y^2/2} (u'' - 2yu' - u + \cancel{y^2 u})$$

$$+ e^{-y^2/2} (2\varepsilon u - \cancel{y^2 u}) = 0$$

$$\Rightarrow e^{-y^2/2} (u'' - 2yu' + (2\varepsilon - 1) u) = 0$$

$$\Rightarrow \quad u'' - 2yu' + (2\varepsilon - 1) u = 0$$

(d) $u'' - 2yu' + \underbrace{(2\varepsilon - 1)}_{-\lambda} u = 0$

$$u'' - 2yu' = \lambda u$$

$$e^{-y^2}u'' - 2ye^{-y^2}u' = \lambda e^{-y^2}u.$$

$$e^{-y^2}(u')' + (e^{-y^2})'u' = \lambda e^{-y^2}u.$$

$$(e^{-y^2}u')' = \lambda e^{-y^2}u.$$

$$\therefore \boxed{\frac{1}{e^{-y^2}}(e^{-y^2}u')' = \lambda u}$$

this is in the Sturm - Liouville form

weight function is ~~w(x)~~ $\boxed{w(y) = e^{-y^2}}$.

(e) try $u(y) = \sum_{n=0}^{\infty} a_n y^n$

~~$u'' = $~~ ~~$u' = \sum_{n=1}^{\infty} n a_n y^{n-1} = \sum_{n=0} n a_n y^{n-1}$~~ $\sum_{n=0}^{\infty}$

~~$u'' = \sum_{n=2}^{\infty} n(n-1) a_n y^{n-2} = \sum_{n=0}^{\infty} n(n-1) a_n y^{n-1}$~~

~~$u' = \sum_{n=1}^{\infty} n a_n y^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} y^m = \sum_{n=0}^{\infty} (n+1) a_{n+1} y^n$~~

~~$u'' = \sum_{n=2}^{\infty} n(n-1) a_n y^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} y^m = \sum_{m=0}^{\infty} (n+2)(n+1) a_{n+2} y^n$~~

~~$u' = \sum_{n=1}^{\infty} n a_n y^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} y^m = \sum_{n=0}^{\infty} (n+1) a_{n+1} y^n.$~~

~~$u'' = \sum_{n=2}^{\infty} a_n n(n-1) y^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} y^m$~~

~~$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} y^n$~~

(e) try $u(y) = \sum_{n=0}^{\infty} a_n y^n$

$$u' = \sum_{n=1}^{\infty} n a_n y^{n-1} = \sum_{n=0}^{\infty} n a_n y^{n-1}$$

$$u'' = \sum_{n=2}^{\infty} n(n-1) a_n y^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} y^m$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} y^n$$

$$u'' - 2y u' + (2\varepsilon - 1) u = 0$$

$$\therefore \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} y^n - 2 \sum_{n=0}^{\infty} n a_n y^n + (2\varepsilon - 1) \sum_{n=0}^{\infty} a_n y^n = 0$$

Because of the linear independence of polynomial y^n

$$\therefore (n+2)(n+1) a_{n+2} - 2n a_n + (2\varepsilon - 1) a_n = 0$$

$$\therefore \boxed{a_{n+2} = a_n \frac{(2n+1-2\varepsilon)}{(n+2)(n+1)}}$$

(f) ~~for $n=0$, $\varepsilon = \frac{1}{2}$~~

Starting from $(a_0, a_1) = (1, 0)$ for even n.

and $(a_0, a_1) = (0, 1)$ for odd

~~for $n=0$, $\varepsilon = \frac{1}{2}$ $a_0 = 1$, $a_{n \neq 0} = 0$~~

~~$\therefore$ $\cancel{H_0(x) = 1}$ $\cancel{H_0(y) = 1}$ $\boxed{H_0(x) = 1}$~~

~~for $n=1$, $\varepsilon = \frac{3}{2}$ $\cancel{a_0 = 0}$ $a_1 = 1$, $a_3 = 0$, $a_5 = 0 \ldots$~~

~~$\therefore$ $\cancel{H_1(x) =}$ $H_1(y) = y$~~

~~$\therefore$ $H_1(x) =$~~

for $n=0$, $\varepsilon = \frac{1}{2}$ $\quad (a_0, a_1) = (1, 0)$

$\therefore \quad a_0 = 1 \quad \therefore \cancel{H_0(x) = e^{-}} \quad \boxed{H_0(x) = 1}$

for $n=1$, $\varepsilon = \frac{3}{2}$ $\quad a_0 = a_2 = \cdots = 0$

$a_1 = 1, \quad a_3 = a_5 = \cdots = 0$

$\therefore \quad \boxed{H_1(x) = x}$

for $n=2$, $\varepsilon = \frac{5}{2}$ $\quad a_0 = 1, \quad \cancel{a_2 = \frac{3-5}{(1)(3)}} \quad a_2 = \frac{-4}{(2)} = -2$

$a_4 = a_6 = \cdots = 0, \quad a_1 = a_3 = \cdots = 0$

$\therefore \quad \boxed{H_2(x) = (1 - 2x^2)}$

for $n=3$, $\varepsilon = \frac{7}{2}$ $\quad a_0 = a_2 = \cdots = 0$

$a_1 = 1 \quad a_3 = \frac{3-7}{6} = -\frac{2}{3}$

$\therefore \quad \boxed{H_3(x) = (x - \frac{2}{3}x^3)}$

(g)

$$H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = \boxed{1}$$

$$H_1(x) = (-1)^1 e^{x^2} (-2x e^{-x^2})$$

$$= \boxed{2x}$$

$$H_2(x) = (-1)^2 e^{x^2} \frac{d}{dx} (-2x e^{-x^2})$$

$$= (-1)^2 e^{x^2} [(-2x)(-2x) e^{-x^2} - 2e^{-x^2}]$$

$$= \boxed{4x^2 - 2}$$

$$H_3(x) = (-1)^3 e^{x^2} \frac{d}{dx} ((4x^2-2) e^{-x^2})$$

$$= (-1) e^{x^2} (8x e^{-x^2} + (4x^2-2)(-2x) e^{-x^2})$$

$$= \boxed{8x^3 - 12x}$$

These are consistent with what we found in (f)

up to rescaling

(h)

Given the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

we have

$$\frac{dH_n(x)}{dx} = (-1)^n \frac{d}{dx}\left(e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\right)$$

$$= (-1)^n 2x e^{x^2} \frac{d^n}{dx^n} e^{-x^2} + (-1)^n e^{x^2} \frac{d^n}{dx^n}(-2x e^{-x^2})$$

$$= \cancel{(-1)^n 2x e^{x^2} \frac{d^n}{dx^n} e^{-x^2}} + (-1)^n e^{x^2} \Big(\cancel{-2x \frac{d^n}{dx^n} e^{-x^2}} \underbrace{- (n)(2) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} + 0 + 0 + \cdots}\Big)$$

~~Lebniz~~ Leibniz' formula for differentiation

$$= \cancel{(-1)^n} (2n)(-1)^{n-1} e^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}$$

$$= 2n H_{n-1}$$

$$\therefore \underline{H_n'(x) = 2n H_{n-1}(x)}$$

Now consider $\int_{-\infty}^{\infty} dy\, e^{-y^2} H_n(y) H_l(y)$

~~For n for n≠l~~

$$\int_{-\infty}^{\infty} dy\, e^{-y^2} H_n(y) H_l(y) = \int_{-\infty}^{\infty} dy\, H_n(y) (-1)^l \cancel{e^{y^2}} \left(\frac{d^l}{dx^l} e^{-y^2}\right) \cancel{e^{-y^2}}$$

$$= (-1)^l \underbrace{\left[H_n(y) \frac{d^{l-1}}{dy^{l-1}} e^{-y^2} \right]_{-\infty}^{\infty}}_{=0} - (-1)^l \int_{-\infty}^{\infty} dy\, H_n'(y) \frac{d^{l-1}}{dx^{l-1}} e^{-y^2}.$$

$$= 2n \int_{-\infty}^{\infty} dy\, e^{-y^2} H_{n-1}(y) \Big[\underbrace{(-1)^{l-1} e^{y^2} \frac{d^{l-1}}{dy^{l-1}} e^{-y^2}}_{H_{l-1}(y)}\Big]$$

$$= 2n \int_{-\infty}^{\infty} dy\, e^{-y^2} H_{n-1}(y) H_{l-1}(y).$$

Now if $n \neq l$, let's assume $n < l$ ~~for~~ without lossing generality.

If we repeat the process n times we get

$$\int_{-\infty}^{\infty} dy\, e^{-y^2} H_n(y) H_l(y) = 2^n n! \int_{-\infty}^{\infty} dy\, e^{-y^2} \underbrace{H_0(y)}_{1} H_{l-n}(y).$$

$$= 2^n n! \int_{-\infty}^{\infty} dy\, e^{-y^2} H_m(y) \qquad (m = l-n)$$

$$= 2^n n! \int_{-\infty}^{\infty} dy\, \cancel{e^{-y^2}} (-1)^m \cancel{e^{y^2}} \frac{d^m}{dy^m} (e^{-y^2})$$

$$= 2^n n! (-1)^m \int_{-\infty}^{\infty} d \left(\frac{d^{m-1}}{dy^{m-1}} e^{-y^2} \right)$$

$$= 2^n n! (-1)^m \underbrace{\left[\frac{d^{m-1}}{dy^{m-1}} e^{-y^2} \right]_{-\infty}^{\infty}}_{0} = 0$$

If $n = l$.

we repeat the process n times to get

$$\int_{-\infty}^{\infty} dy\, e^{-y^2} H_n(y) H_m(y) = 2^n n! \int_{-\infty}^{\infty} dy\, e^{-y^2} \underbrace{H_0(y)}_{1} \underbrace{H_0(y)}_{1}$$

$$= 2^n n! \underbrace{\int_{-\infty}^{\infty} dy\, e^{-y^2}}_{\text{Gaussian integral} = \sqrt{\pi}} = \boxed{2^n n! \sqrt{\pi}}$$

Overall:

$$\int_{-\infty}^{\infty} dy\, e^{-y^2} H_n(y) H_\ell(y) = \underline{\delta_{n\ell} \sqrt{\pi}\, 2^n n!}$$

9. (a)

$$G(x,t) = e^{2tx-t^2}$$

$$= 1 + (2tx - t^2) + \frac{1}{2!}(2tx - t^2)^2 + \frac{1}{3!}(2tx - t^2)^3 + \cdots$$

$$= 1 + 2tx - t^2 + \frac{1}{2!}(4t^2x^2) + O(t^3).$$

$$= \cancel{1 + 2x}\ (1)\left(\frac{t^0}{0!}\right) + (2x)\frac{t^1}{1!} + (4x^2-2)\frac{t^2}{2!} + O(t^3).$$

$\therefore$ $\underline{H_0(x) = 1}$ $\quad$ $\underline{H_1(x) = 2x}$ $\quad$ $\underline{H_2(x) = 4x^2 - 2}$

(b)

$$\frac{\partial G(x,t)}{\partial t} = e^{2tx-t^2}(2x-2t).$$

$$= 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$

Also $\displaystyle \frac{\partial G(x,t)}{\partial t} = \sum_{n=1}^{\infty} H_n(x)\, n\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty} H_n(x)\, n\frac{t^{n-1}}{n!}$

$$\therefore\ 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n(x)\, n\frac{t^{n-1}}{n!}$$

QED.

$$\Rightarrow \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} - \sum_{n=1}^{\infty} H_n(x)\, n\frac{t^{n-1}}{n!} = 0$$

$$\begin{pmatrix} m = n+1 \\ n = m-1 \end{pmatrix} \qquad \begin{pmatrix} k = n-1 \\ n = k+1 \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} - \sum_{k=0}^{\infty} H_{k+1}(x)\frac{t^k}{(k+1)!}(k+1)$$

$$= \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2nH_{n-1}(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} H_{n+1}(x)\frac{t^n}{n!}$$

$\because$ t^n's are linearly independent.

$$\therefore\ 2xH_n(x) - 2nH_{n-1}(x) - H_{n+1}(x) = 0$$

$$\therefore\ \underline{H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)}$$

(c)

$$\frac{\partial G(x,t)}{\partial x} = 2t\, e^{2tx-t^2} = 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$\frac{\partial G(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$\therefore \sum_{n=0}^{\infty} 2H_n(x) \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!}$$

$$\therefore \left(\begin{matrix} m = n+1 \\ n = m-1 \end{matrix} \quad (m-1)! = \frac{m!}{m} \right).$$

$$\Rightarrow \sum_{m=0}^{\infty} \cancel{2mH_{m}(x)}\ 2mH_{m-1}(x) \frac{t^m}{m!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

$$\Rightarrow \sum_{n=0}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Due to linear independence of t^n's

we have $\cancel{H_n' =}$ $\underbrace{H_n'(x) = 2nH_{n-1}(x)}$

(d)

~~from (c) we have~~

$\cancel{H_n'(x) = 2nH_{n-1}(x)}$, $\cancel{H_{n+1}'(x) = 2nH_{n-2}(x)}$

~~from (b) we have~~

$\cancel{H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)}$

$\cancel{\therefore H_{n+2}(x) = 2xH_{n+1}(x) - 2nH_n(x)}$

$\cancel{H_{n+2} = 2xH_{n+1} - 2(n+1)H_n}$

From (c) we know that

$$H_n'' = 2nH_{n-1}' = (2n)(2(n-1))H_{n-2}.$$

$$2xH_n' = (2x)(2n)H_{n-1}$$

$$2nH_n = (2n)H_n$$

$$\therefore H_n'' - 2xH_n' + 2nH_n = 2n(\underbrace{2(n-1)H_{n-2} - 2xH_{n-1} + H_n}_{=0 \text{ by (b)}}) = 0$$

QED

10. $$\frac{1}{|\vec{r}_2-\vec{r}_1|} = \frac{1}{\sqrt{r_2^2+r_1^2-2r_1r_2\cos\theta_{12}}} = \left[r_2^2\left(1-2\cos\theta_{12}\left(\frac{r_1}{r_2}\right)+\left(\frac{r_1}{r_2}\right)^2\right)\right]^{-1/2}$$

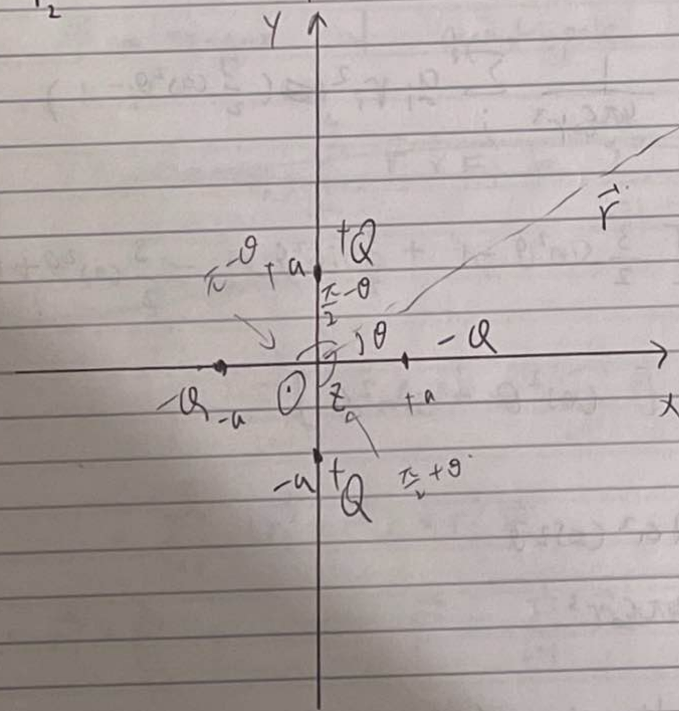
$$= \frac{1}{r_2}\left(1-\left(2\cos\theta_{12}\left(\frac{r_1}{r_2}\right)-\left(\frac{r_1}{r_2}\right)^2\right)\right)^{-\frac{1}{2}}$$

$$= \frac{1}{r_2}\left(1+\frac{1}{2}\left(2\cos\theta_{12}\left(\frac{r_1}{r_2}\right)-\left(\frac{r_1}{r_2}\right)^2\right)+\frac{3}{8}\left(2\cos\theta_{12}\left(\frac{r_1}{r_2}\right)-\left(\frac{r_1}{r_2}\right)^2\right)^2+\cdots\right)$$

$$= \frac{1}{r_2}\left(1+\cos\theta_{12}\left(\frac{r_1}{r_2}\right)-\frac{1}{2}\left(\frac{r_1}{r_2}\right)^2+\left(\frac{3}{8}\right)\left(4\cos^2\theta_{12}\left(\frac{r_1}{r_2}\right)^2\right)+O\left(\left(\frac{r_1}{r_2}\right)^3\right)\right)$$

$$= \frac{1}{r_2}\left(1+\left(\frac{r_1}{r_2}\right)\cos\theta_{12}+\left(\frac{r_1}{r_2}\right)^2\left(\frac{1}{2}(3\cos^2\theta_{12}-1)\right)+\cdots\right).$$

$$= \frac{1}{r_2}\left(1+\left(\frac{r_1}{r_2}\right)P_1(\cos\theta_{12})+\left(\frac{r_1}{r_2}\right)^2P_2(\cos\theta_{12})+\cdots\right).$$



The multipole expansion above can be summarized as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\vec{r}-\vec{r}_i|} = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{2}\frac{1}{r^{l+1}}\sum_i r_i^l q_i P_l(\cos\theta_i)$$

$\therefore$ monopole term $l=0$:

$$V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r}\sum q_i = \frac{1}{4\pi\varepsilon_0 r}\underbrace{(Q+Q-Q-Q)}_{0} = 0$$

Dipole term :

$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2} \sum_i q_i r_i \cos\theta_i$$

$$= \frac{1}{4\pi\varepsilon_0 r^2}\Big[+Q(a)\cos(\tfrac{\pi}{2}-\theta) + Q(a)\cos(\tfrac{\pi}{2}+\theta)$$

$$- Q(a)\cos\theta - Q(a)\cos(\pi-\theta)\Big)$$

$$= \frac{Qa}{4\pi\varepsilon_0 r^2}\Big[\underbrace{\cos(\tfrac{\pi}{2}-\theta)}_{\cancel{\sin\theta}} + \underbrace{\cos(\tfrac{\pi}{2}+\theta)}_{\cancel{-\sin\theta}} - \underbrace{\cos\theta}_{\cancel{-\cos\theta}} - \underbrace{\cos(\pi-\theta)}_{\cancel{+\cos\theta}}\Big)$$

$$= 0$$

Quadrupole term :

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3} \sum_i q_i r_i^2 (\tfrac{3}{2}\cos^2\theta_i - 1)$$

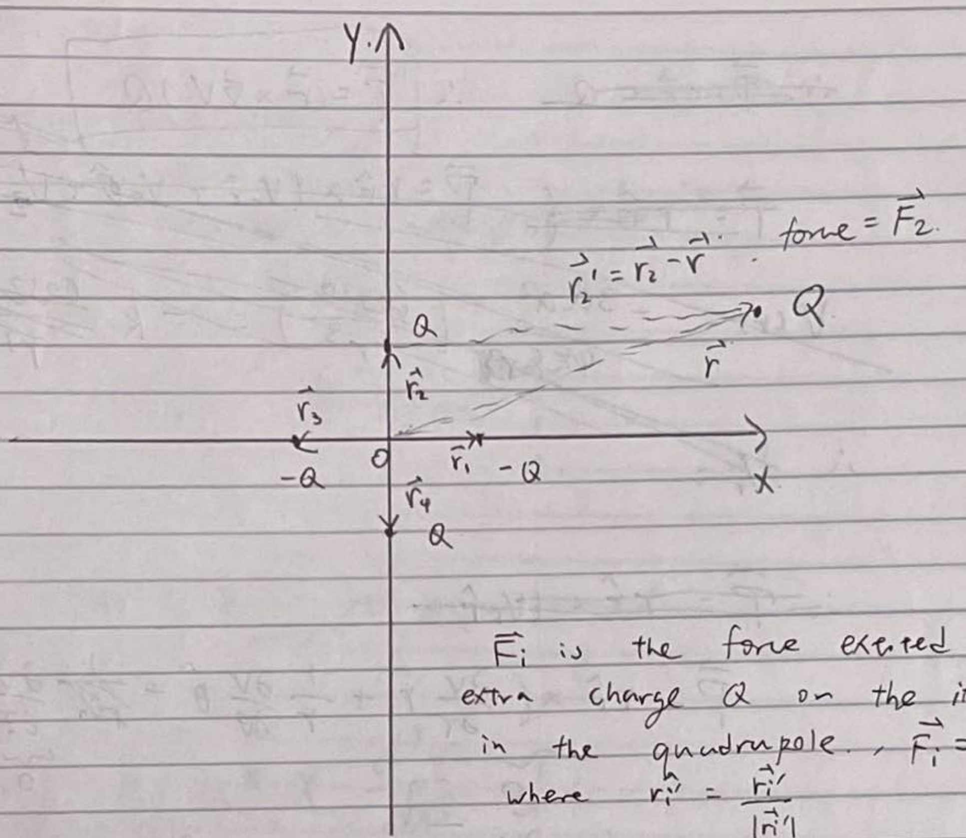
$$= \frac{Qa^2}{4\pi\varepsilon_0 r^3}\Big[\tfrac{3}{2}\sin^2\theta \cancel{-1} + \tfrac{3}{2}\sin^2\theta \cancel{-1} - \tfrac{3}{2}\cos^2\theta \cancel{+1} - \tfrac{3}{2}\cos^2\theta \cancel{+1}$$

$$= \frac{-3Qa^2}{4\pi\varepsilon_0 r^3}\big[\cos^2\theta - \sin^2\theta\big].$$

$$= \frac{-3Qa^2\cos2\theta}{4\pi\varepsilon_0 r^3}$$

$\therefore$ So $V(\vec{r}) = V_1 + V_2 + V_3$

$$\therefore \boxed{V(\vec{r}) = \frac{-3Qa^2\cos2\theta}{4\pi\varepsilon_0 r^3}}$$



$\vec{F_i}$ is the force exerted by the extra charge Q on the ith charge in the quadrupole, $\vec{F_i} = F_i \hat{r_i}'$ where $\hat{r_i}' = \frac{\vec{r_i}'}{|\vec{r_i}'|}$

$\vec{T}$ = torque of quadrupole about O

$$\cancel{\vec{T} = \sum_{i=1}^{4} \vec{r_i} \times \vec{F_i} = \sum_{i=1}^{4} (\vec{r_i} - \vec{r}) \times \hat{r_i}' F_i}$$

$$\cancel{= \sum_{i=1}^{4}}$$

$$\vec{T} = \sum_{i=1}^{4} \vec{r_i} \times \vec{F_i} = \sum_{i=1}^{4} \vec{r_i} \times \hat{r_i}' F_i$$

$$= \sum_{i=1}^{4} (\vec{r_i}' + \vec{r}) \times \hat{r_i}' F$$

$\because \vec{r_i}' \times \hat{r_i}' = 0 \quad \therefore \quad \vec{T} = \sum_{i=1}^{4} \vec{r} \times \hat{r_i}' F = \vec{r} \times \sum_{i=1}^{4} \vec{F_i}$

$\sum_{i=1}^{4} \vec{F_i}$ = total force exerted by Q on quadrupole

= − total force exerted by quadrupole on Q

$= \cancel{-(-Q \vec{\nabla} V)} = -Q\vec{E} = -Q(-\vec{\nabla} V) = Q \vec{\nabla} V$

$$\cancel{\therefore \vec{P} = \vec{r} \times Q} \qquad \therefore \boxed{\vec{T} = (\vec{r} \times \vec{\nabla} V) Q.}$$

$$\cancel{\vec{T} = r\hat{r} \times (} \quad \cancel{\vec{P} = r\hat{r} \times (V_r \hat{r} + V_\theta \hat{\theta} + V_z \hat{z}) Q =}$$

$$\cancel{V(r) = \frac{-3Qa^2}{4\pi\varepsilon_0} \left(\frac{\cos 2\theta}{r^3} \right) = -k \frac{\cos 2\theta}{r^3}}$$

$$\cancel{\therefore V_r =}$$

$$\cancel{\therefore \vec{T} = r\hat{r} \times (V_r \hat{r} +}$$

$$\vec{T} = r\hat{r} \times \left(\underbrace{\frac{\partial V}{\partial r} \hat{r}}_{0} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \underbrace{\frac{\partial V}{\partial z}}_{0} \hat{z} \right) Q.$$

$$= Q \frac{\partial V}{\partial \theta} (\hat{r} \times \hat{\theta}) = Q \frac{\partial V}{\partial \theta} \hat{z}$$

$$\therefore \vec{T} = Q \left(\frac{-3Qa^2}{4\pi\varepsilon_0 r^3} \right) (-2\sin 2\theta) \cdot \hat{z}$$

$$\therefore \boxed{\vec{T} = \frac{3Q^2 a^2 \sin 2\theta}{2\pi\varepsilon_0 r^3} \hat{z}}$$

When $\vec{T} = \vec{0}$, $\sin 2\theta = 0$ $(0 \le \theta < 2\pi)$

$$\therefore \boxed{\theta = 0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi}$$

Energy of Quadrupole in the field of Q is.

$$U = Q \sum_{i=1}^{4} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|r - r_i|} = QV = \frac{-3Q^2 a^2}{4\pi\varepsilon_0 r^3} \cos 2\theta$$

U has minimum when $\cos 2\theta = 1 \Rightarrow \theta = 0, \pi$

U has maximum when $\cos 2\theta = -1 \Rightarrow \theta = \frac{\pi}{2}, \frac{3}{2}\pi$.

$\therefore$ $\theta = 0, \pi$ are stable equilibrium.

$\theta = \frac{\pi}{2}, \frac{3}{2}\pi$ are unstable equilibrium.

$$\hat{L} y_n = \lambda_n y_n$$

real — orthogonal basis

$ix\frac{d^3}{dx^3}$ hermitian ?

$$\int u^* \left(ix\frac{d^3}{dx^3}\right) v \, dx$$

$$= i\int u^* x \frac{d^3 v}{dx^3} \, dx.$$

$$= i\underbrace{\left[u^* x \frac{d^2 v}{dx^2}\right]_{-\infty}^{\infty}}_{0} - i\int_{-\infty}^{\infty} \frac{d}{dx}(u^* x)\frac{d^2 v}{dx^2} \, dx$$

$$= -i\int_{-\infty}^{\infty} \frac{d^3}{dx^3} \underbrace{(u^* x)}\, v \, dx + \cancel{[\;]}_{0}.$$

$$\left(\frac{d^3 u^*}{dx^3}\right)x + 3\frac{d^2 u^*}{dx^2} + \ldots$$

$(A^\dagger)^\dagger = A$

$$\int u^* (A^\dagger)^\dagger v \, dx \overset{?}{=} \int u^* A v \, dx.$$

$$\int u^* (A^\dagger)^\dagger v \, dx = \int (A^\dagger u)^* v \, dx.$$

$$= \left[\int (A^\dagger u) v^* \right]^*$$

$$= \left[\int v^* (A^\dagger u) \right]^* = \left[\int (Av)^* u \right]^*$$

$$= \int u^* A v \, dx.$$

$$\langle u|A|v\rangle = \langle v|A^\dagger|u\rangle^*$$

$$\langle v|A|u\rangle = \langle u|A^\dagger|v\rangle^*.$$

$$h y_n(x) = \lambda_n y_n(x)$$

$$h y_m(x) = \lambda_m y_m(x)$$

$$\int y_m(x) h y_n(x) = \lambda_n \int y_n(x) y_m(x)$$

$$-\int y_n(x) h y_m(x) = \lambda_m \int y_m(x) y_n(x)$$

$$\int y_m(x) h y_n(x) - \int y_n(x) h y_m(x) = (\lambda_m - \lambda_n) \int dx\, y_m(x) y_n(x)$$

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x).$$

$$\Rightarrow \int y_m \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) y_n dx - \int y_n \frac{d}{dx}\left(p \frac{d}{dx}\right) y_m\, dx.$$

$$\cancel{T(\theta) - T(\theta_0) = \frac{\partial E}{\partial \theta}(\theta - \theta_0)}$$

$$T(\theta) = T(\theta_0) + \frac{\partial \tau}{\partial \theta}(\theta - \theta_0)$$

$$T(\theta) - T(\theta_0) = \frac{\partial \tau_v}{\partial \theta}(\theta - \theta_0).$$

if equilibrium than when $(\theta - \theta_0)$ ~~increases~~ > 0

then $T(\theta) - T(\theta_0) < 0$.

and vice versa.

if not than when $(\theta - \theta_0) > 0$

$T(\theta) - T(\theta)$ also > 0.

For equilibrium you want a

torque to bring it back.