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Quantum Mechanics 5 Ziyan Li

1,

The Pauli Matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Computing directly shows that

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k$$

$$\cancel{\Rightarrow \sigma_i\sigma_j - \sigma_j\sigma_j = 2i\varepsilon_{ijk}\sigma_k}$$

, also that $\cancel{\sigma_i\sigma_j = 2\delta_{ij}I + 2i}$

$$\sigma_i\sigma_j = \delta_{ij}I + i\varepsilon_{ijk}\sigma_k$$

∴ Anticommutator :

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i$$

$$= 2\delta_{ij}I + i(\varepsilon_{ijk}\sigma_k + \underbrace{\varepsilon_{jik}}_{-\varepsilon_{ijk}}\sigma_k) = 2\delta_{ij}I$$

$$= 2\delta_{ij}$$

2. $\rightarrow$ We are free to orient our axis. so we are free to choose any general direction $\underline{\hat{n}}$ as our z direction $(\underline{\hat{n}} = \underline{\hat{z}})$

Pauli matrices satisfy $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I$

Since any direction $\underline{\hat{n}}$ can be ~~co~~ chosen as the $\underline{\hat{z}}$ (or $\underline{\hat{x}}$, $\underline{\hat{y}}$) direction, then $(\underline{\hat{n}} \cdot \underline{\sigma})^2$ must also be I

$\rightarrow$ For $\underline{\hat{m}} \cdot \underline{\hat{n}} = 0$, we are free to orient the axis such that $\underline{\hat{m}} = \underline{\hat{x}}$ and $\underline{\hat{n}} = \underline{\hat{y}}$, hence the relation that we have already known, $[\sigma_x, \sigma_y] = 2i\sigma_z$ requires that in general we have $[\underline{\hat{m}} \cdot \underline{\sigma}, \underline{\hat{n}} \cdot \underline{\sigma}] = 2i(\underline{\hat{m}} \times \underline{\hat{n}}) \cdot \underline{\sigma}$ because $\underline{\hat{x}} \times \underline{\hat{y}} = \underline{\hat{z}}$

· General Properties for Pauli matrices:

$\sigma_i \sigma_j = \cancel{2I\delta_{ij}}\ I\delta_{ij} + i\varepsilon_{ijk}\sigma_k$,

let $\underline{\hat{n}} = (n_x, n_y, n_z)^T$ with $\sum_i n_i^2 = 1$ $(\underline{\hat{n}} \cdot \underline{\hat{n}} = 1)$

$\rightarrow (\underline{n} \cdot \underline{\sigma})^2 = (n_i \sigma_i)^2 = \cancel{n_i \sigma_i n_j \sigma_j}\ n_i \sigma_i n_j \sigma_j$

$= n_i n_j (I\delta_{ij} + i\varepsilon_{ijk}\sigma_k)$

$= (n_i n_j \delta_{ij}) I + i\varepsilon_{ijk} n_i n_j \sigma_k$

$= \underbrace{n_i n_i}_{1} I + i\varepsilon_{ijk} n_i n_j \sigma_k$

$= I + i\varepsilon_{ijk} n_i n_j \sigma_k$

~~$\varepsilon_{ijk}\, n_i n_j \sigma_k = \varepsilon_{ikj} \sigma_k A$~~

~~$= \varepsilon_{ikj} A$~~

$$\varepsilon_{ijk}\, n_i n_j \sigma_k = \varepsilon_{jik}\, n_j n_i \sigma_k = \underbrace{-\varepsilon_{ijk}\, n_i n_j \sigma_k}_{} = 0$$

$A = -A$
$\Rightarrow A = 0$

$\therefore \; (\hat{n}\cdot\underline{\sigma})^2 = I$ ✓ $\Rightarrow$ physically $\Rightarrow$ only $\pm$ spin possible

$$\longrightarrow \quad [\hat{\underline{m}}\cdot\hat{\underline{\sigma}}, \hat{\underline{n}}\cdot\hat{\underline{\sigma}}] = (\hat{\underline{m}}\cdot\hat{\underline{\sigma}})(\hat{\underline{n}}\cdot\hat{\underline{\sigma}}) - (\hat{\underline{n}}\cdot\hat{\underline{\sigma}})(\hat{\underline{m}}\cdot\underline{\sigma})$$

$$= m_i\sigma_i n_j\sigma_j - n_j\sigma_j m_i\sigma_i = m_i n_j (\delta_{ij} I + i\varepsilon_{ijk}\sigma_k)$$

$$- n_j m_i (I\delta_{ij} + i\varepsilon_{jik}\sigma_k) = \cancel{m_i n_i I} + i\,\varepsilon_{ijk}\, m_i n_j \sigma_k$$

$$- \cancel{m_i n_i I} - i\,\varepsilon_{jik}\, m_i n_j \sigma_k = i\underbrace{(\varepsilon_{ijk} - \varepsilon_{jik})}_{2\varepsilon_{ijk}} m_i n_j \sigma_k$$

$$= 2i\,\varepsilon_{ijk}\, m_i n_j \sigma_k = 2i(\hat{\underline{m}}\times\hat{\underline{n}})\cdot\underline{\sigma}$$ ✓

3. $\underline{\hat{n}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$

$\sigma_{\hat{n}}$ $\quad \underline{\hat{n}}\cdot\underline{\sigma} = \sin\theta\cos\phi\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\sigma_x} + \sin\theta\sin\phi\underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\sigma_y}$

$+\cos\theta\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\sigma_z}$

$$= \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$\sigma_{\hat{n}}|\psi\rangle = \lambda|\psi\rangle \Rightarrow \det(\sigma_{\hat{n}} - \lambda I) = 0$

$\Rightarrow \det\begin{vmatrix} \cos\theta-\lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta-\lambda \end{vmatrix} = 0$

$\Rightarrow (\cos\theta-\lambda)(-\cos\theta-\lambda) - \sin^2\theta = 0$

$\Rightarrow -\cos^2\theta - \cancel{\lambda\cos\theta} + \cancel{\lambda\cos\theta} + \lambda^2 - \sin^2\theta = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

let $|\psi\rangle = (a, b)^T$

For $\lambda = 1$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow a\cos\theta + b\sin\theta e^{-i\phi} = a$

$$\therefore \frac{a}{b} = \frac{\sin\theta e^{-i\phi}}{1-\cos\theta} = \frac{\sin\theta e^{-i\phi}}{2\sin^2\frac{\theta}{2}}$$

$$= \frac{\cancel{2\sin\theta}\ \cancel{2\sin\frac{\theta}{2}}\cos\frac{\theta}{2}e^{-i\phi}}{\cancel{2}\cancel{\sin\frac{\theta}{2}}\sin\frac{\theta}{2}} = \frac{\cos\frac{\theta}{2}e^{-i\phi}}{\sin\frac{\theta}{2}}$$

$$\therefore \quad |\psi\rangle = |+,\hat{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2} \end{pmatrix}$$

Multiply by a common phase factor $e^{+i\phi/2}$ gives $\boxed{|+,\hat{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}}$, which is also an eigenvector with eigenvalue 1., and is properly normalised

For $\lambda = -1$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

$$-(\cos\theta + 1)a = \sin\theta e^{-i\phi} b.$$

$$\therefore \quad \frac{a}{b} = -\frac{\sin\theta e^{-i\phi}}{1+\cos\theta} = -\frac{\sin\theta e^{-i\phi}}{2\cos^2\frac{\theta}{2}}$$

$$= -\frac{2\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}e^{-i\phi}}{2\cos\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}} = -\frac{e^{-i\phi}\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\therefore \quad \begin{pmatrix} a \\ b \end{pmatrix} = |\psi\rangle = |-,\hat{n}\rangle = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi} \\ -\cos\frac{\theta}{2} \end{pmatrix}.$$

multiply by common phase $e^{i\phi/2}$ gives correctly normalised

$$\boxed{|-,\hat{n}\rangle = \begin{pmatrix} \sin\frac{\theta}{2}e^{-i\phi/2} \\ -\cos\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}}$$

$\cancel{\hat{S}_z = \frac{\hbar}{2}|+\rangle}$. $\hat{S}_z = \frac{\hbar}{2}\sigma_z$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\therefore \hat{S}_z|+\rangle = \frac{\hbar}{2}|+\rangle \Rightarrow |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{S}_z|-\rangle = -\frac{\hbar}{2}|-\rangle \Rightarrow |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow |+,\hat{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

$$= \boxed{\cos\frac{\theta}{2} e^{-i\phi/2}|+\rangle + \sin\frac{\theta}{2} e^{i\phi/2}|-\rangle.}$$

$$\Rightarrow |-,\hat{n}\rangle = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\phi/2} \\ -\cos\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$$

$$= \boxed{\sin\frac{\theta}{2} e^{-i\phi/2}|+\rangle - \cos\frac{\theta}{2} e^{i\phi/2}|-\rangle}$$

When $\theta = \pi/2$ $\sin\frac{\theta}{2} = \frac{1}{\sqrt{2}}$ $\cos\frac{\theta}{2} = \frac{1}{\sqrt{2}}$

$$\therefore |+,\hat{n}\rangle = \frac{1}{\sqrt{2}} e^{-i\phi/2}|+\rangle + \frac{1}{\sqrt{2}} e^{i\phi/2}|-\rangle$$

~~When $\theta = \pi/2$ lies on the x-y plane; in this direction the spin of z-com~~

~~In this case the $\hat{n}$ lies on~~

When $\theta = \pi/2$, $\hat{n}$ lies on the x-y plane, we are only certain about the component of spin along $\hat{n}$, ~~but have~~ so we have no information about the ~~spin~~ z-component of the spin.

Thus the particle has equal probability of having z-component of spin in +z direction or -z direction.

When $\theta = \pi$.

$$|+, n\rangle = |-\rangle$$

When $\theta = \pi$, $\hat{n}$ is along $-\hat{z}$ direction.

We are certain that the spin is along -z direction. ∴ The spin state is certainly $|-\rangle$

4. $\underline{\hat{J}} = \underline{\hat{S}} = (\hat{S_x}, \hat{S_y}, \hat{S_z})^T = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)^T$

$= \frac{\hbar}{2}\hat{\underline{\sigma}}$

$U(\underline{\alpha}) = \exp\left(-\frac{i\underline{\alpha}\cdot\hat{\underline{J}}}{\hbar}\right) = \exp\left(-\frac{i\underline{\alpha}\cdot\hat{\underline{S}}}{\hbar}\right)$

$= \cancel{\exp\left(-\frac{i\alpha\cdot\hat{\sigma}}{2}\right.} \quad \exp\left(-i\left(\frac{\alpha}{2}\right)\cdot\hat{\underline{\sigma}}\right)$

$= I + \left(-i\left(\frac{\alpha}{2}\right)\cdot\hat{\underline{\sigma}}\right) + \frac{1}{2!}\left(-i\left(\frac{\alpha}{2}\right)\cdot\hat{\underline{\sigma}}\right)^2$

$+ \frac{1}{3!}\left(-i\left(\frac{\alpha}{2}\cdot\hat{\underline{\sigma}}\right)\right)^3 + \cdots$

$= I - i\left(\frac{\vec{\alpha}}{2}\right)\cdot\hat{\sigma} - \frac{1}{2!}\left(\left(\frac{\vec{\alpha}}{2}\right)\cdot\vec{\sigma}\right)^2 + \frac{1}{3!}\, i\left(\left(\frac{\vec{\alpha}}{2}\right)\cdot\vec{\sigma}\right)^3 + \cdots$

$\vec{\alpha} = \alpha\hat{\alpha} \quad \therefore \quad \frac{\vec{\alpha}}{2} = \frac{\alpha}{2}\hat{\alpha}.$

$(\hat{\alpha}\cdot\hat{\sigma})^2 = I$ (see Q1)

$\therefore \quad (\hat{\alpha}\cdot\hat{\sigma})^{2n} = I \qquad (\hat{\alpha}\cdot\hat{\sigma})^{2n+1} = \hat{\alpha}\cdot\hat{\sigma}$

for $n \in \mathbb{Z}$

$\therefore \quad U(\vec{\alpha}) = \left(I - \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2\underbrace{(\hat{\alpha}\cdot\hat{\sigma})^2}_{I} + \frac{1}{4!}\left(\frac{\alpha}{2}\right)^4\underbrace{(\hat{\alpha}\cdot\hat{\sigma})^4}_{I} - \cdots\right)$

$- i\left(\left(\frac{\alpha}{2}\right)(\hat{\alpha}\cdot\hat{\sigma}) - \frac{1}{3!}\left(\frac{\alpha}{2}\right)^3\underbrace{(\hat{\alpha}\cdot\hat{\sigma})^3}_{(\hat{\alpha}\cdot\hat{\sigma})} + \frac{1}{5!}\left(\frac{\alpha}{2}\right)^5\underbrace{(\hat{\alpha}\cdot\hat{\sigma})^5}_{(\hat{\alpha}\cdot\hat{\sigma}).}\right.$

$\left. - \cdots \right)$

$= I\left(1 - \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2 + \frac{1}{4!}\left(\frac{\alpha}{4}\right)^2 - \cdots\right) - i\hat{\alpha}\cdot\hat{\underline{\sigma}}\left(\left(\frac{\alpha}{2}\right) - \frac{1}{3!}\left(\frac{\alpha}{2}\right)^2\right.$

$\left. + \frac{1}{5!}\left(\frac{\alpha}{2}\right)^5 - \cdots\right)$

$$= I\cos\left(\frac{\alpha}{2}\right) - i\underline{\hat{\alpha}}\cdot\underline{\hat{\sigma}}\sin\left(\frac{\alpha}{2}\right)$$

when $\alpha = 2\pi$. $\quad \sin\frac{\alpha}{2} = \sin\pi = 0$

$$\cos\left(\frac{\alpha}{2}\right) = \cos(\pi) = -1$$

$$\therefore \quad U(\alpha) = -I$$

This means that ~~for~~ 2π rotation does not bring the system back to where it starts with, ~~It reverses the system~~. You get minus what you originally have.

5. The magnetic moment of an ~~electron~~ spin-$\frac{1}{2}$ particle is

$$\hat{\underline{\mu}} = g\frac{q}{2m}\hat{\underline{S}} = \gamma\hat{\underline{S}} \qquad \boxed{\gamma = g\frac{q}{2m}}$$

γ is the gyromagnetic ratio. q and m are the charge and mass of the ~~electron~~ particle

g is the g-factor, $g = 2 +$ corrections

Hamiltonian of the particle

$$\hat{H} = -\hat{\underline{\mu}}\cdot\hat{\underline{B}} = \boxed{-\gamma\hat{\underline{S}}\cdot\underline{B}}$$

$$\hat{\underline{S}} = \frac{\hbar}{2}\underline{\sigma} = \frac{\hbar}{2}(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$$

B lies on z $\Rightarrow$ $\underline{B} = (B_x, B_y, B_z) = (0, 0, B)$

$$\therefore \quad \hat{H} = -\gamma\hat{\underline{S}}\cdot\underline{B} = -\frac{\hbar}{2}\gamma B\sigma_z$$

It's eigenstates are $|\uparrow_z\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow_z\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Any superposition of $|\uparrow_z\rangle$ and $|\uparrow_z\rangle$ solves the Schrödinger equation with that Hamiltonian

at $t=0$. State of particle

$$|\psi\rangle = |+, x\rangle = |\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle).$$

Energy eigenvalue associated with $|\uparrow_z\rangle$ is

$E_{\uparrow_z} = -\frac{\hbar}{2}\gamma B$, and that associated with

$|\downarrow_z\rangle$ is $E_{\downarrow_z} = +\frac{\hbar}{2}\gamma B$

~~$\therefore |\psi(t)\rangle = \frac{1}{}$~~

$$\therefore |\psi(t)\rangle = \frac{1}{\sqrt{2}} e^{-iE_{\uparrow_z}t/\hbar} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} e^{-iE_{\downarrow_z}t/\hbar} |\downarrow_z\rangle$$

$$= \frac{1}{\sqrt{2}} e^{i\gamma Bt/2} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} e^{-i\gamma Bt/2} |\downarrow_z\rangle$$

At time $t > 0$

$$\langle \hat{S}_x \rangle = \langle \psi(t) | \hat{S}_x | \psi(t) \rangle$$

$$= \left(\frac{1}{\sqrt{2}} e^{-i\gamma Bt/2} \langle \uparrow_z | + \frac{1}{\sqrt{2}} e^{i\gamma Bt/2} \langle \downarrow_z | \right) \left(\frac{\hbar}{2} \hat{\sigma}_x \right)$$

$$\left(\frac{1}{\sqrt{2}} e^{i\gamma Bt/2} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} e^{-i\gamma Bt/2} |\downarrow_z\rangle \right).$$

$$\langle \uparrow_z | \hat{\sigma}_x | \uparrow_z \rangle = \langle \downarrow_z | \sigma_x | \downarrow_z \rangle = 0$$
$$\langle \uparrow_z | \hat{\sigma}_x | \downarrow_z \rangle = \langle \downarrow_z | \sigma_x | \uparrow_z \rangle = 1$$
$$\left(\because \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$\therefore \langle \hat{S}_x \rangle = \frac{\hbar}{4} \left(e^{-i\gamma Bt} + e^{i\gamma Bt} \right) \cancel{= \frac{\hbar}{4} \cos}$$

$$= \boxed{\frac{\hbar}{2} \cos(\gamma Bt)}$$

$$\langle \hat{S}_y \rangle = \langle \psi(t) | \hat{S}_y | \psi(t) \rangle$$

$$= \left(\frac{1}{\sqrt{2}} e^{-i\gamma B t/2} \langle \uparrow_z | + \frac{1}{\sqrt{2}} e^{i\gamma B t/2} \langle \downarrow_z | \right) \left(\frac{\hbar}{2} \hat{\sigma}_y \right)$$

$$\left(\frac{1}{\sqrt{2}} e^{i\gamma B t/2} | \uparrow_z \rangle + \frac{1}{\sqrt{2}} e^{-i\gamma B t/2} | \downarrow_z \rangle \right)$$

$$\langle \uparrow_z | \sigma_y | \uparrow_z \rangle = \cancel{\langle \downarrow_z | \sigma_y | \uparrow_z \rangle}$$

$$= \langle \downarrow_z | \sigma_y | \downarrow_z \rangle = 0$$

$$\langle \uparrow_z | \sigma_y | \downarrow_z \rangle = -i \quad , \langle \downarrow_z | \sigma_y | \uparrow_z \rangle = i$$

$$\therefore \langle \hat{S}_y \rangle = \frac{\hbar}{4} \left(i e^{i\gamma B t} - i e^{-i\gamma B t} \right)$$

$$= \frac{i\hbar}{4} \left(2i \sin(\gamma B t) \right)$$

$$= \boxed{-\frac{\hbar}{2} \sin(\gamma B t)}$$

→ spin precession at $\omega = \gamma B$

6. spin-1 particles

$$\hat{S_x} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \hat{S_y} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$S_z = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

eigenvalues of $\hat{S_x}$ are $\hbar$, 0, $-\hbar$, let eigenvector be $(a, b, c)^T$

For $\hbar$:

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\Rightarrow \cancel{\frac{\hbar}{\sqrt{2}} a = \hbar a} \quad \frac{\hbar}{\sqrt{2}} b = \hbar a, \quad \frac{\hbar}{\sqrt{2}} a + \frac{\hbar}{\sqrt{2}} c = \hbar b$$

$$\frac{\hbar}{\sqrt{2}} b = c$$

$$\Rightarrow a = c, \quad b = \sqrt{2} a = \sqrt{2} c$$

$$\therefore |+, x\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

For 0:

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow b = 0, \quad a + c = 0 \Rightarrow a = -c, \ b = 0$$

$$\Rightarrow |0, x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

~~$|-,x\rangle$~~ For $-\hbar$:

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=-\hbar\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\frac{\hbar}{\sqrt{2}}b=-\hbar a \qquad \frac{\hbar}{\sqrt{2}}b=-\hbar c$$

$$\frac{\hbar}{\sqrt{2}}a+\frac{\hbar}{\sqrt{2}}c=-\hbar b \implies a=c=-\frac{\sqrt{2}}{2}b$$

$$\implies \quad |-,x\rangle=\frac{1}{2}\begin{pmatrix}1\\-\sqrt{2}\\1\end{pmatrix}$$

$|+\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad |0\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\quad |-\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}$ are

eigenvectors of $\hat{S}_z$ with eigenvalues $\hbar, 0, -\hbar$

$$\implies \quad |+\rangle=\frac{1}{2}|+,x\rangle+\frac{1}{\sqrt{2}}|0,x\rangle+\frac{1}{2}|-,x\rangle$$

$$\implies \quad |+,x\rangle=\frac{1}{2}|+\rangle+\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{2}|-\rangle$$

① first filter :

Maximising Shannon entropy $\implies$ equal probability for each state $\implies \frac{1}{3}$ probability to have $J_z=\hbar$

$\implies \frac{1}{3}$ gets through

② second filter :

$\therefore \quad |+\rangle = \frac{1}{2}|+,x\rangle + \frac{1}{\sqrt{2}}|0,x\rangle + \frac{1}{2}|-,x\rangle$

$\therefore$ $\cancel{(\frac{1}{2})^2 = \frac{1}{4}}$ ~~g. has~~ to have $J_x = \hbar$

$$P(+,x) = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$\Rightarrow \frac{1}{4}$ gets through

③ Third filter :

$$|+,x\rangle = \tfrac{1}{2}|+\rangle + \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{2}|-\rangle$$

~~P(-)~~ To have $J_z = -\hbar$

$$P(-) = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$\therefore \frac{1}{4}$ gets through

In total : $\frac{1}{3} \times \frac{1}{4} \times \frac{1}{4} = \boxed{\frac{1}{48}}$ gets through.

7. $\sqrt{s(s+1)}\hbar = \sqrt{6}\hbar \implies s = 2$

$\therefore \hat{S_z}$ is given by

$$\hat{S_z} = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} = \hbar\, diag(2, 1, 0, -1, -2)$$

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \\ a_4' \\ a_5' \end{pmatrix} = |\psi'(\phi)\rangle = \hat{U}(\underline{\phi})|\psi(0)\rangle$$

$$= \hat{U}(\underline{\phi}) \sum_{m=-2}^{2} a_m |2, m\rangle = \hat{U}(\underline{\phi}) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}$$

$\therefore$ The required matrix is

$$\hat{U}(\underline{\phi}) = \exp(-\tfrac{i}{\hbar}\underline{\phi}\cdot\hat{\underline{S}}) = \exp(-\tfrac{i}{\hbar}\phi\hat{\underline{z}}\cdot\hat{\underline{S}})$$

$$= \exp(-\tfrac{i}{\cancel{\hbar}}\phi\cancel{\hbar}\, diag(+2, +1, 0, -1, -2))$$

$$= \exp(diag(-2i\phi, -i\phi, 0, i\phi, 2i\phi))$$

$$= diag(e^{-2i\phi}, e^{-i\phi}, 1, e^{i\phi}, e^{2i\phi})$$

$$= \begin{pmatrix} e^{-2i\phi} & 0 & 0 & 0 & 0 \\ 0 & e^{-i\phi} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & e^{i\phi} & 0 \\ 0 & 0 & 0 & 0 & e^{2i\phi} \end{pmatrix}$$

8. A and B are non-interacting $\Rightarrow$ $|ab\rangle = |a\rangle|b\rangle$

is the dynamical state for AB

For A: $i\hbar \frac{\partial}{\partial t}|a\rangle = \hat{H}_A|a\rangle$

For B: $i\hbar \frac{\partial}{\partial t}|b\rangle = \hat{H}_B|b\rangle$

Time evolution of AB:

$$i\hbar \frac{\partial}{\partial t}|ab\rangle = i\hbar \frac{\partial}{\partial t}\left(|a\rangle|b\rangle\right) = i\hbar\left(\frac{\partial|a\rangle}{\partial t}|b\rangle + |a\rangle\frac{\partial|b\rangle}{\partial t}\right)$$

$= (\hat{H}_A|a\rangle)|b\rangle + \cancel{H_B}\, |a\rangle(\hat{H}_B|b\rangle)$

$= \hat{H}_A|a\rangle|b\rangle + \hat{H}_B|a\rangle|b\rangle$

($\hat{H}_B|a\rangle|b\rangle = |a\rangle\hat{H}_B|b\rangle$ because $\hat{H}_B$ does not act on $|a\rangle$)

$= (\hat{H}_A + \hat{H}_B)|a\rangle|b\rangle = (\hat{H}_A + \hat{H}_B)|ab\rangle$

$\Rightarrow$ ~~State~~ TDSE automatically satisfied

Every operator ~~on~~ of A commutes with every operator of B because they act on different systems

$\Rightarrow [\hat{H}_A, \hat{H}_B] = 0$

$[\hat{H}_A, \hat{H}_B]|a\rangle|b\rangle$

$= \hat{H}_A\hat{H}_B|a\rangle|b\rangle - \hat{H}_B\hat{H}_A|a\rangle|b\rangle$

$= \hat{H}_A|a\rangle\hat{H}_B|b\rangle - \hat{H}_A|a\rangle\hat{H}_B|b\rangle = 0$

With interaction:

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_{int} \quad , \text{ and } |ab\rangle = \sum_{ij} c_{ij} |a_i\rangle |b_j\rangle$$

TDSE

$$i\hbar \frac{\partial}{\partial t} |ab\rangle = \cancel{i\hbar} (\hat{H}_A + \hat{H}_B + \hat{H}_{int}) |ab\rangle$$

$$= \sum_{ij} c_{ij} \left(\hat{H}_A \cancel{|a_i b_j\rangle} |a_i\rangle |b_j\rangle + \hat{H}_B |a_i\rangle |b_j\rangle + \hat{H}_{int} |a_i\rangle |b_j\rangle \right) \quad ①$$

Differentiating yields:

$$i\hbar \frac{\partial}{\partial t} |ab\rangle = \cancel{i\hbar \sum_{ij}} \; i\hbar \frac{\partial}{\partial t} \sum_{ij} c_{ij} |a_i\rangle |b_j\rangle$$

$$= i\hbar \sum_{ij} \frac{dc_{ij}}{dt} |a_i\rangle |b_j\rangle + c_{ij} \left(\frac{\partial |a_i\rangle}{\partial t} |b_j\rangle + |a_i\rangle \frac{\partial |b_j\rangle}{\partial t} \right)$$

$$\because \; i\hbar \frac{\partial |a_i\rangle}{\partial t} = \hat{H}_A |a_i\rangle \quad , \; i\hbar \frac{\partial |b_j\rangle}{\partial t} = \hat{H}_B |b_j\rangle$$

$$\therefore \; i\hbar \frac{\partial}{\partial t} |ab\rangle = i\hbar \sum_{ij} \frac{dc_{ij}}{dt} |a_i\rangle |b_j\rangle + c_{ij} \hat{H}_A |a_i\rangle |b_j\rangle$$

$$+ c_{ij} |a_i\rangle \hat{H}_B |b_j\rangle \quad ②$$

Compare ①, ② gives

$$i\hbar \sum_{ij} \frac{dc_{ij}}{dt} |a_i\rangle |b_j\rangle = \sum_{ij} c_{ij} \hat{H}_{int} |a_i\rangle |b_j\rangle$$

~~$\langle a_k|$~~ $\langle b_l | \langle a_k | \Rightarrow$ ~~$\Rightarrow i\hbar \frac{dc_{ij}}{dt}$~~ $\quad i\hbar \frac{dc_{kl}}{dt} = \sum_{ij} c_{ij}$ $\langle b_l | \langle a_k | \hat{H}_{int} |a_i\rangle |b_j\rangle$

The TDSE changed as the Hamiltonian has changed by $\hat{H}_{int}$. The coefficients c_{ij} for any general states must satisfy the equation above.

If A, B are harmonic oscillators connected by a spring, then

$$\hat{H_A} = \frac{\hat{P_A}^2}{2m_A} + \frac{1}{2} m_A \omega^2 X_A^2 , \quad \hat{H_B} = \frac{\hat{P_B}^2}{2m_B} + \frac{1}{2} m_B \omega^2 \hat{X_B}^2 , \quad \hat{H}_{int} = \frac{1}{2} \mu \omega^2 \hat{X}_{AB}^2$$

(where $\mu = \frac{m_A m_B}{m_A + m_B}$ = reduced mass)

($\hat{X}_{AB} = \hat{x_A} - \hat{x_B}$)

Since $\hat{H_A}$ contains $\hat{P_A}$ and $\hat{H}_{int}$ contains $\hat{x_A}$

and $[\hat{P_A}, \hat{x_A}] \neq 0$

$\therefore \quad [\hat{H_A}, \hat{H}_{int}] \neq 0$

Physically we are unable to know the interaction energy and energy of one of the subsystems simultaneously.

9.

The physical state of system A is correlated with the state of system B means that we cannot write state $|a_i b_j\rangle$ as the product $|a_i\rangle|b_j\rangle$, and the probability of ~~A~~ system being in $|a_i b_j\rangle$ is not equal to the product of probabilities of A being in $|a_i\rangle$ and B being in $|b_j\rangle$. In correlated states the states of different subsystems cannot be described separately. The probability of A being in state $|a_i\rangle$ ~~is dependent~~ given that B is in state $|b_j\rangle$ is dependent upon which $|b_j\rangle$ the system B is in.

(i) Momenta of cars in London's circular motorway ~~is corr~~ describe correlated states because the momentum of one car ~~depend~~ depends on that of other cars (traffic congestion)

(ii) Momenta of cars ~~o~~ in the road over the Nullarbor Plain ~~is~~ describes uncorrelated states as the momenta of cars are independent (no congestion, cars are free to be driven)

10. The force $F = -c|x-y|$ which resembles a spring force

$\Rightarrow$ The corresponding spring potential is

$$V = \frac{c}{2}(x-y)^2$$

$$\therefore \quad \hat{H} = \frac{\hat{p}_1}{2m} + \frac{\hat{p}_2}{2m} + \frac{c}{2}(x-y)^2$$

with the additional $V(x,y) = \frac{c}{2}(x+y)^2$

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + \frac{c}{2}(x-y)^2 + \frac{c}{2}(x+y)^2$$

$$= \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{c}{2}(x^2+y^2+\cancel{2xy}+x^2+y^2-\cancel{2xy})$$

$$= \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + c(x^2+y^2)$$

$$\Rightarrow \Phi(x,y) = c(x^2+y^2)$$

11.

The state of the composite system is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+,\underline{b}\rangle|-\rangle - |-,\underline{b}\rangle|+\rangle\right)$$

Given that Alice measure $+\frac{1}{2}$ for the electron, the state collapsed to

$$|-,b\rangle|+\rangle$$

Bob must measure that positron in the state

$$|-,b\rangle = \cos(\theta/2)e^{i\phi/2}|-\rangle - \sin(\theta/2)e^{-i\phi/2}|+\rangle$$

Amplitude of obtaining $-\frac{1}{2}$ $(|-\rangle)$ is $= \cos\frac{\theta}{2}e^{i\phi/2}$

$$\text{Probability} = \left|\cos\frac{\theta}{2}e^{i\phi/2}\right|^2 = \boxed{\cos^2\frac{\theta}{2}}$$