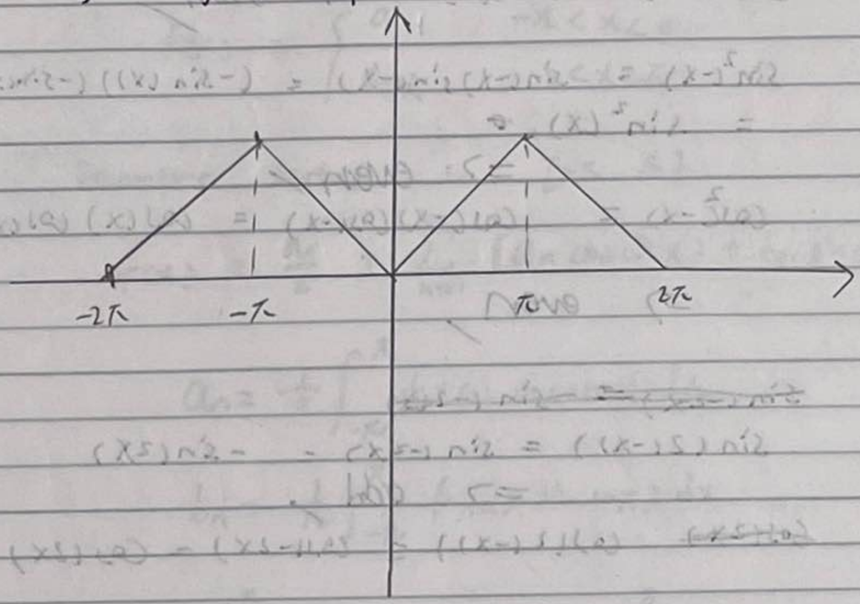
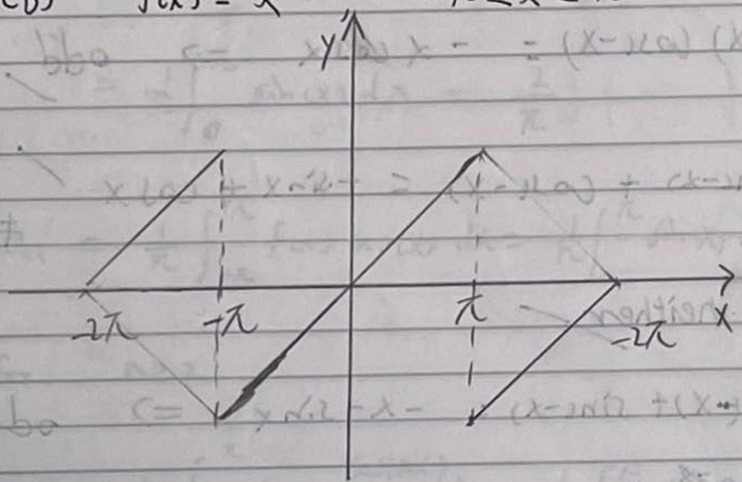
To: Ana Lopez

Mathematical Methods 3 Ziyan Li

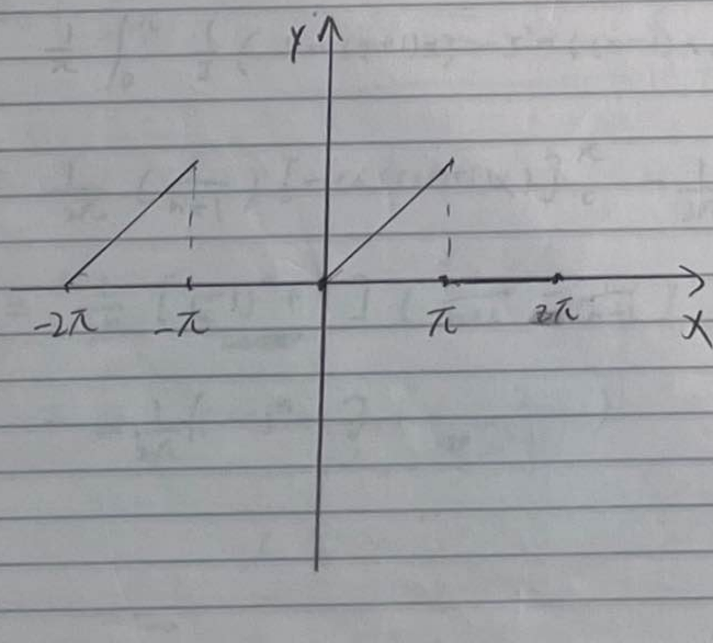
1. (a) $f(x) = |x|$, $-\pi < x < \pi$



(b) $f(x) = x$ $-\pi < x < \pi$.



(c) $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$.



2 (a) $\sin(-x) = -\sin(x) \Rightarrow$ odd

(b) $\cos(-x) = \cos(x) \Rightarrow$ even

(c) $\sin^2(-x) = \sin(-x)\sin(-x) = (-\sin(x))(-\sin(x))$
$= \sin^2(x)$
$\Rightarrow$ even

(d) $\cos^2(-x) = \cos(-x)\cos(-x) = \cos(x)\cos(x) = \cos^2(x)$

$\Rightarrow$ even

(e) ~~sin(-2x) = -sin(-2x)~~
$\sin(2(-x)) = \sin(-2x) = -\sin(2x)$
$\Rightarrow$ odd

(f) ~~cos(2x)~~ $\cos(2(-x)) = \cos(-2x) = \cos(2x)$

$\Rightarrow$ even

(g) $(-x)\cos(-x) = -x\cos x \Rightarrow$ odd

(h) $\sin(-x) + \cos(-x) = -\sin x + \cos x$ $\neq -\sin x - \cos x$
$\neq \sin x + \cos x$

$\therefore$ neither

(i) $(-x) + \sin(-x) = -x - \sin x \Rightarrow$ odd

(j) e^{-x} $\neq e^{x}$, $\neq -e^{x}$ $\Rightarrow$ neither

3. In region $[-\pi, \pi]$

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$

Fourier series in $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin(n\pi)\, dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(0)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(x)\, dx = \frac{2}{\pi}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(x)\, dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(x)\, dx = 0$$

for $n \geq 2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi}\int_0^{\pi} \sin(x)\cos(nx)\, dx$$

$$\cancel{= \frac{-1}{\pi}\int_1^{-1} \cos(nx)\, d(\cos(x)) = \frac{1}{\pi}\int_{-1}^{1} \cos(nx)}$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\left(\sin((n+1)x) - \sin((n-1)x)\right) dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{n+1}\right)\left[-\cos((n+1)x)\right]_0^{\pi} - \frac{1}{2\pi}\left(\frac{1}{n-1}\right)\left[-\cos((n-1)x)\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[(-1)^n + 1\right]\left(\frac{1}{n+1} - \frac{1}{n-1}\right)$$

$$= \frac{1}{2\pi}\left[(-1)^n + 1\right]\left(\frac{2}{n^2-1}\right)$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin^2(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1-\cos 2x}{2}\,dx = \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\pi - 0 - 0 + 0\right] = \frac{1}{2}$$

for $n \geq 2$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(x)\sin(nx)\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \cos((n-1)x) - \cos((n+1)x)\,dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{n-1}\right)\left[\sin((n-1)x)\right]_0^{\pi} - \frac{1}{2\pi}\left(\frac{1}{n+1}\right)\left[\sin((n+1)x)\right]_0^{\pi}$$

$$= 0$$

$\therefore$ $a_0 = \frac{2}{\pi}$, $a_1 = 0$, $b_1 = \frac{1}{2}$, $a_n = -\frac{1}{2\pi}[(-1)^n + 1]\left(\frac{2}{n^2-1}\right)$

$b_n = 0$ for $n \geq 2$

$$\cancel{\therefore f(x) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \sum_{n=2}^{\infty} \frac{1}{2\pi}[(-1)^n+1]\left(\frac{2}{n^2-1}\right)\cos(nx)}$$

$$\therefore f(x) = \frac{1}{\pi} + \frac{1}{2}\sin(x) - \sum_{n=2}^{\infty} \frac{1}{2\pi}[(-1)^n+1]\left(\frac{2}{n^2-1}\right)\cos(nx) \quad \text{for } -\pi < x < \pi$$

4. $f(x) = x\sin x, \quad 0 < x < \pi$.

$$f(-x) = -x\sin(-x) = (-x)(-\sin x) = x\sin x = f(x)$$

$\therefore$ $f(x)$ is an even function.

The fourier cosine series ?

$$f(x) = x\sin x \qquad -\pi < x < \pi.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\, dx = \cancel{\frac{1}{\pi}}$$

$$\cancel{\int \sin x} \quad \cancel{\int x\sin x\, dx = \int \sin x\, d\left(\frac{x^2}{2}\right)}$$

$$\int x\sin x\, dx = \int (-x)\, d(\cos x) = -x\cos x - \int \cos x\, d(-x)$$

$$= -x\cos x + \int \cos x\, dx = -x\cos x + \sin x$$

$$\therefore \cancel{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}} \qquad a_0 = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_{-\pi}^{\pi} = \boxed{2}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\, dx = \frac{1}{\pi}\left(-\frac{1}{4}\right)\left[x\cos 2x - \frac{1}{2}\sin 2x\right]_{-\pi}^{\pi} = 0$$

$$= \frac{1}{\pi}\left(-\frac{1}{4}\right)(2\pi) = \boxed{-1/2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos(nx)\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin(n+1)x - x\sin(n-1)x\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}\left[x\cos(n+1)x - \frac{1}{n+1}\sin(n+1)x\right]_{-\pi}^{\pi} - \frac{-1}{n-1}\left[x\cos(n-1)x - \frac{1}{n-1}\sin(n-1)x\right]_{-\pi}^{\pi}\right]$$

$$= \frac{-1}{2\pi}\left[\frac{1}{n+1}(2\pi)(-1)^{n+1} - \frac{1}{n-1}(2\pi)(-1)^{n+1}\right]$$

$$= (-1)(-1)^{n+1}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \cancel{\frac{2(-1)^n}{n^2-1}} \quad \boxed{\frac{2(-1)^{n+1}}{n^2-1}}$$

∴ Cosine series is

~~$f(x) = 2 - \frac{1}{2}\cos\theta$~~ $$\boxed{f(x) = 2 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos(nx)}$$

$\frac{a_0}{2}$

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The fourier sine series:

$$f(x) = \begin{cases} x\sin x & 0<x<\pi \\ -x\sin x & -\pi<x<0 \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\,dx}_{f(x)\text{ is odd}} = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(x)\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin^2 x\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin^2 x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin^2 x\,dx = \frac{2}{\pi}\left[\frac{x^2}{4} - \frac{1}{4}\left(x\sin 2x + \frac{1}{2}\cos 2x\right)\right]_0^{\pi}$$

$$= \left(\frac{2}{\pi}\right)\left(\frac{\pi^2}{4}\right) = \frac{\pi}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = -\frac{1}{\pi}\int_{-\pi}^{0} x\sin x\sin nx + \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos(n-1)x - x\cos(n+1)x\,dx.$$

$$= \frac{1}{\pi(n-1)}\left[x\sin(n-1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^{\pi} - \frac{1}{\pi(n+1)}\left[x\sin(n+1)x + \frac{1}{n+1}\cos(n+1)x\right]_0^{\pi}$$

$$= \frac{1}{\pi(n-1)}\left[(-1)^{n+1} - 1\right]\cdot\left(\frac{1}{n-1}\right) - \frac{1}{\pi(n+1)}\left[(-1)^{n+1} - 1\right]\cdot\left(\frac{1}{n+1}\right)$$

$$= \frac{1}{\pi}\left[(-1)^{n+1} - 1\right]\cdot\left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2}\right) = \frac{1}{\pi}\left[(-1)^{n+1} - 1\right]\left(\frac{4n}{(n^2-1)^2}\right)$$

∴ Fourier sine series:

$$\boxed{f(x) = \frac{\pi}{2}\sin x + \sum_{n=2}^{\infty} \frac{1}{\pi}\left[(-1)^{n+1} - 1\right]\left(\frac{4n}{(n^2-1)^2}\right)\sin(nx)}$$

5. (a) $f(x) = x^2$ is even ~~over [0,2]~~.

$\therefore$ ~~cos~~ sine series ;

$$f(x) = \begin{cases} x^2 & 0<x<2 \\ -x^2 & -2<x<0 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right)$$ ~~L=2~~ $L = 4$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx.$$ $L=4$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\tfrac{1}{2}\pi n x\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\tfrac{1}{2}\pi n x\right)dx$$

$a_0 = 0 \Leftarrow f(x)$ is odd.

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{\pi n}{2}x\right)dx$$

$$= \frac{1}{2}\int_{-2}^{0} -x^2\sin\left(\frac{n\pi}{2}x\right)dx + \frac{1}{2}\int_{0}^{2} x^2\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \frac{1}{2}\left[\int_0^2 x^2\sin\left(\frac{n\pi}{2}x\right)dx - \int_{-2}^{0} x^2\sin\left(\frac{n\pi}{2}x\right)dx\right].$$

$$\underbrace{=}_{\because\, x^2\sin(\frac{n\pi}{2}x) \text{ is odd}} \left(\cancel{\tfrac{1}{2}}\right)\left(\cancel{2}\right)\int_0^2 x^2\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \int_0^2 x^2\sin\left(\frac{n\pi}{2}x\right)dx$$

$$= \left[-\frac{2}{n\pi}x^2\cos\left(\frac{n\pi}{2}x\right) + \frac{8}{n^2\pi^2}x\sin\left(\frac{n\pi}{2}x\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi}{2}x\right)\right]_0^2$$

$$= \underline{\frac{8}{n\pi}(-1)^{n+1} + \frac{16}{n^3\pi^3}\left((-1)^n - 1\right)}$$

$\therefore$ The Fourier Sine series is

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{8}{n\pi}(-1)^{n+1} + \frac{16}{n^3\pi^3}((-1)^n - 1)\right] \cancel{\sin\frac{2\pi n}{L}} \sin(\tfrac{\pi n}{2}x)$$

(b) The cosine series $\Rightarrow$ $f(x) = x^2$ $(-2 < x < 2)$

$\because L = 4 \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi n x}{L}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{1}{2}\pi n x)$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos(\tfrac{2\pi n}{L}x)\,dx$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\,\cancel{\cos(\tfrac{2\pi n}{L})}\cos(\tfrac{1}{2}\pi n x)\,dx$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{8}{3} \Rightarrow \frac{a_0}{2} = \frac{4}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} x^2\cos(\tfrac{n\pi}{2}x)\,dx$$

$$= \left[\frac{1}{n\pi}x^2\sin(\tfrac{n\pi}{2}x) + \frac{4}{n^2\pi^2}x\cos(\tfrac{n\pi}{2}x) - \frac{8}{n^3\pi^3}\sin(\tfrac{n\pi}{2}x)\right]_{-2}^{2}$$

$$= \frac{16(-1)^n}{n^2\pi^2}$$

$\therefore$ The cosine series is

$$f(x) = \frac{4}{3} + \sum_{n=1}^{\infty}\left[\frac{16(-1)^n}{n^2\pi^2}\right]\cos(\tfrac{n\pi}{2}x)$$

6. In the space $L^2_w(-\infty, \infty)$

the inner product of two functions $f(x)$ and $g(x)$

is defined to be $\langle f|g\rangle = \int_{-\infty}^{\infty} dx\, w(x)\, f^*(x)\, g(x)$

$\because w(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$

$\therefore \langle f|g\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f^*(x) g(x) e^{-x^2} dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x) g(x) e^{-x^2} dx.$

($\because f$ is real)

$|f|^2 = \langle f|f\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} |f(x)|^2 e^{-x^2} dx$

Gaussian integral

$f_i = x^i \Rightarrow$

$E_0 = 1 \qquad e_0(x) = \frac{E_0}{|E_0|} = \frac{f_0}{|f_0|}$

$|E_0|^2 = \langle E_0|E_0\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} (1)^2 e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{=\sqrt{\pi}} = 1$

$\therefore e_0(x) = \boxed{1}$

~~$e_1(x) =$~~ $E_1(x) = f_1 - \langle f_1|e_0\rangle e_0$

$= x - (1)\left[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x e^{-x^2} dx\right]$

$= x - \frac{1}{\sqrt{\pi}}\left[\frac{-1}{2} e^{-x^2}\right]_{-\infty}^{\infty} = x - 0 = x.$

$|E_1|^2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} x\, d(-\frac{1}{2}e^{-x^2})$

$= \frac{1}{\sqrt{\pi}}\left[\underbrace{[-\frac{1}{2} x e^{-x^2}]_{-\infty}^{\infty}}_{0} + \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} e^{-x^2} dx}_{\sqrt{\pi}}\right]. = \frac{1}{2}$

$\therefore |E_1| = \frac{1}{\sqrt{2}}$

$e_1 = \frac{E_1}{|E_1|} = \frac{x}{1/\sqrt{2}} = \boxed{\sqrt{2}\, x}$

$\cancel{E_2 = f_2 - \langle E_2|e_1\rangle e_1 - \langle E_2|e_0\rangle e_0}$

$$E_2 = f_2 - \langle f_2|e_1\rangle e_1 - \langle f_2|e_0\rangle e_0$$

$$= x^2 - \sqrt{2}x\,\langle f_2|e_1\rangle - (1)\langle f_2|e_0\rangle$$

$$\langle f_2|e_1\rangle = \frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \underbrace{x^3 e^{-x^2}}_{\text{odd function}} dx = 0$$

$$\langle f_2|e_0\rangle = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \underbrace{x^2 e^{-x^2} dx}_{\frac{1}{2}\sqrt{\pi}} = \frac{1}{2}$$

$$e_0 = 1$$

$$\therefore \quad E_2 = x^2 - \frac{1}{2}$$

$$|E_2|^2 = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} (x^2 - \tfrac{1}{2})^2 e^{-x^2} dx$$

$$= \frac{1}{\sqrt{\pi}}\left[\int_{-\infty}^{\infty} (x^4 - x^2 + \tfrac{1}{4}) e^{-x^2} dx\right].$$

$\cancel{\int_{-\infty}^{\infty} x^4 e^{-x^2} dx = ?}$

we know $\int x^3 e^{-x^2} dx = \frac{1}{2}\int x^2 e^{-x^2} d(x^2)$ $\quad (y = x^2)$.

$$= \frac{1}{2}\int y e^{-y} dy = \frac{-(y+1)e^{-y}}{2} = -\frac{1}{2}(x^2+1)e^{-x^2}.$$

$$\therefore \int_{-\infty}^{\infty} x^4 e^{-x^2} dx = \int_{-\infty}^{\infty} x\,(x^3 e^{-x^2})\,dx$$

$$= -\frac{1}{2}x(x^2+1)e^{-x^2}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{2}(x^2+1)e^{-x^2} dx.$$

$$= 0 + \frac{1}{2}\left[\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2} + \int_{-\infty}^{\infty} dx\, e^{-x^2}\right].$$

$$= 0 + \frac{1}{2}\left[\cdot \frac{3}{2}\sqrt{\pi}\right] = \frac{3}{4}\sqrt{\pi}.$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

$$\frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}.$$

$$\therefore |E_2|^2 = \frac{1}{\sqrt{\pi}}\left[\cdot \frac{3}{4}\sqrt{\pi} - \frac{1}{2}\sqrt{\pi} + \frac{1}{4}\sqrt{\pi}\right] = \frac{1}{2}$$

$$\therefore |E_2| = \frac{1}{\sqrt{2}}.$$

$$e_2(x) = \frac{E_2}{|E_2|} = \frac{(x^2 - \frac{1}{2})}{1/\sqrt{2}} = \sqrt{2}\,(x^2 - \frac{1}{2})$$

$$= \boxed{\frac{1}{\sqrt{2}}(2x^2 - 1)}$$

$\because$ $e_0(x), e_1(x), e_2(x) \cdots$ form an orthonormal basis

$\therefore$ Any function ~~f(x)~~ $g(x)$ can be expressed as a linear combination of this basis

let $g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n e_n(x)$

and let $f(x) = g(x)\, e^{-\frac{1}{2}x^2}$ $\left(g(x) = f(x)\, e^{\frac{1}{2}x^2} \right)$,

$\because$ g is arbitrary $\therefore$ f is arbitrary too

~~then~~

$\therefore$ any function f can be expressed as

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n e_n(x)\, e^{-\frac{1}{2}x^2}$$

$$\therefore\ g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n e_n(x)$$

$$\therefore\ \langle e_k | g \rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} a_n \underbrace{\langle e_k | e_n \rangle}_{\delta_{kn}} = \frac{1}{\sqrt{2\pi}} a_k$$

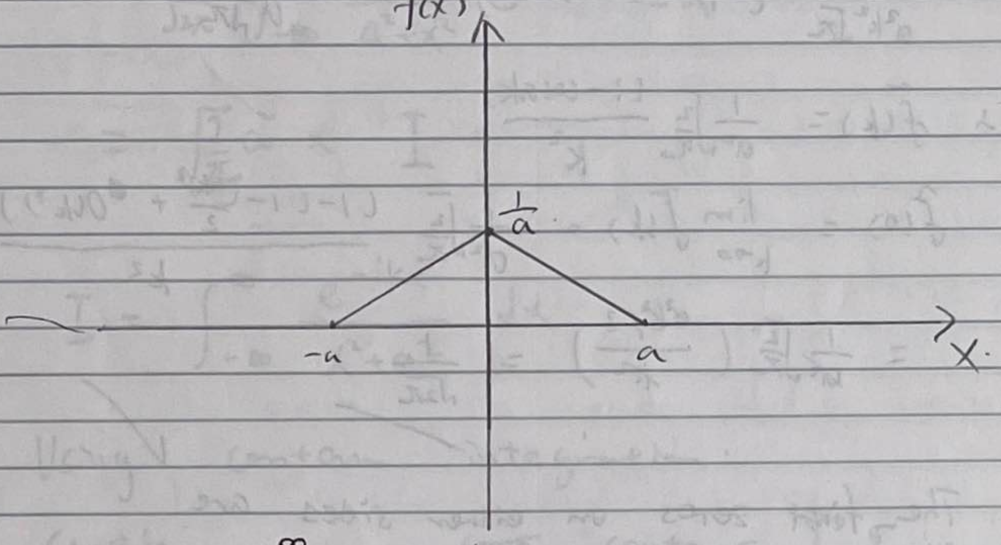
$$\therefore\ a_k = \sqrt{2\pi}\, \langle e_k | g \rangle = (\sqrt{2\pi})\left(\frac{1}{\sqrt{\pi}}\right) \int_{-\infty}^{\infty} dx\, e^{-x^2} e_k(x)\, g(x)$$

$$= \sqrt{2} \int_{-\infty}^{\infty} dx\, e^{-x^2} e_k(x) f(x)\, e^{\frac{1}{2}x^2}$$

$$= \boxed{\sqrt{2} \int_{-\infty}^{\infty} dx\, f(x)\, e_k(x)\, e^{-\frac{1}{2}x^2}}$$

7.

$$f(x) = \begin{cases} (a-x)/a^2 , & 0 \le x \le a \\ (a+x)/a^2 , & -a \le x \le 0 \\ 0 , & \text{otherwise} \end{cases}$$



$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \frac{1}{a^2\sqrt{2\pi}} \left[\int_{-a}^{0} (a+x) e^{-ikx} dx + \int_{0}^{a} (a-x) e^{-ikx} dx \right]$$

$$= \frac{1}{a^2\sqrt{2\pi}} \left[-\frac{a}{ik} e^{-ikx} - \frac{x}{ik} e^{-ikx} + \frac{1}{k^2} e^{-ikx} \right]_{-a}^{0}$$

$$+ \frac{1}{a^2\sqrt{2\pi}} \left[-\frac{a}{ik} e^{-ikx} + \frac{x}{ik} e^{-ikx} - \frac{1}{k^2} e^{-ikx} \right]_{0}^{a}$$

$$= \frac{1}{a^2\sqrt{2\pi}} \left[\cancel{-\frac{a}{ik}} + \frac{1}{k^2} + \cancel{\frac{a}{ik} e^{ika}} - \cancel{\frac{a}{ik} e^{ika}} - \frac{1}{k^2} e^{ika} \right.$$

$$\left. - \cancel{\frac{a}{ik} e^{-ika}} + \cancel{\frac{a}{ik} e^{-ika}} - \frac{1}{k^2} e^{-ika} + \cancel{\frac{a}{ik}} + \frac{1}{k^2} \right]$$

$$\cancel{= \frac{1}{a^2k^2\sqrt{2\pi}} \left[2a e^{ika} \right. \quad \frac{1}{a^2k^2\sqrt{2\pi}} \left[\frac{2}{k^2} \right.}$$

$$= \frac{2}{a^2k^2\sqrt{2\pi}} \cancel{\left[1 - \right.}$$

$$= \frac{1}{a^2\sqrt{2\pi}} \left(\frac{2}{k^2} - \frac{1}{k^2} \left(e^{ika} + e^{-ika} \right) \right)$$

$$= \frac{1}{a^2\sqrt{2\pi}}\left[\frac{2}{k^2} - \frac{2}{k^2}\cos ka\right]$$

$$= \frac{\sqrt{2}}{a^2k^2\sqrt{\pi}}(1-\cos ka) = \frac{\sqrt{2}}{a^2\sqrt{\pi}} \frac{(1-\cos ka)}{k^2}$$

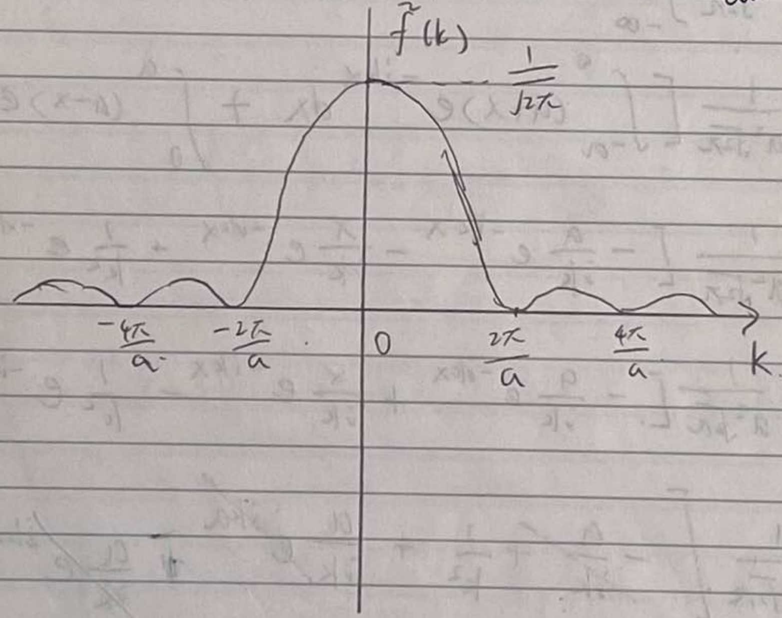
$$\therefore \tilde{f}(k) = \frac{1}{a^2}\sqrt{\frac{2}{\pi}}\frac{(1-\cos ak)}{k^2}$$

(a) $$\tilde{f}(0) = \lim_{k\to 0}\tilde{f}(k) = \frac{1}{a^2}\sqrt{\frac{2}{\pi}}\frac{(1-(1-\frac{(ak)^2}{2}+O(k^4)))}{k^2}$$

$$= \frac{1}{a^2}\sqrt{\frac{2}{\pi}}\left(\frac{a^2k^2/2}{k^2}\right) = \frac{1}{\sqrt{2\pi}}$$

The first zeros on either sides are

when $\cos(ka) = 1$ but $k \neq 0$ $\therefore$ $ak = \pm 2\pi \Rightarrow k = \pm\frac{2\pi}{a}$



Total area under $f(x)$ is $[a-(-a)] \times (\frac{1}{a}) \times (\frac{1}{2}) = 1$

$$\therefore \cancel{\tilde{f}(0)}\ \tilde{f}(0) = \frac{1}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty} f(x)\,dx}_{\text{area under } f(x) = 1} = \frac{1}{\sqrt{2\pi}}$$

$$\therefore \tilde{f}(0) = \frac{1}{\sqrt{2\pi}} \times \text{total area under } f(x)$$

8. (a)

$$\check{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

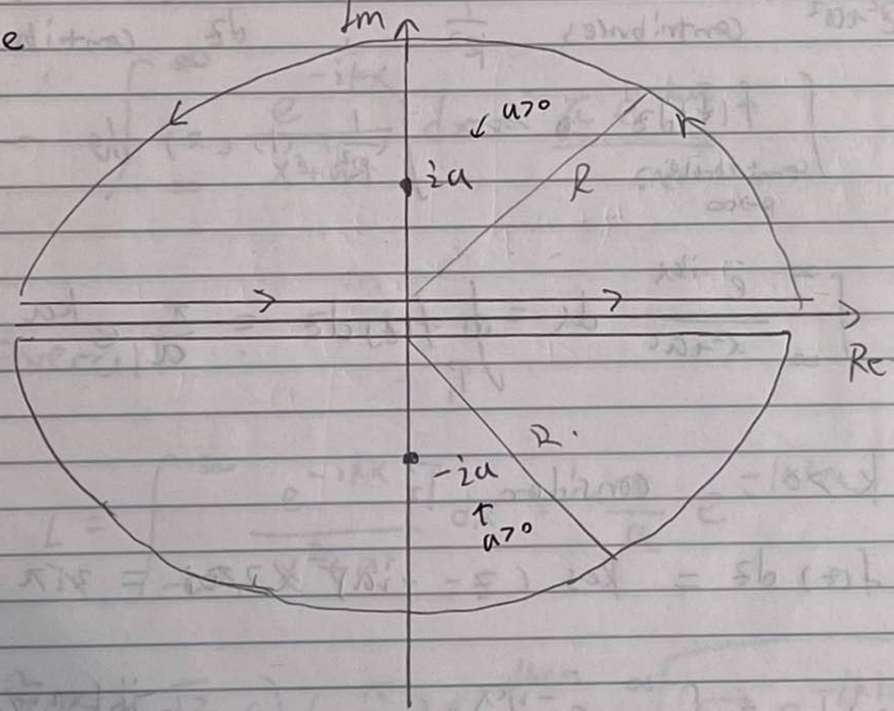
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2a}{a^2+x^2}\, e^{-ikx}\, dx$$

$$= \sqrt{\frac{2}{\pi}}\, a \times I$$

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx$$

Using contour integration:

Consider the ~~com~~ contour Γ_1, Γ_2 in the complex plane



~~If $k<0$, consider Γ_1~~

$$f(z) = \frac{e^{-ikz}}{z^2+a^2} = \cancel{\frac{e^{-ikz}}{(z+ia)(z-ia)}} \quad \frac{e^{-ikz}}{(z+ia)(z-ia)}$$

poles at $z = ia$, $z = -ia$

$$\text{Res}(z=ia) = \frac{e^{-ik(ia)}}{2ia} = \frac{e^{ka}}{2ia}.$$

$$\text{Res}(z=-ia) = \frac{e^{-ik(-ia)}}{\cancel{2ia}\; 2(-ia)} = \frac{e^{-ka}}{-2ia}$$

If $k \leq 0$, consider Γ_1

$$\oint_{\Gamma_1} f(z)\,dz = \underbrace{\mathrm{Res}\,(z = ia)}_{\text{residue theorem}} \times 2\pi i = 2\pi i \times \frac{e^{ka}}{2ia} = \frac{\pi e^{ka}}{a}$$

$$\oint_{\Gamma_1} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\,dx + \int_{\substack{\text{semicircle}\\ R\to\infty}} \frac{e^{-ik(x+iy)}}{z^2+a^2}\,\underbrace{iRe^{i\theta}\,d\theta}_{\substack{dz\\ (z=Re^{i\theta})}}$$

$$|e^{-ik(x+iy)}| = |e^{ky}e^{-ikx}| = e^{ky}$$

upper half plane $\Rightarrow$ $y > 0$

$\because k \leq 0$, $y > 0$ $\therefore ky \leq 0$ $\therefore e^{ky} \leq 1$

z^2+a^2 contributes $\frac{1}{R^2}$, dz contributes R

$$\therefore \int_{\substack{\text{semicircle}\\ R\to\infty}} f(z)\,dz \sim \int \left(\frac{1}{R^2}\right)\cdot(1)(R)\,d\theta \sim \int \frac{1}{R}\,d\theta \to 0$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\,dx = \oint_{\Gamma_1} f(z)\,dz = \frac{\pi}{a}e^{ka}$$

if $k > 0$, consider Γ_2

$$\oint_{\Gamma_2} f(z)\,dz = \mathrm{Res}\,(z = -ia) \times 2\pi i = 2i\pi \times \frac{e^{-ka}}{-2ia} = \frac{-\pi e^{-ka}}{a}$$

$$\oint_{\Gamma_2} f(z)\,dz = \int_{\infty}^{-\infty} \frac{e^{-ikx}}{x^2+a^2}\,dx + \int_{\substack{\text{semicircle}\\ R\to\infty}} \frac{e^{-ik(x+iy)}}{z^2+a^2}\,iRe^{i\theta}\,d\theta$$

$$\frac{1}{z^2+a^2} \Rightarrow \frac{1}{R^2}, \quad dz \Rightarrow R$$

$$|e^{-ik(x+iy)}| = |e^{ky}e^{-ikx}| = e^{ky}$$

$\therefore k>0$, $y<0$ (lower half plane)

$\therefore$ $e^{ky} < 1$ $\therefore \Rightarrow 1$

$$\therefore \int_{\substack{\text{semicircle} \\ R\to\infty}} \cancel{\frac{e^{-ikx}}{x^2+a^2}} f(z)\, dz \sim \int \left(\frac{1}{R^2}\right)(1)(R)$$

$$\sim \int 0 \to 0 \text{ as } R\to\infty$$

$$\therefore \oint_{\pm\infty} \int_{\infty}^{-\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx = -\frac{\pi e^{-ka}}{a}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx = \frac{\pi e^{-ka}}{a}$$

Overall :

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+a^2}\, dx = \frac{\pi}{a} e^{-|k|a}$$

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}}\, a \left(\frac{\pi}{a}\right) e^{-|k|a} = \boxed{\sqrt{2\pi}\, e^{-|k|a}}$$

(b)

$$f(x) = \exp(-a|x|)$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a|x| + ikx)} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+ik)x} \, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a-ik) \cdot x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{a+ik}\right) \left[e^{-(a+ik)} \right] \Big|_{0}^{\infty}$$

$$+ \frac{1}{\sqrt{2\pi}} \frac{1}{a-ik} \left[e^{(a-ik)} \right] \Big|_{-\infty}^{0}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+ik} + \frac{1}{a-ik} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+k^2} = \boxed{\sqrt{\frac{2}{\pi}} \frac{a}{a^2+k^2}}$$ ✓

(c)

$$f(x) = \frac{1}{\cosh(x)}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\, dx.$$

Consider $f(z) = \dfrac{e^{-ikz}}{\cosh(z)}$

poles at $\cosh(z) = 0$

$\Rightarrow \cosh(z) = 0 \;\; \cancel{\Rightarrow e^{z} + e^{-z}}$

$\Rightarrow \dfrac{e^{z} + e^{-z}}{2} = 0 \;\Rightarrow\; e^{z} + e^{-z} = 0 \qquad (w = e^{z})$

$\Rightarrow w + \dfrac{1}{w} = 0 \;\Rightarrow\; w^2 + 1 = 0$

$\Rightarrow w = i$ or $-i \;\Rightarrow\; e^{z} = e^{i(\pi/2 + 2n\pi)}$

or $e^{z} = e^{i(-\pi/2 + 2n\pi)}$

$\therefore \; z = i(\frac{\pi}{2} + 2n\pi)$ or $z = i(-\frac{\pi}{2} + 2n\pi)$

$\therefore z = \; \dots \; -\frac{9}{2}\pi i, -\frac{7}{2}\pi i, -\frac{5}{2}\pi i, -\frac{3}{2}\pi i, -\frac{1}{2}\pi i, \frac{1}{2}\pi i, \frac{3}{2}\pi i, \frac{5}{2}\pi i, \frac{7}{2}\pi i, \frac{9}{2}\pi i \dots$

$\Rightarrow \cancel{z = \frac{2k+1}{2}\pi} \quad \therefore \; z = (n + \frac{1}{2})\pi i$

$$f(z) = \frac{e^{-ikz}}{\cosh(z)} = \cancel{\frac{1 - ikz + \frac{(-ikz)^2}{2!} + \frac{(-ikz)^3}{3!} + \dots}{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots}}$$

has a simple pole at $z = \frac{\pi}{2} i$

we evaluate the residue

$$\mathrm{Res}\left(z=\frac{\pi i}{2}\right) = \lim_{z\to\infty}\left(z-\frac{\pi i}{2}\right)\frac{e^{-ikz}}{\cosh(z)}$$

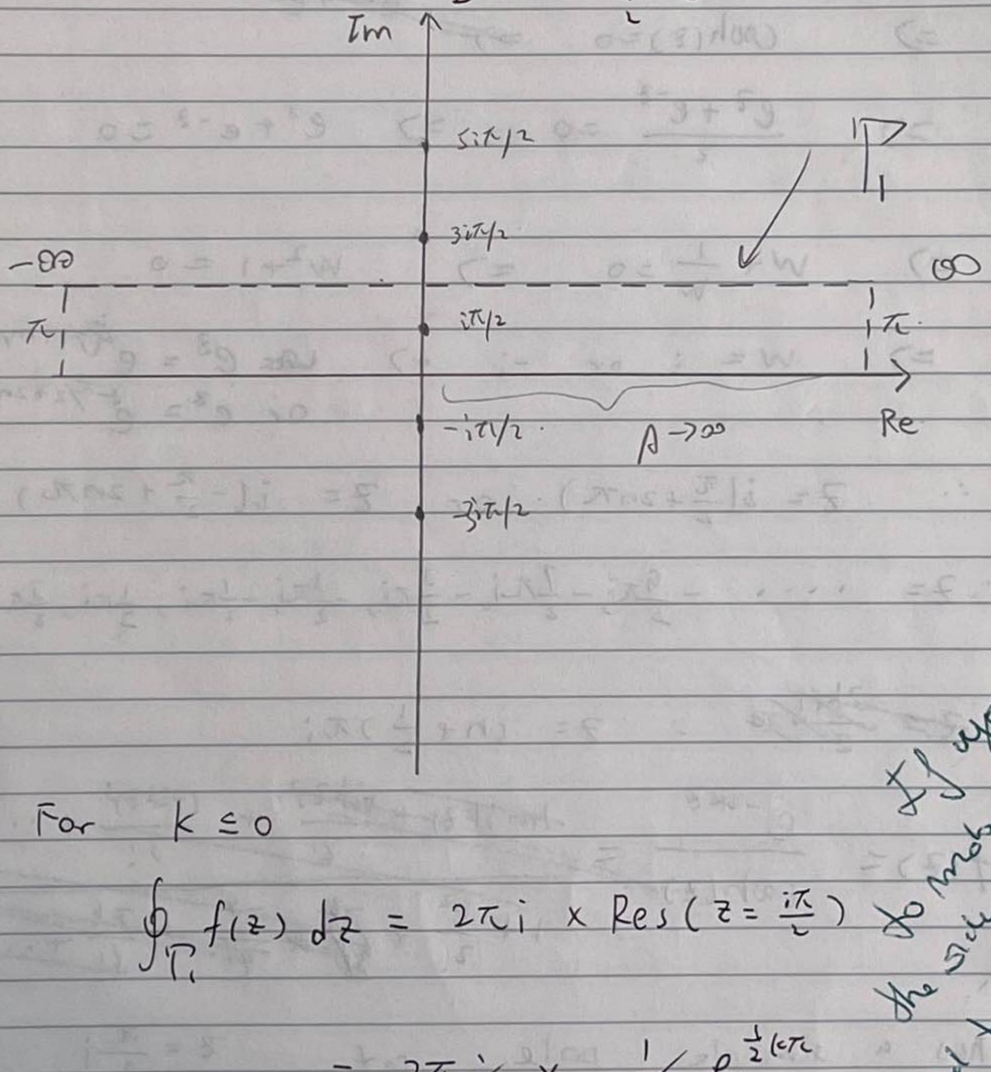
$$= \cancel{e^{\frac{1}{2}k\pi}}\; e^{\frac{1}{2}k\pi}\lim_{z\to\frac{i\pi}{2}}\frac{z-\frac{\pi i}{2}}{\cosh(z)}$$

$$= e^{\frac{1}{2}k\pi}\lim_{z\to\frac{i\pi}{2}}\frac{1}{\sinh(z)} = \frac{e^{\frac{1}{2}k\pi}}{\sinh\left(\frac{i\pi}{2}\right)}$$

L'hopital

$$\sinh\left(\frac{i\pi}{2}\right) = \frac{e^{i\pi/2}-e^{-i\pi/2}}{2} = \frac{i-(-i)}{2} = i$$

$$\therefore\ \mathrm{Res}\left(z=\frac{\pi i}{2}\right) = \frac{1}{i}e^{\frac{1}{2}k\pi}$$



For $k \le 0$

$$\oint_{\Gamma_1} f(z)\,dz = 2\pi i \times \mathrm{Res}\left(z=\frac{i\pi}{2}\right)$$

$$= 2\pi\cancel{i} \times \frac{1}{\cancel{i}}e^{\frac{1}{2}k\pi}$$

$$= 2\pi e^{\frac{1}{2}k\pi}.$$

If you take the limit to make integrals on the sides vanish, don't you need to add over all residues? Not just the first one?

$$\oint_{\Gamma_1} f(z)\,dz = \overbrace{\int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx}^{①} + \overbrace{\int_{\infty+i\pi}^{-\infty+i\pi} \frac{e^{-ikz}}{\cosh(z)}\,dz}^{②}$$

$$+ \underbrace{\int_{\infty}^{\infty+i\pi} f(z)\,dz}_{③} + \underbrace{\int_{0}^{-\infty+i\pi} f(z)\,dz.}_{④}$$

③ $\Rightarrow$ $\cosh(z) = \frac{1}{2}(e^{z} + e^{-z})$

$|e^{z}| = |e^{x+iy}| = |e^{x}e^{iy}| = e^{x} \to \infty$ as $x \to \infty$

④ $\therefore\ |\cosh(z)| \to 0$ as $x \to \infty$

$\therefore$ ③ $\to 0$, similarly ④ $\to 0$

② $\Rightarrow$
$$\lim_{A\to\infty}\int_{A+i\pi}^{-A+i\pi} \frac{e^{-ikz}}{\cosh(z)}\,dz = \int_{z=A+i\pi}^{z=-A+i\pi} \frac{e^{-ik(x+i\pi)}}{\cosh(x+i\pi)}\,dx.$$

$$\underbrace{z = x+i\pi \quad dz = dx.}$$

$$= \cancel{\int_{A+i\pi}^{-A+i\pi}} \int_{x=A}^{x=-A} \frac{e^{-i\pi k}e^{-ikx}}{\frac{e^{x}e^{i\pi} + e^{-x}e^{-i\pi}}{2}}\,dx$$

$(e^{i\pi} = e^{-i\pi} = -1)$

$$= e^{k\pi}\int_{\infty}^{-\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx = e^{k\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{\cosh(x)}\,dx.$$

$$= e^{k\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{\cosh(x)}\,dx.$$

$$\cancel{\therefore\ 2\pi e^{\frac{1}{2}k\pi} = \oint_{\Gamma_1} f(z)\,dz = 2\int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx.}$$

$$\cancel{\therefore \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx = \pi e^{\frac{1}{2}k\pi}}$$

$$2\pi e^{\frac{1}{2}k\pi} = \int_{\Gamma_1} f(z)\,dz = \left(1+e^{k\pi}\right)\int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-ikx}}{\cosh(x)}\,dx = \frac{2\pi e^{\frac{1}{2}k\pi}}{1+e^{k\pi}}$$

$$= \frac{e^{\frac{1}{2}k\pi}}{e^{\frac{1}{2}k\pi}} \frac{\pi}{\frac{e^{\frac{1}{2}k\pi}+e^{-\frac{1}{2}k\pi}}{2}} = \boxed{\frac{\pi}{\cosh(\frac{1}{2}k\pi)}}$$

$$\therefore \boxed{\hat{f}(k) = \frac{\sqrt{\pi/2}}{\cosh(\frac{1}{2}k\pi)}}$$

9.

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$$

(a) $\underline{\widetilde{f(ax)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{-ikx} \, dx$

$\left(\begin{array}{l} y = ax \\ x = y/a \\ dy = a\,dx \\ dx = \frac{1}{a} dy. \end{array} \right)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky/a} \frac{1}{a} dy$$

$$= \frac{1}{a} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-i\left(\frac{k}{a}\right) y} dy \right.$$

$$= \underline{\frac{1}{a} \tilde{f}\left(\frac{k}{a}\right)}$$

(b) $\underline{\widetilde{f(a+x)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(a+x) e^{-ikx} \, dx.$

$\left(\begin{array}{l} y = a+x \\ dy = dx \\ x = y-a \end{array} \right)$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-ik(y-a)} dy$$

$$= e^{ika} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y) e^{-iky} dy \right]$$

$$= \underline{e^{ika} \tilde{f}(k)}$$

(c) $\underline{\widetilde{e^{iqx} f(x)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} f(x) e^{-ikx} \, dx.$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(k-q)x} dx$$

$$= \underline{\tilde{f}(k-q)}$$

d) $\widetilde{\left(\frac{df}{dx}\right)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{df}{dx} e^{-ikx} \, dx.$

$= \cancel{\frac{1}{\sqrt{2\pi}} \int_{-\infty}} = \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} e^{-ikx} \, d(f(x)) \right\}$

$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\left[e^{-ikx} f(x) \right]_{-\infty}^{\infty}}_{\to 0} - \underbrace{\int_{-\infty}^{\infty} (-ik) e^{-ikx} f(x) \, dx.}_{-ik\tilde{f}(k)} \right\}$

→ 0 Assuming $f(x)$ converges ~~diverges~~ at $\pm\infty$

(otherwise $\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$ would diverge)

$= ik\tilde{f}(k)$

(e) $\widetilde{xf(x)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x f(x) e^{-ikx} \, dx$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{dk} (e^{-ikx}) \left(\frac{1}{-i}\right) dx$

$\nearrow i$

$= i \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{dk} (e^{-ikx}) \, dx \right]$

$= i \left[\frac{1}{\sqrt{2\pi}} \cancel{\frac{d}{dk}} \int_{-\infty}^{\infty} \frac{\partial}{\partial k} (f(x) e^{-ikx}) \, dx \right.$

$= i \frac{d}{dk} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx \right].$

$= i \frac{d\tilde{f}(k)}{dk}$

10.

$$\widetilde{\frac{d^2\Phi}{dx^2}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{d^2\Phi}{dx^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{\infty} e^{-ikx} d\left(\frac{d\Phi}{dx}\right). \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{-ikx} \frac{d\Phi}{dx} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \cancel{e^{-ikx}} \frac{d\Phi}{dx} d(e^{-ikx}) \right\}$$

$\rightarrow 0$ as $\frac{d\Phi}{dx}$ must converge at $\pm\infty$

$$= \frac{1}{\sqrt{2\pi}} \left\{ ik \int_{-\infty}^{\infty} e^{-ikx} \frac{d\Phi}{dx} dx. \right\}$$

$$= ik \underbrace{\widetilde{\frac{d\Phi}{dx}}}_{\text{refer to Q9}} = (ik)(ik\tilde{\Phi}) = -k^2\tilde{\Phi}(k)$$

$$\widetilde{k^2\Phi} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k^2 \Phi e^{-ikx} dx. = k^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(x) e^{-ikx} dx$$

$$= k^2 \tilde{\Phi}(k)$$

$$\widetilde{f(x)} = \tilde{f}(k)$$

Taking the fourier transform of both sides of the differential equation

$$\frac{d^2\Phi(x)}{dx^2} - k^2\Phi(x) = f(x) \quad \text{gives, ..}$$

$$-k^2 \tilde{\Phi}(k) - K^2 \tilde{\Phi}(k) = \tilde{f}(k)$$

$$\therefore \tilde{\Phi}(k) = -\frac{\tilde{f}(k)}{k^2 + K^2}$$

$$\Phi(x) = \widetilde{\tilde{\Phi}(k)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{\Phi}(k)\, dk$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} \frac{\tilde{f}(k)}{k^2 + K^2}$$

11 ~~(b)~~

$$\hat{I}(\theta) = \left|\tilde{f}(k\sin\theta)\right|^2$$

(a)

$$f(x) = \begin{cases} 1 & , |x| < b/2 \\ 0 & , \text{otherwise} \end{cases}$$

$$\tilde{f}_{(a)}(k\sin\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx\sin\theta} f(x)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-b/2}^{b/2} dx\, e^{-ik\sin\theta x} = \frac{1/\sqrt{2\pi}}{-ik\sin\theta}\left[e^{-ik\sin\theta x}\right]_{-b/2}^{b/2}$$

$$= \frac{e^{i\frac{1}{2}kb\sin\theta} - e^{-i\frac{1}{2}kb\sin\theta}}{ik\sin\theta}$$

$$= \frac{2i\sin(\frac{1}{2}kb\sin\theta)}{\sqrt{2\pi}\, ik\sin\theta} = \frac{2\sin(\frac{1}{2}kb\sin\theta)}{\sqrt{2\pi}\, k\sin\theta}$$

$$= \frac{b}{\sqrt{2\pi}}\frac{\sin(\frac{1}{2}kb\sin\theta)}{\frac{1}{2}kb\sin\theta} = \underline{b\,\mathrm{sinc}(\tfrac{1}{2}kb\sin\theta)}/\sqrt{2\pi}$$ ✓

$$\underline{\hat{I}(\theta)_{(a)} = b^2\,\mathrm{sinc}^2(\tfrac{1}{2}kb\sin\theta)/2\pi} = \underline{\frac{2}{\pi}\left(\frac{\sin(\frac{1}{2}kb\sin\theta)}{k\sin\theta}\right)^2}$$

(b) $f(x) = \delta(x + a/2) + \delta(x - a/2)$

$$\tilde{f}_{(b)}(k\sin\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx\sin\theta} f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-ikx\sin\theta}\left(\delta(x+\tfrac{a}{2}) + \delta(x-\tfrac{a}{2})\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left\{e^{i\frac{1}{2}ka\sin\theta} + e^{-i\frac{1}{2}ka\sin\theta}\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\cos(\tfrac{1}{2}ka\sin\theta)$$

$$\therefore \underline{\hat{I}(\theta)_{(b)} = \frac{2}{\pi}\cos^2(\tfrac{1}{2}ka\sin\theta)}$$

(c) $\cancel{\tilde{f}(k\sin\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx\sin\theta} f(x)}$

$$f(x) = \begin{cases} 1, & \frac{a-b}{2} \le x \le \frac{a+b}{2}. \\ 1, & -\frac{a+b}{2} \le x \le -\frac{a-b}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{f}_{(c)}(k\sin\theta) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx\sin\theta} f(x)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\frac{a+b}{2}}^{-\frac{a-b}{2}} e^{-ikx\sin\theta}\,dx + \int_{\frac{a-b}{2}}^{\frac{a+b}{2}} e^{-ikx\sin\theta}\,dx\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\frac{1}{-ik\sin\theta}\exp(-ikx\sin\theta)\Big|_{-\frac{a+b}{2}}^{-\frac{(a-b)}{2}} + \frac{1}{-ik\sin\theta}\exp(-ikx\sin\theta)\Big|_{\frac{a-b}{2}}^{\frac{a+b}{2}}\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-ik\sin\theta}\right)\Big[\exp(ik\sin\theta\,\tfrac{a-b}{2}) - \exp(ik\sin\theta\,\tfrac{a+b}{2})$$

$$+ \exp(-ik\sin\theta\,\tfrac{a+b}{2}) - \exp(-ik\sin\theta\,\tfrac{a-b}{2})\Big].$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-ik\sin\theta}\right)\Big[\big(\exp(ik\sin\theta\,\tfrac{a-b}{2}) - \exp(-ik\sin\theta\,\tfrac{a-b}{2})\big)$$

$$- \big(\exp(ik\sin\theta\,\tfrac{a+b}{2}) - \exp(-ik\sin\theta\,\tfrac{a+b}{2})\big)\Big]$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{-ik\sin\theta}\right)\left[2i\right]\left[\sin(\tfrac{1}{2}k\sin\theta(a-b)) - \sin(\tfrac{1}{2}k\sin\theta(a+b))\right]$$

$$= \frac{2}{\sqrt{2\pi}}\frac{1}{k\sin\theta}\left[\sin(\tfrac{1}{2}k\sin\theta(a+b)) - \sin(\tfrac{1}{2}k\sin\theta(a-b))\right].$$

$$\sin(A+B) + \sin(A-B)$$

$$= \sin A \cos B + \cancel{\cos A \sin B} + \sin A \cos B - \cancel{\cos A \sin B}$$

$$= 2 \sin A \cos B$$

$$\therefore \quad \tilde{f}_{(c)}(k\sin\theta) = \frac{2}{\sqrt{2\pi}} \frac{1}{k\sin\theta} \left[2 \sin\left(\tfrac{1}{2} k\sin\theta\, a\right) \cos\left(\tfrac{1}{2} k\sin\theta\, b\right) \right]$$

$$\cancel{= \frac{4}{\sqrt{2\pi}} \left(\frac{1}{k\sin\theta}\right) \sin\left(\tfrac{1}{2} k\sin a\right.}$$

$$= \frac{4}{\sqrt{2\pi}} \frac{\sin\left(\tfrac{1}{2} ka\sin\theta\right) \cos\left(\tfrac{1}{2} kb\sin\theta\right)}{k\sin\theta}$$

$$\therefore \quad \hat{I}_c(\theta) = \frac{8}{\pi} \left(\frac{\sin\left(\tfrac{1}{2} ka\sin\theta\right) \cos\left(\tfrac{1}{2} kb\sin\theta\right)}{k\sin\theta} \right)^2$$

Notice that $\hat{I}_c(\theta) = 2\pi\, \hat{I}_{(a)}(\theta)\, \hat{I}_{(b)}(\theta)$

Convolution of $f_{(a)}(x)$ and $f_{(b)}(x)$ is

$$f_{(a)} * f_{(b)}(x) = \int_{-\infty}^{\infty} f_{(b)}(x-x') f_{(a)}(x')\, dx'$$

$$= \int_{-\infty}^{\infty} \left(\delta(x-x'+a/2) + \delta(x-x'-\tfrac{a}{2}) \right) f_{(a)}(x')\, dx'$$

$$= f_{(a)}(x+\tfrac{a}{2}) + f_{(a)}(x-\tfrac{a}{2}) = \underbrace{f_{(c)}(x)}_{\text{clear from diagram}}$$

$$\therefore \quad f_{(c)}(x) = f_{(a)}(x) * f_{(b)}(x)$$

by convolution theorem $\therefore$

$$\cancel{f_{(b)}} \; \tilde{f}_{(c)}(k\sin\theta) = \sqrt{2\pi}\, \tilde{f}_{(a)}(k\sin\theta)\, \tilde{f}_{(b)}(k\sin\theta)$$

$$\therefore \Rightarrow \quad \underline{\hat{I}_{c}(\theta) = 2\pi\, \hat{I}_{(a)}(\theta)\, \hat{I}_{(b)}(\theta)}$$

Fourier

'1 → piecewise continuous

2 → square integrable

$\delta(x-x')$

Q6:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_n a_n e_n(x) e^{-\frac{1}{2}x^2}$$

$$\sqrt{2\pi} f(x) e^{\frac{1}{2}x^2} = \sum_n a_n e_n(x)$$

$$\int (e_m^*(x) w(x)\, dx) (\sqrt{2\pi} f(x) e^{\frac{1}{2}x^2}) = \int \sum_n \frac{e_m^*(x) w(x)}{a_n e_n(x)} dx$$

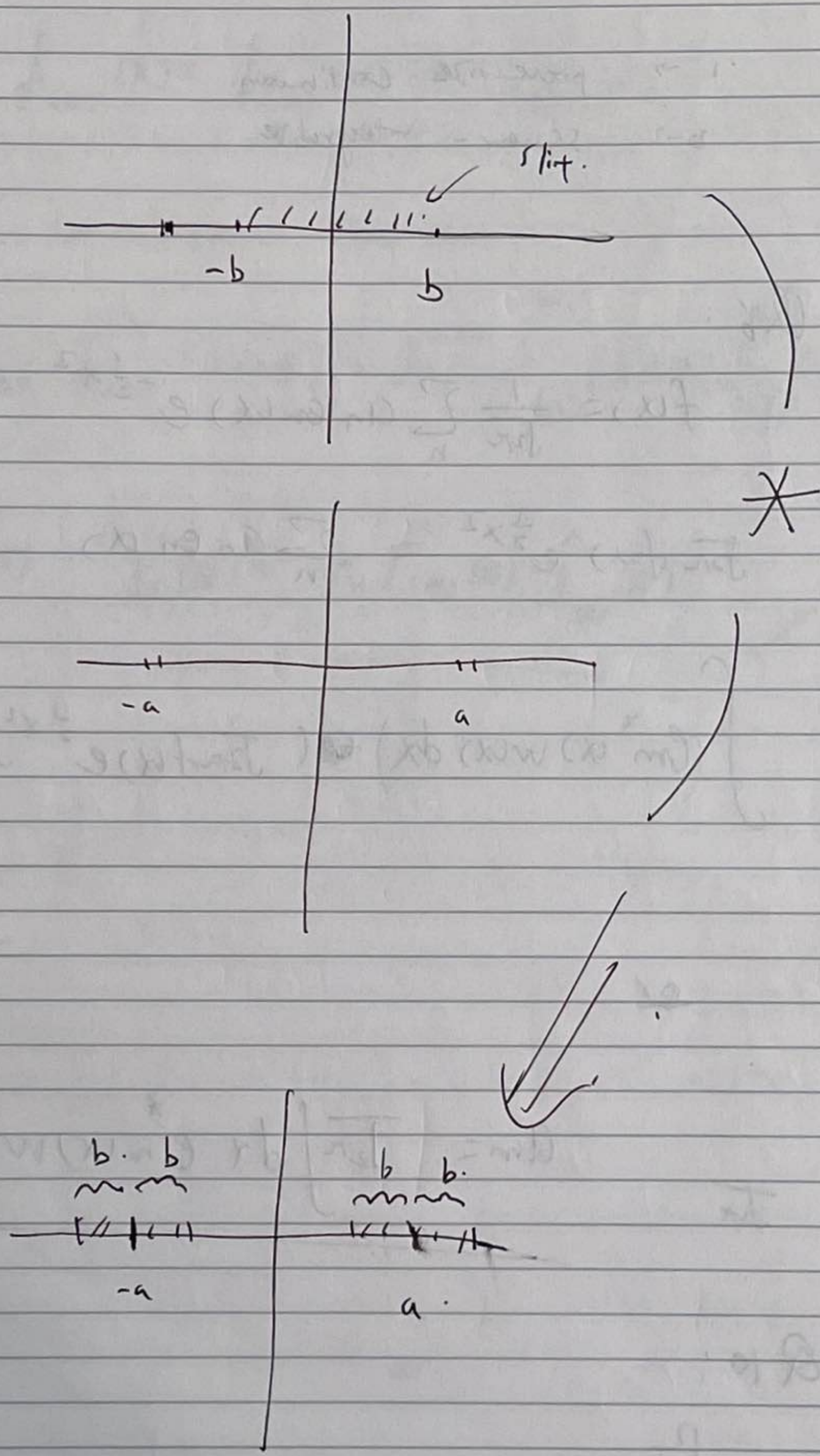
$$= \sum_n a_n \delta_{nm} = a_m.$$

$$a_m = \sqrt{2\pi} \int dx\, e_m^*(x) w(x) f(x) e^{+\frac{1}{2}x^2}.$$

$\sqrt{2\pi}$

Q10:

$$FT \begin{pmatrix} \frac{d^2\Phi}{dx^2} \\ -k^2\Phi \\ f(x) \end{pmatrix} = \begin{pmatrix} -k^2\Phi(k) \\ \\ \\ \end{pmatrix}$$



$$f_3(x) = f_1(x+\tfrac{a}{2}) + f_1(x-\tfrac{a}{2})$$

$$\hat{f}_3(k) = e^{ika/2}\hat{f}_1(k) + e^{-ika/2}\hat{f}_1(k)$$

$$= (e^{ika/2} + e^{-ika/2})\hat{f}_1(k)$$

$$= 2\cos(\tfrac{1}{2}ka)\hat{f}_1(k).$$

$$e^{-ikz} = e^{-ik(x+iy)}$$

$$\delta(x-x') = \int_{-\infty}^{\infty} e^{-ik(x-x')}\, dk$$

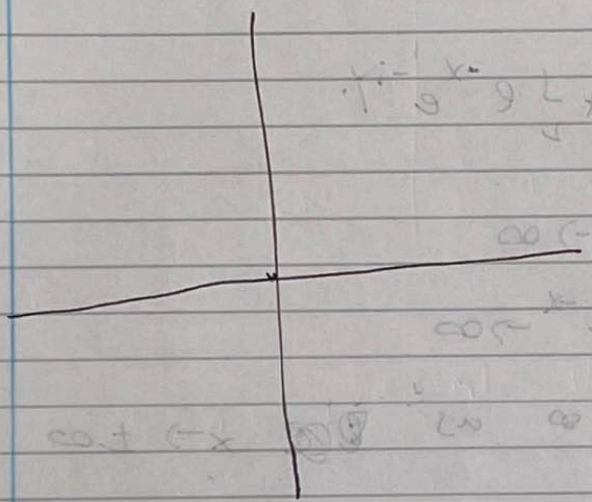
e^{y}

$$\delta(x) = \int_{-\infty}^{\infty} e^{-ikx}\, dk$$

$\delta(x)$

y

$$e^{-ikz} = e$$



$$\frac{e^{-ikz}}{\cancel{\cos}\, e^{z}}$$

$$= e^{x} e^{iy}$$

For 8 (c)

Integral along the sides vanish because

$$|e^{-ikz}| = |e^{-ik(x+iy)}| = |e^{-ikx} e^{y}| = e^{y}$$

y is from 0 to π. $\therefore e^{y}$ is finite between $y = [0, \pi]$

$$\cosh(z) = \frac{1}{2}(e^{z} + e^{-z})$$

$$= \frac{1}{2} e^{x} e^{iy} + \frac{1}{2} e^{-x} e^{-iy}.$$

as $x \to \infty$ $\quad e^{x} \to \infty$

as $x \to -\infty$ $\quad e^{-x} \to \infty$

$\therefore$ $|\cosh(z)| \to \infty$ as $x \to \pm\infty$

$\therefore$ $\left| \frac{e^{-ikz}}{\cosh(z)} \right| \to 0$ as $x \to \pm\infty$.

$\therefore$ Integral vanish.