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Quantum Mechanics 3

1.

$\frac{1}{2}(\hat{p}^2+\hat{x}^2)(\hat{x}\mp i\hat{p})|\psi\rangle$

$= \frac{1}{2}\hat{p}^2\hat{x}|\psi\rangle + \frac{1}{2}\hat{x}^3|\psi\rangle \mp \frac{i}{2}\hat{p}^3|\psi\rangle \mp \frac{i}{2}\hat{x}^2\hat{p}|\psi\rangle$

$\because [\hat{x},\hat{p}] = i \quad \therefore [\hat{x},\hat{p}^2] = 2[\hat{x},\hat{p}]\hat{p} = 2i\hat{p}$

$[\hat{p},\hat{x}^2] = 2[\hat{p},\hat{x}]\hat{x} = -2i\hat{x}$

$\therefore \hat{x}\hat{p}^2 - \hat{p}^2\hat{x} = 2i\hat{p}, \quad \hat{p}\hat{x}^2 - \hat{x}^2\hat{p} = -2i\hat{x}$

$\therefore \hat{p}^2\hat{x} = \hat{x}\hat{p}^2 - 2i\hat{p}, \quad \hat{x}^2\hat{p} = \hat{p}\hat{x}^2 + 2i\hat{x}$

$\therefore \frac{1}{2}(\hat{p}^2+\hat{x}^2)(\hat{x}\mp i\hat{p})|\psi\rangle$

$= \frac{1}{2}(\hat{x}\hat{p}^2 - 2i\hat{p})|\psi\rangle + \frac{1}{2}\hat{x}^3|\psi\rangle \mp \frac{i}{2}\hat{p}^3|\psi\rangle \mp \frac{i}{2}(\hat{p}\hat{x}^2 + 2i\hat{x})|\psi\rangle$

$= \hat{x}\,\frac{1}{2}(\hat{p}^2+\hat{x}^2)|\psi\rangle - i\hat{p}|\psi\rangle \mp (i\hat{p}\,\frac{1}{2}(\hat{p}^2+\hat{x}^2) - \hat{x})|\psi\rangle$

$= (\hat{x}\mp i\hat{p})(\frac{1}{2}(\hat{p}^2+\hat{x}^2))|\psi\rangle - i\hat{p}|\psi\rangle \pm \hat{x}|\psi\rangle$

$= (\hat{x}\mp i\hat{p})\hat{H}|\psi\rangle \pm (\hat{x}\mp i\hat{p})$

$= (\hat{x}\mp i\hat{p})(\hat{H}|\psi\rangle \pm |\psi\rangle)$

$= (\hat{x}\mp i\hat{p})(E\pm 1)|\psi\rangle = \boxed{(E\pm 1)(\hat{x}\mp i\hat{p})|\psi\rangle}$

This algebra shows that $(\hat{x} \mp i\hat{p})|\psi\rangle$ is also a eigenfunction of the hamiltonian operator $\hat{H}$ ~~and~~ with the eigenvalues

$E \pm 1$

Knowing that the minimum energy level is $E = \frac{1}{2}$, we can then conclude that the energy eigenvalues are non-negative integers plus $\frac{1}{2}$. ~~$E=0, 1, 2,$~~ $E_n = n + \frac{1}{2}$

Proof: Define $a = \hat{x} + i\hat{p}$ then $\because \hat{x}, \hat{p}$ are hermitian

$\therefore \hat{a}^{\dagger} = \hat{x} - i\hat{p}$ $\therefore \hat{H}\hat{a}|\psi\rangle = (E-1)\hat{a}|\psi\rangle$

$\hat{H}\hat{a}^{\dagger}|\psi\rangle = (E+1)\hat{a}^{\dagger}|\psi\rangle$. let $\hat{n} = \hat{a}^{\dagger}\hat{a}$

then $\hat{n} = (\hat{x} - i\hat{p})(\hat{x} + i\hat{p}) = \hat{x}^2 + \hat{p}^2 + i\underbrace{[\hat{x}, \hat{p}]}_{i} = \hat{x}^2 + \hat{p}^2 - 1$

$= 2\hat{H} - 1$

$\therefore \hat{n}|\psi\rangle = (2\hat{H} - 1)|\psi\rangle = (2E - 1)|\psi\rangle$

let $\hat{n}|\psi\rangle = n|\psi\rangle$ Then $n = \langle\psi|\hat{n}|\psi\rangle = \langle\psi|\hat{a}^{\dagger}\hat{a}|\psi\rangle$

$= \| \hat{a}|\psi\rangle \|^2 \geq 0$

$\because$ ~~$\hat{H}\hat{a}|\psi\rangle = (E-1)\hat{a}|\psi\rangle$~~ $\therefore$ lowest energy state $n = 0$

$\because$ lowest energy state $2E_0 - 1 = 0$ $\therefore E_0 = \frac{1}{2}$

$\because E + 1$ are also energy eigenvalues when E is an eigenvalue

$\therefore$ $\boxed{E_n = n + \frac{1}{2}}$ where n is a non-negative integer

2.

$$\hat{a} \equiv \frac{m\omega\hat{x}+i\hat{p}}{\sqrt{2m\hbar\omega}} \quad \because \hat{x}, \hat{p} \text{ are hermitian}$$

$$\therefore \hat{a}^{\dagger} = \frac{m\omega\hat{x}-i\hat{p}}{\sqrt{2m\hbar\omega}}$$

define $\hat{n} = \hat{a}^{\dagger}\hat{a} = \frac{1}{2m\hbar\omega}(m\omega\hat{x}-i\hat{p})(m\omega\hat{x}+i\hat{p})$

$$= \frac{1}{2m\hbar\omega}(m^2\omega^2\hat{x}^2 + \hat{p}^2 + i\underbrace{[\hat{x},\hat{p}]}_{i\hbar} m\omega)$$

$$= \frac{1}{2m\hbar\omega}(m^2\omega^2\hat{x}^2 + \hat{p}^2 - \hbar m\omega)$$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2m\hbar\omega} - \frac{m\omega}{2m\omega}$$

$$= \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2m\hbar\omega} - \frac{1}{2}$$

$$\therefore \hat{n} + \frac{1}{2} = \frac{m\omega}{2\hbar}\hat{x}^2 + \frac{\hat{p}^2}{2m\hbar\omega}$$

$$\therefore \hbar\omega(\hat{n}+\frac{1}{2}) = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hat{H}$$

$\because E_n = (n+\frac{1}{2})\hbar\omega \quad \therefore \hat{H}|\psi\rangle = (n+\frac{1}{2})\hbar\omega|\psi\rangle$

$\therefore \hbar\omega(\hat{n}+\frac{1}{2})|\psi\rangle = (n+\frac{1}{2})\hbar\omega|\psi\rangle \implies \hat{n}|\psi\rangle = n|\psi\rangle$

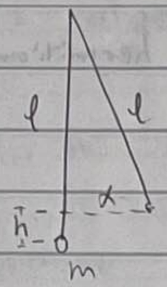
let $|\psi\rangle = |n\rangle$, $\quad \hat{n}|n\rangle = n|n\rangle$

$\therefore \quad n = \langle n|\hat{n}|n\rangle = \langle n|\hat{a}^{\dagger}\hat{a}|n\rangle = ||\hat{a}|n\rangle||^2$

$= \alpha^2 \underbrace{\langle n-1|n-1\rangle}_{1} = \alpha^2 \implies \boxed{\alpha = \sqrt{n}}$

($\alpha = |\alpha|e^{i\theta}$, we are free to choose θ so we choose $\theta = 0$. $\alpha = |\alpha|$ = real and positive)

3.



$$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\therefore\ l = \frac{gT^2}{4\pi^2} = \frac{(9.8)(1)^2}{4\pi^2} = 0.248\,m$$

$x = 0.030\,m$, $h = l - \sqrt{l^2 - x^2} = 0.0018\,m$

Total energy of pendulum $E_n = mgh = (0.2)(9.8)(0.0018)$

$$= 0.00353\,J$$

$E_n = \hbar\omega(n + \frac{1}{2})$, $\omega = \frac{2\pi}{T} = 2\pi\ s^{-1}$

$$\therefore\ n = \frac{E_n}{\hbar\omega} - \frac{1}{2} = \frac{0.00353}{6.63\times10^{-34}} - \frac{1}{2} \approx \boxed{5.32\times10^{30}}$$

4. $E(p, x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, $xp = \frac{\hbar}{2}$. $p = \frac{\hbar}{2x}$ $\frac{dp}{dx} = -\frac{\hbar}{2x^2}$

$p^2 = \frac{\hbar^2}{4x^2}$

$$\frac{dE}{dx} = \frac{p}{m}\frac{dp}{dx} + m\omega^2 x = 0 \quad \text{at extreme values}$$

$$\therefore\ \cancel{\frac{p}{m}\frac{\hbar}{2m}} - \frac{p}{m}\frac{\hbar}{2x^2} + m\omega^2 x = 0$$

$$\therefore\ \frac{\hbar^2}{4mx^3} = m\omega^2 x \quad \therefore\ x^4 = \frac{\hbar^2}{4m^2\omega^2} \Rightarrow x^2 = \frac{\hbar}{2m\omega}$$

$$\Rightarrow p^2 = \frac{\hbar^2}{2\hbar/m\omega} = \frac{\hbar m\omega}{2}$$

$$\therefore\ E_{min} = \frac{1}{2m}\left(\frac{\hbar m\omega}{2}\right) + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)$$

$$= \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \boxed{\frac{1}{2}\hbar\omega}$$

or... from classical point of view, if x & p cannot simultaneously be zero, then there's nonzero min energy for harmonic oscillator

For $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, $\langle x\rangle = 0$, $\langle p\rangle = 0$ $\therefore\ x \sim \Delta x$, $p \sim \Delta p$ $\therefore$ $xp = \frac{\hbar}{2}$ resembles $\Delta x\Delta p = \frac{\hbar}{2}$. When the uncertainty principle takes up the equality sign, the equilibrium (minimum) energy of oscillator is $\frac{1}{2}\hbar\omega$ (Ground state energy)

5. $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$ $\quad \hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \qquad \hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\therefore |n\rangle = \frac{(\hat{a}^{+})^n|0\rangle}{\sqrt{n!}}$

$\hat{a}|0\rangle = \vec{0} \Rightarrow \langle x|\hat{a}|0\rangle = 0$

$\Rightarrow \langle x|\hat{x} + \frac{i\hat{p}}{m\omega}|0\rangle = 0$

$\hat{x}|x\rangle = x|x\rangle$, $\langle x|\hat{x} = x\langle x|$ ($\hat{x}$ is hermitian)

$\therefore x\langle x|0\rangle + \frac{i}{m\omega}\langle x|-i\hbar\frac{\partial}{\partial x}|0\rangle = 0$

$\therefore x\psi_0(x) + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\psi_0(x) = 0$

$\Rightarrow \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}x\,dx$

$\therefore \psi_0(x) = A e^{-\frac{m\omega x^2}{2\hbar}}$

$|n\rangle = \frac{(\hat{a}^{+})^n|0\rangle}{\sqrt{n!}} \Rightarrow \psi_n(x) = \langle x|n\rangle = \frac{(\hat{a}^{+})^n\langle x|0\rangle}{\sqrt{n!}}$

$= \frac{1}{\sqrt{n!}}(\hat{a}^{+})^n\psi_0(x)$

For $\psi_0(x)$, normalise gives

$\int_{-\infty}^{\infty}|\psi_0(x)|^2\,dx = 1 \quad \therefore A^2\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx = 1$

$\therefore$ ~~A^2~~ let $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

then $A^2\int_{-\infty}^{\infty} e^{-\alpha^2 x^2}\,dx = \cancel{g}\ 1$

$$\therefore A^2 \frac{\sqrt{\pi}}{\alpha} = 1$$

$$\therefore A^2 = \frac{\alpha}{\sqrt{\pi}} \qquad \therefore A = \sqrt{\alpha}\,\pi^{-\frac{1}{4}}$$

$$\cancel{\therefore A = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}}$$

$$\cancel{\therefore \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)}$$

$$\psi_n(x) = \frac{1}{\sqrt{n}}\hat{a}^\dagger \psi_{n-1}(x) = \frac{1}{\sqrt{n}}\frac{\alpha}{\sqrt{2}}\left(\hat{x} - \frac{i}{m\omega}(-i\hbar)\frac{d}{dx}\right)\psi_{n-1}(x)$$

$$= \frac{1}{\sqrt{2n}}\left(\alpha x - \frac{1}{\alpha}\frac{d}{dx}\right)\psi_{n-1}(x)$$

let $\xi = \alpha x$, $\psi_n(\xi) = \frac{1}{\sqrt{2n}}\left(\xi - \frac{d}{d\xi}\right)\psi_{n-1}(\xi)$

$$\therefore \psi_n = (2^n n!)^{-\frac{1}{2}}\left(\xi - \frac{d}{d\xi}\right)^n \psi_0$$

$$= \left(\frac{\alpha}{\sqrt{\pi}\,2^n n!}\right)^{\frac{1}{2}}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2}$$

~~Now we prove that~~ $\cancel{\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2} = (-1)^n \frac{d^n}{d\xi^n} e^{-\xi^2}}$

~~Prove by Induction~~

Now we prove that

$$e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2} = e^{\xi^2}(-1)^n\frac{d^n}{d\xi^n}e^{-\xi^2}$$

prove by induction

Base case when $n=1$

$$LHS = e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)e^{-\xi^2/2} = \left(\xi e^{-\xi^2/2} - e^{-\xi^2/2}(-\xi)\right)e^{\xi^2/2}$$

$$= 2\xi e^{-\xi^2/2} e^{+\xi^2/2} = \cancel{2\xi^2}\ 2\xi$$

$$RHS = e^{\xi^2}(-1)^1 \frac{d}{d\xi} e^{-\xi^2} = e^{\xi^2}\left((-1)(-2\xi)\right)e^{-\xi^2}$$

$$= 2\xi$$

RHS = LHS $\Rightarrow$ $n=1$ proved

Induction: assume proposition true for $n=k$, then for $n=k+1$

$$\cancel{e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^k} \quad e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^{k+1} e^{-\xi^2/2}$$

$$= e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)\left(\xi - \frac{d}{d\xi}\right)^k e^{-\xi^2/2}$$

$$\cancel{= e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)\left[\left(\xi - \frac{d}{d\xi}\right)^k e^{-\xi^2/2}\right]}$$

~~inductive assumption~~

$$\cancel{= e^{\xi^2/2}(-1)^k\left(\xi - \frac{d}{d\xi}\right)e^{+\xi^2/2}\frac{d^k}{d\xi^k}e^{-\xi^2}}$$

$$\cancel{= e^{\xi^2/2}(-1)^k \xi e^{\xi^2/2}\frac{d^k}{d\xi^k}e^{-\xi^2} - e(-1)^k e^{\xi^2}\frac{d^{k+1}}{d\xi^{k+1}}}$$

$$= e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)\left[\left(\xi - \frac{d}{d\xi}\right)^k e^{-\xi^2/2}\right]$$

inductive assumption

$$= e^{\xi^2/2}(-1)^k\left(\xi - \frac{d}{d\xi}\right)e^{\xi^2/2}\frac{d^k}{d\xi^k}e^{-\xi^2}$$

$$= e^{\xi^2/2}(-1)^k \cancel{\xi e^{\xi^2/2}\frac{d^k}{d\xi^k}e^{-\xi^2}}$$

$$- e^{\xi^2/2}(-1)^k e^{\xi^2/2}\frac{d^{k+1}}{d\xi^{k+1}}e^{-\xi^2}$$

$$\cancel{- e^{\xi^2/2}(-1)^k \xi e^{\xi^2/2}\frac{d^k}{d\xi^k}e^{-\xi^2}}$$

$$= e^{\xi^2}(-1)^{k+1}\frac{d^{k+1}}{d\xi^{k+1}}e^{-\xi^2}$$ proven.

$$\therefore \quad e^{\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2} = e^{\xi^2}(-1)^n \frac{d^n}{d\xi^n}e^{-\xi^2}$$

$$\therefore \quad \left(\xi - \frac{d}{d\xi}\right)^n e^{-\xi^2/2} = e^{\xi^2/2}(-1)^n \frac{d^n}{d\xi^n}e^{-\xi^2}$$

$$\therefore \quad \psi_n(\xi) = \left(\frac{\alpha}{\sqrt{\pi}\,2^n n!}\right)^{\frac{1}{2}} e^{\xi^2/2}(-1)^n \frac{d^n}{d\xi^n}e^{-\xi^2}$$

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let $N_n = \left(\frac{\alpha}{\sqrt{\pi}\,2^n n!}\right)^{\frac{1}{2}}$

Now we investigate the number ~~of~~ of nodes (~~zero~~ distinct real roots) of $\psi_n(\xi) = 0$

$n=0$, ~~ψ_0~~ $\psi_0(\xi) = N_0 e^{-\xi^2/2}$ has 0 nodes

$n=1$, $\psi_1(\xi) = N_1(2\xi)e^{-\xi^2/2}$ has 1 distinct real root at $\xi = 0$

$\therefore$ We propose that $\psi_n(\xi)$ has n nodes.

Prove by induction.

Base case has ~~pr~~ been proven ~~before~~ above

Induction : Assume ψ_n has n nodes for $n=k$,

~~then for $n=k+1$~~

(This means that $\psi_k(\xi) = N_k e^{\xi^2/2} (-1)^k \frac{d^k}{d\xi^k} e^{-\xi^2}$

has k distinct real roots, which means

that $f(k,\xi) = \frac{d^k}{d\xi^k} e^{-\xi^2}$ has k distinct real roots)

Then for $n = k+1$

$$\frac{d^{k+1}}{d\xi^{k+1}} e^{-\xi^2} = \frac{d}{d\xi}\left(\frac{d^k}{d\xi^k} e^{-\xi^2}\right)$$

$$= \frac{d}{d\xi} f(k,\xi) = f(k+1,\xi)$$

$\because$ Exponential decay faster than polynomial grows

$\therefore$ $f(k,\xi)$ goes to 0 as $\xi \to \pm\infty$

* Rolle's Theorem states that between

any two zeros of a differentiable

was only expecting:

$|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$ • $\hat{a}^+ \sim \hat{x} \Rightarrow (\hat{a}^+)^n \sim \hat{x}^n$

$\Rightarrow |n\rangle$ is n^{th} order polynomial times gaussian $\Rightarrow n$ roots

function there is at least one zero

of its derivative

~~This~~ There ~~are~~ are k distinct real

roots for $f(k, \xi)$ so ~~there~~ we

already have $k-1$ distinct real roots ~~for~~

$\frac{d}{d\xi} f(k, \xi)$.

Also since $f(k, \xi) \to 0$ as $\xi \to \pm\infty$

~~so there must~~, by Rolle's Theorem there must be 2 addition roots ~~, one between the maximum root of fit~~ for $\frac{d}{d\xi} f(k, \xi)$, one

between the maximum root ~~a~~ of $f(k, \xi)$ and $+\infty$, the other between the minimum root of $f(k, \xi)$ and $-\infty$,

$\therefore$ This gives a total of at least $k-1+2 = k+1$ roots for $f(k+1, \xi) = \frac{d}{d\xi} f(k, \xi)$

$\therefore \frac{d^{k+1}}{d\xi^{k+1}} e^{-\xi^2}$ has ~~???~~ at least $k+1$ distinct real roots

$\because \frac{d^{k+1}}{d\xi^{k+1}} e^{-\xi^2}$ is a polynomial of degree $k+1$ times $e^{-\xi^2}$

$\therefore$ it has at most $k+1$ distinct real roots

$\therefore \frac{d^{k+1}}{d\xi^{k+1}} e^{-\xi^2}$ has exactly $k+1$ distinct real roots

$\therefore \frac{d^n}{d\xi^n} e^{-\xi^2}$ has n distinct real roots $\Rightarrow \psi_n(\xi)$ has n nodes

$\Rightarrow \boxed{\psi_n(x) \text{ has } n \text{ nodes}}$

Wow Well done.

6. $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right) \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$

$$l \equiv \sqrt{\frac{\hbar}{2m\omega}} \quad \frac{1}{l} = \sqrt{\frac{2m\omega}{\hbar}}$$

$$\therefore \sqrt{\frac{m\omega}{2\hbar}} = \frac{1}{2}\sqrt{\frac{2m\omega}{\hbar}} = \frac{1}{2l}$$

$$\sqrt{\frac{m\omega}{2\hbar}}\frac{i\hat{p}}{m\omega} = \sqrt{\frac{m\omega}{2\hbar}}\frac{i(-i)\hbar\frac{\partial}{\partial x}}{m\omega} = \sqrt{\frac{\hbar}{2m\omega}} = l$$

$$\therefore \hat{a} = \frac{1}{2l}x + l\frac{d}{dx} \qquad \hat{a}^\dagger = \frac{1}{2l}x - l\frac{d}{dx}$$

$$\cancel{\psi_n(x) = \frac{(\hat{a}^\dagger)^n \psi_0(x)}{\sqrt{n!}} \Rightarrow \psi}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \quad \therefore \hat{a}^\dagger|0\rangle = |1\rangle \quad \hat{a}^\dagger|1\rangle = \sqrt{2}\,|2\rangle$$

$$\hat{a}|0\rangle = \vec{0} \Rightarrow \langle x|\hat{a}|0\rangle = \langle x|\vec{0}\rangle = 0$$

$$\hat{a}\langle x|0\rangle = 0 \Rightarrow \hat{a}\psi_0(x) = 0$$

$$\therefore \frac{1}{2l}x\psi_0 + l\frac{d\psi_0}{dx} = 0$$

$$\therefore \frac{d\psi_0}{\psi_0} = -\frac{x}{2l^2}dx$$

$$\therefore \psi_0 = Ae^{-x^2/4l^2}$$

Normalise $\Rightarrow \int_{-\infty}^{\infty}|\psi_0|^2\,dx = 1$

$$A^2 \int_{-\infty}^{\infty} e^{-x^2/2l^2}\,dx = 0 \Rightarrow A^2 \sqrt{\pi}\sqrt{(2l^2)} = 0$$

$$\therefore A = (2\pi l^2)^{-\frac{1}{4}}$$

$$\cancel{\Rightarrow A^2 \frac{\sqrt{\pi}}{\sqrt{2}l} = 0} \quad \cancel{= A^2 \sqrt{\frac{2}{\pi}}}$$

$$\therefore \cancel{A = \sqrt{}} \quad \cancel{A = l^{\frac{1}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{4}}} \quad A \quad \sqrt{\pi}$$

$$|1\rangle = \hat{a}^+|0\rangle \Rightarrow \psi_1(x) = \hat{a}^+\psi_0(x)$$

$$\psi_1(x) = \left(\frac{1}{2l}x - l\frac{d}{dx}\right)A e^{-x^2/4l^2}$$

$$= A\frac{1}{2l}x e^{-x^2/4l} - Al\left(-\frac{x}{2l^2}e^{-x^2/4l^2}\right)$$

$$= \cancel{\frac{A}{2l}} \cancel{\frac{Ax}{2l}} = \frac{A}{l}x e^{-x^2/4l^2}$$

$$|2\rangle = \frac{\hat{a}^+|1\rangle}{\sqrt{2}} \Rightarrow \psi_2(x) = \frac{1}{\sqrt{2}}\hat{a}^+\psi_1(x)$$

$$= \frac{A}{\sqrt{2}l}\left(\frac{1}{2l}x - l\frac{d}{dx}\right)\left(x e^{-x^2/4l^2}\right)$$

$$= \frac{A}{\sqrt{2}l}\left(\frac{1}{2l}\right)x^2 e^{-x^2/4l} - \left(\frac{A}{\sqrt{2}l}\right)(l)e^{-x^2/4l^2}$$

$$- \left(\frac{A}{\sqrt{2}l}\right)(l)\,x\left(-\frac{x}{2l^2}\right)e^{-x^2/4l^2}$$

$$= \frac{A}{\sqrt{2}}\left(\frac{x^2}{l^2} - 1\right)e^{-x^2/4l^2} \cancel{= \frac{(2\pi l^2)^{-\frac{1}{4}}}{2^{\frac{1}{2}}} = \frac{(2\pi l^2)^{-\frac{1}{4}}}{(2^2)^{\frac{1}{4}}}}$$

$$\cancel{= l^{\frac{1}{2}}(2\pi)^{\frac{1}{4}}\left(\frac{x^2}{l^2} - 1\right)e^{-x^2/4l^2}} \quad \left(\frac{A}{\sqrt{2}} = \frac{(2\pi l^2)^{-\frac{1}{4}}}{(2^2)^{\frac{1}{4}}} = (8\pi l^2)^{-\frac{1}{4}}\right)$$

$$= \boxed{(8\pi l^2)^{-\frac{1}{4}}\left(\frac{x^2}{l^2} - 1\right)e^{-x^2/4l^2}}$$

$$\boxed{\text{Constant} = (8\pi l^2)^{-\frac{1}{4}}}$$

7. $$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}^\dagger + \hat{a}) = l(\hat{a}^\dagger + \hat{a})$$

In energy representation ($\hat{H}|i\rangle = \cancel{i|i\rangle}\ \hbar\omega(i+\frac{1}{2})|i\rangle$)

$$\hat{x}_{jk} = \langle j|\hat{x}|k\rangle = \langle j|\,l(\hat{a}+\hat{a}^\dagger)\,|k\rangle$$

$$= l\,\langle j|\hat{a}^\dagger + \hat{a}|k\rangle$$

$$= l\left[\langle j|\hat{a}^\dagger|k\rangle + \langle j|\hat{a}|k\rangle\right]$$

$$= l\left[\langle j|\sqrt{k+1}\,|k+1\rangle + \langle j|\sqrt{k}\,|k-1\rangle\right]$$

$$= l\left[\sqrt{k+1}\,\langle j|k+1\rangle + \sqrt{k}\,\langle j|k-1\rangle\right]$$

$$= l\left[\sqrt{k+1}\,\delta_{j,k+1} + \sqrt{k}\,\delta_{j,k-1}\right]$$

$$\therefore\ \hat{x}_{jk} = \begin{array}{c|c} {}_{j=}\diagdown^{k=} & 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & \sqrt{1} & & & & \\ \sqrt{1} & 0 & \sqrt{2} & \cdots & 0 & \cdots \\ & \sqrt{2} & 0 & \sqrt{3} & & \\ \vdots & & \sqrt{3} & 0 & \ddots & \\ & & & \ddots & 0 & \ddots \\ 0 & & & & \ddots & 0 \\ \vdots & & & & & \ddots \end{bmatrix} \end{array}$$

$\hat{p}$ ~~$\hat{a}^\dagger = \frac{1}{2\hbar}$~~ $\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$, $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)$

$\Rightarrow \quad \hat{p} = i\sqrt{m\hbar\omega/2}\,(\hat{a}^\dagger - \hat{a})$

$$\hat{P}_{jk} = \langle j | \hat{P} | k \rangle = i\sqrt{m\hbar\omega/2}\,\langle j | \hat{a}^\dagger - \hat{a} | k \rangle$$

$$= i\sqrt{m\omega\hbar/2}\,[\langle j | \hat{a}^\dagger | k \rangle - \langle j | \hat{a} | k \rangle].$$

$$= i\sqrt{\frac{m\omega\hbar}{2}}\,[\sqrt{k+1}\,\delta_{j,k+1} - \sqrt{k}\,\delta_{j,k-1}].$$

$\therefore$

$$P_{jk} = i\sqrt{m\omega\hbar/2}\begin{bmatrix} 0 & -\sqrt{1} & & & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & & & \\ & \sqrt{2} & 0 & -\sqrt{3} & & \\ \vdots & & \sqrt{3} & 0 & & \\ 0 & & & & 0 & \ddots \\ \vdots & & & \ddots & & \ddots \\ \vdots & & & & \ddots & \\ & & & & & \ddots \end{bmatrix}$$

8 $|\psi(0)\rangle = \frac{1}{2}|N-1\rangle + \frac{1}{\sqrt{2}}|N\rangle + \frac{1}{2}|N+1\rangle$

$E_{N-1} = \hbar\omega(N-1+\frac{1}{2}) = \hbar\omega(N-\frac{1}{2})$
$E_N = \hbar\omega(N+\frac{1}{2})$
$E_{N+1} = \hbar\omega(N+1+\frac{1}{2}) = \hbar\omega(N+\frac{3}{2})$

$$\therefore |\psi(t)\rangle = \frac{1}{2}|N-1\rangle e^{-iE_{N-1}t/\hbar} + \frac{1}{\sqrt{2}}|N\rangle e^{-iE_Nt/\hbar} + \frac{1}{2}|N+1\rangle e^{-iE_{N+1}t/\hbar}$$

$\langle x\rangle = \langle\psi|\hat{x}|\psi\rangle$

$$= \left\langle \left(\frac{1}{2}e^{iE_{N-1}t/\hbar}\langle N-1| + \frac{1}{\sqrt{2}}e^{iE_Nt/\hbar}\langle N| + \frac{1}{2}e^{iE_{N+1}t/\hbar}\langle N+1|\right) \middle| \hat{x} \middle| \left(\frac{1}{2}e^{-iE_{N-1}t/\hbar}|N-1\rangle + \frac{1}{\sqrt{2}}e^{-iE_Nt/\hbar}|N\rangle + \frac{1}{2}e^{-iE_{N+1}t/\hbar}|N+1\rangle\right)\right\rangle$$

$$= \frac{1}{4}\langle N-1|\hat{x}|N-1\rangle + \frac{1}{2\sqrt{2}}e^{i(E_{N-1}-E_N)t/\hbar}\langle N-1|\hat{x}|N\rangle$$

$$+ \frac{1}{4}e^{i(E_{N-1}-E_{N+1})t/\hbar}\langle N-1|\hat{x}|N+1\rangle$$

$$+ \frac{1}{2\sqrt{2}}e^{i(E_N-E_{N-1})t/\hbar}\langle N|\hat{x}|N-1\rangle + \frac{1}{2}\langle N|\hat{x}|N\rangle$$

$$+ \frac{1}{2\sqrt{2}}e^{i(E_N-E_{N+1})t/\hbar}\langle N|\hat{x}|N+1\rangle$$

$$+ \frac{1}{4}e^{i(E_{N+1}-E_{N-1})t/\hbar}\langle N+1|\hat{x}|N-1\rangle$$

$$+ \frac{1}{2\sqrt{2}}e^{i(E_{N+1}-E_N)t/\hbar}\langle N+1|\hat{x}|N\rangle + \frac{1}{4}\langle N+1|\hat{x}|N+1\rangle$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^{\dagger}) = l\,(\hat{a}+\hat{a}^{\dagger}) \qquad l \equiv \sqrt{\frac{\hbar}{2m\omega}}$$

$$\therefore \quad \langle N|\hat{x}|N\rangle = l\langle N|\hat{a}+\hat{a}^{\dagger}|N\rangle$$

$$= l\langle N|\hat{a}|N\rangle + l\langle N|\hat{a}^{\dagger}|N\rangle$$

$$= l\sqrt{N}\underbrace{\langle N|N-1\rangle}_{0} + l\sqrt{N+1}\underbrace{\langle N|N+1\rangle}_{0} = 0$$

Similarly $\quad \langle N-1|\hat{x}|N-1\rangle = \langle N+1|\hat{x}|N+1\rangle = 0$

$$\langle N|\hat{x}|N+1\rangle = l\langle N|\hat{a}|N+1\rangle + l\langle N|\hat{a}^{\dagger}|N+1\rangle$$

$$= l\sqrt{N+1}\underbrace{\langle N|N\rangle}_{1} + \cancel{l\langle N|} \, l\sqrt{N+2}\underbrace{\langle N|N+2\rangle}_{0}$$

$$= l\sqrt{N+1}$$

$\langle N+1|\hat{x}|N\rangle = \cancel{l\langle N+1|\hat{a}^{\dagger}|N\rangle}\ l\sqrt{N+1}$ [$\hat{x}$ is hermitian]

Similarly $\quad \langle N-1|\hat{x}|N\rangle = l\sqrt{N}$

$\langle N|\hat{x}|N-1\rangle = l\sqrt{N}$

$\cancel{\langle N|\hat{x}|N}$

$$\langle N-1|\hat{x}|N+1\rangle = l\langle N-1|\hat{a}|N+1\rangle + \cancel{l\langle N-1|\hat{x}} \; l\langle N-1|\hat{a}^{\dagger}|N+1\rangle$$

$$= l\sqrt{N+1}\underbrace{\langle N-1|N\rangle}_{0} + \cancel{l\langle N-1|} \, l\sqrt{N+2}\underbrace{\langle N-1|N+2\rangle}_{0}$$

$$= 0$$

$$\therefore \quad \langle N+1|\hat{x}|N-1\rangle = 0$$

$$\therefore \langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \frac{l\sqrt{N}}{2\sqrt{2}} e^{i(E_{N-1}-E_N)t/\hbar}$$

$$+ \frac{l\sqrt{N}}{2\sqrt{2}} e^{-i(E_{N-1}-E_N)t/\hbar}$$

$$+ \frac{l\sqrt{N+1}}{2\sqrt{2}} e^{i(E_{N+1}-E_N)t/\hbar} + \frac{l\sqrt{N+1}}{2\sqrt{2}} e^{-i(E_{N+1}-E_N)t/\hbar}$$

$$E_{N+1} - E_N = E_N - E_{N-1} = \hbar\omega$$

$$\therefore \langle x \rangle = \cancel{\frac{l\sqrt{N}}{2\sqrt{2}}} \frac{l}{2\sqrt{2}} (\sqrt{N} + \sqrt{N+1}) (e^{i\omega t} + e^{-i\omega t})$$

$$= \frac{l}{\cancel{2}\sqrt{2}} (\sqrt{N} + \sqrt{N+1}) (\cancel{2} \cos \omega t)$$

$$= \boxed{\frac{1}{2}\sqrt{\frac{\hbar}{m\omega}} (\sqrt{N} + \sqrt{N+1}) \cos(\omega t)}$$

The expectation value of x oscillates sinusoidally with time. The amplitude increases with the energy number N and decreases with the frequency ω.

9.

$$|\psi\rangle = \sum_{n=-\infty}^{\infty} a_n |n\rangle , \quad a_n = \frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{|n|/2} e^{in\pi}$$

(a) $P_n = |a_n|^2 = a_n^* a_n$

$$= \cancel{\left(\frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{|n|/2}\right)^2 e^{-in\pi} e^{in\pi}}$$

$$\cancel{= \frac{1}{2}(}$$

$$\cancel{a_n^*} = \quad a_n = \frac{1}{\sqrt{2}}\left(\frac{-i}{3}\right)^{|n|/2} e^{in\pi}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{3}\right)^{|n|/2} \left(e^{-i\pi/2}\right)^{|n|/2} e^{in\pi}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{3}\right)^{|n|/2} e^{-i\left(\frac{|n|}{4}\pi - n\pi\right)}$$

$$\therefore a_n^* = \frac{1}{\sqrt{2}}\left(\frac{1}{3}\right)^{|n|/2} e^{i\left(\frac{|n|}{4}\pi - n\pi\right)}$$

$$\therefore P_n = \left(\frac{1}{\sqrt{2}}\left(\frac{1}{3}\right)^{|n|/2}\right)^2 = \boxed{\frac{1}{2}\cdot\left(\frac{1}{3}\right)^{|n|}}$$

(b) $P_{n\geq 0} = P_0 + P_1 + P_2 + \cdots$

$$= \frac{1}{2}\left[\left(\frac{1}{3}\right)^0 + \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \cdots\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{3}}\right] = \frac{1}{2}\times\frac{3}{2} = \boxed{\frac{3}{4}}.$$

10. $\hat{H} = \hbar\omega\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\hat{B} = b\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(a) Yes, $\hat{A}$ and $\hat{B}$ are hermitian

since $\hat{H}^\dagger = \hat{H}$, $\hat{B}^\dagger = \hat{B}$

(b) Eigenvalues:

For H, $H_1 = \hbar\omega$, $H_2 = -\hbar\omega$

For B: $\det\begin{pmatrix} b & 0 & 0 \\ 0 & -B & b \\ 0 & b & -B \end{pmatrix} = 0$

$\Rightarrow B^2 - b^2 = 0 \Rightarrow B = b, -b$

let eigenvector $|v\rangle = (a, b, c)^T$, the up to rescaling:

$[\hat{H}, H = \hbar\omega] \Rightarrow \hbar\omega\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}\hbar\omega$

then $a = a$, $-b = b$, $-c = c$

$\therefore$ ~~b~~ $b = c = 0$, $\boxed{|v_{H,\hbar\omega}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$[\hat{H}, H = -\hbar\omega] \Rightarrow \hbar\omega\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\hbar\omega\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\therefore$ $a = -a$, $-b = -b$, $-c = -c$

$\therefore a = 0$, $b = b$, $c = c$

$\boxed{|v_{H,-\hbar\omega}\rangle_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |v_{H,-\hbar\omega}\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

~~Has~~ Degeneracy $= 2$, eigenvector not uniquely specified

$[\hat{B}, B=b] \Rightarrow$

$$b\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=b\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$a=a\ ,\quad c=b\ ,\quad b=c$

$\therefore\ b=c$, a can be anything

$$\therefore\ \cancel{\hat{B}}:\ \boxed{|v_{B,b}\rangle_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\ ,\ |v_{B,b}\rangle_2=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}}$$

Degeneracy = 2, eigenvectors not uniquely specified.

$[\hat{B}, B=-b] \Rightarrow$

$$b\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=-b\begin{pmatrix}a\\b\\c\end{pmatrix}$$

$a=-a$, $c=-b$, $b=-c$

$\therefore\ a=0$, ~~b=b, c=c~~ , $c=-b$

$$\therefore\ \boxed{|V_{B,-b}\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}}$$

Neither matrices uniquely specify eigenvectors because any linear combinations of the eigenvectors of the degenerate eigenvalues ~~becomes a new~~ is also an eigenvector of the corresponding matrix.

$$[\hat{H},\hat{B}]=\hat{H}\hat{B}-\hat{B}\hat{H}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}-\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}=[0]$$

Common basis : $|1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $|2\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}$, $|3\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\end{pmatrix}$

11.

$$|\psi\rangle = \sum_k a_k |E_k, t\rangle$$, then. since

$$\hat{H}|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle$$

$$\Rightarrow \quad \hat{H}\sum_k a_k |E_k, t\rangle = i\hbar\frac{\partial}{\partial t}\sum_k a_k |E_k, t\rangle$$

since $\hat{H}|E_k, t\rangle = E_k|E_k, t\rangle$

$$\Rightarrow \quad \sum_k a_k E_k|E_k, t\rangle = i\hbar\left(\sum_k \frac{da_k}{dt}|E_k, t\rangle + \sum_k a_k \frac{d}{dt}|E_k, t\rangle\right)$$

$$\hat{H}|E_k, t\rangle = i\hbar\frac{\partial}{\partial t}|E_k, t\rangle = E_k|E_k, t\rangle$$

$$\therefore \quad \frac{d}{dt}|E_k, t\rangle = -\frac{iE_k}{\hbar}|E_k, t\rangle$$

$$\Rightarrow \quad \sum_k a_k \cancel{E_k}|E_k, t\rangle = i\hbar\left(\sum_k \frac{da_k}{dt}|E_k, t\rangle + \sum_k a_k\left(-\cancel{\frac{iE_k}{\hbar}}\right)|E_k, t\rangle\right)$$

$$\Rightarrow \quad \cancel{\sum_k a_k E_k} \quad \sum_k \frac{da_k}{dt}|E_k, t\rangle = 0$$

& $\{|E_k, t\rangle\}$ are linearly independent

$$\Rightarrow \quad \frac{da_k}{dt} = 0$$

$\Rightarrow$ a_k, the ~~coefficient~~ ~~prob~~ probability amplitude, is constant

$\therefore P_k = |a_k|^2$ is time-independent

12 $\psi(x) = \sum_r a_r u_r(x)$ (expansion with u_r basis)

$$\int_{-\infty}^{\infty} u_m^*(x) \psi(x)\, dx = \sum_r a_r \int \underbrace{u_m^*(x) u_r(x)\, dx}_{\delta_{mr}}$$

$$= a_m$$

$$\Rightarrow \quad a_r = \int_{-\infty}^{\infty} u_r^*(x) \psi(x)\, dx$$

$$\therefore \boxed{P(q_r|\psi) = |a_r|^2 = \left| \int_{-\infty}^{\infty} u_r^*(x) \psi(x)\, dx \right|^2}$$

$$1 = \int_{-\infty}^{\infty} \psi^*(x) \psi(x)\, dx = \int_{-\infty}^{\infty} \sum_r a_r^* u_r^*(x) \sum_m a_m u_m(x)\, dx$$

$$= \sum_r \sum_m a_r^* a_m \int \underbrace{u_r^*(x) u_m(x)\, dx}_{\delta_{rm}}$$

$$= \sum_r a_r^* a_r = \sum_r |a_r|^2 = \sum_r P(q_r|\psi)$$

QED

$$\int_{-\infty}^{\infty} \psi^* \hat{Q} \psi \, dx = \int_{-\infty}^{\infty} \sum_r a_r^* u_r^* \sum_m q_m a_m u_m \, dx.$$

$$= \sum_r \sum_m a_r^* q_m a_m \underbrace{\int_{-\infty}^{\infty} u_r^* u_m \, dx}_{\delta_{rm}}$$

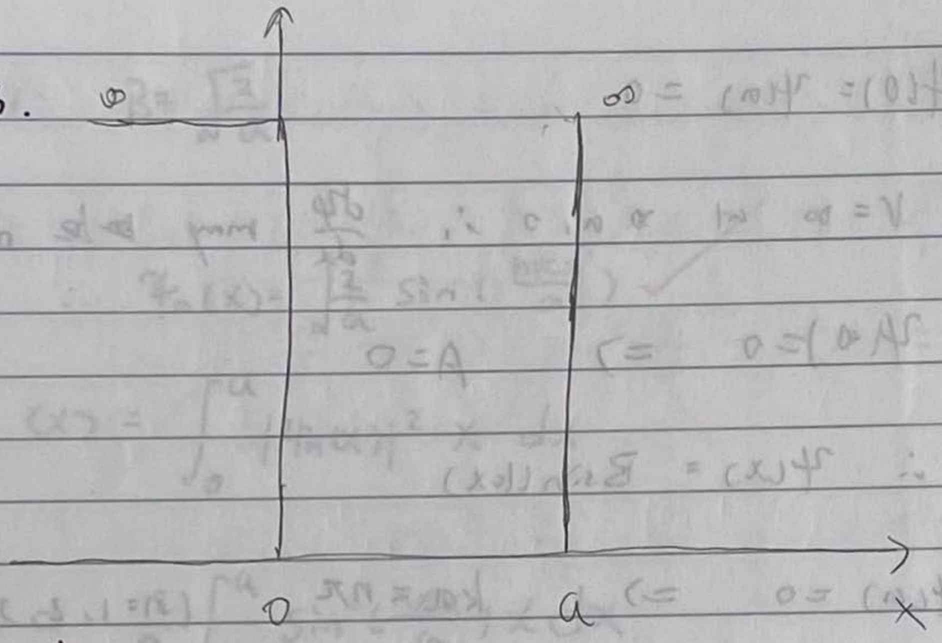
$$= \sum_r \sum_m a_r^* q_m a_m \delta_{rm} = \sum_r a_r^* a_r q_r$$

$$= \sum_r |a_r|^2 q_r \cancel{= \sum_r P(q_r|\psi)} = \sum_r P(q_r|\psi) \, q_r$$

$$= \langle Q \rangle$$

QED

13.



a)

The TISE : $\hat{H}\psi = E\psi$

$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\psi = E\psi$$

$$\therefore \frac{d^2\psi}{dx^2} + \frac{2m(E-V)}{\hbar^2}\psi = 0$$

$$V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & x \leq 0,\ x \geq a \end{cases}$$

$$\therefore \psi = 0 \quad \text{for} \quad x \leq 0,\ x \geq a.$$

For $0 \leq x \leq a$, $\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$

let $\underbrace{k^2 = \frac{2mE}{\hbar}}$ ($E>0$, k is real)

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\therefore \psi(x) = A\cos(kx) + B\sin(kx)$$

$\psi(0) = \psi(a) = 0$,

$\left(\because V = \infty \text{ at } x\ a, 0 \therefore \frac{d\psi}{dx} \text{ may be discontinuous}\right)$

$\therefore \psi(0) = 0 \Rightarrow A = 0$

$\therefore \psi(x) = B\sin(kx)$

$\psi(a) = 0 \Rightarrow ka = n\pi \quad (n = 1, 2, 3, \cdots)$

$\therefore \psi_n(x) = B\sin\left(\frac{n\pi x}{a}\right)$, $ka = n\pi \Rightarrow k^2a^2 = n^2\pi^2$

Energy given by $\frac{2mE_n}{\hbar^2} = \frac{n^2\pi^2}{a^2}$

$$\Rightarrow \boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}} \checkmark$$

b) Normalise $\psi_n(x)$:

$$\int |\psi_n(x)|^2 dx = 1 \Rightarrow 1 = B^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= B^2 \int_0^a \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right) dx$$

$$= B^2 \left[\frac{a}{2} - \frac{1}{2}\int_0^a \cos\left(\frac{2n\pi x}{a}\right) dx\right]$$

$$= B^2 \left[\frac{a}{2} - \frac{1}{2}\left(\frac{a}{2n\pi}\right) \underbrace{\sin\left(\frac{2n\pi x}{a}\right)\Big|_0^a}_{0}\right] = B^2 \frac{a}{2}$$

$\therefore \quad B = \sqrt{\frac{2}{a}}$

$\therefore \quad \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

$$\langle x \rangle = \int_0^a |\psi_n(x)|^2 \, x \, dx$$

$$= \frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) x \, dx$$

$$= \frac{2}{a}\int_0^a \left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right)\right) x \, dx$$

$$= \frac{2}{a}\int_0^a \frac{1}{2} x \, dx - \frac{2}{a}\int_0^a \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right) x \, dx$$

$$= \frac{2}{a}\left(\frac{a^2}{4}\right) - \frac{1}{a}\int_0^a \cos\left(\frac{2n\pi x}{a}\right) x \, dx$$

$$= \frac{a}{2} - \frac{1}{a}\int_0^a \cos\left(\frac{2n\pi x}{a}\right) x \, dx$$

$$\cancel{\int_0^a \cos\left(\frac{2n\pi x}{a}\right) x \, dx}$$

$$= \frac{a}{2} - \frac{1}{a}\left[\frac{a^2}{(2n\pi)^2} \underbrace{x \sin\left(\frac{2n\pi}{a}x\right)}_{0} + \frac{a^2}{(2n\pi)^2} \underbrace{\cos\left(\frac{2n\pi}{a}x\right)}_{0}\right]_0^a$$

$$= \boxed{\frac{a}{2}}$$

c) $\langle (x - \langle x \rangle)^2 \rangle = \langle x^2 - 2x\langle x \rangle + \langle x \rangle^2 \rangle$

$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2$$

$$= \langle x^2 \rangle - \left(\frac{a}{2}\right)^2 = \boxed{\langle x^2 \rangle - \frac{a^2}{4}}$$

$$\langle x^2 \rangle = \int_0^a |\psi_n(x)|^2 x^2 \, dx$$

$$= \frac{2}{a} \int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

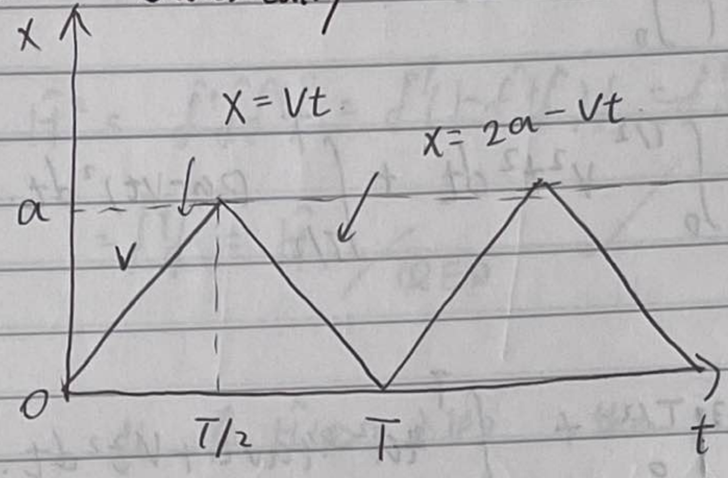
$$= \frac{2}{a} \cdot a^3 \left(\frac{1}{6} - \frac{1}{4n^2\pi^2} \right)$$

$$= a^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right)$$

$\therefore \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \frac{a^2}{4} = \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2}$

$$= \boxed{a^2 \left(\frac{1}{12} - \frac{1}{2n^2\pi^2} \right)}$$

(d) Classically



$T = \frac{2a}{v}$

$$\langle x \rangle_c = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{T}\left[\int_0^{T/2} vt\,dt + \int_{T/2}^{T} (2a - vt)\,dt\right]$$

$$= \frac{1}{T}\left[\frac{1}{2}vt^2\Big|_0^{T/2} + \left(2at - \frac{1}{2}vt^2\right)\Big|_{T/2}^{T}\right]$$

$$= \frac{1}{T}\left[\frac{1}{2}v\frac{T^2}{4} + 2a\frac{T}{2} - \frac{1}{2}v\frac{3}{4}T^2\right]$$

$$= a - \frac{1}{2}vT\left(\frac{1}{2}\right) = a - \frac{1}{2}\cancel{v}\left(\frac{2a}{\cancel{v}}\right)\left(\frac{1}{2}\right)$$

$$= a - \frac{a}{2} = \boxed{\frac{a}{2}}$$

$$\langle (x - \langle x \rangle)^2 \rangle_c = \langle x^2 - 2x\langle x \rangle_c + \langle x \rangle_c^2 \rangle_c$$

$$= \langle x^2 \rangle_c - 2\langle x \rangle_c^2 + \langle x \rangle_c^2 = \langle x^2 \rangle_c - \langle x \rangle_c^2$$

$$= \langle x^2 \rangle_c - \frac{a^2}{4}$$

$$\langle x^2\rangle_c = \frac{1}{T}\int_0^T x^2\,dt$$

$$= \frac{1}{T}\Big[\int_0^{T/2} v^2t^2\,dt + \int_{T/2}^{T} (2a-vt)^2\,dt.$$

$$= \frac{1}{T}\Big[\frac{1}{3}v^2t^3\Big|_0^{T/2} + \int_{T/2}^{T} 4a^2-4avt+v^2t^2\,dt.$$

$$= \frac{1}{T}\Big[\frac{1}{3}v^2\Big(\frac{T^3}{8}\Big) + 4a^2\Big(\frac{T}{2}\Big) - 4av\frac{1}{2}\Big(T^2-\frac{T^2}{4}\Big)+\frac{1}{3}v^2\Big(T^3-\frac{T^3}{8}\Big)\Big]$$

$$= \frac{1}{T}\Big[\frac{1}{3}v^2T^3 + 2a^2T - \frac{3}{2}avT^2\Big].$$

$$= \frac{1}{3}v^2T^2 + 2a^2 - \frac{3}{2}avT.$$

$$= \frac{1}{3}v^2\Big(\frac{4a^2}{v^2}\Big) + 2a^2 - \frac{3}{2}av\Big(\frac{2a}{v}\Big)$$

$$= \frac{4}{3}a^2 + 2a^2 - 3a^2 = \frac{4}{3}a^2 - a^2 = \frac{1}{3}a^2 \checkmark$$

$$\therefore \langle (x-\langle x\rangle)^2\rangle = \frac{1}{3}a^2 - \frac{a^2}{4} = \boxed{\frac{a^2}{12}}. \checkmark$$

As $n\to\infty$:

$$\langle x\rangle = \frac{a}{2} = \langle x\rangle_c \checkmark$$

$$\langle (x-\langle x\rangle)^2\rangle = a^2\Big(\frac{1}{12}-\frac{1}{2n^2\pi^2}\Big) \to \frac{a^2}{12} = \langle (x-\langle x\rangle)^2\rangle_c$$

Simpler way: speed constant between bounces

$\Rightarrow$ equal probabilities everywhere. $\left(P(x)=\frac{1}{a}\right)$

$\langle x\rangle = \int_0^a dx\,\frac{x}{a} = \frac{a}{2}$, $\langle x^2\rangle = \int_0^a dx\,\frac{x^2}{a} = \frac{a^2}{3}$

14. $\hat{H} = \hat{f}^\dagger \hat{f}$, $\hat{f}^2 = \hat{f}\hat{f} = 0$, $\hat{f}\hat{f}^\dagger + \hat{f}^\dagger \hat{f} = 1$

$$\hat{H}^2 = \hat{f}^\dagger \hat{f} \hat{f}^\dagger \hat{f} = \hat{f}^\dagger (1 - \hat{f}^\dagger \hat{f}) \hat{f} = \hat{f}^\dagger \hat{f} - \hat{f}^\dagger \hat{f}^\dagger \underbrace{(\hat{f}\hat{f})}_{0}$$

$$= \hat{f}^\dagger \hat{f} = \hat{H}$$ / QED

If $\hat{H}|\phi\rangle = \lambda|\phi\rangle$ then

$$\hat{H}^2|\phi\rangle = \hat{H}|\phi\rangle \Rightarrow \lambda^2|\phi\rangle = \lambda|\phi\rangle$$

$$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \boxed{\lambda_1 = 1 , \lambda_2 = 0}$$

$\therefore$ let $\hat{H}|0\rangle = 0$ $\hat{H}|1\rangle = |1\rangle$ $(\langle 0|0\rangle = 1 , \langle 1|1\rangle = 1)$

(a) $\hat{H}|0\rangle = 0 \Rightarrow \hat{f}^\dagger \hat{f}|0\rangle = 0$

$\hat{f}^2|0\rangle = 0 \Rightarrow \hat{f}\hat{f}|0\rangle = 0$

$\therefore$ $\hat{f}\hat{f}^\dagger \hat{f}|0\rangle = 0$ ①

$\hat{f}^\dagger \hat{f}\hat{f}|0\rangle = 0$ ②

① + ② $\Rightarrow (\hat{f}\hat{f}^\dagger + \hat{f}^\dagger \hat{f})\hat{f}|0\rangle = 0 \Rightarrow \boxed{\hat{f}|0\rangle = 0}$

(b) $|0\rangle = (\hat{f}\hat{f}^\dagger + \hat{f}^\dagger \hat{f})|0\rangle = \hat{f}\hat{f}^\dagger|0\rangle + \hat{f}^\dagger \underbrace{\hat{f}|0\rangle}_{0} = \hat{f}\hat{f}^\dagger|0\rangle$

$\therefore$ $\hat{f}^\dagger \hat{f}\hat{f}^\dagger|0\rangle = \hat{f}^\dagger|0\rangle \Rightarrow \hat{H}(\hat{f}^\dagger|0\rangle) = \hat{f}^\dagger|0\rangle$ ③

also $|\hat{f}^\dagger|0\rangle|^2 = \langle 0|\hat{f}\hat{f}^\dagger|0\rangle = \langle 0|0\rangle = 1$ ④

③, ④ $\Rightarrow \boxed{\hat{f}^\dagger|0\rangle = |1\rangle}$, which is the eigenvector (normalised) with eigenvalue 1

The Hamiltonian of a fermion has only two energy eigen-values, namely 0 and 1. This corresponds to the Pauli exclusion principle, which states that only 0 or 1 fermion may occupy a particular quantum state.

5. $|n\rangle \propto (a^\dagger)^n |0\rangle$

$$\hat{a}|0\rangle = 0 \Rightarrow u_0(x) = Ae^{-x^2/4l^2}$$

$$\psi_n(x) = (a\hat{x} - i\hat{p}b)^n \, u_0(x)$$

$$= (a^n \hat{x}^n + \beta \hat{x}^{n-1}\hat{p} + \cdots + (-i)^n b^n \hat{p}^n) u_0(x)$$

in position representation.

$$\cancel{\hat{x} = x} \quad \hat{x}|\psi\rangle = x|\psi\rangle$$

$$\hat{x}\, u_0(x) = x\, u_0(x)$$

$$= \underbrace{(a^n x^n + \beta x^{n-1}(-i\hbar \tfrac{\partial}{\partial x} + \cdots + (-i)^n b^n (-i\hbar \tfrac{\partial}{\partial x})^n)}_{} u_0$$

$$\frac{\partial u_0}{\partial x} = \frac{-2x}{4l^2} u_0$$

n^{th} order polynomial

(polynomial

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11. $$\frac{d\langle E\rangle}{dt} = \frac{1}{i\hbar}\langle [\hat{H}, \hat{H}]\rangle = 0.$$

$$0 = \frac{d}{dt}\sum_k P_k E_k$$

$$= \sum_k \frac{dP_k}{dt} E_k = 0$$

$$? \Rightarrow \frac{dP_k}{dt}$$

1.(e)

$$\psi(x, y, z) = \psi(r)$$

$$\partial_x \psi = \frac{\partial \psi}{\partial r} \frac{\partial r}{\partial x}$$

$$= \frac{\partial \psi}{\partial r} \frac{x}{r}$$

$$\hat{L}_x \psi = y p_z - z p_y$$

$$= i\hbar (z \partial_y - y \partial_z) \psi$$

$$= i\hbar \frac{1}{r} \underbrace{(zy - yz)}_{0} \psi$$

$$= 0$$