To: Michael Barnes

Quantum Mechanics 7 Ziyan Li

1.
$$ {}^{3}_{1}H \rightarrow {}^{3}_{2}He^{+} + {}^{0}_{-1}e^{-} $$

1 proton, 2 neutron, 1 electron
charge neutral

2 proton, 1 neutron, 1 electron
charge +1

speed of the departing electron:

$$ E_1 = \frac{1}{2} m_e V_1^2 \rightarrow V_1 = \sqrt{\frac{2E_1}{m_e}} = \left(\frac{2 \times 16 \times 1000 \times 1.6 \times 10^{-19} J}{9.11 \times 10^{-31} kg} \right)^{\frac{1}{2}} $$

$$ = 7.5 \times 10^{7} \ m/s $$

Size of atom is on the scale of the Bohr radius

$$ a_0 = 5.29 \times 10^{-11} \ m \quad \text{for Hydrogen} $$

$\therefore$ typical time taken for the electron to leave is

$$ t_1 \sim \frac{a_0}{V_1} = \frac{5.29 \times 10^{-11} m}{7.5 \times 10^{7} m/s} \approx 7.1 \times 10^{-19} s $$

Consider the Bohr Model: the orbital speed of the orbiting electron is

$$ \frac{e^2}{4\pi\varepsilon_0 a_0^2} = \frac{V_2^2 m_e}{a_0} \rightarrow V_2 = \sqrt{\frac{e^2}{4\pi\varepsilon_0 a_0 m_e}} $$

$$ = \sqrt{\frac{(1.6 \times 10^{-19})^2}{4\pi \times 8.854 \times 10^{-12} \times 5.29 \times 10^{-11} \times 9.11 \times 10^{-31}}} $$

$$ = 2.2 \times 10^{6} \ m/s $$

Period of orbit is

$$ t_2 = \frac{2\pi a_0}{V_2} = \frac{2\pi \times 5.29 \times 10^{-11} m}{2.2 \times 10^{6} m/s} = 1.5 \times 10^{-16} s $$

$\frac{t_1}{t_2} = 0.47\%$ is very small

$\therefore$ As the departing electron leaves the atom, the orbiting ~~of~~ electron has not moved by a significant amount.

$\rightarrow$ The departure can be treated as sudden.

Use the sudden approximation:

$\because$ The atom is in the ground state of ${}^3_1H$ $|1,0,0;Z=1\rangle$ before the depature, it will remain in the state $|1,0,0;Z=1\rangle$ after the departure.

$\rightarrow$ Expand $|1,0,0;Z=1\rangle$ in terms of the ~~states~~ energy eigenstates of ${}^3_2He^+$, then the modulus square of probability amplitude represents the probability

$\therefore$ $\langle 1,0,0;Z=2|1,0,0;Z=1\rangle$

$$= \int_0^\infty r^2 dr\, U_{n=1}^{l=0}(r, Z=2) \times U_{n=1}^{l=0}(r, Z=1)$$

$$= \frac{2}{(a_0)^{3/2}} \times \frac{2}{\left(\frac{a_0}{2}\right)^{3/2}} \int_0^\infty dr\, r^2 e^{-r/a_0} e^{-2r/a_0}$$

$$= 2^{7/2} a_0^{-3} \int_0^\infty dr\, r^2 e^{-3r/a_0}$$

$u = \frac{3}{a_0} r$

$du = \frac{3}{a_0} dr$

$dr = \frac{a_0}{3} du$

$r^2 = \left(\frac{a_0}{3}\right)^2 u^2$

$$= 2^{7/2} a_0^{-3} \left(\frac{a_0}{3}\right)^3 \int_0^\infty \underbrace{u^2 e^{-u}\, du}_{2!}$$

$$= 2^{9/2} \left(\frac{1}{3}\right)^3 = \cancel{\frac{2^9}{3^6}}$$

$$P(\text{excited}) = 1 - P(\text{ground})$$

$$= 1 - \left| \langle 1,0,0 \,;\, Z=2 \mid 1,0,0 \,;\, Z=1 \rangle \right|^2$$

$$= 1 - \frac{2^9}{3^6} = \boxed{0.298}$$

2. $\hat{L}^2 = (\hat{\underline{x}} \times \hat{\underline{p}}) \cdot (\hat{\underline{x}} \times \hat{\underline{p}}) = \sum_{ijklm} \varepsilon_{ijk} \hat{x}_j \hat{p}_k \varepsilon_{ilm} \hat{x}_l \hat{p}_m$

$= \varepsilon_{ijk} \varepsilon_{ilm} \hat{x}_j \hat{p}_k \hat{x}_l \hat{p}_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \hat{x}_j \hat{p}_k \hat{x}_l \hat{p}_m$

$= \hat{x}_j \hat{p}_k \hat{x}_j \hat{p}_k - \hat{x}_j \hat{p}_k \hat{x}_k \hat{p}_j$

$= \hat{x}_j (\hat{x}_j \hat{p}_k - [\hat{x}_j, \hat{p}_k]) \hat{p}_k - \hat{x}_j \hat{p}_k (\hat{p}_j \hat{x}_k + [\hat{x}_k, \hat{p}_j])$

$= \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - \hat{x}_j (i\hbar \delta_{jk}) \hat{p}_k - \hat{x}_j \hat{p}_k \hat{p}_j \hat{x}_k - \hat{x}_j \hat{p}_k (i\hbar \delta_{kj})$

$= \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - i\hbar \hat{x}_j \hat{p}_j - \cancel{\hat{x}_j \hat{p}_j \hat{x}_k} \hat{x}_j \hat{p}_j \hat{p}_k \hat{x}_k$
$- i\hbar \hat{x}_j \hat{p}_j$

$= \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - 2i\hbar \hat{x}_j \hat{p}_j - \hat{x}_j \hat{p}_j (\hat{x}_k \hat{p}_k - [\hat{x}_k, \hat{p}_k])$

$= \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - 2i\hbar \hat{x}_j \hat{p}_j - \hat{x}_j \hat{p}_j \hat{x}_k \hat{p}_k + 3i\hbar \hat{x}_j \hat{p}_j$

$= \hat{x}_j \hat{x}_j \hat{p}_k \hat{p}_k - \hat{x}_j \hat{p}_j \hat{x}_k \hat{p}_k + i\hbar \hat{x}_j \hat{p}_j$

$\cancel{= \hat{r} \hat{r}}$

$= (\underline{r} \cdot \underline{r})(\underline{p} \cdot \underline{p}) - (\underline{r} \cdot \underline{p})(\underline{r} \cdot \underline{p}) + i\hbar (\underline{r} \cdot \underline{p})$

$= r^2 p^2 - (\underline{r} \cdot \underline{p})^2 + i\hbar (\underline{r} \cdot \underline{p})$

$\rightarrow \quad p^2 = \frac{L^2}{r^2} + \frac{1}{r^2} \left\{ (\underline{r} \cdot \underline{p})^2 - i\hbar (\underline{r} \cdot \underline{p}) \right\}$

$(\hat{\underline{x}} = \underline{r})$

$\hat{\underline{r}} = \frac{\underline{r}}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}, \quad \partial_i r = \frac{\partial r}{\partial x_i} = \frac{2x_i}{2r} = \frac{x_i}{r},$

$\cancel{(\underline{p} \cdot \hat{\underline{r}} - \hat{\underline{r}} \cdot \underline{p})\psi = p_i x_i \; p_i} \qquad \partial_i x_j = \delta_{ij}, \quad \hat{x}_i = x_i$

$\cancel{\Rightarrow p_i \left(\frac{x_i}{r}\right) = -i\hbar \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = -3i\hbar \frac{1}{r} - i\hbar x_i \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right)}$

$\cancel{= -3i\hbar \frac{1}{r} + i\hbar x_i \frac{1}{r^2} \frac{x_i}{r} = \frac{-2i\hbar}{r}}$

$(\underline{p} \cdot \hat{\underline{r}} - \hat{\underline{r}} \cdot \underline{p})\psi = \cancel{p_i x_i} \; \hat{p}_i \frac{\hat{x}_i \psi}{r} - \frac{\hat{x}_i}{r} \hat{p}_i \psi$

$= -i\hbar \partial_i \left(\frac{x_i \psi}{r}\right) + \frac{x_i}{r} i\hbar \partial_i \psi$

Could also use $\underline{p} = -i\hbar \nabla$

$$= -i\hbar \cancel{\frac{x_i}{r}\partial_i\psi} - i\hbar\,\partial_i\left(\frac{x_i}{r}\right)\psi + i\hbar \cancel{\frac{x_i}{r}\partial_i\psi}$$

$$= \left[-i\hbar\frac{1}{r}\underbrace{\partial_i x_i}_{=\delta_{ii}=3} - i\hbar\, x_i\, \partial_i\left(\frac{1}{r}\right)\right]\psi$$

$$= \left[\frac{-3i\hbar}{r} - i\hbar\, x_i\left(-\frac{1}{r^2}\right)\underbrace{\partial_i r}_{\frac{x_i}{r}}\right]\psi$$

$$= -\frac{3i\hbar}{r}\psi + i\hbar\frac{1}{r}\underbrace{\left(\frac{x_i x_i}{r^2}\right)}_{1}\psi$$

$$= \left[-\frac{3i\hbar}{r} + \frac{i\hbar}{r}\right]\psi = -\frac{2i\hbar}{r}\psi$$

$$\to \quad \underline{p}\cdot\hat{\underline{r}} - \hat{\underline{r}}\cdot\underline{p} = -\frac{2i\hbar}{r} \qquad ①$$

$$\hat{p}_r = \frac{1}{2}\left(\hat{\underline{r}}\cdot\underline{p} + \underline{p}\cdot\hat{\underline{r}}\right) \quad \to \quad \underline{p}\cdot\hat{\underline{r}} + \hat{\underline{r}}\cdot\underline{p} = 2\hat{p}_r \qquad ②$$

$$② - ① \quad \to \quad 2\hat{\underline{r}}\cdot\underline{p} = 2\left(\hat{p}_r + \frac{i\hbar}{r}\right)$$

$$\to \quad \hat{\underline{r}}\cdot\underline{p} = \hat{p}_r + \frac{i\hbar}{r}$$

$$\to \quad \underline{r}\cdot\underline{p} = r\hat{p}_r + i\hbar$$

$$\hat{p}_r = \frac{1}{2}\left(\hat{\underline{r}}\cdot\underline{p} + \underline{p}\cdot\hat{\underline{r}}\right) = -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)$$

$$\hat{p}_r^2 = -\frac{\hbar^2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) = -\hbar^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)$$

$$p^2 = \frac{L^2}{r^2} + \frac{1}{r^2}\left\{(\underline{r}\cdot\underline{p})^2 - i\hbar(\underline{r}\cdot\underline{p})\right\}$$

$$\to \quad p^2 - \frac{L^2}{r^2} = \frac{1}{r^2}\left\{(r\hat{p}_r + i\hbar)(r\hat{p}_r + i\hbar) - i\hbar(r\hat{p}_r + i\hbar)\right\}$$

$$= \frac{1}{r^2}\left\{r\hat{p}_r(r\hat{p}_r) + 2i\hbar\, r\hat{p}_r + \cancel{(i\hbar)^2} - i\hbar\, r\hat{p}_r - \cancel{(i\hbar)^2}\right\}$$

$$= \frac{1}{r^2}\left\{ r\hat{p_r}(r\hat{p_r}) + i\hbar\, r\hat{p_r} \right\}$$

$$\cancel{= \frac{1}{r^2}}$$

$$\rightarrow \quad (p^2 - \frac{L^2}{r^2})\psi = \frac{1}{r^2}\left\{ r\hat{p_r}(r\hat{p_r}) + i\hbar\, r\hat{p_r} \right\}\psi$$

$$= \frac{1}{r^2}\left\{ r(-i\hbar)^2\left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\left(r\frac{\partial \psi}{\partial r} + \psi\right) + i\hbar(-i\hbar)\left[r\frac{\partial \psi}{\partial r} + \psi\right] \right\}$$

$$= \frac{1}{r^2}\left\{ \cancel{i\hbar}(-i\hbar)^2 r\left[r\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{\partial r} + \frac{\partial \psi}{\partial r} + \frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right] + \hbar^2 r\left[\frac{\partial \psi}{\partial r} + \frac{\psi}{r} \right] \right\}$$

$$= \frac{1}{r^2}\left\{ -\hbar^2 r\left[r\frac{\partial^2 \psi}{\partial r^2} + 3\frac{\partial \psi}{\partial r} \cancel{- \frac{\partial \psi}{\partial r}} + \cancel{\frac{\psi}{r}} - \frac{\partial \psi}{\partial r} - \cancel{\frac{\psi}{r}} \right] \right\}$$

$$= \frac{1}{r^2}\left\{ -\hbar^2 r\left[r\frac{\partial^2 \psi}{\partial r^2} + 2\frac{\partial \psi}{\partial r} \right] \right\}$$

$$= \frac{1}{\cancel{r^2}} \cdot \cancel{r^2} \cdot \left[-\hbar^2\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) \right]$$

$$= -\hbar^2\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \right) = \hat{p_r}^2$$

$$\rightarrow \quad \boxed{\hat{p}^2 = \hat{p}_r^2 + \frac{\hat{L}^2}{r^2}}$$

$$\rightarrow \quad \frac{\hat{p}^2}{2m} = \frac{\hat{p_r}^2}{2m} + \frac{\hat{L}^2}{2mr^2}$$

$\rightarrow$ This equation represents that the total kinetic energy equals to the radial kinetic energy plus the orbital (angular) kinetic energy.

3. An electron is in a magnetic field B along the z - axis.

$\rightarrow$ The Hamiltonian $\hat{H} = -\gamma \hat{\underline{S}} \cdot \hat{\underline{B}}$

where $\gamma = \underbrace{g \frac{q}{2m}}_{\text{for electrons}} = 2 \times \frac{e}{2m_e} = \frac{e}{m_e}$

$\rightarrow \hat{H} = -\frac{e}{m_e} \underline{\hat{S}} \cdot \underline{\hat{B}} = \boxed{-\frac{eB}{m_e} \hat{S}_z}$

the eigenvalues $E_+ = -\frac{eB}{m_e}\frac{\hbar}{2} = -\frac{e\hbar B}{2m_e}$

with eigenstate $|+, z\rangle$

and $E_- = +\frac{e\hbar B}{2m_e}$ with eigenstate $|-, z\rangle$

At $t=0$ the spin state of electron is

$|\psi, t=0\rangle = |\cancel{+}, \cancel{x}\rangle = \frac{1}{\sqrt{2}}|+, z\rangle + \frac{1}{\sqrt{2}}|-, z\rangle$

let $\boxed{\omega = \frac{eB}{2m_e}}$, $E_+ = -\hbar\omega$, $E_- = \hbar\omega$

$\rightarrow$ The evolution in time of this state is

$|\psi, t\rangle = \frac{1}{\sqrt{2}} \exp(-iE_+t/\hbar)|+, z\rangle + \frac{1}{\sqrt{2}} \exp(-iE_-t/\hbar)|-, z\rangle$

$= \frac{1}{\sqrt{2}} \exp(i\omega t)|+, z\rangle + \frac{1}{\sqrt{2}} \exp(-i\omega t)|-, z\rangle$

(i) $P(+x, t) = |\langle +, x|\psi, t\rangle|^2$

$= \left| \frac{1}{\sqrt{2}} \exp(i\omega t) \underbrace{\langle +, x|+, z\rangle}_{= \frac{1}{\sqrt{2}}(1,1)\binom{1}{0} = \frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} \exp(-i\omega t) \langle +, x|-, z\rangle \right|^2$

$= \frac{1}{\sqrt{2}}(1,1)\binom{0}{1} = \frac{1}{\sqrt{2}}$

$= \left| \frac{1}{2}[\exp(i\omega t) + \exp(-i\omega t)] \right|^2 = \boxed{\cos^2(\omega t)}$

(ii) $P(-x, t) = |\langle -, x | \psi, t\rangle|^2$

$$= \left| \frac{1}{\sqrt{2}} \exp(i\omega t) \underbrace{\langle -, x | +, z\rangle}_{\frac{1}{\sqrt{2}}(1,-1)\binom{1}{0} = \frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{2}} \exp(-i\omega t) \underbrace{\langle -, x | -, z\rangle}_{= \frac{1}{\sqrt{2}}(1,-1)\binom{0}{1} = -\frac{1}{\sqrt{2}}} \right|^2$$

$$= \left| \frac{1}{2} (\exp(i\omega t) - \exp(-i\omega t)) \right|^2 = |i \sin(\omega t)|^2$$

$$= \boxed{\sin^2(\omega t)}$$

(iii) $P(+z, t) = |\langle +, z | \psi, t\rangle|^2$

$$= \left| \frac{1}{\sqrt{2}} \exp(i\omega t) \underbrace{\langle +, z | +, z\rangle}_{1} + \frac{1}{\sqrt{2}} \exp(-i\omega t) \underbrace{\langle +, z | -, z\rangle}_{0} \right|^2$$

$$= \left| \frac{1}{\sqrt{2}} \exp(i\omega t) \right|^2 = \boxed{\frac{1}{2}}$$

4. For $l = 1$:

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi} \qquad Y_1^{0} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_1^{1} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\phi}$$

$$\rightarrow \sum_{m=-1}^{1} |Y_1^m|^2 = \frac{3}{8\pi}\sin^2\theta + \frac{3}{4\pi}\cos^2\theta + \frac{3}{8\pi}\sin^2\theta$$

$$= \frac{3}{4\pi}[\sin^2\theta + \cos^2\theta] = \frac{3}{4\pi} = \text{a constant}$$

For $l=2$:

$$Y_2^{-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{-2i\phi} \qquad Y_2^{-1} = \sqrt{\frac{15}{32\pi}} \sin 2\theta \, e^{-i\phi}$$

$$Y_2^{0} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) \qquad Y_2^{1} = -\sqrt{\frac{15}{32\pi}} \sin 2\theta \, e^{i\phi} \qquad Y_2^{2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta \, e^{2i\phi}$$

$$\rightarrow \sum_{m=-2}^{2} |Y_2^m|^2 = \frac{15}{32\pi}\sin^4\theta \times 2 + \frac{15}{32\pi}\sin^2 2\theta \times 2 + \cancel{\frac{10}{32\pi}}$$

$$+ \frac{10}{32\pi}(3\cos^2\theta - 1)^2$$

$$= \frac{15}{16\pi}(\sin^4\theta + 4\sin^2\theta\cos^2\theta) + \cancel{10}\,\frac{5}{16\pi}(9\cos^4\theta - 6\cos^2\theta + 1)$$

$$= \frac{5}{16\pi}[3\sin^4\theta + 12\sin^2\theta\cos^2\theta + 9\cos^4\theta - 6\cos^2\theta + 1]$$

$$\cancel{= \frac{5}{16\pi}[3(\sin^4\theta + 4\sin^2\theta\cos^2\theta + 4\cos^4\theta}$$

$$= \frac{5}{16\pi}[3(\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta) \cancel{+ 8\cos^4\theta - 6\cos^2\theta + 1]}$$

$$+ 6\sin^2\theta\cos^2\theta + 6\cos^4\theta - 6\cos^2\theta + 1]$$

$$\cancel{= \frac{5}{16\pi}[3(\sin^2\theta + \cos^2\theta)^2}$$

$$= \frac{5}{16\pi}[3(\underbrace{\sin^2\theta + \cos^2\theta}_{=1})^2 + [\underbrace{6(\sin^2\theta + \cos^2\theta) - 6}_{=6-6=0}]\cos^2\theta$$

$$+ 1]$$

$$= \frac{5}{16\pi} \times 4 = \frac{5}{4\pi} = \text{a constant}$$

This result, $\sum_{m=-l}^{l} |Y_l^m|^2 = $ a constant, shows that at a given radius r, the sum of probabili. of all possible Hydrogen wavefunctions of the same l is independent of θ and ϕ

This leads to the Unsöld's Theorem, which states that atoms containing only half-filled and fully filled orbitals are spherically symmetric.

5. $\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$

$\langle j,j|\hat{J}_x|j,j\rangle = \frac{1}{2}\langle j,j|\underbrace{\hat{J}_+|j,j\rangle}_{=0} + \frac{1}{2}\langle j,j|\hat{J}_-|j,j\rangle$

$\cancel{= \frac{1}{2}\langle j,j|j,j\rangle} = \frac{1}{2}\sqrt{j(j+1) - j(j-1)}\ \underbrace{\langle j,j|j,j-1\rangle}_{0}$

$\sqrt{2j}$

$= 0$

$\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$

$\langle j,j|\hat{J}_y|j,j\rangle = \frac{1}{2i}\underbrace{\langle j,j|\hat{J}_+|j,j\rangle}_{0} - \frac{1}{2i}\underbrace{\langle j,j|\hat{J}_-|j,j\rangle}_{0}$

$= 0$

$\langle j,j|\hat{J}_x^2 + \hat{J}_y^2|j,j\rangle = \langle j,j|\hat{J}^2 - \hat{J}_z^2|j,j\rangle$

$= \langle j,j|\hat{J}^2|j,j\rangle - \langle j,j|\hat{J}_z \cdot \hat{J}_z|j,j\rangle$

$= j(j+1)\hbar^2\langle j,j|j,j\rangle - j^2\hbar^2\langle j,j|j,j\rangle$

$= j(j+1-j)\hbar^2 = j\hbar^2$

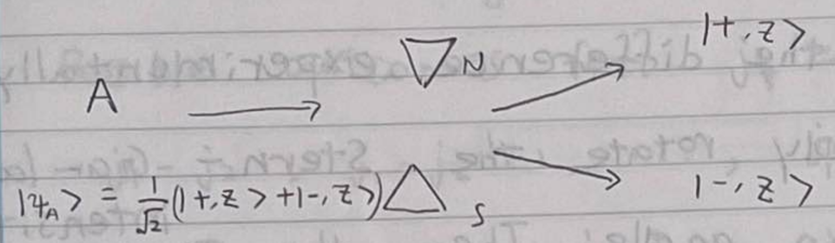
We don't know the orientation of the angular momentum component in the x-y plane

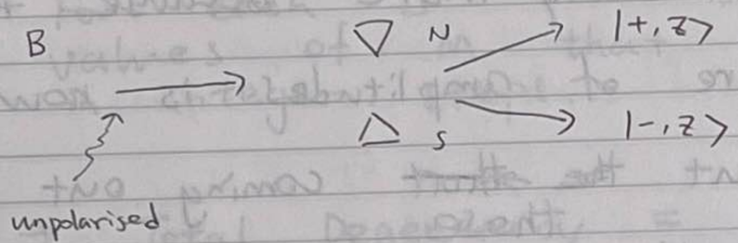
When the angular momentum vector $\underline{J}$ is maximally aligned with z, the ~~par~~ proportion that is uncertain in orientation is given by

$$\sqrt{\frac{j\hbar^2}{j(j+1)\hbar^2}} = \frac{1}{\sqrt{j+1}}$$

When j increases the uncertainty in orientation decreases.

6.





The outcome of the 2 cases is the same. ~~≠~~ Two emerging beams have equal number of electrons (probabilities). One emerging beam has state $|+,z\rangle$ and the other has state $|-,z\rangle$. The difference is that ~~the~~ in case A the beam A splits into two beams of equal intensity because the amplitude of $|+,z\rangle$ and $|-,z\rangle$ is the same, whereas in case B, the beam B splits into two beams of equal intensity because of the assumption of equal a priori probabilities based on maximisation of shanon ~~ent~~ entropy.

To detect the difference experimentally, We can simply rotate the Stern - Garlach filter by an angle. Then the ~~intensity~~ intensities of two beams coming out of beam A will not be equal any more because the ~~modulu~~ modulus square of amplitudes is now ~~different~~ different, but ~~the that~~ those coming out of beam B will ~~be~~ still be the same. because equal a priori probability assump- -tion does not care about the orientation of the filter.

what about detecting difference after beams passed through z-aligned SG filter?

See tute.

7. Add 2 Angular Momenta j_1 and j_2:

Possible outcomes for $j_1 \geq j_2$ are

$j_1 - j_2$, $j_1 - j_2 + 1$, ..., $j_1 + j_2$

$\rightarrow$ There are $j_1 + j_2 - (j_1 - j_2) + 1 = 2j_2 + 1$ number of possible values of J

For each J there are $2J+1$ possible values of M that run from $-J$ to $+J$ in integer steps.

$$\rightarrow \text{Total Degeneracy} = \sum_{J=j_1-j_2}^{j_1+j_2} 2J+1$$

$= \frac{1}{2}$ (first term + last term) × number of terms

$$= \frac{1}{2} \times [2(j_1 - j_2) + 1 + 2(j_1 + j_2) + 1] \times [2j_2 + 1]$$

$$= \frac{1}{2}(4j_1 + 2)(2j_2 + 1)$$

$$= (2j_1 + 1)(2j_2 + 1)$$

Before the momenta were combined:

Number of states of $j_1 = 2j_1 + 1$

Number of States of $j_2 = 2j_2 + 1$

Total number of possible combinations of j_1 and j_2

$$= (2j_1 + 1)(2j_2 + 1)$$

$\longrightarrow$ Consistent

8. The lowering Operator for angular momentum

$$\hat{J_-} = \hat{J_x} - i\hat{J_y}$$ and

$$\hat{J_-}|j, m\rangle = \hbar\sqrt{j(j+1) - m(m-1)}\,|j, m-1\rangle$$

For a box containing 2 spin-1 particles A and B, the ~~st~~ state of the whole box with maximal alignment in Z-direction is:

$$|2,2\rangle = |1,1\rangle_A |1,1\rangle_B$$

~~Appta~~ Apply the lowering operator of the combined angular momentum:

$$\hat{J_-} = \hat{J_-^A} + \hat{J_-^B}$$

$$\rightarrow \hat{J_-}|2,2\rangle = (\hat{J_-^A} + \hat{J_-^B})|1,1\rangle_A |1,1\rangle_B$$

$$\rightarrow \hbar\sqrt{2(3)-(2)(1)}\,|2,1\rangle = \hbar\sqrt{1(2)-1(0)}\,|1,0\rangle_A|1,1\rangle_B + \hbar\sqrt{1(2)-1(0)}\,|1,1\rangle_A|1,0\rangle_B$$

$$\rightarrow 2|2,1\rangle = \sqrt{2}\left(|1,0\rangle_A|1,1\rangle_B + |1,1\rangle_A|1,0\rangle_B\right)$$

$$\rightarrow |2,1\rangle = \frac{1}{\sqrt{2}}|1,0\rangle_A|1,1\rangle_B + \frac{1}{\sqrt{2}}|1,1\rangle_A|1,0\rangle_B$$

and $|2,1\rangle$ is the state corresponds to $J = 2$ and $M = 1$

$$\rightarrow P(A, m=1) = \left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{2}}$$, and

$$P(A, m=0) = \left(\frac{1}{\sqrt{2}}\right)^2 = \boxed{\frac{1}{2}}$$, and

$$P(A, m=-1) = \boxed{0}$$

9. Start with the state of combined system with maximal alignment in the z-direction

$$|1,1\rangle = |+,e\rangle|+,p\rangle$$

Apply lowering operator $\hat{J}_- = \hat{J}_-^e + \hat{J}_-^p$ to the above equation

$\rightarrow$ $\hat{J}_-|1,1\rangle = \hat{J}_-^e |+,e\rangle|+,p\rangle + \hat{J}_-^p|+,e\rangle|+,p\rangle$

$\rightarrow$ $\sqrt{(1)(2)-(1)(0)} \times |1,0\rangle = \sqrt{\frac{1}{2}(\frac{3}{2})-(\frac{1}{2})(-\frac{1}{2})}\ |-,e\rangle|+,p\rangle + \sqrt{\frac{1}{2}(\frac{3}{2})-(\frac{1}{2})(-\frac{1}{2})}\ |+,e\rangle|-,p\rangle$

$\rightarrow$ $\boxed{|1,0\rangle = \frac{1}{\sqrt{2}}|-,e\rangle|+,p\rangle + \frac{1}{\sqrt{2}}|+,e\rangle|-,p\rangle}$

$|+,e\rangle$ and $|-,e\rangle$ are eigenvectors of operator $\hat{S}_z^e = \frac{\hbar}{2}\hat{\sigma}_z^e = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_e$

$\rightarrow$ $|+,e\rangle = \begin{pmatrix}1\\0\end{pmatrix}_e$ with eigenvalue $\frac{\hbar}{2}$

$|-,e\rangle = \begin{pmatrix}0\\1\end{pmatrix}_e$ with eigenvalue $-\frac{\hbar}{2}$

$|x+,e\rangle$ and $|x-,e\rangle$ are eigenvectors of operator $\hat{S}_x^e = \frac{\hbar}{2}\hat{\sigma}_x^e = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_e$

$\rightarrow$ $|x+,e\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_e$ with eigenvalue $\frac{\hbar}{2}$

$\rightarrow$ $|x-,e\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}_e$ with eigenvalue $-\frac{\hbar}{2}$

Hence we have

$$\frac{1}{\sqrt{2}}(|+,e\rangle \pm |-,e\rangle) = \cancel{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}\,\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_e \pm \begin{pmatrix}0\\1\end{pmatrix}_e\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm 1\end{pmatrix}_e = |x\pm,e\rangle$$

$$\rightarrow |x\pm,e\rangle = \frac{1}{\sqrt{2}}(|+,e\rangle \pm |-,e\rangle)$$ Q.E.D.

$$\cancel{|1,0\rangle = |x+,e\rangle\langle x+,e|1,0\rangle + |x-,e\rangle\langle x-,e|1,0\rangle}$$

$$\cancel{= |x+,e\rangle}$$

Similarly $|x\pm,p\rangle = \frac{1}{\sqrt{2}}(|+,p\rangle \pm |-,p\rangle)$

$$|1,0\rangle = |x+,e\rangle\langle x+,e|1,0\rangle + |x-,e\rangle\langle x-,e|1,0\rangle$$

$$= |x+,e\rangle \frac{1}{\sqrt{2}}(\langle +,e| + \langle -,e|)\cdot\frac{1}{\sqrt{2}}(|-,e\rangle|+,p\rangle + |+,e\rangle|-,p\rangle)$$

$$+ |x-,e\rangle \frac{1}{\sqrt{2}}(\langle +,e| - \langle -,e|)\frac{1}{\sqrt{2}}(|-,e\rangle|+,p\rangle + |+,e\rangle|-,p\rangle)$$

$$= |x+,e\rangle\cdot\frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}(|+,p\rangle + |-,p\rangle)}_{|x+,p\rangle}$$

$$+ |x-,e\rangle\cdot\frac{1}{\sqrt{2}}\cdot\underbrace{\frac{1}{\sqrt{2}}(|-,p\rangle - |+,p\rangle)}_{-|x-,p\rangle}$$

$$= \boxed{\frac{1}{\sqrt{2}}(|x+,e\rangle|x+,p\rangle - |x-,e\rangle|x-,p\rangle)}$$

The physical significance of that is that in the state $|1,0\rangle$ the two particles have identical components of spin along x, so even the z components of spin are antiparallel, the state of atom still ~~have~~ has maximum angular momentum because the x components are parallel.