To: Michael Barnes

Mathematical Method 2 Ziyan Li

1)

(a) $b_j = A_{ji} a_i \Rightarrow \boxed{\underline{b} = A\underline{a}}$

(b) $b_t = A^*_{kt} a_k = (A^\dagger)_{tk} a_k \Rightarrow \boxed{\underline{b} = A^\dagger \underline{a}}$

(c) $b_j A_{jk} b_k = \boxed{\underline{b}^T A \underline{b}}$

(d) $C_{ij} = A_{kj} B_{kj} = A^T_{ik} B_{kj} \Rightarrow \boxed{C = A^T B}$

(e) $F_{ij} = B_{jk} A_{ki} = B^T_{kj} A^T_{ik} = A^T_{ik} B^T_{kj}$

$\Rightarrow \boxed{F = A^T B^T}$

(f) $A_{ij} \delta_{jk} B_{ki} = A_{ij} B_{ji} = (AB)_{ii} = \boxed{tr(AB)}$

2)

$$AB = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = [0]$$

$\because \det(AB) = \det(A)\det(B)$

$\Rightarrow \because \det(AB) = \det([0]) = 0$

$\therefore \det(A)\det(B) = 0$

$\therefore \det(A) = 0$ or $\det(B) = 0$

$\therefore$ A or B is singular.

3) if $[A, B] = 0$ then $AB = BA$

$$(A+B)^n = A^n + A^{n-1}B + \underbrace{A^{n-2}BA + A^{n-3}BA^2 + \dots + BA^{n-1}}_{n^{th}}$$

$$+ A^{n-2}B^2 + \underbrace{\cancel{A} + \dots + B^2A^{n-2}}_{\binom{n}{2}^{th}} + \dots\dots + B^2$$

~~if $AB = BA$, then $A^m B^p A^q$ ... A~~

~~then $A^m B^p A^{n-m-p}$ = $A^m (BA)$~~

~~= A^{n-m-p}~~

~~then $A^m B^p A^{n-m-p} = A^m B^p A^p A^{n-m-2p}$~~

~~Consider the matrix product~~

~~AA ... ABAAB~~

if $AB = BA$, then any product of m A's and $n-m$ B's can be transformed to $A^m B^{n-m}$

$\therefore$ $(A+B)^n = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i}$. by rules of binomial expansion.

could show...

$$\exp(A)\exp(B) = \left(I + A + \frac{A^2}{2!} + \cdots \right)\left(I + B + \frac{B^2}{2!} \right)$$

$$= I + (A+B) + \frac{1}{2!}\left(A^2 + \frac{2!}{1!1!}AB + B^2 \right)$$

$$+ \frac{1}{3!}\left(A^3 + \frac{3!}{2!1!}A^2B + \frac{3!}{2!1!}AB^2 + B^3 \right)$$

$$+ \frac{1}{4!}\left(A^4 + \frac{4!}{3!1!}A^3B + \frac{4!}{2!2!}A^2B^2 + \frac{4!}{1!3!}AB^3 + B^4 \right)$$

$$\cancel{+ \cdots + \frac{1}{m!}\left(\sum_{m=0}^{n} \binom{n}{m} A^m B^{n-m} + \cdots \right.}$$

$$\cancel{= I + (A+B) + \frac{1}{2!}\left(A^2 + \binom{2}{1} AB + B^2 \right.}$$

$$+ \frac{1}{m!}\sum_{m=0}^{n} \binom{n}{m} A^m B^{n-m} + \cdots$$

$$= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \cdots$$

using the statement proved before

$$= \exp(A+B)$$ QED //

4. σ^x and σ^z are symmetric

~~Q~~ all matrices are hermitian

$$\sigma^x\sigma^y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$\sigma^y\sigma^x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$\therefore [\sigma_x, \sigma_y] = 2i\sigma_z \qquad [\sigma_y, \sigma_x] = -2i\sigma_z$

$$\cancel{\sigma_y, \sigma_z}\ \sigma_y\sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\sigma_z\sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$\therefore [\sigma_y, \sigma_z] = 2i\sigma_x \qquad [\sigma^z, \sigma^y] = \cancel{0} -2i\sigma^x$

$$\cancel{\sigma_z}\ \sigma^z\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\sigma^x\sigma^z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\therefore [\sigma^z, \sigma^x] = 2i\sigma^y \qquad [\sigma^x, \sigma^z] = -2i\sigma^y$

$$\therefore \boxed{[\sigma^a, \sigma^b] = 2\varepsilon_{abc}\sigma^c}$$

$$\text{let } \alpha G = i\alpha\sigma^y = i\alpha\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

$$\cancel{\exp(i\alpha G)}\ \exp(i\alpha\sigma^y) = \exp(\alpha G)$$

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad G^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$\therefore \exp(\alpha G) = I + \alpha G + \frac{1}{2!}\alpha^2 G^2 + \frac{1}{3!}\alpha^3 G^3 + \cdots$$

$$= I + \alpha G - \frac{1}{2!}\alpha^2 I - \frac{1}{3!}\alpha^3 G + \frac{1}{4!}\alpha^4 I + \cdots$$

$$\cancel{= \left(I - \frac{1}{2!}\alpha^2 I + \frac{1}{4!}\alpha^4 \right.}$$

$$= I\left(1 - \frac{1}{2!}\alpha^2 + \frac{1}{4!}\alpha^4 - \cdots\right) + G\left(\alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 - \cdots\right)$$

$$= I\cos\alpha + G\sin\alpha$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

This is a rotation matrix of ~~counter~~ direction clockwise of angle α.

let $H = \alpha\sigma^y$ and $H = \begin{pmatrix} 0 & -\alpha i \\ \alpha i & 0 \end{pmatrix}$ is hermitian

$\therefore H^\dagger = H$, we know $(\exp(A))^\dagger = \left(I + A + \frac{A^2}{2!} + \cdots\right)^\dagger$

$$= I + A^\dagger + \frac{(A^\dagger)^2}{2!} + \cdots = \exp(A^\dagger)$$

$$\therefore \left(\exp(i\alpha\sigma^y)\right)^\dagger = \left(\exp(iH)\right)^\dagger = \exp((iH)^\dagger) = \underbrace{\exp(-iH)}_{H^\dagger = H.}$$

$$\therefore \left(\exp(i\alpha\sigma^y)\right)^\dagger \exp(i\alpha\sigma^y) = \cancel{\exp(\ldots)} \exp(-iH)\exp(iH)$$

$$\cancel{(iH)\ldots} \because [-iH, iH] = 0$$

$$\therefore \left(\exp(i\alpha\sigma^y)\right)^\dagger \exp(i\alpha\sigma^y) = \exp(-iH)\exp(iH)$$

$$= \exp(-iH + iH) = \exp([0]) = I$$

$\therefore \exp(i\alpha\sigma^y)$ is unitary.

5. $\langle v_+|A|v_+\rangle - \langle v_-|A|v_-\rangle + \langle v_+'|A|v_+'\rangle - \langle v_-'|A|v_-'\rangle$

$= (\langle v_1| + \langle v_2|) A (|v_1\rangle + |v_2\rangle) - (\langle v_1| - \langle v_2|) A (|v_1\rangle - |v_2\rangle)$

$+ ((\langle v_1| - i\langle v_2|) A (|v_1\rangle + i|v_2\rangle) i - (\langle v_1| + i\langle v_2|) A (|v_1\rangle - i|v_2\rangle) i$

$= \cancel{\langle v_1|A|v_1\rangle} + \underline{\langle v_2|A|v_1\rangle} + \cancel{\langle v_1|A|v_2\rangle} + \cancel{\langle v_2|A|v_2\rangle}$

$- \cancel{\langle v_1|A|v_1\rangle} + \underline{\langle v_2|A|v_1\rangle} - \cancel{\langle v_2|A|v_2\rangle} + \cancel{\langle v_1|A|v_2\rangle}$

$+ i\langle v_1|A|v_1\rangle + \langle v_2|A|v_1\rangle - \cancel{\langle v_1|A|v_2\rangle} + i\langle v_2|A|v_2\rangle$

$- i\cancel{\langle v_1|A|v_1\rangle} - \cancel{\langle v_1|A|v_2\rangle} + \underline{\langle v_2|A|v_1\rangle} - i\cancel{\langle v_2|A|v_2\rangle}$

$= 4\langle v_2|A|v_1\rangle$ // QED ✓

(i) $\langle v|A|v\rangle = 0$ for all $|v\rangle \in V$

$\Rightarrow \because 4\langle v_2|A|v_1\rangle = \underbrace{\langle v_+|A|v_+\rangle}_{0} - \underbrace{\langle v_-|A|v_-\rangle}_{0} + i\underbrace{\langle v_+'|A|v_+'\rangle}_{0} - i\underbrace{\langle v_-'|A|v_-'\rangle}_{0}$

$\therefore \langle v_2|A|v_1\rangle = 0$ for any $|v_2\rangle, |v_1\rangle \in V$

$\Rightarrow$ choose $|v_2\rangle = |e_i\rangle$, $|v_1\rangle = |e_j\rangle$ $(i = 1, \dots, n;\ j = 1, \dots, n)$

then ~~eef~~

$0 = \langle e_i|A|e_j\rangle = A_{ij}$ ✓

$\Rightarrow A_{ij} = 0 \Rightarrow A = [0]$ // QED.

well done ✓

(ii) $\langle v|A|v\rangle$ is real for all $|v\rangle \in V$

$\Leftrightarrow \langle v|A|v\rangle - \langle v|A|v\rangle^* = 0$

$\Leftrightarrow \langle v|A|v\rangle = \langle v|A|v\rangle^* = \langle v|A|v\rangle^\dagger = \langle v|A^\dagger|v\rangle$

$\Leftrightarrow \langle v|A - A^\dagger|v\rangle = 0$ for all $|v\rangle \in V$

using conclusion in (i) $\Leftrightarrow A - A^\dagger = 0$

$\Leftrightarrow A = A^\dagger \Leftrightarrow A$ is Hermitian ✓

6. The relation is

$$|e_1'\rangle = \cos\alpha\,|e_1\rangle + \sin\alpha\,|e_2\rangle$$
$$|e_2'\rangle = -\sin\alpha\,|e_1\rangle + \cos\alpha\,|e_2\rangle$$

$\therefore$ we have $\quad |e_i'\rangle = U_{ji}\,|e_j\rangle$ (summation over j implied).

where $\quad U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

Both Bases are orthonormal bases and $\quad U^T U = 1$

In the $|e_1\rangle$, $|e_2\rangle$ basis:

① rotation 45° counterclockwise

$$\Rightarrow \quad A_1 = \begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

② reflection with respect to the horizontal

$$\Rightarrow \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\therefore \quad \underline{A} = A_2 A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \boxed{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}$$

For an arbitrary $|v\rangle \in V$ that satisfies $|v\rangle = a_i |e_i\rangle$

and $|v\rangle = a_i' |e_i'\rangle$, ~~then~~ also $|e_k'\rangle = U_{ik}|e_i\rangle$

$$a_j = \langle e_j | v\rangle = \langle e_j | I | v\rangle = \langle e_j | e_k'\rangle\langle e_k' | v\rangle$$
$$= U_{ik}\langle e_j | e_i\rangle\langle e_k' | v\rangle \ \cancel{= U_{ik}\delta_{i1}\langle e_k' | e_i}$$
$$= U_{ik}\delta_{ij}\,a_k' = U_{jk}\,a_k'$$

$\therefore$ the coordinate vectors have the relation

$$|a\rangle = U|a'\rangle \quad \therefore \quad |a'\rangle = U^{-1}|a\rangle = U^T|a\rangle$$

In basis $|e_1'\rangle$, $|e_2'\rangle$:

~~A~~ the map $\hat{A}$ in original basis (A) now becomes A' , and its elements are

$$A'_{ij} = \langle e_i' | \hat{A} | e_j' \rangle$$

$$= \langle e_i' | e_k \rangle \langle e_k | \hat{A} | e_l \rangle \langle e_l | e_j' \rangle$$

$$\langle e_k | \hat{A} | e_l \rangle = A_{kl} \qquad \langle e_l | e_j' \rangle = \langle e_l | U_{mj} | e_m \rangle$$

$$= U_{mj} \langle e_l | e_m \rangle = U_{mj} \delta_{lm}$$

$$= U_{lj}$$

~~$\langle e_i | e_i \rangle = P_{in}$~~ ~~$|e_i'\rangle = P_{ni} |e_n\rangle$~~

~~$\Rightarrow \langle e_i' | = P_{ni} \langle e_n | = (P^T)_{in} \langle e_n |$~~

~~$\therefore \langle e_i' | e_k \rangle = (P^T)_{in} \langle e_n | e_k \rangle = (P^T)_{in} \delta_{nk}$~~ $= P^T$

$$|e_i'\rangle = U_{ni} |e_n\rangle \Rightarrow \langle e_i' | = U_{ni} \langle e_n | = U^T_{in} \langle e_n |$$

$$\therefore \langle e_i' | e_k \rangle = U^T_{in} \langle e_n | e_k \rangle = U^T_{in} \delta_{nk} = U^T_{ik}$$

$$\therefore A'_{ij} = U^T_{ik} A_{kl} U_{lj} \Rightarrow \boxed{A' = U^T A U}$$

$$\therefore A' = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\alpha - \sin\alpha & -\sin\alpha - \cos\alpha \\ -\sin\alpha - \cos\alpha & \sin\alpha - \cos\alpha \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2} \begin{pmatrix} \cos^2\alpha - \sin\alpha\cos\alpha + \sin^2\alpha - \sin\alpha\cos\alpha & -\cos\alpha\sin\alpha - \cos^2\alpha + \sin^2\alpha - \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \sin^2\alpha - \sin\alpha\cos\alpha - \cos^2\alpha & \sin^2\alpha + \sin\alpha\cos\alpha - \cancel{\sin^2\alpha} + \sin\alpha\cos\alpha - \cos^2\alpha \end{pmatrix}$$

$$\boxed{= \frac{\sqrt{2}}{2} \begin{pmatrix} \cos^2\alpha - \sin^2\alpha - 2\sin\alpha\cos\alpha & \cancel{\sin^2\alpha} \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha \\ \sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha & \sin^2\alpha - \cos^2\alpha + 2\sin\alpha\cos\alpha. \end{pmatrix}}$$

Fejer

Kemal

7) (1)

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 2 & -1 \end{pmatrix}\begin{array}{l} R_1' \\ R_2' \\ R_3' \end{array}$$

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 - R_1 \end{array}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & -2 \end{pmatrix}\begin{array}{l} R_1' \\ R_2' \\ R_3' \end{array}$$

$$\begin{array}{r} R_1 \\ R_2 \\ 2R_2 - 3R_3 \end{array}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}\begin{array}{l} R_1' \\ R_2' \\ \ \end{array}$$

$\therefore$ rank $= 3$ ✓

(2)

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -1 \end{pmatrix}\begin{array}{l} R_1' \\ R_2' \\ R_3' \end{array}$$

$$\begin{array}{r} R_1 \\ R_2 \\ R_3 - R_2 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix}\begin{array}{l} 1' \\ 2' \\ 3' \end{array}$$

$$\begin{array}{r} R_1 \\ R_1 - R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

$$\begin{array}{r} R_1 \\ R_2 \\ 2R_2 - 3R_3 \end{array}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

rank $= 3$ ✓

(3)
$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 5 & -2 & -1 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3-5R_2 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \\ 0 & 3 & 4 \end{pmatrix}$$

$$\begin{array}{c} R_1 \\ R_1-R_2 \\ R_3 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 3 & 4 \end{pmatrix}$$

rank = 2

$$\begin{array}{c} R_1 \\ R_2 \\ R_2-R_3 \end{array}\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

(4)

① $\begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix}\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$ $\begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ 0 & x^2-y & xy-1 \end{pmatrix}$

if $x=y=0$ then

$$\begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix}\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \Rightarrow \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ 0 & x^2-y & xy-1 \end{pmatrix}\begin{array}{c} R_1 \\ R_2 \\ xR_1-R_3 \end{array}$$

~~if $x=y=0$, then rank = 2~~

$$\Rightarrow \begin{pmatrix} 1 & x & y \\ 0 & 3x^2-2y & 3xy-1 \\ 0 & x^2-y & xy-1 \end{pmatrix}$$

(4) $\begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix}$ if

① if $x = y = 0$ $\Rightarrow$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

rank = 2

② if $x = 0$, $y \neq 0$ $\Rightarrow$ $\begin{array}{c|c} \begin{pmatrix} 1 & 0 & y \\ 0 & 2y & 1 \\ 0 & y & 1 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \end{array}$

$\Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & 0 & y \\ 0 & 2y & 1 \\ 0 & 0 & -1 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_2 - 2R_3 \end{matrix} \end{array}$ $\therefore$ rank = 3

③ if $x \neq 0$, $y = 0$ $\Rightarrow$ $\begin{array}{cc} \begin{pmatrix} 1 & x & 0 \\ 3x & 0 & 1 \\ x & 0 & 1 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \end{array}$

$\Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & x & 0 \\ 3x & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_2 - 3R_3 \end{matrix} \end{array} \Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & x & 0 \\ 0 & 3x^2 & -1 \\ 0 & 0 & -2 \end{pmatrix} & \begin{matrix} R_1 \\ 3xR_1 - R_2 \\ R_3 \end{matrix} \end{array}$

$\therefore$ rank = 3

if $x \neq 0$, $y \neq 0$

$\begin{array}{cc} \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ x & y & 1 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \end{array} \Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ 0 & x^2 - y & xy - 1 \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ xR_1 - R_3 \end{matrix} \end{array}$

$\Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & x & y \\ 0 & 3x^2 - 2y & 3xy - 1 \\ 0 & x^2 - y & xy - 1 \end{pmatrix} & \begin{matrix} R_1 \\ 3xR_1 - R_2 \\ R_3 \end{matrix} \end{array} \Rightarrow \begin{array}{cc} \begin{pmatrix} 1 & x & y \\ 0 & 3x^2 - 2y & 3xy - 1 \\ 0 & 3x^2 - 2y & \frac{3x^2 - 2y}{x^2 - y}(xy - 1) \end{pmatrix} & \begin{matrix} R_1 \\ R_2 \\ \frac{3x^2 - 2y}{x^2 - y} R_3 \end{matrix} \end{array}$

$$\Rightarrow \begin{array}{c} \begin{pmatrix} 1 & x & y \\ 0 & 3x^2-2y & 3xy-1 \\ 0 & 0 & 3xy-1-\frac{3x^2-2y}{x^2-y}(xy-1) \end{pmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_2-R_3 \end{array}$$

$$3xy-1-\frac{3x^2-2y}{x^2-y}(xy-1)=0$$

$$\Rightarrow (x^2-y)(3xy-1)=(3x^2-2y)(xy-1)$$

$$\cancel{3x^3y}-3xy^2-x^2+y=\cancel{3x^3y}-2xy^2-3x^2+2y$$

$$\therefore xy^2-2x^2+y=0$$

$$\boxed{\begin{array}{l} \text{if } xy^2-2x^2+y\neq 0 \text{ then rank} = 3 \\ \text{if } xy^2-2x^2+y=0 \text{ then } \cancel{\text{rank}=2} \text{ rank}=2. \end{array}}$$

$\cancel{3x^2-2y=0 \Rightarrow y=\frac{3}{2}x^2 \quad 3xy=1}$

~~if $3x^2-2y$ or $3xy-1\neq 0$~~

~~the rank $=2$~~

~~if $3x^2-2y=0$, $3xy-1\neq 0$ then~~

~~rank $=1$~~

~~in this case~~

~~$3x^2=2y$, $3xy=1$ $\Rightarrow$ $y=\frac{3}{2}x^2$~~

~~$\therefore \frac{9}{2}x^3=1 \quad \therefore x^3=\frac{2}{9}$~~

~~$\therefore x=\left(\frac{2}{9}\right)^{1/3} \quad y=\frac{3}{2}\left(\frac{2}{9}\right)^{2/3}$~~

8. (i)

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

$$e_i A = (0, \cdots, \underset{i\text{th}}{1}, \cdots, 0) \begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & \ddots & \\ \vdots & & \ddots \end{pmatrix}$$

$= (A_{i1}, A_{i2}, \cdots, A_{in})$ = the ith row of A

$\therefore A = \begin{pmatrix} e_1 A \\ e_2 A \\ \vdots \\ e_n A \end{pmatrix}$, similarly $B = \begin{pmatrix} e_1 B \\ \vdots \\ e_n B \end{pmatrix}$

$\cancel{(AB)_{ij} = \sum_k A_{ik} B_{kj}}$

the ith row of $AB = (AB_{i1}, AB_{i2}, \cdots, AB_{in})$

$= (A_{i1}B_{11} + A_{i2}B_{21} + \cdots + A_{in}B_{n1},\ A_{i1}B_{12} + A_{i2}B_{22} + \cdots + A_{in}B_{n2}$

$, \cdots, A_{i1}B_{1n} + A_{i2}B_{2n} + \cdots + A_{in}B_{nn})$

$= A_{i1}(B_{11}, B_{12}, \cdots, B_{1n}) + A_{i2}(B_{21}, B_{22}, \cdots, B_{2n})$

$+ \cdots + A_{in}(B_{n1}, B_{n2}, \cdots, B_{nn})$

$= \cancel{\sum_i A_i} \sum_j A_{ij} e_j B$

$$\therefore AB = \begin{pmatrix} \sum_{j_1} A_{1j_1} e_{j_1} B \\ \vdots \\ \sum_{j_n} A_{nj_n} e_{j_n} B \end{pmatrix}$$

Lemma: $\det(A^T) = \det(A)$

$~~\det(A) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)~~$ $\det(A) = \sum_{p} \mathrm{sgn}(p) A_{1,p(1)} \cdots A_{n,p(n)}$

set ~~$j(a) = p(a)$~~ $j_a = p(a)$, then

~~$\det(A) = \sum_p \mathrm{sgn}(p) A_{1,p(1)} \cdots A_{n,p(n)}$~~

$$A_{1,p(1)} \cdots A_{n,p(n)} = A_{1j_1} \cdots A_{nj_n}$$

$$= A_{p^{-1}(j_1)j_1} \cdots A_{p^{-1}(j_n)j_n} = \underbrace{A_{p^{-1}(1),1} \cdots A_{p^{-1}(n),n}}_{\text{reordering}}$$

$$\underline{\det(A)} = \sum_p \mathrm{sgn}(P) A_{p^{-1}(1),1} \cdots A_{p^{-1}(n),n}$$

$$\underbrace{=}_{\mathrm{sgn}(p) = \mathrm{sgn}(p^{-1})} \sum_{p^{-1}} \mathrm{sgn}(p^{-1}) A_{p^{-1}(1),1} \cdots A_{p^{-1}(n),n}$$

$$\underbrace{=}_{\rho = p^{-1}} \sum_\rho \mathrm{sgn}(\rho) (A^T)_{1\rho(1)} \cdots (A^T)_{n\rho(n)} = \underline{\det(A^T)}$$

(ii) $\quad (\underline{a} = \vec{a}^T \quad \underline{b} = \vec{b}^T \quad \underline{c_i} = \vec{c_i}^T)$

$$\det \begin{pmatrix} \alpha\vec{a} + \beta\vec{b} \\ \vec{c_2} \\ \vdots \\ \vec{c_n} \end{pmatrix} \underbrace{=}_{\text{transpose}} \det(\alpha\underline{a} + \beta\underline{b}, \underline{c_2}, \cdots, \underline{c_n})$$

$$\underbrace{=}_{\text{multilinearity}} \alpha \det(\underline{a}, \underline{c_2}, \cdots, \underline{c_n}) + \beta \det(\underline{b}, \underline{c_2}, \cdots, \underline{c_n})$$

$$\underbrace{=}_{\text{transpose}} \alpha \det \begin{pmatrix} \vec{a} \\ \vec{c_2} \\ \vdots \\ \vec{c_n} \end{pmatrix} + \beta \det \begin{pmatrix} \vec{b} \\ \vec{c_2} \\ \vdots \\ \vec{c_n} \end{pmatrix}$$

QED

(iii)

$$\underline{\det(AB)} = \det\begin{pmatrix} \sum_{j_1} A_{1j_1} e_{j_1} B \\ \vdots \\ \vdots \\ \sum_{j_n} A_{nj_n} e_{j_n} B \end{pmatrix}$$

$$= \sum_{j_1} A_{1j_1} \det\begin{pmatrix} e_{j_1} B \\ \sum_{j_2} A_{2j_2} e_{j_2} B \\ \vdots \\ \sum_{j_n} A_{nj_n} e_{j_n} B \end{pmatrix}$$

$$= \sum_{j_1} \cdots \sum_{j_n} A_{1j_1} \cdots A_{nj_n} \det\begin{pmatrix} e_{j_1} B \\ \vdots \\ e_{j_n} B \end{pmatrix}$$

QED.

(iv) We only consider ~~$(j_1, \dots, j_n)$~~ distinct $(j_1, \dots, j_n)$ because if any ~~four~~ two of them are same then

$$\det\begin{pmatrix} e_{j_1} B \\ \vdots \\ e_{j_n} B \end{pmatrix} = 0$$

~~set $j_a = P(a)$~~ $P(a) = j_a$

~~$\det(AB) =$~~ $$\det\begin{pmatrix} e_{j_1} B \\ \vdots \\ e_{j_n} B \end{pmatrix} = \det\begin{pmatrix} e_{P(1)} B \\ \vdots \\ e_{P(n)} B \end{pmatrix} = \operatorname{sgn}(P) \det\begin{pmatrix} e_1 B \\ \vdots \\ e_n B \end{pmatrix}.$$

~~A_{j_1}~~ $$A_{1j_1} \cdots A_{nj_n} = A_{1P(1)} \cdots A_{nP(n)}$$

$$\therefore\ \underline{\det(AB)} = \sum_{j_1 \cdots j_n} A_{1j_1} \cdots A_{nj_n} \det\begin{pmatrix} e_{j_1} B \\ \vdots \\ e_{j_n} B \end{pmatrix}$$

$$= \underbrace{\sum_{P} A_{1P(1)} \cdots A_{nP(n)} \operatorname{sgn}(P)}_{\det(A)} \underbrace{\det\begin{pmatrix} e_1 B \\ \vdots \\ e_n B \end{pmatrix}}_{\det(B)}$$

$$= \underline{\det(A) \det(B)}$$

QED

9. (i) $\det(I+\varepsilon A) = \det\begin{pmatrix} 1+\varepsilon A_{11} & \varepsilon A_{12} & \varepsilon A_{13} & \cdots \\ \varepsilon A_{21} & \ddots & & \\ \varepsilon A_{31} & & \ddots & \\ \vdots & & & 1+A_{nn} \end{pmatrix}$

$\because$ The determinant is the sum of $n!$ terms. Each term is formed by the sign of permutation times the product of n elements of the matrix, where no two elements can be in the same row or ~~column~~ ~~column~~ column.

As a result, ~~So, any~~ Any term except the term $(1+\varepsilon A_{11})\cdots(1+\varepsilon A_{nn})$ (i.e. the product of diagonal ~~terms~~ elements) will have at least two off-diagonal ~~terms~~ elements ε, because if $(1+\varepsilon A_{11}), (1+\varepsilon A_{22}), \cdots, (1+\varepsilon A_{n-1\,n-1})$ are in the term, then the ~~last~~ last term must be $(1+\varepsilon A_{nn})$.

So $\underline{\det(I+\varepsilon A)} = (1+\varepsilon A_{11})(1+\varepsilon A_{22})\cdots(1+\varepsilon A_{nn}) + O(\varepsilon^2)$

$= 1+\varepsilon(A_{11}+A_{22}+\cdots+A_{nn}) + O(\varepsilon^2)$

$= \underline{1+\mathrm{tr}(A) + O(\varepsilon^2)}$ ✓ / QED.

(ii) $\exp(A) = \lim_{N\to\infty}\left(1+\frac{A}{N}\right)^N$

$\therefore \det(\exp(A)) = \det\left(\lim_{N\to\infty}\left(1+\frac{A}{N}\right)^N\right) = \lim_{N\to\infty}\det\left(\left(1+\frac{A}{N}\right)^N\right)$

$= \lim_{N\to\infty}\left(\det\left(1+\frac{A}{N}\right)\right)^N = \lim_{N\to\infty}\left(1+\frac{1}{N}\mathrm{tr}(A)+O\left(\frac{1}{N^2}\right)\right)^N$

* ~~$\lim_{n\to\infty}\left[\left(1+\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right)^n-\left(1+\frac{a}{n}\right)^n\right] = \lim_{n\to\infty}\left(1+\frac{a}{n}+O\left(\frac{1}{n^2}\right)-\left(1+\frac{a}{n}\right)\right)\left(\left(1+\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right)^{n-1}\left(1+\frac{a}{n}\right)+\cdots+\left(1+\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1+\frac{a}{n}\right)^{n-1}\right)$~~

~~$= \lim_{n\to\infty}\left(O\left(\frac{1}{n^2}\right)\right)(1) = 0 \Rightarrow \lim_{n\to\infty}\left(1+\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right)^n = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n$~~

$\therefore \underline{\det(\exp(A))} = \lim_{N\to\infty}\left(1+\frac{1}{N}\mathrm{tr}(A)+O\left(\frac{1}{N^2}\right)\right)^N \overset{*}{=} \lim_{N\to\infty}\left(1+\frac{1}{N}\mathrm{tr}(A)\right)^N$

$= \underline{\exp(\mathrm{tr}(A))}$ ✓ QED.

10. $tr(px) = (px)_{ii} = P_{ij}X_{ji} = X_{ji}P_{ij} = (Xp)_{jj} = tr(Xp)$

$\therefore$ $tr[p, x] = tr(px - xp) = tr(px) - tr(xp) = 0 \neq -i\hbar$

$\therefore$ there is no ~~p and~~ finite-dimensional matrices p and x such that $[p, x] = -i\hbar$

When dimensions of p and x become ~~infinit,~~ infinite, $[p, x] = \infty - \infty$, which cannot be defined.

$\therefore$ $[p, x]$ is not ~~nesessi~~ nesessarily 0

11.

$$A = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix}.$$

$$A^TA = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= I$

$$\det(A) = -\frac{1}{2}\left((\tfrac{1}{2})(1) - (0)(0)\right) - \left(-\frac{\sqrt{3}}{2}\right)\left((-\tfrac{\sqrt{3}}{2})(1) - (0)(0)\right) + 0 = -1$$

$\therefore$ A is <u>not</u> a rotation.

$\therefore$ A is a rotation followed by ~~plus~~ a <u>reflection</u> in the x coordinate directions. ~~in one of~~

$$A = FR = \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{F \text{ is a reflection}} \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{R \text{ is a rotation}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^TB = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & +\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & -\sqrt{\frac{3}{8}} \\ -\sqrt{\frac{3}{8}} & +\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\det(B) = \left(\frac{1}{4}\right)\left(\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right) - \left(\sqrt{\frac{3}{8}}\right)\left(-\sqrt{\frac{3}{8}}\right)\right) - \left(\frac{3}{4}\right)\left(\left(\frac{3}{4}\right)\left(-\frac{1}{2}\right) - \left(\sqrt{\frac{3}{8}}\right)\left(\sqrt{\frac{3}{8}}\right)\right) + \left(-\sqrt{\frac{3}{8}}\right)\left(\left(\frac{3}{4}\right)\left(-\sqrt{\frac{3}{8}}\right) - \left(\frac{1}{4}\right)\left(\sqrt{\frac{3}{8}}\right)\right)$$

$$= 1$$

$\therefore$ B is a rotation

The axis of rotation $\vec{n}$ is such that $B\vec{n} = \vec{n}$

$(\vec{n} = (n_1, n_2, n_3)^T)$,

$$\begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\therefore \left.\begin{aligned} \tfrac{1}{4}n_1 + \tfrac{3}{4}n_2 - \sqrt{\tfrac{3}{8}}n_3 &= n_1 \\ \tfrac{3}{4}n_1 + \tfrac{1}{4}n_2 + \sqrt{\tfrac{3}{8}}n_3 &= n_2 \\ \sqrt{\tfrac{3}{8}}n_1 - \sqrt{\tfrac{3}{8}}n_2 - \tfrac{1}{2}n_3 &= n_3 \end{aligned}\right\}$$

$\Rightarrow \quad n_1 = n_2, \quad n_3 = 0$

$$\therefore \vec{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$tr(B) = 2\cos\theta + 1$ (θ = angle of rotation)

$$\therefore \theta = \arccos\left(\frac{tr(B) - 1}{2}\right) = \arccos\left(-\frac{1}{2}\right) = \underline{120^\circ}$$

$\therefore$ Axis of rotation $\vec{n} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Angle of rotation $\theta = 120^\circ$

well done

12 (a)

$$AB = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & \cos\phi\sin\theta + \sin\phi\cos\theta \\ -\sin\phi\cos\theta - \cos\phi\sin\theta & -\sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi+\theta) & \sin(\phi+\theta) \\ -\sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}.$$

which is a rotation of ϕ after a rotation of θ

(b) $$A^TA = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\phi + \sin^2\phi & \cos\phi\sin\phi - \sin\phi\cos\phi \\ \sin\phi\cos\phi - \cos\phi\sin\phi & \cancel{\cos^2\phi}\ \sin^2\phi + \cos^2\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$A^T = A^{-1}$. $A^{-1}A = I$ $\Rightarrow$ rotation of ϕ counterclockwise and rotation of ϕ ~~rotation matrices are orthogonal~~ clockwise returns ~~the~~ to the original ~~vect~~ position.

$$A^TB = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cancel{\begin{pmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta \end{pmatrix}}$$

$$= \begin{pmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta - \cos\phi\sin\theta & \sin\phi\sin\theta + \cos\phi\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta-\phi) & \sin(\theta-\phi) \\ -\sin(\theta-\phi) & \cos(\theta-\phi) \end{pmatrix}$$

$A^T = A^{-1}$ $A^TB = A^{-1}B$ = rotation of θ counterclockwise followed by rotation of ϕ clockwise.

(c)

$$A\vec{v} = \lambda\vec{v} \implies \det(A - \lambda I) = 0$$

$$\therefore \det\begin{pmatrix} \cos\phi - \lambda & \sin\phi \\ -\sin\phi & \cos\phi - \lambda \end{pmatrix} = 0$$

$$\therefore (\cos\phi - \lambda)^2 + \sin^2\phi = 0$$

$$\implies \cos^2\phi - 2\cos\phi\,\lambda + \lambda^2 + \sin^2\phi = 0$$

$$\implies \lambda^2 - 2\cos\phi\,\lambda + 1 = 0$$

$$\therefore \cancel{\frac{2 \pm \sqrt{4\cos^2\phi - 4}}{2}} \quad \lambda = \frac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2}$$

$$= \cos\phi \pm i\sqrt{1-\cos^2\phi} = \cos\phi \pm i\sqrt{\sin^2\phi}$$

$$= \cos\phi \pm i\sin\phi$$

$$\therefore \underline{\lambda_1 = e^{i\phi}, \quad \lambda_2 = e^{-i\phi}}$$ ✓

λ_1, λ_2 are not real because if λ_1 is the solution of the characteristic polynomial, then $\lambda_2 = \lambda_1^*$ is also a solution.

$\therefore \lambda_1, \lambda_2$ are complex conjugates of each other,

They don't have to be real.

λ_1, λ_2 are not necessarily real because the matrix A is not hermitian ✓

eigenvectors?

13. (a) $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. $\sigma^y \underline{u} = \lambda \underline{u}$ $u = (a, b)^T$

then

$$\det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$$

$\therefore \lambda^2 - (i)(-i) = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1$ ✓

$\lambda_1 = 1 \Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow -ib = a, \quad ia = b$

$\therefore \underline{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ ✓

$\lambda_2 = -1 \Rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$

$\Rightarrow -ib = -a, \quad ia = -b$

$\therefore \underline{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$ ✓

$\underline{u}_1^\dagger \underline{u}_1 = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) (1, -i) \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{2}{2} = 1.$

$\underline{u}_2^\dagger \underline{u}_2 = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) (-i, 1) \begin{pmatrix} i \\ 1 \end{pmatrix} = \frac{2}{2} = 1$ ✓

$\underline{u}_1^\dagger \cdot \underline{u}_2 = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) (1, -i) \begin{pmatrix} i \\ 1 \end{pmatrix} = 0$

$U = (\underline{u}_1 \ \underline{u}_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ $U^\dagger U = (\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}$ ✓

$U^\dagger \sigma^y U = ?$

(b)

① $\cancel{\vec{v} = \alpha v_1}$ $\quad \underline{v} = \alpha \underline{u_1} + \beta \underline{u_2}$

$\Rightarrow \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \alpha \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{\sqrt{2}} \beta \begin{pmatrix} i \\ 1 \end{pmatrix}$

$\therefore \quad \frac{1}{\sqrt{2}}(\alpha + i\beta) = c_1 \quad \Rightarrow \quad \alpha = \sqrt{2} c_1 - i\beta$

$\frac{1}{\sqrt{2}}(i\alpha + \beta) = c_2.$

$\Rightarrow \quad \cancel{i(c_1 - i\beta) + \beta = c_2} \quad i(\sqrt{2} c_1 - i\beta) + \beta = \sqrt{2} c_2.$

$\cancel{\Rightarrow \quad i c_1 + \beta + \beta = c_2 \quad \Rightarrow \quad \beta = \frac{1}{2}(}$

$\Rightarrow \quad i\sqrt{2} c_1 + 2\beta = \sqrt{2} c_2.$

$\therefore \quad \beta = \frac{1}{\sqrt{2}}(c_2 - i c_1)$

$\alpha = \sqrt{2} c_1 - \frac{i}{\sqrt{2}}(c_2 - i c_1)$

$= \frac{1}{\sqrt{2}}(c_1 - i c_2)$

$\therefore \quad \underline{\alpha = \frac{1}{\sqrt{2}}(c_1 - i c_2) \qquad \beta = \frac{1}{\sqrt{2}}(c_2 - i c_1)}$ ✓

②

$\underline{v} = \alpha \underline{u_1} + \beta \underline{u_2}$

$\underline{v_1}^\dagger \underline{v} = \alpha \underbrace{\underline{v_1}^\dagger \underline{v_1}}_{1} + \beta \underbrace{\underline{v_1}^\dagger \underline{v_2}}_{0} = \alpha$

$\underline{v_2}^\dagger \underline{v} = \alpha \underbrace{\underline{v_2}^\dagger \underline{v_1}}_{0} + \beta \underbrace{\underline{v_2}^\dagger \underline{v_2}}_{1} = \beta$

$\therefore \quad \alpha = \frac{1}{\sqrt{2}}(1, -i) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \boxed{\frac{1}{\sqrt{2}}(c_1 - i c_2)}$ ✓

$\beta = \frac{1}{\sqrt{2}}(-i, 1) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \boxed{\frac{1}{\sqrt{2}}(c_2 - i c_1)}$

14.

$$\det\begin{pmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = 0 \qquad \underline{A}\underline{u} = \lambda\underline{u}$$

$\cancel{u = (u_1, u_2, u_3)}$

$\underline{u} = (a, b, c)^T$

$$\Rightarrow (-\lambda)(-\lambda)(2-\lambda) \cancel{-(1)(1)(2)} - (1)(1)(2-\lambda) = 0$$

$$\Rightarrow \lambda^2(2-\lambda) - (2-\lambda) = 0$$

$$\Rightarrow (\lambda^2 - 1)(2-\lambda) = 0 \Rightarrow (\lambda+1)(\lambda-1)(\lambda-2) = 0$$

$$\therefore \underline{\lambda_1 = -1, \quad \lambda_2 = 1, \quad \lambda_3 = 2.}$$

$$\lambda_1 = -1 \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} b = -a \\ \cancel{b =} \\ a = -b \\ 2c = -c \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a + b = 0 \\ c = 0 \end{array}\right.$$

$$\Rightarrow \underline{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = 1 \Rightarrow \left.\begin{array}{l} b = a \\ a = b \\ 2c = c \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a = b \\ c = 0 \end{array}\right.$$

$$\underline{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 2 \Rightarrow \left.\begin{array}{l} b = 2a \\ a = 2b \\ 2c = 2c \end{array}\right. \Rightarrow \left\{\begin{array}{l} a = 0 \\ b = 0 \quad c \in \mathbb{C} \\ \cancel{c \in \mathbb{R}} \end{array}\right.$$

$$\therefore \underline{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

$$R = (\underline{u}_1, \underline{u}_2, \underline{u}_3) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

$$R^T A R = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, = \text{a diagonal matrix.}$$

15.

$$\hat{A} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad R = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \quad R^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$$

$$A = R\hat{A}R^T = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 0 & 0 & \sqrt{2} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix}.$$

10.

16. $F\underline{v} = \lambda \underline{v}$ $\quad \det\begin{pmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix} = 0.$

$$(4-\lambda)(1-\lambda) - (-2)(-2) = 0 \qquad \underline{v} = (v_1, v_2)^T$$

$$\lambda^2 - 5\lambda + 4 - 4 = 0 \implies \lambda^2 - 5\lambda = 0.$$

$$\therefore \lambda_1 = 0, \quad \lambda_2 = 5.$$

$\lambda_1 = 0$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$4v_1 = 2v_2. \quad \cancel{\implies v_1 = 2v} \implies v_2 = 2v_1.$$

$$\therefore \cancel{v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \quad \underline{v_1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$\lambda_2 = 0.$

$$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5v_1 \\ 5v_2 \end{pmatrix}$$

$$4v_1 - 2v_2 = 5v_1 \implies v_1 = -2v_2.$$
$$-2v_1 + v_2 = 5v_2 \implies v_1 = -2v_2.$$

$$\underline{v_2} = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$P = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \qquad P^T P = I$$

$$\hat{F} = P^T F P = \mathrm{diag}(0, 5) \qquad F = P\hat{F}P^T$$

$$\boxed{F^n} = P\hat{F}\underbrace{P^T P}_{I}\underbrace{\hat{F}P^T \dots P\hat{F}P}_{n} = P\hat{F}^n P^T$$

$$= P(\mathrm{diag}(0,5))^n P^T = P\,\mathrm{diag}(0^n, 5^n)\,P^T$$

$$= \left(\frac{1}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)\begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 5^n \end{pmatrix}\begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -2\times 5^n & 5^n \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 4\times 5^n & -2\times 5^n \\ -2\times 5^n & 5^n \end{pmatrix} = \cancel{4\times 5^{n-1} \quad -2\times 5^{n-1}} \quad \boxed{5^{n-1} \times \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}}$$

17.

$\cancel{R_1 = (}$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

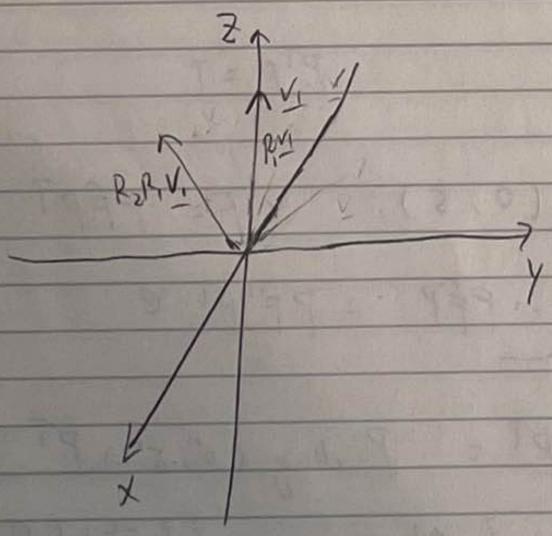
$$\therefore \quad R_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

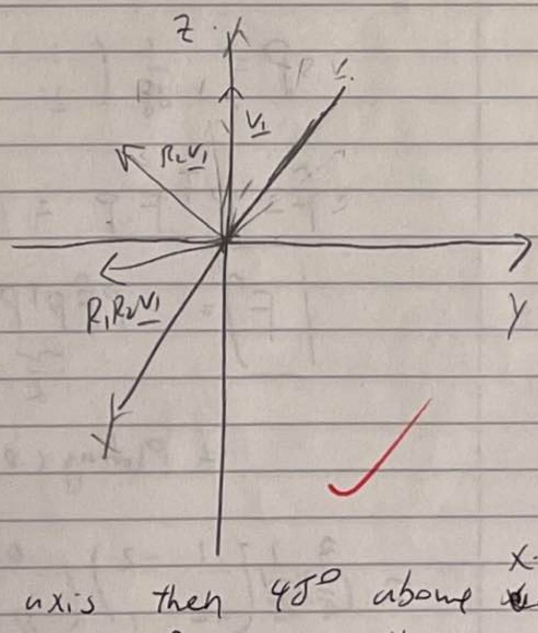
$$Q_1 = R_1R_2 = \begin{pmatrix} \sqrt{2}/2 & -1/2 & 1/2 \\ \sqrt{2}/2 & 1/2 & -1/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

$$Q_2 = R_2R_1 = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1/2 & 1/2 & -\sqrt{2}/2 \\ 1/2 & 1/2 & \sqrt{2}/2 \end{pmatrix}.$$

Q_2



Q_1



$\therefore$ Rotating 45° about z axis then 45° about x-axis is different from rotating 45° about ~~then~~ x-axis then 45° about z axis.

18. ~~Let λ~~ $H|v\rangle = \lambda|v\rangle \Rightarrow \det\begin{pmatrix} 10-\lambda & 3i \\ -3i & 2-\lambda \end{pmatrix} = 0$

$\therefore (10-\lambda)(2-\lambda) - 9 = 0$

$\therefore \lambda^2 - 12\lambda + 11 = 0$ $\qquad (|v\rangle = (a, b)^T$

$\therefore a_1 = 1 \quad \lambda_2 = 11$

$\lambda_1 = 1 \Rightarrow \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\therefore \begin{matrix} 9a + 3ib = 0 \\ -3ia + b = 0 \end{matrix} \Rightarrow \boxed{|v_1\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3i \end{pmatrix}}$

$\lambda_2 = 11 \Rightarrow \begin{pmatrix} -1 & 3i \\ -3i & -9 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad a = 3ib \Rightarrow \boxed{|v_2\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3i \\ 1 \end{pmatrix}}$

$\boxed{U = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}}$

$D = U^\dagger H U = (\frac{1}{\sqrt{10}})(\frac{1}{\sqrt{10}})\begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix}\begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}$

19. ~~2~~ 2 matrices have the same eigenvectors iff their commutator vanishes.

Notice that for A, B, C, and D.

$[A, B] = [A, D] = [B, D] = 0$
$[A, C] = [B, C] = [D, C] \neq 0$

$\therefore$ A, B, D have the same set of eigenvectors.

$A|v\rangle = \lambda|v\rangle \Rightarrow \det\begin{pmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{pmatrix} = 0$
$|v\rangle = (a, b)^T$

$\therefore (6-\lambda)(9-\lambda) - 4 = 0 \Rightarrow \lambda^2 - 15\lambda + 50 = 0$
$\Rightarrow (\lambda-5)(\lambda-10) = 0 \Rightarrow \lambda_1 = 5,\ \lambda_2 = 10$

$\lambda_1 = 5$.

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow a = 2b \Rightarrow |v_1\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\lambda_2 = 10$

$$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -2a = b \Rightarrow |v_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\therefore$ the common eigenvectors ~~are~~ for A, B, D are

$$|v_1\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

eigenvector

20. $\sigma^+ = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

$\sigma^+ |v\rangle = \lambda |v\rangle = 0 \quad (|v\rangle = (a, b))$

$\Rightarrow \det \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 = 0 \Rightarrow \lambda = 0$

$\therefore \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow b = 0, \ a \in \mathbb{C}$

$\therefore |v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only eigenvector of σ^+ (up to rescaling)

Obviously $|v\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ does not form a basis of the 2-D complex vector space $\mathbb{C}^2$

$\therefore$ Eigenvectors does not form a basis

$\therefore \sigma^+$ cannot be diagonalised.

21. (i) $Q = 4x^2+2y^2+2z^2-2xy+2yz-2zx$

$$= \underbrace{(x, y, z)}_{\vec{\alpha}^T} \underbrace{\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\vec{x}}$$

$\therefore Q = \vec{x}^T V \vec{\alpha}$ let P be the diagonalising basis transformation matrix of V

then $\text{diag}(\lambda_1, \cdots, \lambda_n) = \hat{V} = P^T V P$.

$\therefore V = P\hat{V}P^T$ let $\vec{y} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = P^T\vec{x}$

then $\vec{y}^T \hat{V} y = (P^T\vec{x})^T \hat{V} P^T \hat{x} = \hat{x}^T \underbrace{P\hat{V}P^T}_{V} \vec{x}$

$= \vec{x}^T V \vec{x} = Q$.

$\therefore Q = (x', y', z') \hat{V} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$.

$\det \begin{pmatrix} 4-\lambda & -1 & -1 \\ -1 & 2-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{pmatrix} = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 5$

$\therefore Q = (x', y', z') \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$= x'^2 + 2y'^2 + 5z'^2 \Rightarrow \lambda = 1, \mu = 2, \nu = 5$ ✓

$\vec{y} = P^T\vec{\alpha}$ is the rotation of coordinates

(ii)

$$\sigma = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad \vec{j} = \sigma \vec{E}$$

$$\vec{j} = \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}, \quad \vec{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \hat{\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

$$\hat{\sigma} = P^T \sigma P \implies \sigma = P \hat{\sigma} P^T$$

Now let $\vec{j}' = P^T \vec{j}$, $\vec{E}' = P^T \vec{E}$, then

$$\vec{j} = \sigma \vec{E} \implies P\vec{j}' = P\hat{\sigma}\underbrace{P^T P}_{I}\vec{E}' \implies \vec{j}' = \hat{\sigma}\vec{E}'$$

multiply by P^T on both sides.

$$\therefore \vec{j}' = \hat{\sigma}\vec{E}' \implies$$

$$\therefore \begin{pmatrix} j_1' \\ j_2' \\ j_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} E_1' \\ E_2' \\ E_3' \end{pmatrix}$$

$\therefore$ ~~$j_1' = E_1'$~~ $\quad \tilde{\sigma}_1 = 1 \quad \tilde{\sigma}_2 = 2 \quad \tilde{\sigma}_3 = 5$

$\underline{n=1}$

$(A+B)^1 = A+B$

$\underline{n=2}$

$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

$= A^2 + 2AB + B^2.$

Assume

$$(A+B)^k = \sum_{i=0}^{n} \binom{n}{i} A^i B^{n-i} \quad \text{for } n=k$$

the for $n = k+1$

$$(A+B)^{k+1} = (A+B)^k (A+B)$$

$$= \left(\sum_{i=0}^{n} \binom{k}{i} A^i B^{k-i} \right) (A+B)$$

$$\frac{r\,k! +}{(k-r+1)(k-r)!\,r!} = \sum_{i=0}^{n} \binom{k}{i} A^{i+1} B^{k-i} + \sum_{i=0}^{n} \binom{k}{i} BA^i B^{k-1+i}$$

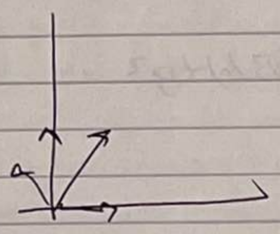
$=$

$$\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$$

$$= \frac{k!}{(k-r+1)!\,(r-1)!} + \frac{k!}{\quad}$$

$k!$

$$\frac{k!}{(k-r+1)(k-r)!\,(r-1)!} + \frac{k!}{(k-r)!\,r!}$$



$$R\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} \Rightarrow \begin{matrix} R_{11} = \cos\alpha \\ R_{21} = \sin\alpha \end{matrix}$$

$$R\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix} \Rightarrow \begin{matrix} R_{12} = -\sin\alpha \\ R_{22} = \cos\alpha \end{matrix}$$

$$A' = \langle e_i' | A | e_j' \rangle = \langle e_i' | e_m \rangle \langle e_m | A | e_n \rangle \langle e_n | e_j' \rangle$$

$$\begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ \alpha & y & 1 \end{pmatrix} \to \begin{pmatrix} 1 & x & y \\ 3x & 2y & 1 \\ 0 & y & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & x & y \\ 0, & 2y-3x^2, & 1-3y\alpha \\ 0 & y & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & x & y \\ 0 & 2y-3x^2 & 1-3yx \\ 0 & 0 & -y(1-3xy)+2(2y-3x^2) \end{pmatrix}$$

$$\operatorname{tr}[P, X] = \operatorname{tr}(PX - XP)$$

$$= \sum_i (PX - XP)_{ii} = \sum_i \sum_j (P_{ij} \cancel{P} X_{ji} - X_{ij} P_{ji})$$

$$= \sum_i \sum_j P_{ij} X_{ji} - \sum_p \sum_q P_{pq} X_{qp}$$

$$= \sum_i \sum_j P_{ij} X_{ji} - \sum_j \sum_i P_{ij} X_{ji}$$

$$= \sum_i \sum_j P_{ij} X_{ji} - \sum_i \sum_j P_{ij} X_{ji}$$

Prove Problem 3 by induction

First (Lemma 1):

$$L = \binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} = R$$

Proof:

$$L = \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$= \frac{(n-r+1)n! + rn!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)\,n!}{(n+1-r)!\,r!} = \frac{(n+1)!}{(n+1-r)!\,r!} = R$$

~~QED.~~

To prove $[A, B] = 0 \Leftrightarrow (A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$

Base case $n=1$

$(A+B)^1 = A + B$ ✓

Assume proposition is true for $n=k$, $\left((A+B)^k = \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \right)$

then for $n = k+1$

$$(A+B)^{k+1} = (A+B)^k (A+B)$$

$$= (A+B)^k A + (A+B)^k B$$

$$= \left(\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \right) A + \left(\sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i} \right) B$$

$$\underbrace{=}_{AB=BA} \sum_{i=0}^{k} \binom{k}{i} A^{i+1} B^{k-i} + \sum_{i=0}^{k} \binom{k}{i} A^i B^{k-i+1}$$ ~~let $s = i+1$~~

$$= \cancel{\binom{k}{0} A} \; \cancel{\binom{k}{k} A^{k+1}} \; \cancel{+ \sum_{i=0}^{k} \binom{k}{i} A^{i+1} B^{k-i} + \sum_{s=1}^{k} \binom{k}{s-1} A^{s-1} B}$$

~~let j, $i = j+1$ $\Rightarrow$ $\binom{k}{j} + \binom{k}{i} = \binom{k+1}{i}$ (lemma 1)~~

$$= \binom{k+1}{k+1} A^{k+1} + \binom{k+1}{0} B^{k+1}$$

$$+ \sum_{i=0}^{k-1} \binom{k}{i} A^{i+1} B^{k-i} + \sum_{i=1}^{k} \binom{k}{i} A^{i} B^{k+1-i}$$

$$= \sum_{j=1}^{k} \binom{k}{j-1} A^{j} B^{k+1-j} + \sum_{i=1}^{k} \binom{k}{i} A^{i} B^{k+1-i} + A^{k+1} + B^{k+1}$$

$$= \sum_{i=1}^{k} \binom{k}{i-1} A^{i} B^{k+1-i} + \sum_{i=1}^{k} \binom{k}{i} A^{i} B^{k+1-i}$$

$$+ A^{k+1} + B^{k+1}$$

$$= A^{k+1} + B^{k+1} + \sum_{i=1}^{k} \Big(\underbrace{\binom{k}{i-1} + \binom{k}{i}}_{\binom{k+1}{i}} \Big) A^{i} B^{k+1-i}$$

$$= A^{k+1} + B^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} A^{i} B^{k+1-i}$$

$$= \sum_{i=0}^{k} \binom{k+1}{i} A^{i} B^{k+1-i}$$

QED.