To: Michael Barnes

Mathematical Method 1 Ziyan Li

1

① $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$. is also a

2×2 matrix

$\therefore$ closed under addition

② $\alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}$ is also a

2×2 matrix

additive inverse
additive identity

$\therefore$ closed under scalar multiplication

$\therefore$ 2×2 matrix. is a vector space

Dim = 4.

Basis : $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

2. Dim = n^2. basis is ~~a_{ij}~~

$$a_{ij} = \begin{cases} 1 & i = I \quad j = J. \\ 0 & i \neq I \text{ or } j \neq J \end{cases}$$

for all ~~$I \neq J$~~ $I = 1, 2, \cdots, n$ and

$J = 1, 2, \cdots, n$

edit. $\boxed{\text{the basis is } |e_i\rangle\langle e_j|}$

81

3.) Dim = n ✓

4.) Dim = 3

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ ✓

5.) (a) $\alpha(1) + \beta(t) = 0$ for all t.

$\alpha = \beta = 0$.

(b) $\alpha t + \beta t^2 = 0$ for all t.

$\alpha = \beta = 0$.

demonstrate by choosing value of t that forces $\alpha = 0$ or $\beta = 0$.

(c) $\alpha e^t + \beta t = 0$ for all t.

✓

$\alpha = \beta = 0$.

d) $\alpha \sin t + \beta \cos t = 0$ for all t.

$\alpha = \beta = 0$.

6). $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$. ✓

7) (a) $\underline{2}$ orthogonality? (b) $\underline{1}$ ✓ ~~(c) 2~~ $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) $\underline{2}$ ✓

$$\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ are orthogonal

(d). $\underline{4}$. (e) $\underline{3}$ $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ are orthogonal,

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ ~~and~~ $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ are orthogonal.

but $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ are not

8) (a)

$\because |V_1\rangle, \cdots, |V_m\rangle$ are linearly dependent

$\therefore \alpha_1|V_1\rangle + \cdots + \alpha_m|V_m\rangle = 0$ has solution that

at least 2 of $\{\alpha_1, \ldots, \alpha_m\} \neq 0$

let j be the largest integer ~~from~~ between 2 to m

such that $\alpha_j \neq 0$, then

$$|V_j\rangle = -\frac{1}{\alpha_j}\sum_{i=1}^{j-1}\alpha_i|V_i\rangle$$

$\therefore$ ~~$|V_j\rangle$ at least~~ $|V_j\rangle \in span(|V_1\rangle, \cdots, |V_{j-1}\rangle)$

$\therefore$ QED

(b) $\because k$ is the smallest index such that

$|V_k\rangle \in span(|V_1\rangle, \cdots, |V_{k-1}\rangle)$

$\therefore |V_j\rangle \notin span(|V_1\rangle, \cdots, |V_{j-1}\rangle)$ if $j < k$

Use mathematical induction

Base case when $n = 1$, $\alpha_1|V_1\rangle = 0 \Rightarrow \because |V_1\rangle \neq 0 \therefore \alpha_1 = 0$

Induction ~~Proof~~ Assumption : Assume ~~A~~ when $n = p < k$,

$\alpha_1|v_1\rangle + \cdots + \alpha_p|v_p\rangle = 0 \Rightarrow \alpha_1 = \cdots = \alpha_p = 0$

then for $n = p+1 < k$:

let $\alpha_1|v_1\rangle + \cdots + \alpha_{p+1}|v_{p+1}\rangle = 0$ ①, then ~~$\alpha_{p+1} = 0$~~

α_{p+1} has to be 0, otherwise

$$|V_{p+1}\rangle = -\frac{1}{\alpha_{p+1}} \sum_{i=1}^{p} \alpha_i |V_i\rangle$$, which is

contradictory to $|V_{p+1}\rangle \notin \text{span}(|V_1\rangle, \cdots, |V_p\rangle)$

$\therefore$ $\alpha_{p+1} = 0$ ~~so~~ , then ① becomes

$\alpha_1 |V_1\rangle + \cdots + \alpha_p |V_p\rangle = 0$, which by induction

assumption ~~is~~ yields

$\alpha_1 = \cdots = \alpha_p = 0$

$\therefore$ $\alpha_1 = \cdots = \alpha_{p+1} = 0$

$\therefore$ As long as $j < k$ ~~and~~ i.e $|V_i\rangle \notin \text{span}(|V_1\rangle, \cdots, |V_{j-1}\rangle)$

then $|V_1\rangle, \cdots, |V_j\rangle$ are linearly independent

The largest such j is $j = k-1$

$\therefore$ $|V_1\rangle, \cdots, |V_{k-1}\rangle$ are linearly independent.

✓

9. We use mathematical induction on k eigenvectors of A

Base case $k=2$

$$A|v_1\rangle = \lambda_1|v_1\rangle \qquad A|v_2\rangle = \lambda_2|v_2\rangle \qquad \lambda_1 \neq \lambda_2.$$

we let $\alpha_1|v_1\rangle + \alpha_2|v_2\rangle = |0\rangle$ ①

then $A(\alpha_1|v_1\rangle + \alpha_2|v_2\rangle) = A|0\rangle = |0\rangle$

$\therefore$ $\alpha_1\lambda_1|v_1\rangle + \alpha_2\lambda_2|v_2\rangle = |0\rangle$ ②

λ_1 ① $\Rightarrow$ $\alpha_1\lambda_1|v_1\rangle + \alpha_2\lambda_1|v_2\rangle = |0\rangle.$ ③

③ $-$ ② $\Rightarrow$ $\alpha_2(\lambda_1 - \lambda_2)|v_2\rangle = |0\rangle$

$\because$ $\lambda_1 \neq \lambda_2$, $|v_2\rangle \neq |0\rangle$

$\therefore$ $\alpha_2 = 0$ $\quad\therefore$ $\alpha_1 = 0$

$\therefore$ $\alpha_1 = \alpha_2 = 0$ $\quad\therefore$ $|v_1\rangle$ and $|v_2\rangle$ are linearly independant

$\therefore$ ~~proof~~ proposition shown for $k=2$.

Assumption : proposition is true for $k = m-1$

then for $k=m$

let $\sum_{i=1}^{m} \alpha_i|v_i\rangle = |0\rangle$ ④, then $A\sum_{i=1}^{m}\alpha_i|v_i\rangle = |0\rangle.$

$\therefore$ $\sum_{i=1}^{m}\alpha_i\lambda_i|v_i\rangle = |0\rangle.$ ⑤

λ_1 ④ $\Rightarrow$ $\sum_{i=1}^{m}\alpha_i\lambda_1|v_i\rangle = |0\rangle$ ⑥

⑤ $-$ ⑥ $\Rightarrow$ $\sum_{i=2}^{m}\alpha_i(\lambda_i - \lambda_1)|V_i\rangle = |0\rangle$ ⑦

The L.H.S of 3 is a sum of ~~m-1~~ scalar multiples of $m-1$ eigenvectors of A, and by inductive assumption all the ~~factors~~ coefficients are 0 ($|v_2\rangle, \cdots, |v_m\rangle$ are linearly independent, all $\alpha_i(\lambda_i-\lambda_1)$'s are 0)

$\therefore \quad \alpha_i(\lambda_i-\lambda_1) = 0 \quad$ for $i = 2, \cdots, m$

$\because$ all eigenvalues are distinct, i.e. $\lambda_i - \lambda_1 \neq 0$

$\therefore \quad \alpha_2 = \alpha_3 = \cdots = \alpha_m = 0$

$\therefore \alpha_1 = 0 \quad$ (substitute into 4)

$\therefore \quad \alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$

$\therefore \quad |v_1\rangle, \cdots, |v_m\rangle$ are linearly independent.

10. let $\sum_{i=1}^{n} \alpha_i \vec{a}_i = 0$ for $\alpha_1, \dots \alpha_n \in \mathbb{R}$

$\because$ $\vec{a}_1, \dots, \vec{a}_n$ are mutually perpendicular (orthogonal)

$\therefore$ $\vec{a}_i \cdot \vec{a}_j = \begin{cases} 0 & i \neq j \\ |\vec{a}_j|^2 & i = j \end{cases}$

$0 = \vec{a}_j \cdot \left(\sum_{i=1}^{n} \alpha_i \vec{a}_i \right) = \alpha_j |\vec{a}_j|^2$

$\because \vec{a}_j \neq \vec{0}$ $\therefore$ $\alpha_j = 0$ for all $j = 1, \dots, n$

Hence $\vec{a}_1, \dots, \vec{a}_n$ are linearly independent

11. $\underline{u}^{+} \underline{u} = 1 \Rightarrow \alpha^2 (\cancel{1+i}\ 1-i \quad 1+i) \begin{pmatrix} 1+i \\ 1-i \end{pmatrix} = 1$

$\Rightarrow \alpha^2 (2+2) = 1$ $\therefore$ $\alpha^2 = \frac{1}{4} \Rightarrow \alpha = \pm\frac{1}{2}$

$\underline{v}^{+} \underline{v} = 1 \Rightarrow \beta^2 (1+i \quad 1-i) \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} = 1$

$\therefore \beta^2 (4) = 1$ $\therefore \beta^2 = \frac{1}{4} \Rightarrow \beta = \pm\frac{1}{2}$

we choose $\underline{\alpha = \beta = \frac{1}{2}}$

$\therefore$ $\underline{u} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$ $\underline{V} = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

$\therefore \cancel{u^{+} v = \frac{1}{4} (1+i \quad 1-i) \begin{pmatrix} 1-i \\ 1+i \end{pmatrix} =}$

$u^{+} v = \frac{1}{4} (1-i \quad 1+i) \begin{pmatrix} 1-i \\ 1+i \end{pmatrix}$

$= \frac{1}{4} \left((1-i)^2 + (1+i)^2 \right)$

$= \frac{1}{4} (-2i + 2i) = 0$

let $\alpha \underline{u} + \beta \underline{v} = 0$

$\therefore \frac{1}{2}(1+i)\alpha + \frac{1}{2}(1-i)\beta = 0$ ①

$\frac{1}{2}(1-i)\alpha + \frac{1}{2}(1+i)\beta = 0$ ②

$\cancel{① \Rightarrow \alpha = \frac{1+i}{1-i}\beta = \frac{2i}{2}\beta = i\beta}$

~~Subsit~~ substitute into $\cancel{② \Rightarrow \frac{1}{2}(1-i)\alpha + \frac{1}{2}(1+i)i\beta}$

① $\beta = -\frac{1+i}{1-i}\alpha = -\frac{2i}{2}\alpha = -i\alpha$

$\rightarrow$ ② $\Rightarrow \frac{1}{2}(1-i)\alpha + \frac{1}{2}(1+i)(-i\alpha) = 0$

$\Rightarrow \frac{1}{2}(1-i)\alpha + \frac{1}{2}(1-i)\alpha = 0$

$\therefore (1-i)\alpha = 0 \Rightarrow \underline{\alpha = 0 \Rightarrow \beta = 0}$

$\therefore$ $\underline{u}$ and $\underline{v}$ are linearly independant

$\rightarrow$ Yes there are further linearly independent vectors.

2D complex vector space has only sets of 2 LI vectors

for example $\cancel{\begin{pmatrix} 1 \\ i \end{pmatrix}}$ ~~and~~ $\cancel{\begin{pmatrix} i \\ 1 \end{pmatrix}}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2i-1 \end{pmatrix}$, then

$$\begin{pmatrix} 1 \\ i \end{pmatrix} = \left(\frac{1}{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\frac{1}{2}\right)\begin{pmatrix} 1 \\ 2i-1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ i \end{pmatrix} = a\underline{u} + b\underline{v}$

a and b?

12. First we prove the Cauchy-Schwarz inequality.

$$\cancel{|\langle v|w\rangle|^2 \le |\langle v|v\rangle||\langle w|w}$$

$$|\langle v|w\rangle|^2 \le \langle v|v\rangle \langle w|w\rangle$$

let $|u\rangle = |v\rangle + \lambda|w\rangle$

then $0 \le \langle u|u\rangle = (\langle v| + \lambda^* \langle w|)(|v\rangle + \lambda|w\rangle)$

$$= \langle v|v\rangle + \lambda\langle v|w\rangle + \lambda^*\langle w|v\rangle + |\lambda|^2\langle w|w\rangle$$

Now we choose ~~$|w\rangle$ to be~~ λ to be

$$\lambda = -\frac{\langle w|v\rangle}{\langle w|w\rangle} \quad \text{then} \quad \lambda^* = -\frac{\langle v|w\rangle}{\langle w|w\rangle}$$

$$0 \le \langle v|v\rangle - \frac{\langle w|v\rangle\langle v|w\rangle}{\langle w|w\rangle} - \cancel{\frac{\langle v|w\rangle\langle w|v\rangle}{\langle w|w\rangle}}$$

$$+ \cancel{\frac{\langle w|v\rangle\langle v|w\rangle}{\langle w|w\rangle^2}\langle w|w\rangle}$$

$$\therefore 0 \le \langle v|v\rangle - \frac{|\langle v|w\rangle|^2}{\langle w|w\rangle}$$

$$\therefore |\langle v|w\rangle|^2 \le \langle v|v\rangle\langle w|w\rangle$$

only for $\langle v|w\rangle$ real. See complex case in tutorial.

$$\therefore \langle v|w\rangle \le ||v\rangle|\,||w\rangle| \quad \text{(taking the square root}$$

Then, for the triangle inequality

$$(||v\rangle| + ||w\rangle|)^2 = ||v\rangle|^2 + ||w\rangle|^2 + 2||v\rangle|\,||w\rangle|$$

$$\ge ||v\rangle|^2 + ||w\rangle|^2 + 2\langle v|w\rangle$$

} the Cauchy-Schwarz inequality

$$= \langle v|v\rangle + \langle w|w\rangle + 2\langle v|w\rangle = (\langle v| + \langle w|)(|v\rangle + |w\rangle)$$

$$= ||v\rangle + |w\rangle|^2$$

$$\therefore \underline{||v\rangle + |w\rangle| \le ||v\rangle| + ||w\rangle|} \quad \text{QED.}$$

13. let $\underline{u} = (u_1, u_2, u_3)^T \alpha$ to be the required vec

(a) $(1, 2, 3)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow u_1 + 2u_2 + 3u_3 = 0$ ①

$(-1, -1, 1)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow -u_1 - u_2 + u_3 = 0$ ②

① + ② $\Rightarrow u_2 + 4u_3 = 0$ let $u_3 = 1$, $u_2 = -4$

Then $u_1 = 5$ $\therefore$ ~~$\underline{u} = (5, -4, 1)^T$~~

$\therefore \underline{u} = \alpha(5, -4, 1)^T$. to normalise $\alpha = \frac{1}{\sqrt{25+16+1}} = \frac{1}{\sqrt{42}}$

$\therefore \boxed{\underline{u} = \frac{1}{\sqrt{42}}(5, -4, 1)^T}$

$\frac{1}{\sqrt{14}}(1, 2, 3)^T$

$\frac{1}{\sqrt{3}}(-1, -1, 1)^T$

(b) ~~$(1+i\sqrt{3}, 2, 1-i\sqrt{3})^T$~~

$(1-i\sqrt{3}, 2, 1+i\sqrt{3})\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow (1-i\sqrt{3})u_1 + 2u_2 + (1+i\sqrt{3})u_3 = 0$ ①

$(1, -1, 1)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = 0 \Rightarrow u_1 - u_2 + u_3 = 0$ ②

① + 2② $\Rightarrow (3-i\sqrt{3})u_1 + (3+i\sqrt{3})u_3 = 0$.

$\therefore u_1 = -\frac{3+i\sqrt{3}}{3-i\sqrt{3}}u_3 = -\frac{6+6\sqrt{3}i}{12}u_3 = -\frac{1+\sqrt{3}i}{2}u_3$.

$\therefore$ let $u_1 = 1+\sqrt{3}i$, $u_3 = -2 \Rightarrow u_2 = -1+\sqrt{3}i$

$\therefore \underline{u} = \alpha(1+\sqrt{3}i, -1+\sqrt{3}i, -2)$ $\alpha = \frac{1}{2\sqrt{3}}$

$\therefore \boxed{\underline{u} = \frac{1}{2\sqrt{3}}(1+\sqrt{3}i, -1+\sqrt{3}i, -2)^T}$

$\frac{1}{2\sqrt{3}}(1+i\sqrt{3}, 2, 1-i\sqrt{3})^T$, $\frac{1}{\sqrt{3}}(1, -1, 1)^T$

(c) $(1-i, 1, 3i)$, $(1+2i, 2, 1)$

$$\Downarrow$$

$(1+i)u_1 + u_2 - 3iu_3 = 0$ ①

$(1-2i)u_1 + 2u_2 + u_3 = 0$ ②

$\cancel{2① - ② \Rightarrow (3+4i)u_1 - 7iu_3 = 0}$

$\cancel{(3+4i)u_1}$

2① $(2+2i)u_1 + 2u_2 - 6iu_3 = 0$

2① − ② $\Rightarrow$ $(1+4i)u_1 - (1+6i)u_3 = 0$

$\therefore \quad u_1 = \dfrac{1+6i}{1+4i}u_3 = \dfrac{25+2i}{17}u_3$

$\therefore \; u_1 = 25+2i \qquad u_3 = 17 \Rightarrow$

$u_2 = 3iu_3 - (1+i)u_1$

$= 51i - (1+i)(25+2i)$

$= 51i - (25-2+27i) = -23+24i$

$\therefore \; \underline{u} = \alpha(25+2i, -23+24i, 17)$

$\alpha = \dfrac{1}{17\sqrt{7}}$

$$\boxed{\therefore \underline{u} = \frac{1}{17\sqrt{7}}(25+2i, -23+24i, 17)^T}$$ X

$$\boxed{\begin{array}{l}\dfrac{1}{2\sqrt{3}}(1-i, 1, 3i)^T \\[2ex] \dfrac{1}{\sqrt{10}}(1+2i, 2, 1)^T\end{array}}$$

not orthogonal to either of 2 other vectors (check)

easy way is to set $\underline{u}_3 = \underline{u}_1 \times \underline{u}_2$ and normalise.

Q. 14. $\vec{x}_1 = (0,0,1,1)^T$

$$\cancel{\varepsilon = \frac{1}{\sqrt{2}}(0,0,1,1)} \qquad \hat{\varepsilon}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$\vec{x}_2' = \vec{x}_2 - (\vec{x}_2 \cdot \hat{\varepsilon}_1)\,\hat{\varepsilon}_1$$

$$= \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} - \left(\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right) \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} - \cancel{\frac{1}{\sqrt{2}}}\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\-1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}1\\0\\-1/2\\1/2\end{pmatrix}$$

$$\hat{\varepsilon}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$\vec{x}_3' = \vec{x}_3 - (\vec{x}_3 \cdot \hat{\varepsilon}_1)\,\hat{\varepsilon}_1 - (\vec{x}_3 \cdot \hat{\varepsilon}_2)\,\hat{\varepsilon}_2$$

$$= \begin{pmatrix}1\\2\\0\\2\end{pmatrix} - \left(\begin{pmatrix}1\\2\\0\\2\end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right) \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$- \left(\begin{pmatrix}1\\2\\0\\2\end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}\right) \frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\0\\2\end{pmatrix} - \underbrace{\frac{1}{\sqrt{2}}(2)\frac{1}{\sqrt{2}}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{1}{\sqrt{6}}(4)\frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\0\\2\end{pmatrix} - \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1/3\\2\\-1/3\\1/3\end{pmatrix} = \cancel{\frac{1}{\sqrt{39}}\begin{pmatrix}-1\\6\\-1\\1\end{pmatrix}} \qquad \therefore \hat{\varepsilon}_3 = \frac{1}{\sqrt{39}}\begin{pmatrix}-1\\6\\-1\\1\end{pmatrix}$$

$$\vec{x_4}' = \vec{x_4} - (\vec{x_4}\cdot\hat{\varepsilon_1})\hat{\varepsilon_1} - (\vec{x_4}\cdot\hat{\varepsilon_2})\hat{\varepsilon_2} - (\vec{x_4}\cdot\hat{\varepsilon_3})\hat{\varepsilon_3}$$

$$= \begin{pmatrix}2\\1\\1\\1\end{pmatrix} - \left(\begin{pmatrix}2\\1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$- \left(\begin{pmatrix}2\\1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}\right)\frac{1}{\sqrt{6}}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$- \left(\begin{pmatrix}2\\1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{39}}\begin{pmatrix}-1\\6\\-1\\1\end{pmatrix}\right)\frac{1}{\sqrt{39}}\begin{pmatrix}-1\\6\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\1\\1\\1\end{pmatrix} - \cancel{\frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{\cancel{4}\,2}{\cancel{6}\,3}\begin{pmatrix}2\\0\\-1\\1\end{pmatrix}$$

$$- \frac{4}{39}\begin{pmatrix}-1\\6\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2-0-4/3+4/39\\1-0-0-24/39\\1-1+2/3+4/39\\1-1-2/3-4/39\end{pmatrix} = \begin{pmatrix}10/13\\5/13\\10/13\\-10/13\end{pmatrix}$$

$$\therefore \boxed{\hat{\varepsilon_4} = \frac{1}{\sqrt{13}}\begin{pmatrix}2\\1\\2\\-2\end{pmatrix}}$$

$$\therefore \boxed{\begin{aligned}\hat{\varepsilon_1} &= \tfrac{1}{\sqrt{2}}(0,0,1,1)^T & \hat{\varepsilon_2} &= \tfrac{1}{\sqrt{6}}(2,0,-1,1)^T\\ \hat{\varepsilon_3} &= \tfrac{1}{\sqrt{39}}(-1,6,-1,1)^T & \hat{\varepsilon_4} &= \tfrac{1}{\sqrt{13}}(2,1,2,-2)^T\end{aligned}}$$

15.

- $\langle f | g+h \rangle = \int_{-\pi}^{\pi} dt\, f^*(t) \, (g(t) + h(t))$

$= \int_{-\pi}^{\pi} dt\, f^*(t)\, g(t) + \int_{-\pi}^{\pi} dt f^*(t)\, h(t)$

$= \langle f | g \rangle + \langle f | h \rangle$ ①

- $\langle f | \alpha g \rangle = \int_{-\pi}^{\pi} dt f^*(t)\, \alpha\, g(t) = \alpha \int_{-\pi}^{\pi} dt f^*(t) g(t)$

$= \alpha \langle f | g \rangle$ ②

- $\cancel{\langle f | g} \; \langle g | f \rangle = \int_{-\pi}^{\pi} dt\, g^*(t)\, f(t)$

$= \lim_{\Delta x \to 0} \sum_{n=1}^{2\pi/\Delta x} \Delta x\, g^*(n\Delta x)\, f(n\Delta x)$

$= \lim_{\Delta x \to 0} \sum_{n=1}^{2\pi/\Delta x} \left(\Delta x\, g(n\Delta x)\, f^*(n\Delta x) \right)^*$

$= \left(\lim_{\Delta x \to 0} \sum_{n=1}^{2\pi/\Delta x} \Delta x\, f^*(n\Delta x)\, g(\Delta x) \right)^*$

$= \left(\int_{-\pi}^{\pi} dt\, f^*(t)\, g(t) \right)^* = \langle f | g \rangle^*$ ③

- $\langle f | f \rangle = \int_{-\pi}^{\pi} dt\, f^*(t) f(t) = \int_{-\pi}^{\pi} dt |f(t)|^2$

$|f(t)|^2 \geq 0$ in $[\pi, \pi]$ and $= 0$ only if $f(t) = 0$

$\therefore$ $\langle f | f \rangle \geq 0$ and $= 0$ only if $f(t) = 0$ over $[-\pi, \pi]$. ④

~ ①, ②, ③, ④ $\therefore$ it is an inner product

(a) $\langle \sin t | \cos t \rangle = \int_{-\pi}^{\pi} dt \sin t^* \cos t$

$$= \int_{-\pi}^{\pi} dt \sin t \cos t$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dt \sin(2t) = \frac{1}{4}\left[-\cos(2t)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4}\left[-\underset{0}{\cos(2\pi)} + \underset{0}{\cos(2\pi)}\right] = 0$$

$\therefore$ orthogonal

(b)

$\langle \exp(int) | \exp(ikt) \rangle$

$$= \int_{-\pi}^{\pi} dt \left(\exp(int)\right)^* \left(\exp(ikt)\right)$$

$$= \int_{-\pi}^{\pi} \exp(-int) \exp(ikt) dt$$

$$= \int_{-\pi}^{\pi} \exp(i(k-n)t) dt$$

if $k=n$, $\langle \exp(int) | \exp(ikt) \rangle = 2\pi$

if $k \neq n$

$\langle \exp(int) | \exp(ikt) \rangle$

$= \frac{1}{i(k-n)} \exp(i(k-n)t) \Big|_{-\pi}^{\pi}.$

$= \frac{\exp(i(k-n)\pi) - \exp(-i(k-n)\pi)}{i(k-n)}.$

$= \frac{2i\sin((k-n)\pi)}{ik-n}.$

$= \frac{2}{k-n} \sin((k-n)\pi)$

$\because$ n.k are integers

$\therefore$ $k-n$ = integer $\therefore \sin((k-n)\pi) = 0$

$\therefore \langle \exp(int) | \exp(ikt) \rangle = 0$

overall : $\boxed{\langle \exp(int) | \exp(ikt) \rangle = 2\pi\delta_{nk}}$

$\therefore$ orthogonal ~~+f n+k~~

(c) ~~$\langle f | g \rangle =$~~ $\langle t^2 | t^4 \rangle$

$= \int_{-\pi}^{\pi} dt (t^2)^* (t^4) = \int_{-\pi}^{\pi} dt (t^6)$

$= \frac{t^7}{7} \Big|_{-\pi}^{\pi} = \frac{\pi^7 - (-\pi)^7}{7} = \frac{\pi^7 - (-\pi^7)}{7}$

$= \frac{2}{7}\pi^7 \neq 0$

$\therefore$ Not orthogonal.

16.

~~$\int(a, \int(a, b)$~~

① $S(\underline{a}, \underline{b}+\underline{c}) = \underline{a}^{\dagger}M(\underline{b}+\underline{c})$

$= \underline{a}^{\dagger}M\underline{b} + \underline{a}^{\dagger}M\underline{c} = S(\underline{a}, \underline{b}) + S(\underline{a}, \underline{c})$

~~automatically satisfied~~ true if M is any matrix

② $S(\underline{a}, \alpha\underline{b}) = \underline{a}^{\dagger}M(\alpha\underline{b}) = \alpha\,\underline{a}^{\dagger}M\underline{b} = \alpha S(\underline{a}, \underline{b})$

true if M is any matrix

③ ~~$S(\underline{a}, b$~~ $S(\underline{a}, \underline{b}) = (S(\underline{b}, \underline{a}))^{*}$

$\Rightarrow a^{\dagger}Mb = (b^{\dagger}Ma)^{*} = (b^{\dagger}Ma)^{\dagger}$

$= a^{\dagger}M^{\dagger}b$

$\Rightarrow M = M^{\dagger} \Rightarrow M$ is hermitian

④ $S(\underline{a}, \underline{a}) \geq \underline{0}$ and $= \underline{0}$ only if $\underline{a} = 0$

$\Rightarrow \underline{a}^{\dagger}M\underline{a} \geq 0$ for any $\underline{a}$

$\Rightarrow M$ is positive definite

$\because M$ is hermitian $\therefore$ all eigenvalues are real

let $\hat{M} = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ (λ_i are eigenvalues)

and $\hat{M} = P^{\dagger}MP$, then $M = P\hat{M}P^{T}$

$\therefore a^{\dagger}Ma = a^{\dagger}\underbrace{P\hat{M}P^{\dagger}a}_{} = U^{\dagger}\hat{M}U$

~~$U = a$~~ $\underline{U} = P^{\dagger}\underline{a}$

$\underline{u} = (u_1, \cdots, u_n)^{T}$

$\therefore \; a^{+} M a = U^{+} \hat{M} U = \lambda_1 |u_1|^2 + \lambda_2 |u_2|^2 + \cdots + \lambda_n |u_n|^2$

$a^{+} M a \geq 0$ always hold true

iff $\lambda_1, \cdots, \lambda_n > 0$

$\therefore$ Overall, the condition is that

$\boxed{\text{M is hermitian and all eigenvalues of M are positive}}$

17. In real vector space.

① $\langle A|B+C\rangle = tr(A(B+C))$

showed show

$= tr(AB+AC) = tr(AB) + tr(AC)$

$= \langle A|B\rangle + \langle A|C\rangle$ ✓

② $\langle A|\alpha B\rangle = tr(A\alpha B) = tr(\alpha AB)$

$= \alpha\, tr(AB) = \alpha\langle A|B\rangle$ ✓ ✓

③ $\langle A|B\rangle = tr(AB) = (AB)_{ii} = A_{ij}B_{ji}$ ✓

$= B_{ji}A_{ij} = (BA)_{jj} = tr(BA) = \langle B|A\rangle$ ✓

④ $\because$ $A, B \in V$ and V is the vector space of all real symmetric matrix

$\therefore$ all eigenvalues of $A \in V$ are real

$\therefore$ ~~$\langle A|A\rangle = tr(A^2) =$~~ let the diagonalisation of A be $\hat{A} = P^T A P = diag(\lambda_1, \cdots, \lambda_n)$

then $tr(\hat{A}^2) = tr(P\hat{A}\underbrace{P^T P}_{I}\hat{A}P^T) = tr(P(\hat{A}^2P^T))$

$= tr(\cancel{P}A^2\underbrace{P^TP}_{I}) = tr(A^2)$

$\therefore \langle A|A\rangle = tr(A^2) = tr(\hat{A}^2) = \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2$

$\because \lambda_1, \cdots, \lambda_n \in \mathbb{R}$ $\therefore$ $\langle A|A\rangle \geq 0$ and $= 0$ only if $A = [0]$. ✓ /

$\therefore$ $\langle A|B\rangle = tr(AB)$ is a scalar product in V

$\langle A|A\rangle = tr(A^2)$

$tr(AB) = A_{ij}B_{ji}$

$tr(A^2) = A_{ij}A_{ji} = (A_{ij})^2$

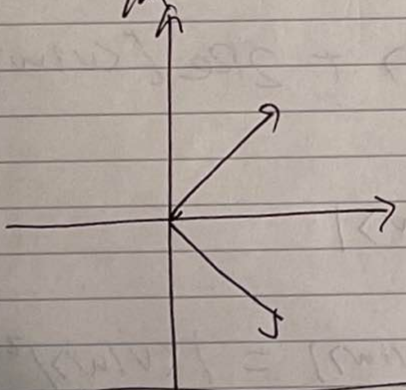
$$\underline{u} = \begin{pmatrix} x_r + ix_i \\ y_r + iy_i \\ z_r + iz_i \end{pmatrix} \qquad \underline{v} = \begin{pmatrix} a_r + ia_i \\ b_r + ib_i \\ c_r + ic_i \end{pmatrix}.$$

$$= \underbrace{\begin{pmatrix} x_r \\ y_r \\ z_r \end{pmatrix}}_{\underline{u_r}} + i\underbrace{\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}}_{\underline{u_i}} \qquad = \underbrace{\begin{pmatrix} a_r \\ b_r \\ c_r \end{pmatrix}}_{\underline{v_r}} + i\underbrace{\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}}_{\underline{v_i}}$$

$\underline{u} \times \underline{v} =$

$$= (\underline{u_r} + i\underline{u_i}) \times (\underline{v_r} + i\underline{v_i})$$

$$= \underline{u_r} \times \underline{v_r} + i(\underline{u_r} \times \underline{v_i} + \underline{u_i} \times \underline{v_r})$$
$$- \underline{u_i} \times \underline{v_i}$$



$\cancel{U^T(\underline{u} \times \underline{v}) = 0}$

$w = u \times v$

$U^T w = 0$

$(U^T w)^\dagger = 0$

$w^\dagger (U^T)^\dagger = 0$

$$\boxed{\frac{1}{\sqrt{128}} \begin{pmatrix} 1+6i \\ -4i-7 \\ 1+5i \end{pmatrix}}$$

same as mine.

$(U^T)^\dagger = (U_1 U_2 U_3)^\dagger = \begin{pmatrix} U_1^* \\ U_2^* \\ U_3^* \end{pmatrix}$

$(U^T)^\dagger$ = complex conjugate each element orthonormal

$|u\rangle$, $|v\rangle$ are basis

$$|w\rangle = I|w\rangle = |u\rangle\langle u|w\rangle + |v\rangle\langle v|w\rangle$$

$$(\langle v| + \langle w|)(|v\rangle + |w\rangle)$$

$$= \langle v|v\rangle + \langle w|w\rangle + \langle v|w\rangle + \langle w|v\rangle$$

$$= \langle v|v\rangle + \langle w|w\rangle + \langle v|w\rangle + \langle v|w\rangle^*$$

$$= \langle v|v\rangle + \langle w|w\rangle + 2Re[\langle v|w\rangle]$$

$$Re[\langle v|w\rangle] \leq |\langle v|w\rangle|$$

$$\because Re^2[\langle v|w\rangle] + Im^2[\langle v|w\rangle] = |\langle v|w\rangle|^2$$

$$\leq \langle v|v\rangle + \langle w|w\rangle + 2|\langle v|w\rangle|$$

$$\leq \langle v|v\rangle\langle w|w\rangle$$

$A|v\rangle = \lambda|v\rangle$

$A^\dagger|v\rangle = a^*|v\rangle$