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Quantum Mechanics 4 Ziyan Li

1. (a) $\hat{L}_i = \varepsilon_{ijk} \hat{x}_j \hat{p}_k$ $\hat{x}_j = x_j$, $\hat{p}_k = -i\hbar \frac{\partial}{\partial x_k} = -i\hbar \partial_k$

$\therefore$ $[\hat{L}_i, \hat{x}_j]\psi = \hat{L}_i \hat{x}_j \psi - \hat{x}_j \hat{L}_i \psi$

$= \cancel{i\hbar \varepsilon_{ij}} - i\hbar \varepsilon_{ikl} x_k \partial_l x_j \psi + i\hbar \varepsilon_{ikl} x_j x_k \partial_l \psi$

$= -i\hbar \varepsilon_{ikl} x_k \psi \underbrace{\partial_l x_j}_{\delta_{lj}} - i\hbar \cancel{\varepsilon_{ikl} x_k x_j \partial_l \psi}$

$+ i\hbar \cancel{\varepsilon_{ikl} x_k x_j \partial_l \psi}$

$= -i\hbar \varepsilon_{ikj} x_k \psi = i\hbar \varepsilon_{ijk} \hat{x}_k \psi$

$\Rightarrow$ $[\hat{L}_i, \hat{x}_j] = i\hbar \varepsilon_{ijk} \hat{x}_k$

$\therefore$ $[\hat{L}_i, \hat{p}_j]\psi = \hat{L}_i \hat{p}_j \psi - \hat{p}_j \hat{L}_i \psi$

$= -\hbar^2 \varepsilon_{ikl} x_k \partial_l \partial_j \psi + \hbar^2 \varepsilon_{ikl} \partial_j x_k \partial_l \psi$

$= -\hbar^2 \cancel{\varepsilon_{ikl} x_k \partial_l \partial_j \psi} + \hbar^2 \cancel{\varepsilon_{ikl} x_k \partial_l \partial_j \psi}$

$+ \hbar^2 \varepsilon_{ikl} \delta_{jk} \partial_l \psi = \cancel{\hbar^2 \varepsilon_{ikl}} \hbar^2 \varepsilon_{ijl} \partial_l \psi$

$= \cancel{-i\hbar}$ $i\hbar \varepsilon_{ijk} (-i\hbar \partial_k)\psi = i\hbar \varepsilon_{ijk} \hat{p}_k \psi$

$\Rightarrow$ $[\hat{L}_i, \hat{p}_j] = i\hbar \varepsilon_{ijk} \hat{p}_k$

(b) $[\hat{L}_x, \hat{L}_y] = [\hat{L}_x, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z]$

$= [\hat{L}_x, \hat{z}\hat{p}_x] - [\hat{L}_x, \hat{x}\hat{p}_z].$

$= \hat{z}[\hat{L}_x, \hat{p}_x] + [\hat{L}_x, \hat{z}]\hat{p}_x - \hat{x}[\hat{L}_x, \hat{p}_z] - \cancel{p_z}[\hat{L}_x, \hat{x}]\hat{p}_z$

$= (\hat{z}(0) + i\hbar(-\hat{y})\hat{p}_x - \hat{x}(-\hat{p}_y)i\hbar - 0(\hat{p}_z))$

$= (\hat{x}\hat{p}_y - y\hat{p}_x)\, i\hbar = i\hbar\hat{L}_z$

(c)

$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = \cancel{y\frac{\partial}{\partial z}} -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})$

$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$

$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$

$(\hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x)f(x,y,z) = -\hbar^2\Big[(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})$

$- (z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})\Big]f$

$= -\hbar^2\Big[(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y})(z\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial z})$

$- (z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z})(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y})\Big].$

$$= -\hbar^2 \Big[y\frac{\partial}{\partial z}\Big(z\frac{\partial f}{\partial x}\Big) - z\frac{\partial}{\partial y}\Big(z\frac{\partial f}{\partial x}\Big) - y\frac{\partial}{\partial z}\Big(x\frac{\partial f}{\partial z}\Big)$$

$$+ z\frac{\partial}{\partial y}\Big(x\frac{\partial f}{\partial z}\Big) - z\frac{\partial}{\partial x}\Big(y\frac{\partial f}{\partial z}\Big) + x\frac{\partial}{\partial z}\Big(y\frac{\partial f}{\partial z}\Big)$$

$$+ z\frac{\partial}{\partial x}\Big(z\frac{\partial f}{\partial y}\Big) - x\frac{\partial}{\partial z}\Big(z\frac{\partial f}{\partial y}\Big)\Big]$$

$$= -\hbar^2 \Big[y\frac{\partial f}{\partial x} + yz\cancel{\frac{\partial^2 f}{\partial z\partial x}} - z^2\cancel{\frac{\partial^2 f}{\partial y\partial x}} - yx\cancel{\frac{\partial^2 f}{\partial z^2}}$$

$$+ zx\cancel{\frac{\partial^2 f}{\partial y\partial z}} - zy\cancel{\frac{\partial^2 f}{\partial x\partial z}} + xy\cancel{\frac{\partial^2 f}{\partial z^2}} + z^2\cancel{\frac{\partial^2 f}{\partial x\partial y}}$$

$$- x\frac{\partial f}{\partial y} - xz\cancel{\frac{\partial^2 f}{\partial z\partial y}} \Big]$$

$$= (-i\hbar)(-i\hbar)\Big[y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big]f$$

$$= i\hbar\Big[-i\hbar\Big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\Big)\Big]f$$

$$= i\hbar[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x]f = i\hbar\hat{L}_z f$$

$$\Rightarrow \underline{[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z}$$

Similar expressions:

$$[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

$$\Rightarrow \boxed{[\hat{L}_i, \hat{L}_j] = i\hbar\,\varepsilon_{ijk}\hat{L}_k}$$

(d) $\hat{L}^2 = \hat{L_x}^2 + \hat{L_y}^2 + \hat{L_z}^2 = \sum_i \hat{L_i}\hat{L_i}$

$$[\hat{L_j}, \hat{L}^2] = \sum_i [\hat{L_j}, \hat{L_i}\hat{L_i}]$$

$$= \sum_i \hat{L_i}[\hat{L_j}, \hat{L_i}] + [\hat{L_j}, \hat{L_i}]\hat{L_i}$$

$$= i\hbar \sum_i (\hat{L_i}\, \varepsilon_{jik} \hat{L_k} + \varepsilon_{jik}\hat{L_k}\hat{L_i})$$

$$= i\hbar \sum_i \cancel{\varepsilon_{jik}} (\varepsilon_{jik} \hat{L_i}\hat{L_k} + \varepsilon_{jik}\hat{L_k}\hat{L_i})$$

$$\underbrace{=}_{\text{swap } i,k} i\hbar \sum_i (\varepsilon_{jki}\hat{L_k}\hat{L_i} + \varepsilon_{jki}\hat{L_i}\hat{L_k})$$

$$\underbrace{=}_{\text{permute } i,k \text{ once}} -i\hbar \sum_i (\varepsilon_{jik}\hat{L_k}\hat{L_i} + \varepsilon_{jik}\hat{L_i}\hat{L_k})$$

$$\underbrace{=}_{A=-A \Rightarrow A=0} -[\hat{L_j}, \hat{L}^2] = 0$$

$$\Rightarrow [\hat{L_x}, \hat{L}^2] = [\hat{L_y}, \hat{L}^2] = [\hat{L_z}, \hat{L}^2] = 0$$

(e) $\hat{L_x} = i\hbar \left[z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \right]$

$\cancel{x = r\cos\theta}$ $x = r\sin\theta\cos\phi$ ~~tan~~ $r^2 = x^2 + y^2 + z^2$

$y = r\sin\theta\sin\phi$ $\tan^2\theta = \frac{x^2+y^2}{z^2}$

$z = r\cos\theta$ $\tan\phi = \frac{y}{x}$

$$\therefore \frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$2r\frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta\sin\phi$$

$\partial \quad z\tan\theta\sec^2\theta\,\frac{\partial\theta}{\partial y} = \frac{2y}{z^2} = \frac{\cancel{2}r\sin\theta\sin\phi}{r^2\cos^2\theta}$

$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{1}{r}\sin\theta\cot\theta\sin\phi$

$\cancel{\sec^2\theta} \frac{\partial\phi}{\partial y}\sec^2\phi = \frac{1}{x} = \frac{1}{r\sin\theta\cos\phi} \cancel{= \frac{1}{r\sin\theta}}$

$$\boxed{\Rightarrow \frac{\partial\phi}{\partial y} = \cancel{\frac{\cos\phi}{r\sin\theta}}\ \frac{\cos\phi}{r\sin\theta}}$$

$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}$

$\frac{\partial r}{\partial z} = \cancel{\frac{r}{z}}\ \frac{z}{r} = \cos\theta \qquad \cancel{\frac{\partial\theta}{\partial z}} =$

$z\tan\theta\sec^2\theta\,\frac{\partial\theta}{\partial z} = -\frac{2(x^2+y^2)}{z^3} = -2\frac{r^2\sin^2\theta}{r^3\cos^3\theta}$

$\frac{\partial\theta}{\partial z} = -\frac{1}{r}\sin\theta$

$\frac{\partial\phi}{\partial z} = 0$

$$\therefore \boxed{\begin{aligned}\frac{\partial}{\partial y} &= \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cot\theta\sin\phi\frac{\partial}{\partial\theta} \\ &\quad + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial z} &= \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\end{aligned}}$$

$$\hat{L}_x = i\hbar\left[z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right]$$

$$= [i\hbar]\left[\cancel{r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r}} + \underbrace{\sin\theta\cos\theta\frac{\cos\theta}{\sin\theta}}_{\cos^2\theta}\sin\phi\frac{\partial}{\partial\theta}\right.$$

$$\left. + \underbrace{\frac{\cos\theta\cos\phi}{\sin\theta}}_{\tan\theta\cos\phi}\frac{\partial}{\partial\phi} - \cancel{r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r}} + \sin^2\theta\sin\phi\frac{\partial}{\partial\theta}\right]$$

$$= \boxed{i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right).}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$$

$$= \cancel{\sin} \quad \frac{\partial r}{\partial x} = \frac{x}{r} = \cancel{\cos}\ \sin\theta\cos\phi$$

$$\cancel{2\tan\theta\sec^2\theta}\frac{\partial\theta}{\partial x} = \frac{2x}{z^2} = \frac{\cancel{2}r\sin\theta\cos\phi}{r^2\cancel{\cos}\theta}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{r}\sin\theta\cot\theta\cos\phi$$

$$\cancel{2}\ \cancel{\sec^2\phi}\frac{\partial\phi}{\partial x} = -\frac{y}{x^2} = -\frac{\cancel{r}\sin\cancel{\theta}\sin\phi}{r^2\sin^2\theta\cancel{\cos^2\phi}}$$

$$= -\frac{1}{r}\frac{\sin\phi}{\sin\theta} \Rightarrow \cancel{\partial}$$

$$\therefore\ \cancel{\hat{L}_y = i\hbar\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]}.$$

$$= \cancel{i\hbar\left[r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r}\right.}$$

$$\boxed{\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cot\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}}$$

$$\hat{L}_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\frac{\cos\theta}{\sin\theta}\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial\theta}$$

$$= i\hbar\Big(\cancel{-r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r}} - \cancel{\cos\theta\sin^2\theta\sin\phi\sin^2\theta\cos\phi\frac{\partial}{\partial\theta}}$$

$$+ \cot\theta\sin\phi\frac{\partial}{\partial\phi} + \cancel{r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r}}$$

$$- \cos\phi\sin^2\theta\frac{\partial}{\partial\theta}\Big)$$

$$= \boxed{i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)}$$

$$\hat{L}_z = i\hbar\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) :$$

~~= iħ~~ And

$$\frac{\partial}{\partial\phi} = \underbrace{\frac{\partial x}{\partial\phi}}_{\substack{\cancel{= r\sin\theta\cos\phi} \\ -r\sin\theta\sin\phi \\ = -y}}\frac{\partial}{\partial x} + \underbrace{\frac{\partial y}{\partial\phi}}_{\substack{r\sin\theta\cos\phi \\ = x}}\frac{\partial}{\partial y} + \underbrace{\frac{\partial z}{\partial\phi}}_{0}\frac{\partial}{\partial z}$$

$$= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{-1}{i\hbar}\hat{L}_z$$

$$\therefore\ \hat{L}_z = \boxed{-i\hbar\frac{\partial}{\partial\phi}}$$

$$\because\ \psi(x, y, z) = \psi(|r|) = \psi(r)$$

$$\therefore\ \frac{\partial\psi}{\partial\theta} = 0,\quad \frac{\partial\psi}{\partial\phi} = 0$$

$$\therefore \quad \hat{L}_x \psi = i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\psi(r) = \boxed{0}$$

$$\hat{L}_y \psi = i\hbar\left(-\cos\phi \frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)\psi(r) = \boxed{0}$$

$$\hat{L}_z \psi = -i\hbar\frac{\partial}{\partial\phi}\psi(r) = \boxed{0}$$

Easier way:

$$\psi(x,y,z) = \psi(r) \Rightarrow \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{x}{r}$$

Similarly, $\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial r}\frac{y}{r}, \ \frac{\partial\psi}{\partial z}\frac{z}{r}$

$$\Rightarrow \hat{L}_x\psi = -i\hbar(y\partial_z - z\partial_y)\psi$$

$$= \frac{i\hbar}{r}(yz - zy)\frac{\partial\psi}{\partial r} = 0$$

etc.

2. (a)

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

$$\hat{L}^2\psi = -\hbar^2\Big[\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial\psi}{\partial\theta} + \cos\phi\cot\theta\frac{\partial\psi}{\partial\phi}\right)$$

$$+\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)\left(-\cos\phi\frac{\partial\psi}{\partial\theta} + \sin\phi\cot\theta\frac{\partial\psi}{\partial\phi}\right)$$

$$+\frac{\partial^2\psi}{\partial\phi^2}\Big].$$

$$= -\hbar^2\Big[\sin^2\phi\frac{\partial^2\psi}{\partial\theta^2} + \cancel{\sin\phi\cos\phi\cot\theta\frac{\partial^2\psi}{\partial\theta\partial\phi}}$$

$$+\cancel{\sin\phi\cos\phi\frac{\partial\psi}{\partial\phi}\left(-\frac{1}{\sin^2\theta}\right)} + \cancel{\cos\phi\sin\phi\cot\theta\frac{\partial^2\psi}{\partial\theta\partial\phi}}$$

$$+\cos\phi\cot\theta\frac{\partial\psi}{\partial\theta}\cos\phi$$

$$+\cos^2\phi\cot^2\theta\frac{\partial^2\psi}{\partial\phi^2} + \cancel{\cos\phi\cot^2\theta\frac{\partial\psi}{\partial\phi}(-\sin\phi)}$$

$$+\cos^2\phi\frac{\partial^2\psi}{\partial\theta^2} - \cancel{\sin\phi\cos\phi\cot\theta\frac{\partial^2\psi}{\partial\phi\partial\theta}}$$

$$-\cancel{\sin\phi\cos\phi\frac{\partial\psi}{\partial\phi}\left(-\frac{1}{\sin^2\theta}\right)}$$

$$-\cancel{\sin\phi\cos\phi\cot\theta\frac{\partial^2\psi}{\partial\phi\partial\theta}} - \sin\phi\cot\theta\frac{\partial\psi}{\partial\theta}(-\sin\phi)$$

$$+\sin^2\phi\cot^2\theta\frac{\partial^2\psi}{\partial\phi^2} + \cancel{\sin\phi\cot^2\theta\frac{\partial\psi}{\partial\theta}(\cos\phi)}$$

$$+\frac{\partial^2\psi}{\partial\phi^2}.$$

$$= -\hbar^2 \left[\frac{\partial^2 \psi}{\partial \theta^2} + \cot\theta \frac{\partial \psi}{\partial \theta} + (\cot^2\theta + 1) \frac{\partial^2 \psi}{\partial \phi^2} \right]$$

$$= \cancel{-\hbar^2 \left[\frac{\partial^2}{\partial \theta^2} \right.}$$

$$\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) = \frac{1}{\sin\theta} \left(\cos\theta \frac{\partial \psi}{\partial \theta} + \sin\theta \frac{\partial^2 \psi}{\partial \theta^2} \right)$$

$$= \frac{\partial^2 \psi}{\partial \theta^2} + \cot\theta \frac{\partial \psi}{\partial \theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cot^2\theta + 1 = \frac{1}{\sin^2\theta}.$$

$$\Rightarrow \hat{L}^2 \psi = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial}{\partial \phi} \right] \psi$$

$$\Rightarrow \boxed{\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]}$$

~~For $\cos\theta$:~~

$$\cancel{\hat{L}^2 \cos\theta = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (\sin\theta\cos\theta) + \frac{1}{\sin^2\theta} \underbrace{\frac{\partial}{\partial \phi} \cos\theta}_{0} \right]}$$

$$\cancel{= -\hbar^2 \frac{1}{\sin\theta} \frac{1}{2} \frac{\partial}{\partial \theta} (\sin 2\theta)}$$

$$\cancel{= -\hbar^2 \frac{1}{\sin\theta} \sin 2\theta = -\hbar^2 \cos\theta.}$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}.$$

- For $\cos\theta$:

$$\hat{L}^2 \cos\theta = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\,(-\sin\theta)) + \frac{1}{\sin^2\theta} \underbrace{\frac{\partial^2}{\partial\phi^2}\cos\theta}_{0} \right]$$

$$= \hbar^2 \left[\frac{1}{\cancel{\sin\theta}}\, 2\cancel{\sin\theta}\cos\theta = \underline{2\hbar^2 \cos\theta}\right.$$

$$\text{eigenvalue} = 2\hbar^2$$

$$\hat{L}_z \cos\theta = -i\hbar \frac{\partial}{\partial\phi}\cos\theta = 0$$

$$\text{eigenvalue} = 0$$

- For $\sin\theta e^{i\phi}$:

$$\hat{L}^2 \sin\theta e^{i\phi} = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\cos\theta e^{i\phi}) + \frac{1}{\sin^2\theta} \sin\theta(-1) e^{i\phi} \right]$$

$$\cancel{= -\hbar^2 \left[\frac{1}{\sin\theta} (\cos \right.}$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} (-\sin^2\theta + \cos^2\theta) e^{i\phi} - \frac{e^{i\phi}}{\sin\theta} \right]$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} (-\sin^2\theta + \cancel{\cos^2\theta} - \sin^2\theta - \cancel{\cos^2\theta}) e^{i\phi} \right]$$

$$= 2\hbar^2 \sin\theta e^{i\phi} \qquad \text{eigenvalue} = 2\hbar^2$$

$$\hat{L}_z \sin\theta e^{i\phi} = -i\hbar \frac{\partial}{\partial\phi} \sin\theta e^{i\phi} = \hbar \sin\theta e^{i\phi}$$

$$\text{eigenvalue} = \hbar$$

$$\hat{L}^2 \sin\theta e^{-i\phi} = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta e^{-i\phi}) + \frac{1}{\sin^2\theta}\sin\theta(-1)e^{-i\phi}\right]$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}(\cos^2\theta - \sin^2\theta)e^{-i\phi} - \frac{1}{\sin\theta}(\cos^2\theta + \sin^2\theta)e^{-i\phi}\right]$$

$$= \cancel{-\hbar^2[} = 2\hbar^2 \sin\theta e^{-i\phi}$$

$$\text{eigenvalue} = 2\hbar^2.$$

$$\hat{L}_z \sin\theta e^{-i\phi} = -i\hbar\frac{\partial}{\partial\phi}\sin\theta e^{-i\phi} = -\hbar \sin\theta e^{-i\phi}$$

$$\text{eigenvalue} = -\hbar$$

(b) ~~$\int_0^{2\pi}$ Normalisation $\int N\psi$~~

~~$\int |N\psi(r,\theta,\phi)|^2 \, d\tau = 1$~~

$$\Rightarrow \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta N^2 |\psi(\theta,\phi)|^2 = 1$$

For $\psi(\theta,\phi) = \cos\theta$

$$\underbrace{\int_0^{2\pi} d\phi}_{2\pi} \int_0^{\pi} d\theta \sin\theta\cos^2\theta N^2 = 1$$

$$\int_0^{\pi} d\theta \sin\theta \cos^2\theta = -\int_1^{-1} d(\cos\theta)\cos^2\theta = \int_{-1}^{1} \cos^2\theta \, d(\cos\theta)$$

$$= \frac{\cos^3\theta}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\therefore \quad (2\pi)\left(\frac{2}{3}\right) N^2 = 1$$

$$\therefore \quad \boxed{N = \sqrt{\frac{3}{4\pi}}}$$

For $\psi(\theta,\phi) = \sin\theta e^{i\phi}$ $\quad |\psi(\theta,\phi)| = \sin\theta$

$$\Rightarrow \quad \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^3\theta \, N^2 = 1$$

$$\int_0^{\pi} d\theta \sin^3\theta = \int_0^{\pi} d\theta \sin\theta (1 - \cos^2\theta)$$

$$= \underbrace{\int_0^{\pi} d\theta \sin\theta}_{= -\cos\theta|_0^{\pi} = 2} - \int_0^{\pi} \underbrace{d\theta \sin\theta \cos^2\theta}_{2/3} = \frac{4}{3}$$

$$\therefore \quad (2\pi)\left(\frac{4}{3}\right) N^2 = 1$$

$$\Rightarrow \quad N^2 = \frac{3}{8\pi}$$

by convention $\quad \boxed{\cancel{N = +\sqrt{\frac{3}{8\pi}}}} \quad \boxed{N = -\sqrt{\frac{3}{8\pi}}}$

For $\psi(\theta,\phi) = \sin\theta\, e^{-i\phi}$, $|\psi(\theta,\phi)|^2 = \sin^2\theta$

$\Rightarrow N^2 = \frac{3}{8\pi}$

By convention $\boxed{N = \sqrt{\frac{3}{8\pi}}}$ ✓

(C) In the symbol $Y_{lm}(\theta,\phi)$

l represents the eigenvalue of $\hat{L}^2$,
m represents the eigenvalue of $\hat{L}_z$.

Hence

$Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$Y_{1-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$ ✓

(d) $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$x^2+y^2 = r^2\sin^2\theta \Rightarrow \sin^2\theta = \frac{x^2+y^2}{x^2+y^2+z^2}$$

$$\sin\theta = \sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}$$

$$e^{i\phi} = \cos\phi + i\sin\phi$$

~~$e^{i\phi} = \cos\phi + i\sin\phi$~~, $\tan\phi = \frac{y}{x}$, $\phi = \arctan\left(\frac{y}{x}\right)$

$$\therefore \cos\phi = \frac{x}{\sqrt{x^2+y^2}} \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}.$$

$$e^{i\phi} = \frac{x+iy}{\sqrt{x^2+y^2}}.$$

$$\therefore Y_{10}(x,y,z) = \boxed{\frac{z}{\sqrt{x^2+y^2+z^2}}\sqrt{\frac{3}{4\pi}}}$$

$$Y_{11}(x,y,z) = \left(\sqrt{\frac{x^2+y^2}{x^2+y^2+z^2}}\right)\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right) = \boxed{\sqrt{\frac{3}{8\pi}}\frac{x+iy}{\sqrt{x^2+y^2+z^2}}}$$

$$Y_{1,-1}(x,y,z) = \boxed{\frac{x-iy}{\sqrt{x^2+y^2+z^2}}\sqrt{\frac{3}{8\pi}}}$$

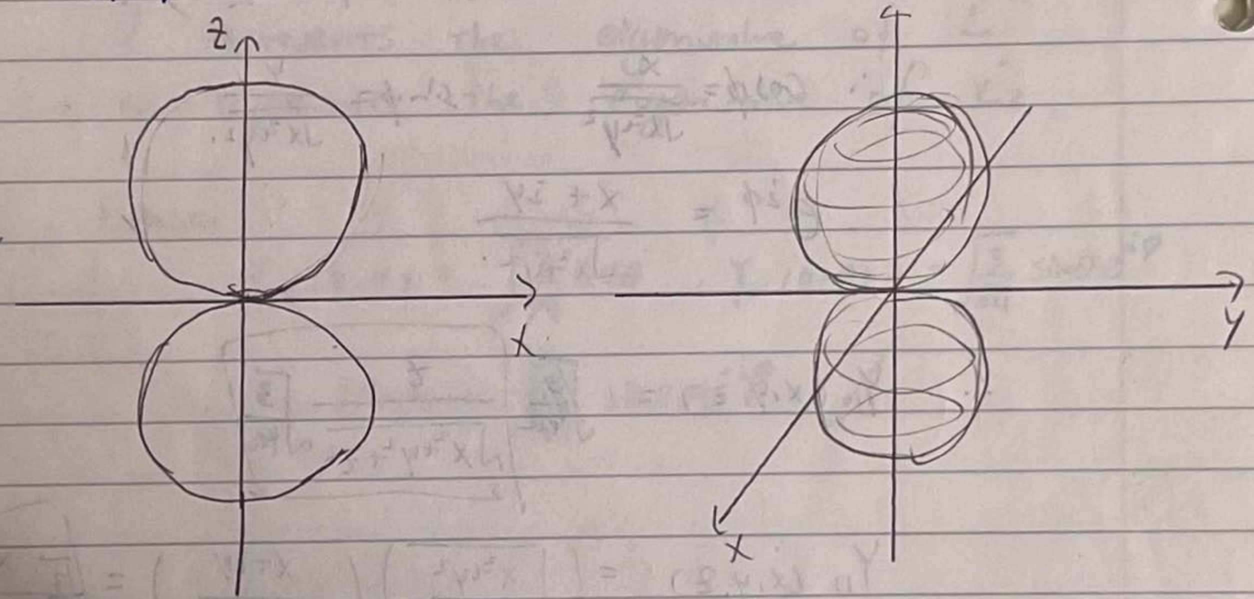
$|Y_{10}|^2 = \frac{3}{4\pi}\cos^2\theta$

$|Y_{1,1}|^2 = |Y_{1,-1}|^2 = \frac{3}{8\pi}\sin^2\theta$

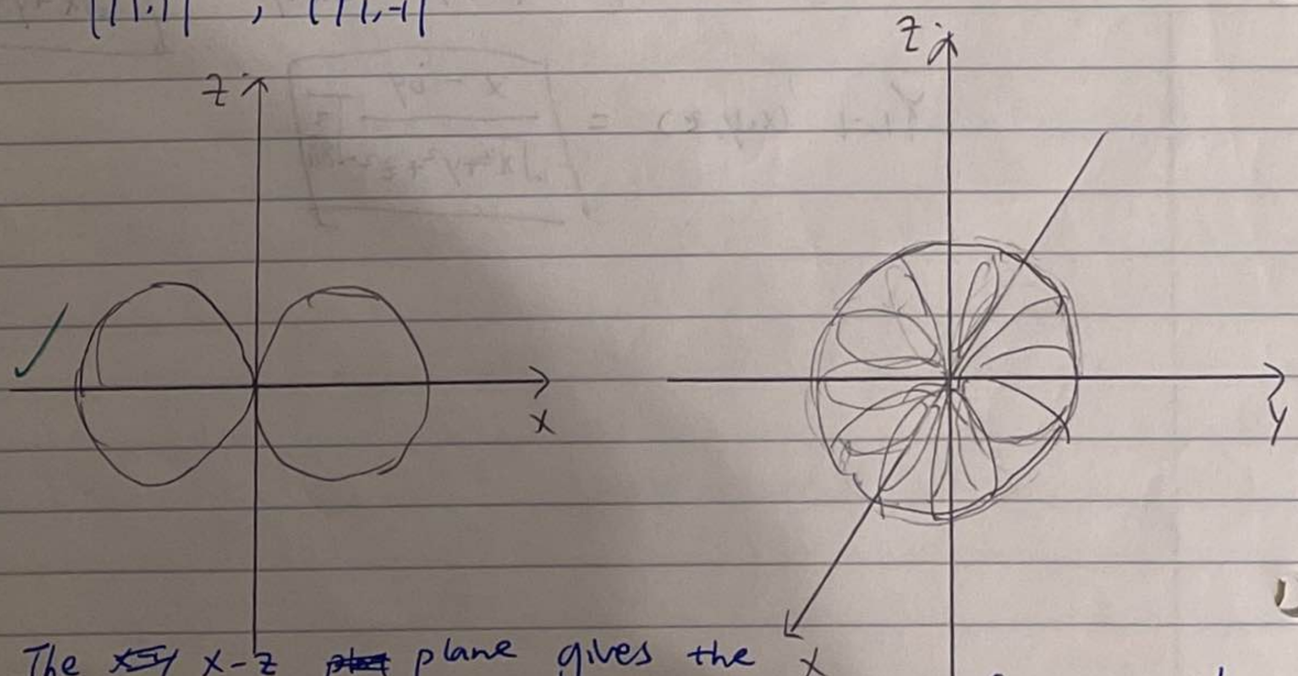
In angular space, the distance between the origin and point (θ, ϕ) is the value of $|Y|^2$

then we have the following plots.

$|Y_{10}|^2$:



$|Y_{1,1}|^2$, $|Y_{1,-1}|^2$:



The ~~x-y~~ x-z ~~plot~~ plane gives the correct ~~co~~ cross-section since the three functions have no ϕ dependence (azimuthal symmetry)

3. $\psi(\theta,\phi) = \langle\theta,\phi|\psi\rangle \propto \sqrt{2}\cos\theta + \sin\theta e^{-i\phi} - \sin\theta e^{i\phi}$

$$\propto a_1\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right) + a_2\left(+\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$$

$$+ a_3\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)$$

$$a_1\sqrt{\frac{3}{4\pi}} : a_2\sqrt{\frac{3}{8\pi}} : a_3\left(-\sqrt{\frac{3}{8\pi}}\right) = \sqrt{2} : 1 : -1$$

$$\Rightarrow \quad a_1 : a_2 : a_3 = 1 : 1 : 1$$

$$\therefore \psi(\theta,\phi) = \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right) + \frac{1}{\sqrt{3}}\left(\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$$

$$+ \frac{1}{\sqrt{3}}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right)$$ after normalisation

$$\therefore \langle\theta,\phi|\psi\rangle = \frac{1}{\sqrt{3}}\langle\theta,\phi|1,0\rangle + \frac{1}{\sqrt{3}}\langle\theta,\phi|1,-1\rangle$$

$$+ \frac{1}{\sqrt{3}}\langle\theta,\phi|1,1\rangle$$

$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{3}}|1,0\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle + \frac{1}{\sqrt{3}}|1,1\rangle$$

(a) only possible $\hat{L}^2$ is $\boxed{2\hbar^2}$

(b)

eigenstate	L_z	Probability
$\|1,0\rangle$	0	$\frac{1}{3}$
$\|1,-1\rangle$	$-\hbar$	$\frac{1}{3}$
$\|1,1\rangle$	$\hbar$	$\frac{1}{3}$

$$\langle L_z \rangle = \langle \psi | \hat{L}_z | \psi \rangle$$

$$= \frac{1}{3}(0)\underbrace{\langle 1,0|1,0\rangle}_{1} + \frac{1}{3}(-\hbar)\underbrace{\langle 1,-1|1,-1\rangle}_{1}$$

$$+ \frac{1}{3}(\hbar)\underbrace{\langle 1,1|1,1\rangle}_{1} = \boxed{0}$$

4. $$\hat{L}^2(\sin^2\theta\, e^{2i\phi}) = -\hbar^2\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}(\sin^2\theta\, e^{2i\phi})\Big)$$
$$+ \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(\sin^2\theta\, e^{2i\phi})\Big].$$

$$= -\hbar^2\Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(2\sin^2\theta\cos\theta\, e^{2i\phi}) - 4e^{2i\phi}\Big]$$

$$= -\hbar^2\Big[\frac{1}{\sin\theta}(-2\sin^3\theta\, e^{2i\phi} + 4\sin\theta\cos^2\theta\, e^{2i\phi}) - 4e^{2i\phi}\Big]$$

$$= -\hbar^2\Big[-2\sin^2\theta\, e^{2i\phi} - 4\underbrace{(1-\cos^2\theta)}_{\sin^2\theta}e^{2i\phi}\Big].$$

$$= 6\hbar^2\,[\sin^2\theta\, e^{2i\phi}]$$

$\therefore$ $\sin^2\theta\, e^{2i\phi}$ is an eigenfunction of $\hat{L}^2$ with eigenvalue $6\hbar^2$

$$\hat{L}_z(\sin^2\theta\, e^{2i\phi}) = -i\hbar\frac{\partial}{\partial\phi}(\sin^2\theta\, e^{2i\phi})$$

$$= -i\hbar(2i)\sin^2\theta\, e^{2i\phi}$$

$$= (2\hbar)\sin^2\theta\, e^{2i\phi}$$

$\therefore$ $\sin^2\theta\, e^{2i\phi}$ is an eigenfunction of $\hat{L}_z$ with eigenvalue $2\hbar$

∴ (a) ~~L_z^2 gives~~

L_z gives $\boxed{2\hbar}$

(b)

L^2 gives $\boxed{6\hbar^2}$

Probability $= 1$ for both (a) and (b)

5. $Y_{00} = \frac{1}{2}\frac{1}{\sqrt{\pi}}$ $\qquad Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$

$$3\cos^2\theta - 1 = 3(\cos^2\theta - 1) + 2$$
$$= -3\sin^2\theta + 2$$

$$\Rightarrow \quad \sin^2\theta = \frac{2}{3} - \frac{1}{3}(3\cos^2\theta - 1)$$

$\because \psi(\theta, \phi) \propto \sin^2\theta$

$\therefore \psi(\theta, \phi) \propto \frac{2}{3} - \frac{1}{3}(3\cos^2\theta - 1)$

let $\psi(\theta,\phi) = c_1\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\right) + c_2\left(\frac{1}{4}\sqrt{\frac{5}{\pi}}\ \cancel{\cos}\ (3\cos^2\theta - 1)\right)$

$$\Rightarrow \quad \frac{2\cancel{\frac{1}{2}}\frac{1}{\cancel{\sqrt{\pi}}}c_1}{\cancel{\frac{1}{4}}\sqrt{\frac{5}{\cancel{\pi}}}c_2} = \frac{2/3}{-1/3} = -2$$

$$\Rightarrow \quad \frac{2c_1}{\sqrt{5}c_2} = -2$$

$$\therefore \quad c_1 = -\sqrt{5}c_2$$

$$c_1^2 + c_2^2 = 1 \Rightarrow 6c_2^2 = 1 \Rightarrow c_2 = -\sqrt{\frac{1}{6}}, \ c_1 = \sqrt{\frac{5}{6}}$$

$$\Rightarrow \underbrace{\psi(\theta,\phi)}_{\langle\theta,\phi|\psi\rangle} = \sqrt{\frac{5}{6}}\underbrace{\left(\frac{1}{2}\frac{1}{\sqrt{\pi}}\right)}_{\langle\theta,\phi|0,0\rangle} - \sqrt{\frac{1}{6}}\underbrace{\left[\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)\right]}_{\langle\theta,\phi|2,0\rangle}$$

$$\Rightarrow \quad |\psi\rangle = \sqrt{\frac{5}{6}}|0,0\rangle - \sqrt{\frac{1}{6}}|2,0\rangle$$

(a)

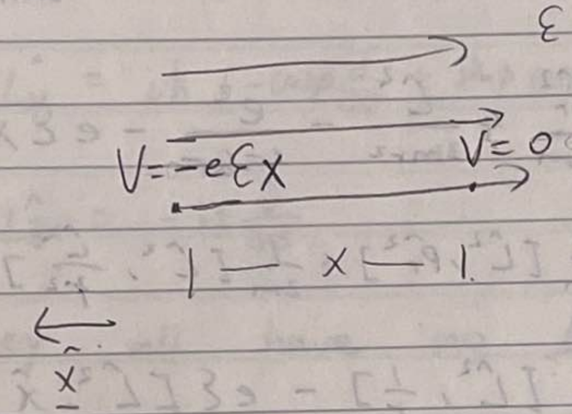
L_z	P
0	1

(b)

L^2	P
0	$\frac{5}{6}$
$6\hbar^2$	$\frac{1}{6}$

6. (a)

Consider an external constant electric field $\vec{\mathcal{E}} = \mathcal{E}\hat{x}$ along ~~...~~ the ~~x-direction~~ minus x-direction



The ~~poten~~ potential energy associated with this field is $V = -e\mathcal{E}x$

Convert to operator gives $\hat{V} = -e\mathcal{E}\hat{x}$

which is the last term in $\hat{H}$

∴ ~~$\hat{H}$~~ The last term represents a constant ~~electric f~~ external electric field.

(b) $[L_i, x_j] = i\hbar \varepsilon_{ijk} \hat{x}_k$

$\Rightarrow [\hat{L}_x, \hat{x}] = \boxed{0}$

$[\hat{L}_y, \hat{x}] = \cancel{-\hat{z}}\ \boxed{-i\hbar \hat{z}}$

$[\hat{L}_z, \hat{x}] = \boxed{i\hbar \hat{y}}$

(c) $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r} - e\mathcal{E}\hat{x}$

$\Rightarrow \cancel{\hat{H}}$ $\because$ $-\frac{\hbar^2}{2m}\nabla^2 = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2}$

$\Rightarrow \hat{H} = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r} - e\mathcal{E}\hat{x}$

$[\hat{L}^2, \hat{H}] = \frac{1}{2m}[\hat{L}^2, \hat{p}_r^2] + \frac{1}{2m}[\hat{L}^2, \frac{\hat{L}^2}{r^2}]$

$- \frac{e^2}{4\pi\varepsilon_0}[\hat{L}^2, \frac{1}{r}] - e\mathcal{E}[\hat{L}^2, \hat{x}]$

$\cancel{[\hat{L}^2}$ $\because$ $\hat{L}^2 = \cancel{\frac{\hbar^2}{s}} -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$

has no r dependence

and $\hat{p}_r^2 = -\hbar^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})\right]$

has no θ and ϕ dependence

$\therefore [\hat{L}^2, \hat{p}_r^2] = 0$

$[\hat{L}^2, \frac{\hat{L}^2}{r^2}] = \cancel{\hat{L}^2[\hat{L}^2, \frac{1}{r^2}] + [\hat{L}^2, \hat{L}^2]\frac{1}{r^2}}$

$\frac{1}{r^2}\underbrace{[\hat{L}^2, \hat{L}^2]}_{0} + \underbrace{[\hat{L}^2, \frac{1}{r^2}]}_{0}\hat{L}^2 = 0$

$[\hat{L}^2, \frac{1}{r}] = 0$

$$\Rightarrow \quad [\hat{L}^2, \hat{H}] = -e\mathcal{E}[\hat{L}^2, \hat{x}]$$

$$\hat{L_x} = i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L_y} = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$\hat{L_z} = -i\hbar\frac{\partial}{\partial\phi}$$

they all have no ~~θ ϕ~~ r dependence

$$\Rightarrow \cancel{[\hat{L_x}, \hat{p_r}^2]} \quad [\hat{L_i}, \hat{p_r}^2] = [\hat{L_i}, \frac{\hat{L}^2}{r^2}]$$
$$= [\hat{L_i}, \frac{1}{r}] = 0 \quad \text{for } i = x, y, z$$

$$\left([\hat{L_i}, \frac{\hat{L}^2}{r}] = 0 \quad \text{since} \quad [\hat{L_i}, \hat{L}^2] = 0\right)$$

$$\therefore [\hat{L_i}, \hat{H}] = \frac{1}{2m}[\hat{L_i}, \hat{p_r}^2] + \frac{1}{2m}[\hat{L_i}, \frac{\hat{L}^2}{r^2}]$$
$$- \frac{e^2}{4\pi\varepsilon_0}[\hat{L_i}, \frac{1}{r}] - e\mathcal{E}[\hat{L_i}, \hat{x}]$$
$$= -e\mathcal{E}[\hat{L_i}, \hat{x}].$$

(i) for $\mathcal{E} = 0$

$$[\hat{L}^2, \hat{H}] = 0 \quad , \quad [\hat{L_i}, \hat{H}] = 0$$

By ~~Ehre~~ Ehrenfest theorem:

$$\frac{d}{dt}\langle\hat{L}^2\rangle = \frac{1}{i\hbar}\langle[\hat{L}^2, \hat{H}]\rangle = 0 \qquad \frac{d}{dt}\langle\hat{L_i}\rangle = \frac{1}{i\hbar}\langle[\hat{L_i}, \hat{H}]\rangle$$
$$= 0$$

$\therefore \quad \hat{L}^2, \hat{L_x}, \hat{L_y}, \hat{L_z}$ are all constants.

(ii) $\quad \varepsilon \neq 0$

$$[\hat{L}^2, \hat{x}] = [\hat{L_x}^2 + \hat{L_y}^2 + \hat{L_z}^2, \hat{x}]$$

$$= \hat{L_x}\underbrace{[\hat{L_x}, \hat{x}]}_{0} + \underbrace{[\hat{L_x}, \hat{x}]}_{0}\hat{L_x} + \hat{L_y}\underbrace{[\hat{L_y}, \hat{x}]}_{\cancel{-i\hbar\hat{z}}\ -i\hbar\hat{z}} + \underbrace{[\hat{L_y}, \hat{x}]}_{-i\hbar\hat{z}}\hat{L_y}$$

$$+ \hat{L_z}\underbrace{[\hat{L_z}, \hat{x}]}_{i\hbar\hat{y}} + \underbrace{[\hat{L_z}, \hat{x}]}_{i\hbar\hat{y}}\cancel{\hat{L_y}}\hat{L_z}$$

$$= -i\hbar[\hat{L_y}\hat{z} + \hat{z}\hat{L_y} - \hat{L_z}\hat{y} - \hat{y}\hat{L_z}] \neq 0$$

$\Rightarrow \quad \underline{\hat{L}^2 \text{ is not constant}}$

$[L_x, \hat{x}] = 0 \quad \Rightarrow \quad \underline{\hat{L_x} \text{ is constant}}$

$[\hat{L_y}, \hat{x}] \cancel{\neq 0} = -i\hbar\hat{z} \neq 0 \quad \Rightarrow \quad \underline{\cancel{\hat{L_z}}\ \hat{L_y} \text{ is not constant}}$

$[\hat{L_z}, \hat{x}] = i\hbar\hat{y} \neq 0 \quad \Rightarrow \quad \underline{\hat{L_z} \text{ is not constant}}$

7. $\hat{x}\cdot\hat{p} = \sum_j \hat{x_j}\hat{p_j}$

$$[\hat{L_i}, \hat{x}\cdot\hat{p}] = \sum_j [\hat{L_i}, \hat{x_j}\hat{p_j}]$$

$$= \sum_j \left(x_j[\hat{L_i}, \hat{p_j}] + [\hat{L_i}, \hat{x_j}]\hat{p_j}\right)$$

$$= i\hbar \sum_{j,k} \left(\hat{x_j}\varepsilon_{ijk}\hat{p_k} + \varepsilon_{ijk}\hat{x_k}\hat{p_j}\right)$$

$$= i\hbar \sum_{j,k} [\varepsilon_{ijk}(\hat{x_j}\hat{p_k} + \hat{x_k}\hat{p_j})]$$

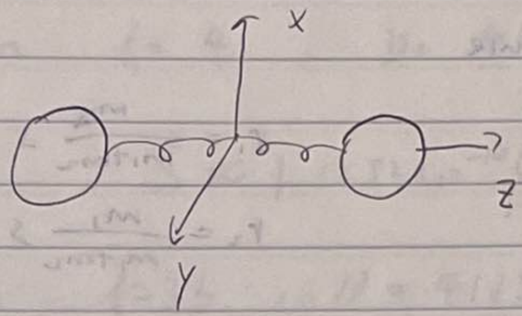
$$= i\hbar \sum_{j,k} [\varepsilon_{ikj}(\hat{x_k}\hat{p_j} + \hat{x_j}\hat{p_k})] \quad \text{(swap j and k)}$$

$$= i\hbar \sum_{j,k} [-\varepsilon_{ijk}(\hat{x_j}\hat{p_k} + \hat{x_k}\hat{p_j})] \quad (\varepsilon_{ikj} = -\varepsilon_{ijk})$$

$$= -[\hat{L_i}, \hat{x}\cdot\hat{p}] = 0 \quad (A = -A \Leftrightarrow A = 0)$$

QED

8.



Hamiltonian $\hat{H} = \frac{\hat{L}_x^2}{2I_x} + \frac{\hat{L}_y^2}{2I_y} + \frac{\hat{L}_z^2}{2I_z}$

where $I_z \ll I_x = I_y = I$

$\cancel{\hat{H} = \frac{\hat{L}^2}{2I} + \hat{L}_z^2}$ $\qquad \hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle$

$\hat{L}_z |l,m\rangle = m\hbar |l,m\rangle$

$\cancel{\Rightarrow \hat{H} = \frac{\hat{L}^2}{2I} + \hat{L}_z^2 \left(\frac{1}{2I} - \frac{1}{2I_z} \right)}$

$$\Rightarrow \hat{H} = \frac{\hat{L}^2}{2I} + \hat{L}_z^2 \left(\frac{1}{2I_z} - \frac{1}{2I} \right)$$

For an eigenstate of $\hat{L}^2$, $\hat{L}_z$ $|l,m\rangle$

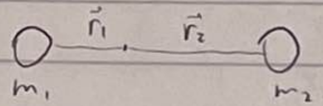
$$\hat{H}|l,m\rangle = \left[\frac{l(l+1)\hbar^2}{2I} + \frac{(m\hbar)^2}{2} \left(\frac{1}{I_z} - \frac{1}{I} \right) \right] |l,m\rangle$$

We focus on the low energy properties of this molecule so we set $m=0$, otherwise the energy cost is very large

$$\Rightarrow \hat{H}|l,m\rangle = \frac{\hbar^2 l(l+1)}{2I} |l,m\rangle$$

$$\Rightarrow \boxed{E_l = \frac{\hbar^2 l(l+1)}{2I}}$$

CO molecule



$$r_1 = \frac{m_2}{m_1+m_2} s$$

$$r_2 = \frac{m_1}{m_1+m_2} s$$

$$\cancel{I = \frac{1}{2} m_1 r_1^2 +} \qquad I = m_1 r_1^2 + m_2 r_2^2$$

$$= m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 s^2 + m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 s^2$$

$$= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2} s^2$$

$$= \cancel{m_1+} \frac{m_1 m_2 \cancel{(m_1+m_2)}}{(m_1+m_2)^{\cancel{2}}} s^2$$

$$\Rightarrow \quad I = \left(\frac{m_1 m_2}{m_1+m_2}\right) s^2$$

$$\Rightarrow \quad \boxed{I = \mu s^2}$$

$$\cancel{\hbar\omega = \hbar(2\pi f) =}$$

$$\hbar\omega = \hbar(2\pi\nu) = E_l - E_{l-1}$$

$$= \frac{\hbar^2}{2I}\left[l(l+1) - l(l-1)\right]$$

$$= \frac{\hbar^2 l}{I} \quad \Rightarrow \quad \nu = \frac{\hbar l}{2\pi I}$$

$$\Rightarrow \nu = \frac{\hbar l}{2\pi \mu s^2} = \frac{7\hbar l}{96\pi m_p s^2} \Rightarrow \boxed{s = \left(\frac{7\hbar l}{96\pi m_p \nu}\right)^{\frac{1}{2}}}$$

$$\left(\mu = \frac{48}{7} m_p\right)$$

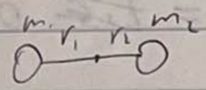
When $l = 4$, $\nu = 461.04077 \times 10^{9}$ Hz

$\Rightarrow \quad S \approx 1.1252 \times 10^{-10}$ m

When $l = 36$, $\nu = 4115.6055 \times 10^{9}$ Hz

$\Rightarrow \quad S \approx 1.1298 \times 10^{-10}$ nm

$\Rightarrow \quad \boxed{S \sim 0.113 \text{ nm}}$



Centripetal force

$$F = \cancel{m_2 \omega} \; m_2 \omega^2 r_2 = m_2 \omega^2 \frac{m_1}{m_1 + m_2} S$$

$$= \frac{m_1 m_2}{m_1 + m_2} \omega^2 S$$

$$= \mu S \omega^2$$

$\because \quad L = I\omega \quad \Rightarrow \quad \omega = \frac{L}{I}$

$\because \quad I = \mu S^2 \quad \Rightarrow \quad \mu S = \frac{I}{S}$

$$\Rightarrow \quad F = \frac{I}{S} \frac{L^2}{I^2} = \frac{L^2}{IS} = \frac{L^2}{\mu S^3} = \boxed{\frac{\hbar^2 l(l+1)}{\mu S^3}}$$

$F_1 = k(x_1 - x_0)$

$F_2 = k(x_2 - x_0)$

$\Rightarrow \quad F_2 - F_1 = k(x_2 - x_1)$

$\Rightarrow \quad \Delta F = k \Delta x$

$$\therefore k = \frac{\Delta F}{\Delta x} = \frac{\Delta F}{\Delta S}$$

$$\Delta S = (1.1298 - 1.1252) \times 10^{-10} = 0.0046 \times 10^{-10}\ m$$

$$\cancel{F_{36} = 8.01 \times 10^{-2}\ 8.41 \times 10^{-10}\ N}$$

$$F_{36} = \frac{\hbar^2 (36)(37)}{\frac{48}{7} m_p (1.1298 \times 10^{-10})^3} = 8.89 \times 10^{-10}\ N$$

$$F_4 = \frac{\hbar^2 (4)(5)}{\frac{48}{7} m_p (1.1252 \times 10^{-10})^3} = 0.135 \times 10^{-10}\ N$$

$$\Delta F = 8.755 \times 10^{-10}\ N$$

$$\therefore k = \frac{\Delta F}{\Delta S} = \frac{8.755 \times 10^{-10}\ N}{0.0046 \times 10^{-10}\ m} = \boxed{1903\ Nm^{-1}}$$

Vibrational frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{N}} = \frac{1}{2\pi}\sqrt{\frac{1903}{\frac{48}{7} \times 1.67 \times 10^{-27}}} = \boxed{6.488 \times 10^{13}\ Hz}$$