To : Caroline Terquem

Electromagnetism 2

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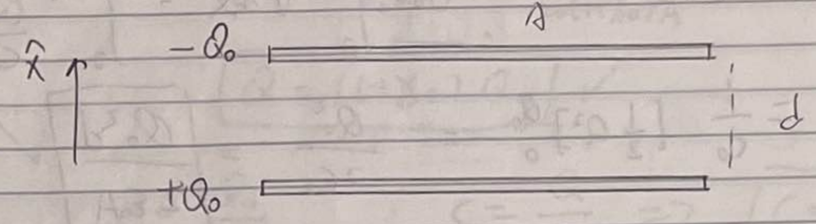
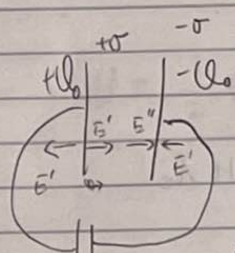
Excellent!

Check the signs for dU in Pb 3

1)



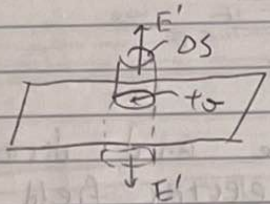
a)



$+\sigma = \dfrac{+Q_0}{A}$ $\quad \therefore \sigma = \dfrac{Q_0}{A}$

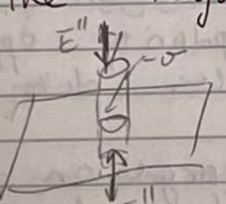
Gauss's Law ~~applied to~~ $\oint \vec{E}\cdot d\vec{s} = \dfrac{Q_{enc}}{\varepsilon_0}$ to

① the +ve plate :



$$E'(2\Delta S) = \frac{\sigma \Delta S}{\varepsilon_0} \quad \therefore E' = \frac{\sigma}{2\varepsilon_0}$$

② the negative plate



$$E''(2\Delta S) = \frac{-\sigma \Delta S}{\varepsilon_0} \quad \therefore E'' = \frac{-\sigma}{2\varepsilon_0}$$

Field between plates is a ~~sup~~ superposition of the two fields produced by two plates

$$\therefore E_0 = E' + (-E'') = \frac{\sigma}{\varepsilon_0} = \frac{Q_0}{A\varepsilon_0}$$

$$\therefore \vec{E}_0 = \boxed{\frac{Q_0}{A\varepsilon_0}\hat{x}} \checkmark$$

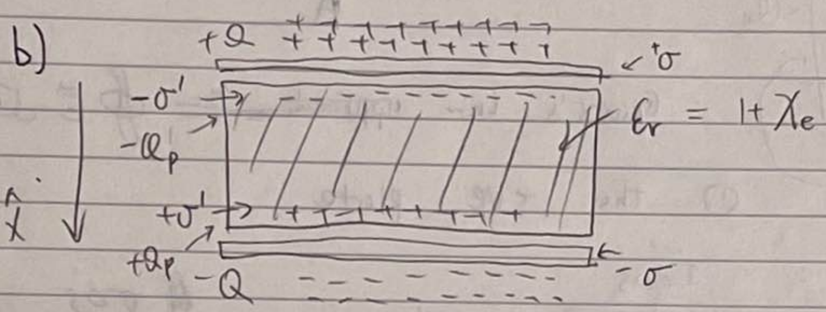
Potential difference $\Delta V_0 = -\displaystyle\int_d^0 \frac{Q_0}{A\varepsilon_0}\hat{x}\cdot d\vec{r} = \int_0^d \frac{Q_0}{A\varepsilon_0}dx = \boxed{\frac{Q_0 d}{A\varepsilon_0}}$ ✓

Capacitance $C_0 = \dfrac{Q_0}{\Delta V_0} = \boxed{\dfrac{\varepsilon_0 A}{d}}$ ✓

$$dU = V\,dQ \Rightarrow U_0 = \int_0^{Q_0} V\,dQ = \int_0^{Q_0} \frac{Q}{C_0}\,dQ$$

$$= \frac{1}{C_0}\left[\frac{1}{2}Q^2\right]_0^{Q_0} = \frac{Q_0^2}{2C_0} = \boxed{\frac{Q_0^2 d}{2\varepsilon_0 A}} \checkmark$$

b)



The potential difference and distance between plates stay the same ⇒ electric field stays the same

∵ polarized charges present on the boundary of dielectric and has polarity opposite to the ~~cha~~ charge on the capacitor plate ~~it~~ next to it

∴ There must be more charge on the capacitor plates to counteract the effect of the polarized charges

yes

∴ $Q > Q_0$

$\vec{P} = P\hat{x}$ ~~$Q_p = \vec{P}\cdot\hat{x} = P$~~ $\sigma' = \vec{P}\cdot\hat{x} = P$

∴ $\sigma = \frac{Q}{A}$

$$E = \frac{\sigma - \sigma'}{\varepsilon_0} = \boxed{\frac{1}{\varepsilon_0}\left(\frac{Q}{A} - P\right)} \checkmark$$

$$P = \varepsilon_0\chi_e E \Rightarrow \varepsilon_0 E = \frac{Q}{A} - \varepsilon_0\chi_e E$$

$$\therefore \underbrace{\varepsilon_0(1+\chi_e)E}_{D} = \frac{Q}{A} \Rightarrow \boxed{\vec{D} = \frac{Q}{A}\hat{x}} \checkmark$$

$$E = E_0 \Rightarrow \frac{Q_0}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0(1+\chi_e)A}$$

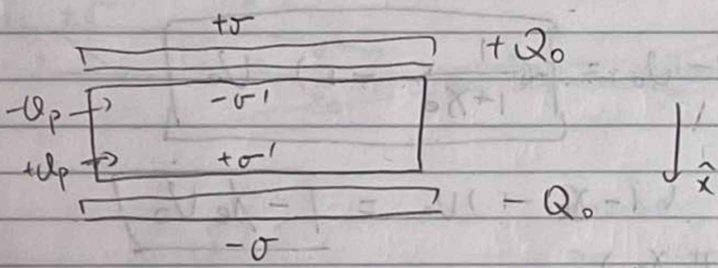
$$\therefore \boxed{Q = (1+\chi_e)\,Q_0} \checkmark$$

$$C_0 = \frac{Q_0}{\Delta V_0}, \quad C = \frac{Q}{\Delta V_0} \Rightarrow \boxed{C = (1+\chi_e)C_0} \checkmark$$

$$U = \frac{1}{2} C \Delta V_0^2 \qquad U_0 = \frac{1}{2} C_0 \Delta V_0^2 \Rightarrow \cancel{U = (1+\chi_e)U_0}$$

$$U = (1+\chi_e)U_0 \Rightarrow \boxed{\Delta U = \chi_e U_0} \checkmark$$

c)



Battery disconnected $\Rightarrow$ Charge Q_0 does not change

the polarization charges produce a E-field

that ~~counter act~~ counteracts the ~~ori~~ original

E-field. $\Rightarrow$ Total electric field decreases

$\Rightarrow$ since separation ℓ stays the same, potential

difference must decrease. yes

$$\sigma' = \vec{P}\cdot\hat{x} = P = \varepsilon_0 \chi_e E$$

$$E = \frac{\sigma - \sigma'}{\varepsilon_0} = \frac{1}{\varepsilon_0}(\sigma - \varepsilon_0 \chi_e E)$$

$$\therefore \varepsilon_0(1+\chi_e)E = \sigma = \frac{Q_0}{A}$$

$$\therefore E = \frac{Q_0}{\varepsilon_0(1+\chi_e)A} \checkmark$$

$$\Delta V = Ed = \frac{Q_0 d}{\varepsilon_0 (1+\chi_e) A}$$

$$C = \frac{Q_0}{\Delta V} = \frac{\varepsilon_0 (1+\chi_e) A}{d} = \boxed{(1+\chi_e) C_0}$$ ✓

$$U = \frac{Q_0^2}{2C} = \cancel{Q_0^2 \varepsilon_0 (1+\chi_e}$$

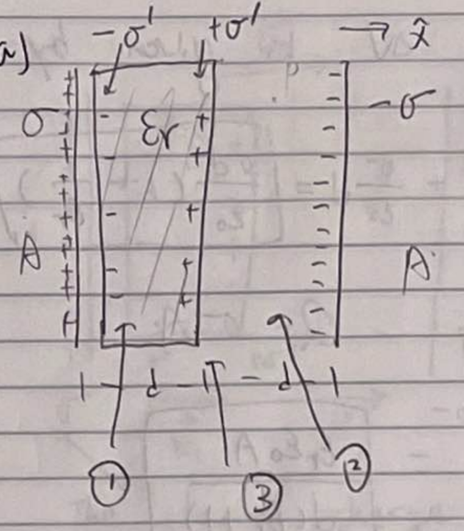
$$= \frac{Q_0^2 d}{2\varepsilon_0 (1+\chi_e) A} = \frac{1}{1+\chi_e} \frac{Q_0^2 d}{2\varepsilon_0 A} = \frac{1}{1+\chi_e} U_0$$

$$\Delta U = U - U_0 = \boxed{\left(\frac{1}{1+\chi_e} - 1\right) U_0}$$ ✓

$$\approx (1 - \chi_e - 1) U_0 = \boxed{-\chi_e U_0}$$

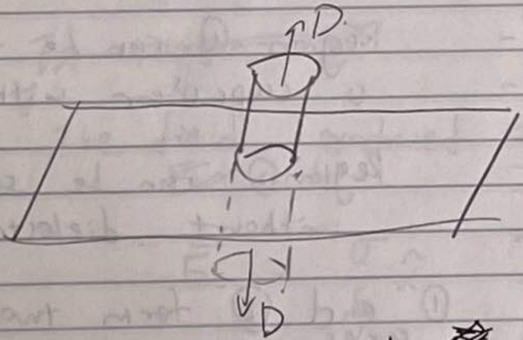
(for small χ_e)

2) a)



~~Consider point P. The effect due to +ve and -ve polarized charges cancel, The field is due to free charges $\pm\sigma$.~~

~~$\therefore \vec{E} = \frac{\sigma}{\varepsilon_0}\hat{x}$ in free region~~



Consider Gaussian surface on one of the capacitor plates.

$$\oint \vec{D}\cdot d\vec{s} = Q_{free}$$

$$\therefore 2\Delta s\, D = \sigma \Delta s$$

$$\therefore D = \frac{\sigma}{2}.$$

Superposition of two plates gives the total $\vec{D}$:

$$\boxed{\vec{D} = \sigma\hat{x}}$$ ✓ for both regions ① and ②

(because there is no free charge between plates, $\vec{D}$ has to be continuous)

(E-field and $\vec{P}$) $\therefore$ In region ①:

$$\sigma\hat{x} = \vec{D} = \varepsilon\vec{E}_1 = \varepsilon_r\varepsilon_0\vec{E}_1 \Rightarrow \boxed{\vec{E}_1 = \frac{\sigma}{\varepsilon_r\varepsilon_0}\hat{x}}$$ ✓

Polarization $\vec{P}_1 = \varepsilon_0\chi_e\vec{E}_1 = \varepsilon_0(\varepsilon_r - 1)\frac{\sigma}{\varepsilon_r\varepsilon_0}\hat{x} = \boxed{(1-\frac{1}{\varepsilon_r})\sigma\hat{x}}$

In region ②:

$$\sigma\hat{x} = \vec{D} = \varepsilon\vec{E}_2 = \varepsilon_0\vec{E}_2 \Rightarrow \boxed{\vec{E}_2 = \frac{\sigma}{\varepsilon_0}\hat{x}}$$ ✓

polarization $\vec{P}_2 = \boxed{\vec{0}}$

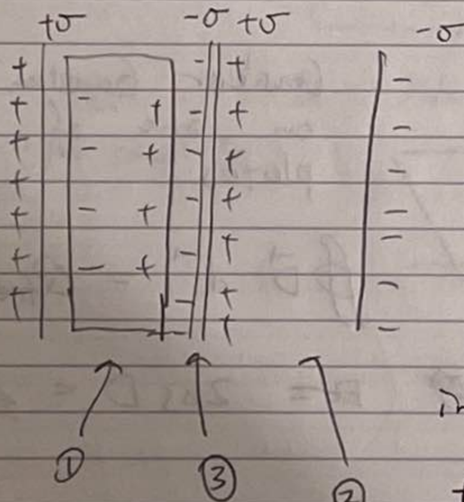
b) The potential difference ΔV is given by

$$\Delta V = E_1 d + E_2 d = \frac{\sigma}{\varepsilon_r \varepsilon_0} d + \frac{\sigma}{\varepsilon_0} d = \boxed{\frac{\sigma d}{\varepsilon_0}\left(1 + \frac{1}{\varepsilon_r}\right)}$$ ✓

capacitance $C = \dfrac{Q}{\Delta V}$, $Q = \sigma A$

$$\therefore \; C = \frac{\sigma A}{\frac{\sigma d}{\varepsilon_0}\left(1+\frac{1}{\varepsilon_r}\right)} = \boxed{\frac{\varepsilon_r \varepsilon_0 A}{d(\varepsilon_r + 1)}}$$ ✓

Add a superposition of ~~positive~~ +ve and −ve charges $(\pm\sigma)$ at boundary ③ (which has no effect at all to the system)



Region ① can be treated as a capacitor with dielectric;

Region ② can be seen as one without dielectric.

① and ② form two capacitors in series ~~parallel~~ as they carry the same free charge (σA), and their voltage add up. This is the definition of series connection.

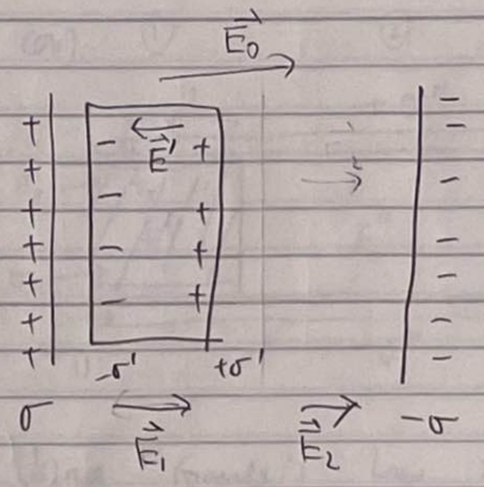
yes

$$\therefore \; C = \left(C_1^{-1} + C_2^{-1}\right)^{-1}, \quad C_1 = \frac{\varepsilon_r \varepsilon_0 A}{d}, \quad C_2 = \frac{\varepsilon_0 A}{d}$$

result in Problem (1)

$$\therefore \; C = \frac{1}{\frac{d}{\varepsilon_r \varepsilon_0 A} + \frac{d}{\varepsilon_0 A}} = \boxed{\frac{\varepsilon_r \varepsilon_0 A}{d(\varepsilon_r + 1)}}$$, same as what we've calculated before.

c)



The bound charge surface density

σ' is given by $\sigma' = \vec{P_1} \cdot \hat{x} = (1 - \frac{1}{\varepsilon_r})\sigma$ ✓

∴ The field produced by +ve and −ve bound charges is $\vec{E'} = -\frac{\sigma'}{\varepsilon_0}\hat{x} = (\frac{1}{\varepsilon_r} - 1)\frac{\sigma}{\varepsilon_0}\hat{x}$

~~In region (1)~~

The field produced by free charges is as usual

$\vec{E_0} = \frac{\sigma}{\varepsilon_0}\hat{x}$

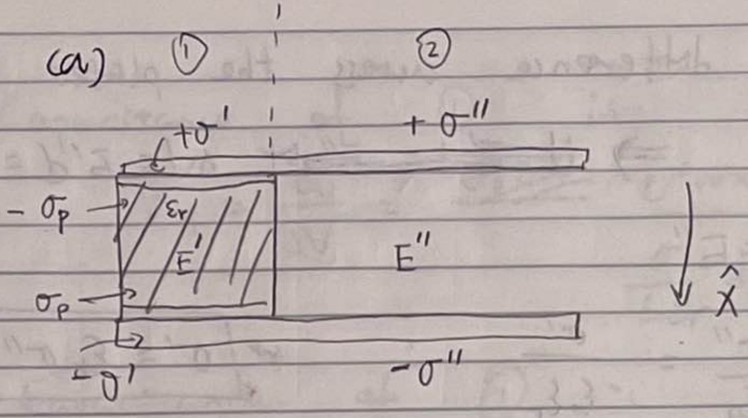
∴ In region (1):

$$\vec{E_1} = \vec{E_0} + \vec{E'} = ((\frac{1}{\varepsilon_r} - 1) + 1)\frac{\sigma}{\varepsilon_0}\hat{x} = \boxed{\frac{\sigma}{\varepsilon_r \varepsilon_0}\hat{x}}$$ ✓

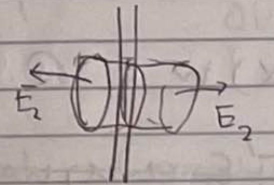
In region (2):

$$\vec{E_2} = \vec{E_0} = \boxed{\frac{\sigma}{\varepsilon_0}\hat{x}}$$ ✓

3) (a)



Using Gauss's Law in ②



$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\varepsilon_0}$$

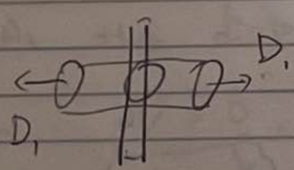
$$2 E_2 \Delta S = \frac{\sigma'' \Delta S}{\varepsilon_0}$$

$$\therefore E_2 = \frac{\sigma''}{2\varepsilon_0}$$

Superposition of +ve and -ve plates gives

$$\boxed{\vec{E}'' = \frac{\sigma''}{\varepsilon_0} \hat{x}} \checkmark$$

Gauss's Law in ①



$$\oint \vec{D} \cdot d\vec{s} = Q_{free}$$

$$\therefore 2 D_1 \Delta S = \sigma' \Delta S \quad \therefore D_1 = \frac{\sigma'}{2}$$

~~E~~ superposition of two plates gives

$$\cancel{D'' = \sigma'} \qquad \vec{D}' = \sigma' \hat{x}$$

$$\vec{D}' = \varepsilon \vec{E}' = \varepsilon_0 \varepsilon_r \vec{E}'$$

$$\Rightarrow \vec{E}' = \frac{\vec{D}'}{\varepsilon_0 \varepsilon_r} = \boxed{\frac{\sigma'}{\varepsilon_0 \varepsilon_r} \hat{x}} \checkmark$$

~~Conservation of total charge on one plate~~

$$\cancel{\Rightarrow \boxed{\sigma' x w + \sigma''(L - x) w = Q}}$$

Potential difference across the plates

is constant $\Rightarrow$ $\cancel{V \; E'd = E''d}$ $\Delta V = E'd = E''d$

$\therefore \quad E' = E''$

$$\therefore \quad \frac{\sigma''}{\varepsilon_0} = \frac{\sigma'}{\varepsilon_0 \varepsilon_r} \quad \therefore \quad \boxed{\sigma' = \varepsilon_r \sigma''}$$

Conservation of charge on one plate

$$\Rightarrow \quad \sigma' x W + \sigma''(L-x)W = Q$$

$$\cancel{\therefore \quad \sigma' x W + \sigma' \varepsilon_r (L-x) W = Q}$$

$$\cancel{\therefore \quad \sigma'' = \frac{Q}{xW + \varepsilon_r (L-x) W}}$$

$$\cancel{\therefore \quad \sigma = \frac{Q}{\varepsilon_r L W - (\varepsilon_r - 1) W x}}$$

$$\cancel{\therefore \quad \sigma'' = \frac{Q}{LW - (1 - \frac{1}{\varepsilon_r}) W x}}$$

$$\Rightarrow \quad \varepsilon_r \sigma'' x W + \sigma''(L-x) W = Q$$

$$\therefore \quad \boxed{\sigma'' = \frac{Q}{(\varepsilon_r - 1) W x + L W}}, \quad \sigma' = \frac{\varepsilon_r Q}{(\varepsilon_r - 1) W x + L W}$$

Potential difference

$$\Delta V = E'' d = \frac{\sigma''}{\varepsilon_0} d = \boxed{\frac{Qd}{\varepsilon_0 (\varepsilon_r - 1) W x + \varepsilon_0 L W}}$$

Capacitance of ① is

$$C_1 = \frac{Q_1}{\Delta V} = \cancel{\sigma' w x} \; \frac{\sigma' w x}{\frac{\sigma'}{\varepsilon_0 \varepsilon_r} d} = \frac{\varepsilon_r \varepsilon_0 W x}{d}$$

Capacitance of ② is

$$C_2 = \frac{Q_2}{\Delta V} = \frac{\sigma''(L-x)w}{\frac{\sigma''}{\varepsilon_0} d} = \frac{\varepsilon_0 (L-x) W}{d}$$

total Capacitance, since C_1 and C_2 are in parallel, is $C = C_1 + C_2$

$$\therefore \quad C = \frac{\varepsilon_0(\varepsilon_r - 1) w x + \varepsilon_0 L W}{d}$$

total energy is $U = \frac{Q^2}{2C} = \boxed{\frac{1}{2} \frac{Q^2 d}{\varepsilon_0(\varepsilon_r - 1) W x + \varepsilon_0 L W}}$ ✓

As the dielectric is pulled out, say $\cancel{F>0}$, $F<0$

then $dU > 0$, $dx < 0$

$$\therefore \quad \boxed{F = + \frac{dU}{dx}}$$

(Force is pulling in)

sign wrong

$$\therefore F = + \frac{dU}{dx} = \boxed{\frac{-Q^2 d \, \varepsilon_0 (\varepsilon_r - 1) W}{2[\varepsilon_0(\varepsilon_r - 1) W x + \varepsilon_0 L W]^2}}$$

To pull the dielectric out a distance dx, we have to exert a force $-F$ and we therefore provide the work $-F dx = dU$

$\Rightarrow F = -\frac{dU}{dx}$

$\because$ we do positive work to the system ($dU > 0$) as we pull the dielectric out against F

$\therefore$ F ~~is~~ pulls the dielectric into the capacitor.

b)

~~$F = \frac{dU}{dx}$~~ $F = \frac{dU}{dx}$

When battery is connected V is constant,

but the battery is also included in the system.

As we move the slab by dx ~~$+dx$~~

the energy of the capacitor is changed by

$d(\frac{1}{2}CV^2) = \frac{1}{2}V^2 dC$, and

~~where dC~~ the batter loses energy

VdQ as it transfers ~~area~~ charge

dQ to the capacitor.

Signs wrong

$\therefore \quad dU = \frac{1}{2}V^2 dC - VdQ$

$$dU = \underbrace{-Fdx}_{\text{work done by us}} + \underbrace{VdQ}_{\text{work given by the battery to keep the potential difference constant}}$$

$$\therefore F = \frac{dU}{dx} = \frac{1}{dx}\left(\frac{1}{2}V^2 dC - VdQ\right)$$

$$= \frac{1}{2}V^2\frac{dC}{dx} - V^2\frac{dC}{dx}$$

$$= -\frac{1}{2}V^2\frac{dC}{dx} = -\frac{1}{2}(V^2)\frac{d}{dx}\left(\frac{\varepsilon_0(\varepsilon_r-1)wx+\varepsilon_0 Lw}{d}\right)$$

$$= -\frac{1}{2}\left(\frac{Q^2 d^2}{[\varepsilon_0(\varepsilon_r-1)wx+\varepsilon_0 Lw]^2}\right)\left(\frac{\varepsilon_0(\varepsilon_r-1)w}{d}\right)$$

$$= \boxed{\frac{-Q^2 d\, \varepsilon_0(\varepsilon_r-1)w}{2[\varepsilon_0(\varepsilon_r-1)wx+\varepsilon_0 Lw]^2}}$$

Same as (a), pulling inwards.

c) if that were the case in reality, there would NOT be a force on the ~~die~~ dielectric. ✓ The force comes from the fringing field around the edges. ✓ When we use energy to calculate the force F, we got the correct answer ~~we~~ assuming the field lines are all straight. This is true because ~~as~~ the energy of fringing field stays constant as we pull the dielectric. The change in energy dU all comes from the change in the energy of field lines that can be approximated as straight.

yes

4)

a)

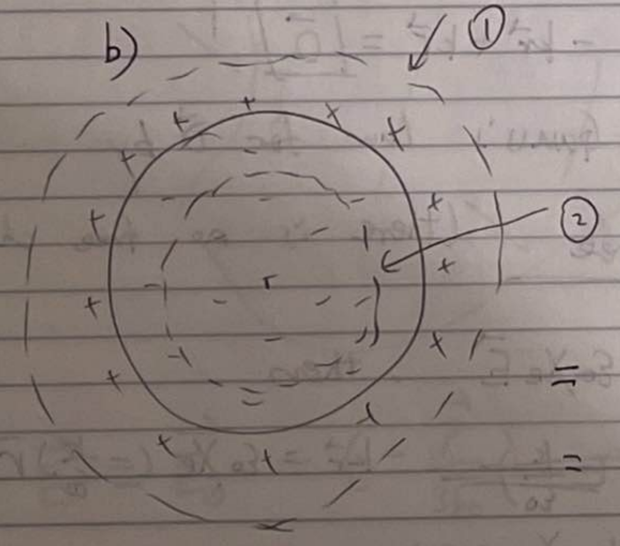
Volume density of polarization charges in the volume is

$$\rho_p = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(k\vec{r}) = -k(\vec{\nabla}\cdot\vec{r}) = \boxed{-3k}$$

surface density of polarization charges appears at the surface is

$$\sigma_p = \vec{P}\cdot\hat{n} = kR\,\hat{r}\cdot\hat{r} = \boxed{kR}$$

b)



Consider Gaussian surface ①

total charge inside

$$Q_{tot} = \iiint_V \rho_p\, dV + \oint\!\!\oint \sigma_p\, dS$$

$$= -3k\left(\frac{4}{3}\pi R^3\right) + kR\,(4\pi R^2)$$

$$= -4\pi R^3 k + 4\pi R^3 k = 0$$

Gauss's ~~surface~~ Law : $\oint\!\!\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}$

$$\therefore\ E(4\pi r^2) = \frac{Q_{tot}}{\varepsilon_0} = 0$$

$\therefore\ \boxed{\vec{E}_{out} = \vec{0}}$ outside the sphere

Consider Gaussian Surface ②

$$\oint\!\!\oint \vec{E}\cdot d\vec{S} = \frac{Q_{enc}}{\varepsilon_0}, \quad Q_{enc} = \iiint_{V_0} \rho_p\, dV$$

$$= (-3k)\left(\frac{4}{3}\pi r^3\right) = -4\pi k r^3$$

$$\therefore E(4\pi r^2) = \frac{-4\pi k r^3}{\varepsilon_0}$$

$$\therefore \cancel{\vec{E} = -kr} \quad E = -\frac{kr}{\varepsilon_0}$$

$$\therefore \boxed{\vec{E}_{in} = -\frac{k\vec{r}}{\varepsilon_0}}$$ ✓ inside sphere

c) Inside sphere

$$\vec{E} = -\frac{k\vec{r}}{\varepsilon_0}, \quad \vec{P} = k\vec{r}$$

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = -k\vec{r} + k\vec{r} = \boxed{\vec{0}}$$ ✓

Outside sphere

$\vec{E} = 0$, $\vec{P} = 0$

$\therefore \boxed{\vec{D} = 0}$ ✓

It satisfies the Gauss's Law for $\vec{D}$ by

$\vec{\nabla}\cdot\vec{D} = 0 = \rho_{free}$ ✓ (there is no free charge)

assume $\vec{P}$

$\vec{P} = \varepsilon_0\chi_e\vec{E}_{before}$

$\vec{P} = \varepsilon_0\chi_e\vec{E}_{after}$

different $\vec{E}$ as you removed the external field $\therefore$ permanent electret

Assume $\vec{P} = \varepsilon_0\chi_e\vec{E}$, then

$$\cancel{\vec{P} \quad k\vec{r} = \varepsilon_0\left(-\frac{k}{\varepsilon_0}\right)} \quad k\vec{r} = \varepsilon_0\chi_e\left(-\frac{k}{\varepsilon_0}\right)\vec{r}$$

$$\therefore \chi_e = -1$$

$\therefore$ relative permittivity $\varepsilon_r = 1 + \chi_e = 1 - 1 = 0$

This is not possible since $\varepsilon_r \geq 1$

$\therefore$ The dielectric is <u>not</u> linear. ✓

explain why

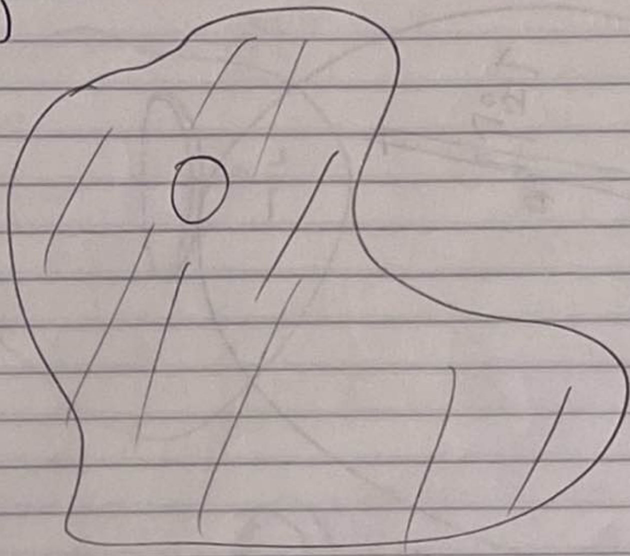
Also if the dielectric is linear,

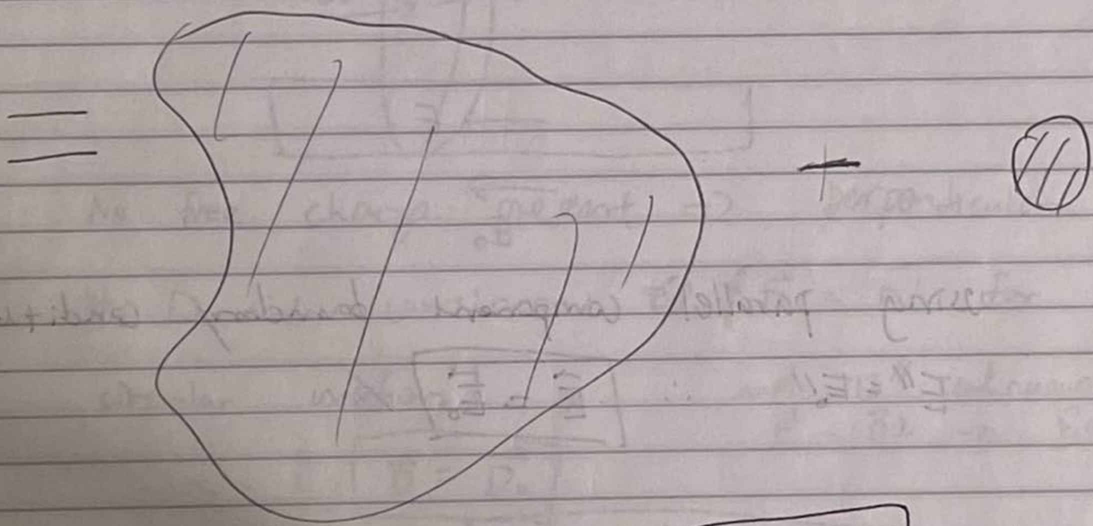
$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot(\varepsilon_0\chi_e\vec{E}) = -\varepsilon_0\chi_e(\vec{\nabla}\cdot\vec{E}) = -\chi_e\frac{\rho_{total}}{\varepsilon_0}$$
$$= -\chi_e(\rho_b + \rho_{free})$$

$$\therefore \rho_b = -\frac{\chi_e}{1+\chi_e}\rho_{free}.$$

If there is no free charge, there is no bound charge inside the material. But the material has $\rho_b = -3k$ and $\rho_{free} = 0$ $\therefore$ It is <u>not</u> linear

5) (a)





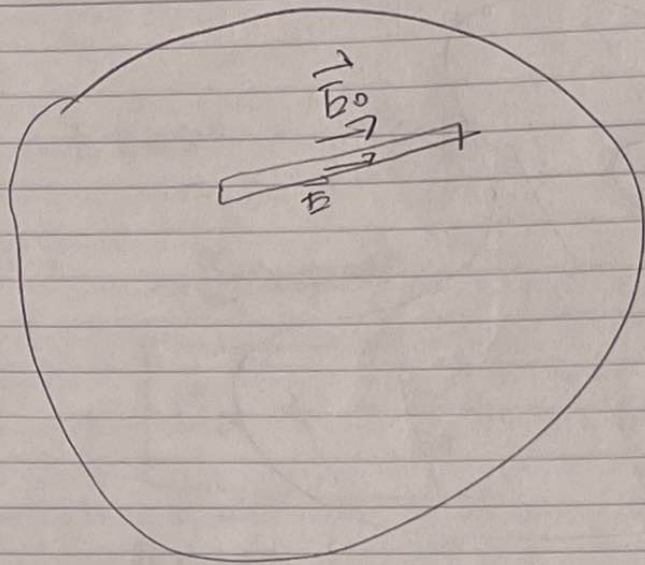
$$\vec{E} = \vec{E}_0 - \left(-\frac{\vec{P}}{3\varepsilon_0}\right) = \boxed{\vec{E}_0 + \frac{\vec{P}}{3\varepsilon_0}}$$

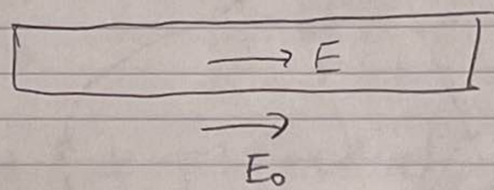
(from Problem set 1, Question 5.)

$\vec{D}$ in ~~cavity~~ uniformly polarized sphere is $\vec{D}' = \varepsilon_0\left(-\frac{\vec{P}}{3\varepsilon_0}\right) + \vec{P} = \frac{2}{3}\vec{P}$

$\therefore$ total $\vec{D}$ is $\vec{D} = \vec{D}_0 - \vec{D}' = \boxed{\vec{D}_0 - \frac{2}{3}\vec{P}}$

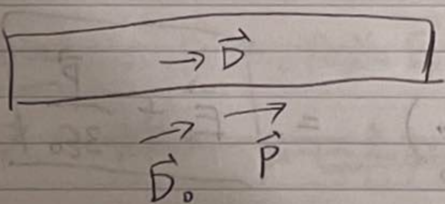
b)





using parallel component boundary condition for $\vec{E}$

$E^{\parallel} = E_0^{\parallel}$ $\quad\therefore\quad$ $\boxed{\vec{E} = \vec{E}_0}$ ✓

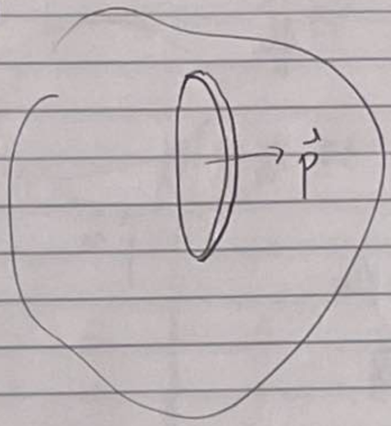


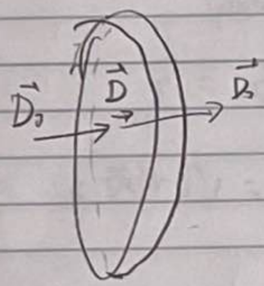
using parallel component boundary condition for $\vec{D}$

$$\vec{D}_0 - \vec{P} = \vec{D} - 0$$

$$\therefore \boxed{\vec{D} = \vec{D}_0 - \vec{P}}$$ ✓

c)





No free charge present $\Rightarrow$ perpendicular component of D is continuous (i.e. $\vec{D} = \vec{D}_\perp$ for the circular wafer)

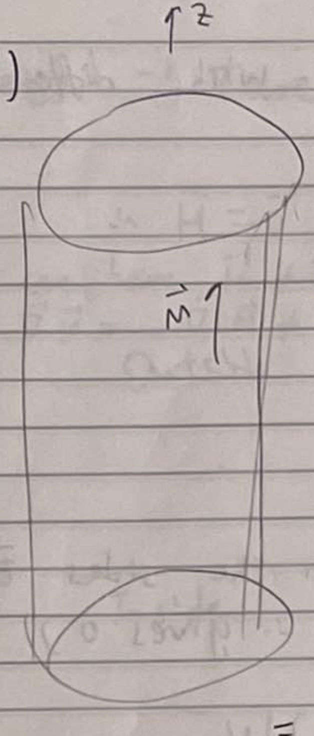
and $\vec{E}_\parallel$ continuous with $\vec{P} = \vec{P}_\perp \Rightarrow \vec{D}_\parallel$ continuous

$$\therefore \boxed{\vec{D} = \vec{D}_0}$$

$$\therefore \vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{\vec{D}_0}{\varepsilon_0} = \frac{\varepsilon_0 \vec{E}_0 + \vec{P}}{\varepsilon_0}$$

$$\therefore \boxed{\vec{E} = \vec{E}_0 + \frac{\vec{P}}{\varepsilon_0}}$$ ✓

6)



$J_f(\vec{r}) = 0$

In cylindrical coordinates.
let $\hat{r} = (\cos\theta, \sin\theta, 0)$

$\vec{M} = kr\hat{z}$

a) surface current density:

$$\vec{K}_m = \vec{M} \times \hat{n} = \vec{M} \times \hat{r} = kR\hat{z} \times \hat{r} = \boxed{kR\hat{\theta}}$$ ✓

Volume current density:

$$\vec{J}_m = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (kr\hat{z})$$

$$= k\vec{\nabla} \times (r\hat{z}) = k(r\underbrace{(\vec{\nabla} \times \hat{z})}_{0} + \underbrace{\underbrace{\vec{\nabla} r}_{\hat{r}} \times \hat{z}}_{-\hat{\theta}})$$

$$= \boxed{-k\hat{\theta}}$$ ✓

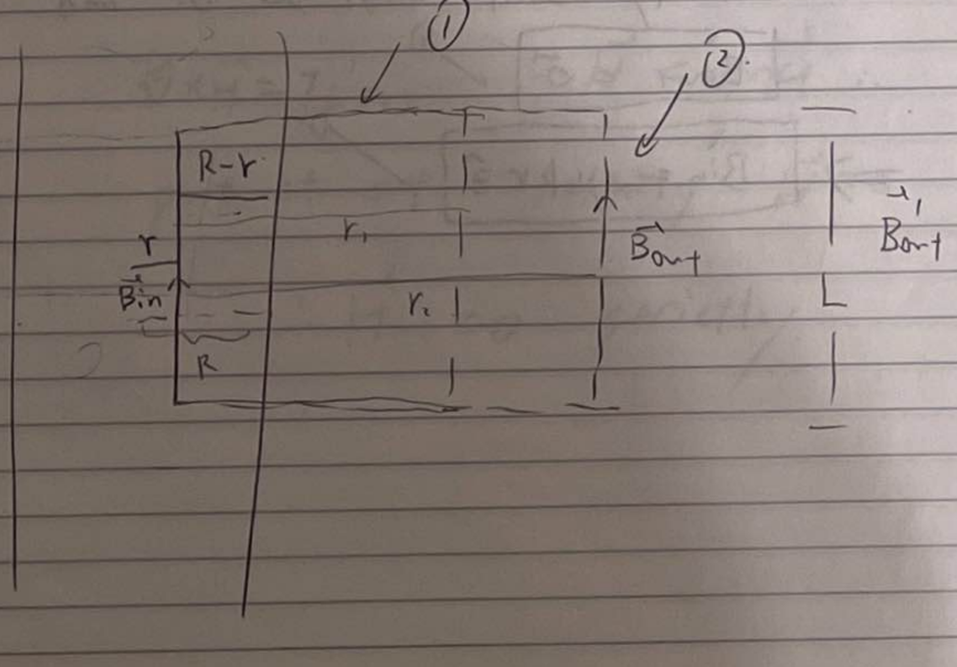
~~$\therefore \vec{K}_m = R\vec{J}_m$~~

~~Outside Total current along~~

~~Outside the material the vector potential~~ is given by.

~~$$A(\vec{r}) = \frac{\mu_0}{4\pi}\iiint_V \frac{J_m(\vec{r}')\,d\tau'}{|r - r'|} + \frac{\mu_0}{4\pi}\oint\!\!\iint_\Sigma \frac{K_m(\vec{r}')\,d\Sigma'}{|\vec{r} - \vec{r}'|}$$~~

By symmetry $\vec{B}$ is along $\pm\hat{z}$ direction both inside and outside



Consider 2 Ampere Loops with different radius r_1 and r_2

Ampere's Law :

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{penetrates}$$

- For ① ⇒ ~~$B_{out} L$~~

$B_{out} L - B_{in} L$ + (on the sides $\vec{B} \perp$ loop ∴ gives 0)

$= (-K_m L + J_m L (R-r)) \mu_0$

⇒ ~~B_{out}~~

$$B_{in} = B_{out} + \mu_0 K_m - \underbrace{\mu_0 J_m R}_{\mu_0 J_m R} \ldots$$

$$B_{in} = B_{out} + \cancel{\mu_0 K_m} \mu_0 K_m - \mu_0 J_m (R-r)$$

$$\Rightarrow B_{in} = B_{out} + \mu_0 J_m r$$

$$\therefore B_{in} = B_{out} + \mu_0 k r$$ ✓

For ②, similarly we get $B_{in} = B'_{out} + \mu_0 k r$

$\therefore B_{out} = B'_{out} \Rightarrow \vec{B}_{out}$ is a constant

∵ At infinity B_{out} must be 0 and B_{out} = const

$$\therefore \boxed{\vec{B}_{out} = \vec{0}}$$ ✓

$$\Rightarrow \boxed{\vec{B}_{in} = \mu_0 k r \hat{z}}$$ ✓

b) ∵ There is no free current anywhere

$\therefore \vec{H} = \vec{0}$ $\quad$ ∵ $\cancel{\vec{B}}$ $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{B} = \mu_0(\vec{H} + \vec{M})$

We can have $\vec{H} \neq \vec{0}$ even when there is no free current,
if $\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \neq 0$

Outside ~~cylinder~~ cylinder $\vec{M} = 0$

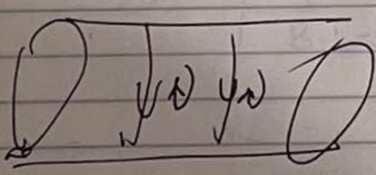
$$\therefore \vec{B} = \mu_0(\vec{0} + \vec{0}) = \boxed{\vec{0}}$$

Inside cylinder

$$\vec{B} = \mu_0(\underbrace{\vec{H}}_{0} + \underbrace{\vec{M}}_{kr\hat{z}}) = \boxed{\mu_0 k r \hat{z}}$$

Bunch of solenoid.

superposed



$$\vec{H} = \frac{\vec{B}}{\mu} - \vec{M}$$

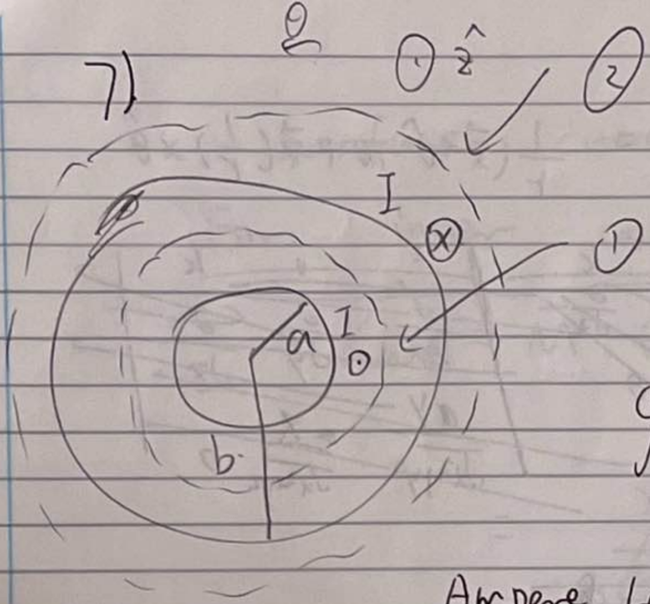
$$\vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\underbrace{\vec{M}}_{0}$$

$$\vec{\nabla}\times\vec{H} = J_f = 0 \quad \therefore \vec{H} = \overrightarrow{\text{const}}$$

put it at infinity $\vec{H} = 0$

$\therefore \vec{H} = 0$ strictly.

7)



$$\oint \vec{H}\cdot d\vec{l} = I_{free}$$

Ampere Loop 1

$$\Rightarrow \quad H(2\pi r) = I$$

$$\therefore \quad \vec{H} = \frac{I}{2\pi r}\hat{\theta}$$

$$\vec{B} = \mu_0(1+\chi_m)\vec{H}$$

$$\Rightarrow \quad \boxed{\vec{B}_{in} = \frac{\mu_0(1+\chi_m)I}{2\pi r}\hat{\theta}}$$

Ampere Loop 2.

$$H(2\pi r) = 0 \quad \Rightarrow \quad \vec{H} = \vec{0}$$

No $\vec{M}$ outside $\therefore \vec{B} = \mu_0 \vec{H}$

$\therefore \cancel{\vec{B} = \vec{0}}$ $\quad \boxed{\vec{B}_{out} = \vec{0}}$

Inside ~~cylinder~~ cable!

$$\vec{M} = \chi_m \vec{H} = \frac{\chi_m I \hat{\theta}}{2\pi r}$$

$\therefore$ Volume current $\vec{J}_m = \vec{\nabla}\times\vec{M} = \frac{\chi_m I}{2\pi}\vec{\nabla}\times\left(\frac{\hat{\theta}}{r}\right)$.

$\cancel{\vec{\nabla}\times\left(\frac{\hat{\theta}}{r^2}\right)}$ $\vec{\nabla}\times\left(\frac{\hat{\theta}}{r}\right) = \frac{1}{r}(\vec{\nabla}\times\hat{\theta}) + \vec{\nabla}\left(\frac{1}{r}\right)\times\hat{\theta}$

$\cancel{\vec{\nabla}\times\hat{\theta} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos\theta & -\sin\theta & \end{vmatrix}}$ $\cancel{\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}}$

use directly the expression of the curl in cylindrical coordinates

$\cancel{= (0,\ 0,\ )}$

$$\vec{\nabla}\times\hat{\theta} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= \left(0,\ 0,\ \frac{1}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\hat{z}}{r}$$

$$\vec{\nabla}\left(\frac{1}{r}\right)\times\hat{\theta} = -\frac{\hat{r}}{r^2}\times\hat{\theta} = -\frac{\hat{z}}{r^2}$$

$$\therefore \vec{\nabla}\times\left(\frac{\hat{\theta}}{r}\right) = \frac{\hat{z}}{r^2} - \frac{\hat{z}}{r^2} = 0$$

$$\therefore \boxed{\vec{J}_m = \vec{0}}\ ✓$$

Surface currents.

For inner cable.

$$\vec{K}_{am} = \vec{M} \cdot (-\hat{r}) = \frac{\chi_m I}{2\pi r} \underbrace{\hat{\theta} \times (-\hat{r})}_{+\hat{z}} \Big|_{r=a}$$

$$= \cancel{\frac{\chi_m I}{2\pi r}} \frac{\chi_m I \hat{z}}{2\pi a}$$

$$I_m = K_{am} (2\pi a) = \chi_m I$$

$\therefore$ Net current is $I_{tot} = I_{free} + I_m = (1+\chi_m) I$.
(in $+\hat{z}$ direction)

Ampere's Law then gives

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{tot}$$

$$\Rightarrow B(2\pi r) = (1+\chi_m) I \mu_0$$

$$\boxed{\vec{B}_{in} = \frac{\mu_0 (1+\chi_m) I \hat{\theta}}{2\pi r}}$$

For outer cable

$$\vec{K}_{bm} = \vec{M} \cdot \hat{r} = \frac{\chi_m I}{2\pi b} \hat{\theta} \times \hat{r} = \frac{-\chi_m I \hat{z}}{2\pi b}$$

$$I_{bm} = (2\pi b) K_{bm} = -\chi_m I$$

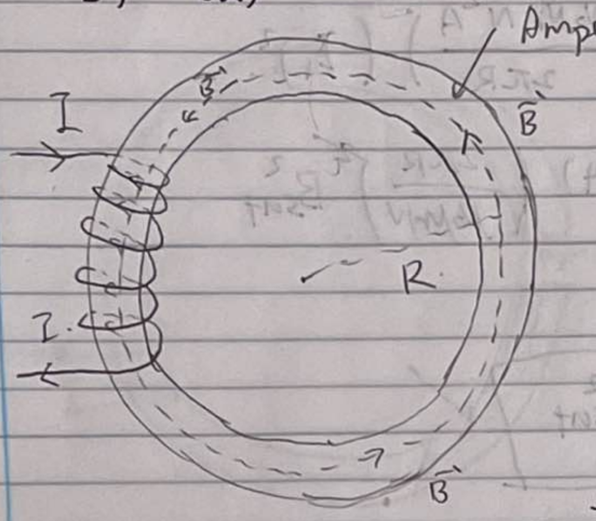
$\therefore$ ~~I_{tot}~~ $\therefore$ Outside ~~of~~ cable:

$$I_{tot} = I - I + \chi_m I - \chi_m I = 0$$

Ampere's Law gives

$$B(2\pi r) = 0$$

$$\Rightarrow \boxed{\vec{B}_{out} = \vec{0}}$$ ✓

8) (a) 

~~A~~ The ferromagnetic ~~φ~~ core concentrates the ~~B field~~ magnetic field.

Consider the Amperian loop shown

By Ampere's Law.

$$\oint_{\Gamma} \vec{H} \cdot d\vec{l} = \sum_i I_{free,i} = NI$$

$$\therefore H(2\pi R) = NI \quad \therefore H = \frac{NI}{2\pi R}.$$

$$\therefore B = \mu H = \frac{\mu N I}{2\pi R} = \cancel{\frac{\mu_0 \mu_r NI}{2\pi R}} \quad \frac{\mu_0 \mu_r N I}{2\pi R}.$$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 \mu_r N I}{2\pi R} \hat{\phi}}$$

surface ~~area~~ Area linking the flux

$S = NA$

$$\therefore \text{Total flux } \Phi = \int \vec{B} \cdot d\vec{S} = BS = \frac{\mu_0 \mu_r N I}{2\pi R}(NA).$$

$$\boxed{L = \frac{\Phi}{I} = \frac{\mu_0 \mu_r N^2 A}{2\pi R}}$$

The maximum value of B is B_{sat}

$$\therefore B_{sat} = \frac{\mu_0 \mu_r N I_1}{2\pi R}$$

$$\therefore \boxed{I_1 = \frac{2\pi R B_{sat}}{\mu_0 \mu_r N}}$$

$$W_1 = \frac{1}{2}LI_1^2 = \frac{1}{2}\left(\frac{\mu_0\mu_r N^2 A}{2\pi R}\right)(I_1)^2$$

$$= \frac{1}{2}\left(\frac{\mu_0\mu_r \cancel{N}}{\cancel{2\pi R}}\right)(NA)\left(\frac{2\pi R}{\mu_0\mu_r N}\right)^{\cancel{2}} B_{sat}^2$$

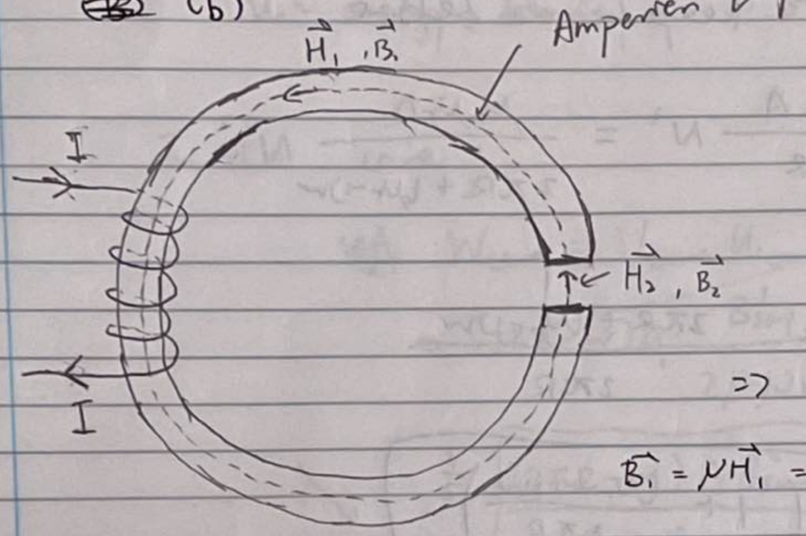
$$\cancel{= \pi R N A}$$

$$= \boxed{\frac{\pi R A}{\mu_0\mu_r} B_{sat}^2}$$

If I is increased beyond I_1, B stays at B_{sat} ∴ $\Phi = B_{sat} NA$

$$\therefore L = \frac{B_{sat} NA}{I}$$, thus L decreases as I increases with $L \propto \frac{1}{I}$ yes

~~B2~~ (b)



Apply boundary conditions:

$$B_1^{\perp} = B_2^{\perp}$$

$$\Rightarrow \vec{B}_1 = \vec{B}_2$$

$$\vec{B}_1 = \mu \vec{H}_1 = \mu_0 \mu_r \vec{H}_1 \quad , \quad \vec{B}_2 = \mu_0 \vec{H}_2$$

$$\Rightarrow \mu_0 \mu_r \vec{H}_1 = \mu_0 \vec{H}_2 \quad \Rightarrow \quad H_2 = \mu_r \vec{H}_1$$

Ampere's Law $\Rightarrow$

$$\cancel{\oint_{\Gamma} \vec{B} \cdot d\vec{l}} = \oint_{\Gamma} \vec{H} \cdot d\vec{l} = \sum_i I_{free,i} = N'I$$

$$\Rightarrow (2\pi R - W) H_1 + W \mu_r H_1 = N'I$$

$$\cancel{\Rightarrow (2\pi R + W\mu_r) H_1 = N'I}$$

~~$(W \ll R)$~~

$$\cancel{\Rightarrow H_1 = \frac{N'I}{2\pi R + W\mu_r} \Rightarrow \vec{B}_1 = \mu_0 \mu_r H_1 = \frac{\mu_0 \mu_r N'I}{2\pi R + W\mu_r} \hat{\phi}}$$

$\therefore$

$$\Rightarrow H_1 = \frac{N'I}{2\pi R + (\mu_r - 1) W}$$

$$\therefore \boxed{\vec{B}_1 = \frac{\mu_0 \mu_r N' I \hat{\phi}}{2\pi R + (\mu_r - 1) W}}$$

(N' is the # of turns now)

$$\Phi = N'AB_1 = \frac{\mu_0 \mu_r N'^2 I A}{2\pi R + (\mu_r - 1) W}$$

$$\therefore L = \frac{\mu_0 \mu_r N'^2 A}{2\pi R + (\mu_r - 1) W}$$

~~tompa Too~~ to keep L as before

$$\Rightarrow \quad \frac{\mu_0 \mu_r A}{2\pi R} N^2 = \frac{\mu_0 \mu_r A}{2\pi R + (\mu_r - 1)w} N'^2$$

$$\therefore \left(\frac{N'}{N}\right)^2 = \frac{2\pi R + (\mu_r - 1)w}{2\pi R}$$

$$\therefore \boxed{N' = N\left(1 + \frac{(\mu_r - 1)w}{2\pi R}\right)^{\frac{1}{2}}}$$

~~$\therefore \mu_r \gg 1, \quad R \gg w$~~

~~$\therefore N' \approx N\left(1 + \frac{\mu_r w}{}\right)$~~

Now ~~$B_{sat} = \mu_0 \mu_r A \left(\frac{N'^2}{2\pi R + (\mu_r - 1)w}\right)$~~

$$B_{sat} = \frac{\mu_0 \mu_r N' I_2}{2\pi R + (\mu_r - 1)w} \Rightarrow I_2 = \frac{(2\pi R + (\mu_r - 1)w) B_{sat}}{\mu_0 \mu_r N'}$$

$$\Rightarrow I_2 = \frac{1}{\mu_0 \mu_r N}(2\pi R + (\mu_r - 1)w)\left(\frac{2\pi R}{2\pi R + (\mu_r - 1)w}\right)^{\frac{1}{2}} B_{sat}.$$

$$\Rightarrow \boxed{I_2 = \frac{B_{sat}\sqrt{(2\pi R)(2\pi R + (\mu_r - 1)w)}}{\mu_0 \mu_r N} > I_1}$$

u = energy density $\Rightarrow$ $u = \frac{1}{2}\vec{B}\cdot\vec{H} = \frac{B^2}{2\mu_0\mu_r}$

In the core :

$$W_{core} = V_{core}\, u_{core}$$

$$= (2\pi R - w)A \cdot \frac{B_{sat}^2}{2\mu_0\mu_r}$$

$$= \boxed{\frac{(2\pi R - w)A B_{sat}^2}{2\mu_0\mu_r}} \checkmark$$

In the gap :

$$W_{gap} = V_{gap}\, u_{gap}$$

$$= \boxed{\frac{wA \cdot B_{sat}^2}{2\mu_0}} \checkmark$$

Total energy

$$W_2 = W_{core} + W_{gap}$$

$$= \cancel{2\pi R - w}\ \boxed{\frac{[2\pi R + (\mu_r - 1)w]A\, B_{sat}^2}{2\mu_r\mu_0}}$$

$$> W_1$$

For $R = 10\,cm = 0.1\,m$ $w = 3\,mm = 0.003\,m$ $\cancel{\mu_r}$

$\mu_r - 1 \approx \mu_r = 1500$ $2\pi R - w \simeq 2\pi R$.

$$W_{core} = \left(\frac{2\pi \times 0.1}{1500}\right)\frac{AB_{sat}^2}{2\mu_0} = 0.00042\,\frac{AB_{sat}^2}{2\mu_0}$$

$$W_{gap} = \cancel{(0.003)}\ 0.003\,\frac{AB_{sat}^2}{2\mu_0}$$

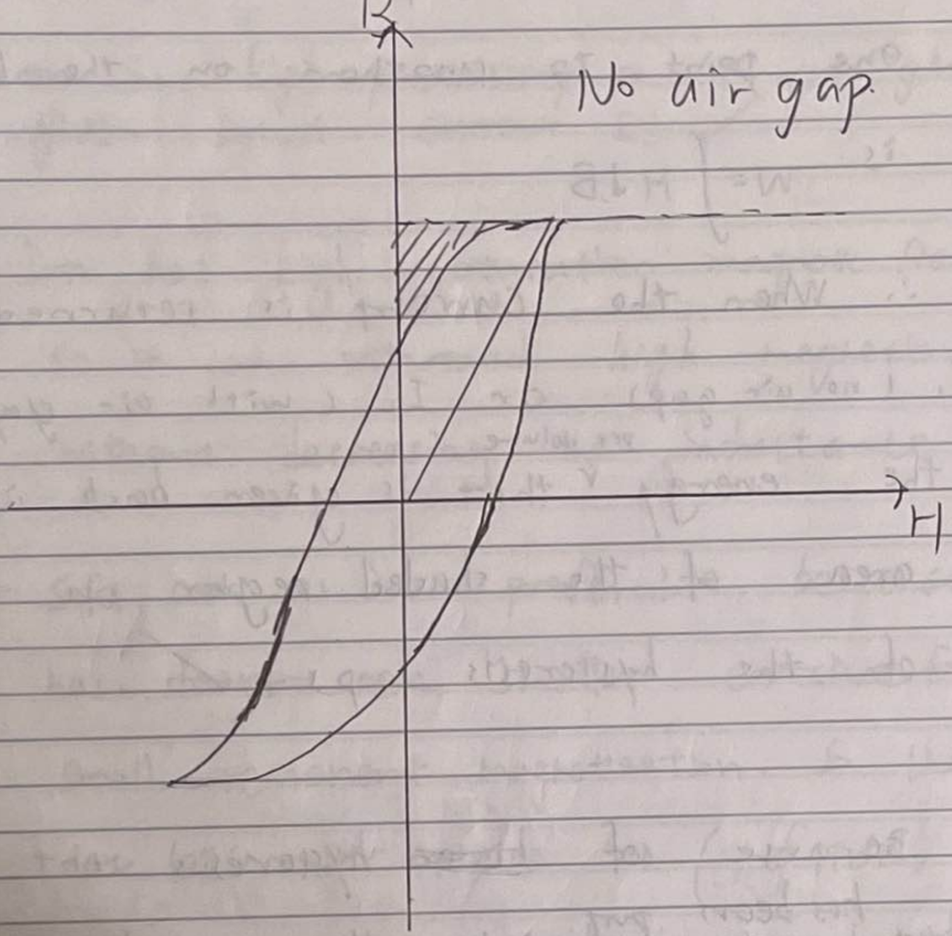
$$\therefore \boxed{W_{gap} > W_{core}}$$

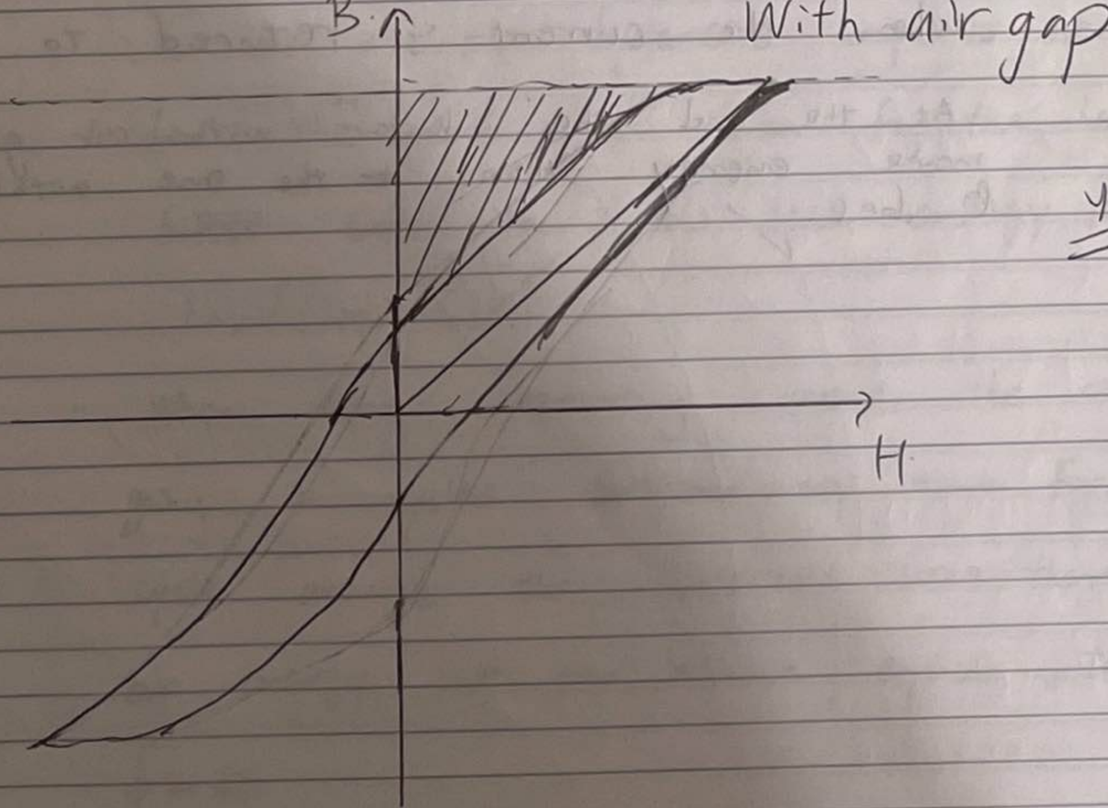
more energy is

$$\frac{W_{gap}}{W_{core}} = ?$$

stored in the gap.

(c)





yes

∵ the work done per unit volume from one point to another on the hysteresis curve is $W = \int H \, dB$

∴ When the current is returned from I_1 (no air gap) or I_2 (with air gap) to 0, the energy per volume that is given back is the area of the shaded region on each of the hysteresis loops. ✓

Because of ~~hister~~ hysteresis not all energy that has been ~~~~ put ~~put~~ into the inductor is released when the current is retured to 0.

At the end the inductor with air gap releases more energy than ~~to~~ the one without air gap does. ✓

(d)

- Iron has high permeability so it can give large B given small current I. ✓

- Iron has high saturation magnetic field B_{sat}, so it can withstand high magnetic field without decreasing the inductance. ✓

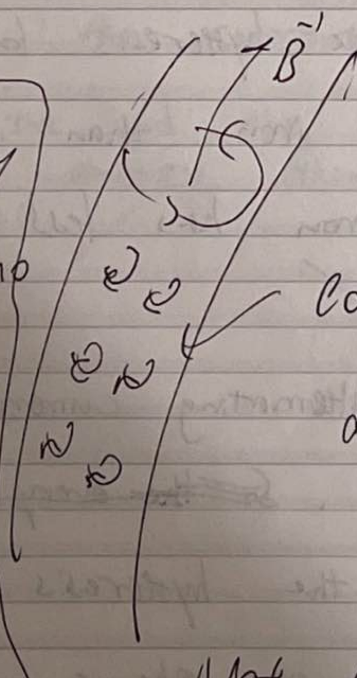
- Soft iron is more preferable because ~~sof~~ it has ~~lower rem~~ smaller coercive force and small remanent magnetization. So the energy ~~los~~ less per cycle ($\oint H\,dB$, which is the area enclosed by the hysteresis loop) is less as the hysteresis loop is ~~na~~ narrower ~~for~~ the soft iron than for the hard iron. ~~Softer~~ Soft iron has less energy loss ~~than~~ than hard iron. ✓

- When using alternating currents the current is going in cycles. ~~So the every time~~ For every cycle around the hysteresis loop there is an energy lost per volume equal to $\oint H\,dB$.

- Also, iron is a ~~conductor~~ conducting material so it has ~~very~~ low resistance. This means that the electromotive force induced by the changing magnetic field generated by the alternating current will induce a relatively large eddy current. Which opposes the original magnetic ~~field~~ ~~fi~~ field and causes energy loss.

yes

use a sheets of insulating material to prevent them from circulating



eddy current

but they have thermo expansion and contraction

⇓

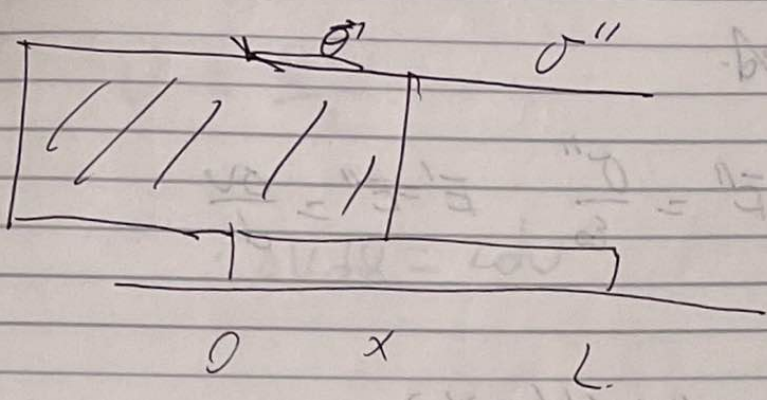
induce acoustic waves.

Why transformers are noisy?

eddy current.

dissipate energy through Joule effect

Not shown in the hysteresis ~~cur~~ curve



Q fixed

$$U = \frac{Q^2 d}{2\varepsilon_0 w[L + x(\varepsilon_r - 1)]}$$

If Q is fixed

to pull the dielectric by a distance dx, I have to exert force $-F$ ⇒ I have to give the work $-F\,dx = dU$

$$\therefore F = -\frac{dU}{dx}$$

$F>0$ $dx<0$. $dU>0$

$F<0$ ~~$dx>0$~~ $dx<0$ ~~$dU>$~~ $dU<0$.

If ΔV is fixed.

$$E' = \frac{\sigma'}{\varepsilon_0 \varepsilon_r} \qquad E'' = \frac{\sigma''}{\varepsilon_0} \qquad E' = E'' = \frac{\Delta V}{d}$$

$$Q = \underbrace{\sigma'}_{\varepsilon_r \sigma''} wL + \sigma'' w(L - x)$$

$$= \cancel{w\varepsilon_0\Delta x}\ \frac{w\varepsilon_0\Delta V}{d}(L + x(\varepsilon_r - 1))$$

$$U = \frac{Q\Delta V}{2} = \frac{\varepsilon(\Delta V)^2 w}{2d}(L + x(\varepsilon_r - 1))x.$$

When the dielectric is pulled out a distance dx, the charge varies by dQ so that ΔV ~~state~~ stays constant

$\therefore$ battery has to give work.

$\Delta V\, dQ.$

$$dU = -F\,dx + \Delta V\, dQ$$

$$U = \frac{Q\Delta V}{2}$$

$$\therefore \Delta V\, dQ = 2dU$$

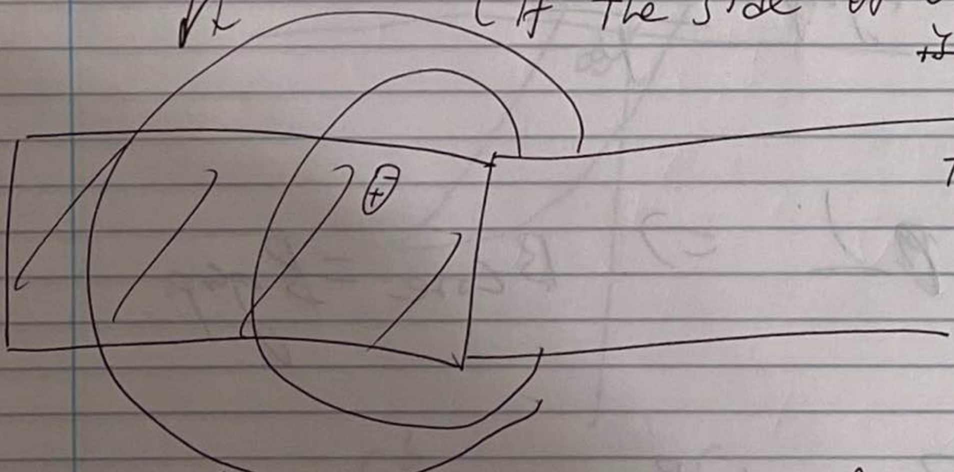
$$\therefore dU = -F dx + \underbrace{\Delta V\, dQ}_{2dU}$$

$$\therefore dU = F dx$$

$$\therefore F = \frac{dU}{dx}$$

energy of this part stays constant.

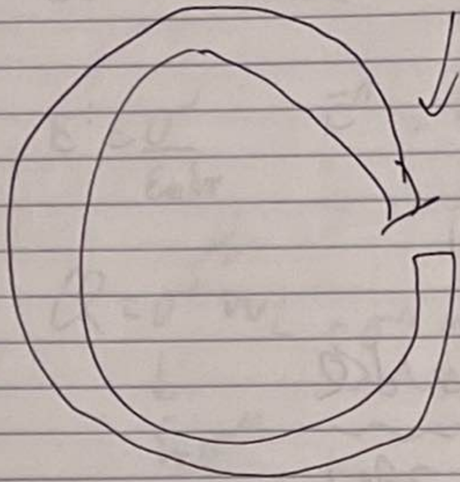
(if the side of dielectric far enough ~~is away of~~ from the fringing then ~~over~~ $\vec{E}$ is constant & energy)



non-uniform field, gives ~~to~~ a dipole non-uniform

This is still an approximation

neglect fringing field for small gap



$2\pi r H = NI$

$B = \mu_0 \mu_r H$

$2\pi(R-w)H_{core} + wH_{gap} = N'I$

$$H_{core} = \frac{B_{core}}{\mu_0 \mu_r}$$

$$H_{gap} = \frac{B_{gap}}{\mu_0}$$

B is $\perp$ $\Rightarrow$ $B_{core} = B_{gap}$

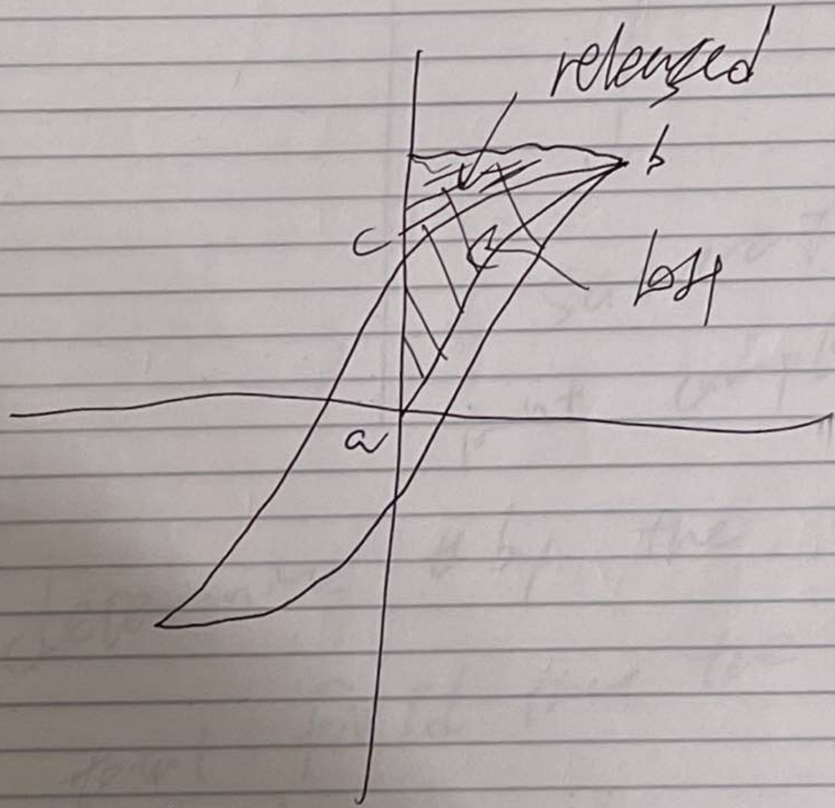
$$\frac{2\pi R B}{\mu_0 \mu_r} + \frac{wB}{\mu_0} = N'I$$

$$L = \frac{\phi}{I} = \frac{BAN'}{I}$$

$$W = \frac{1}{2} L I^2$$

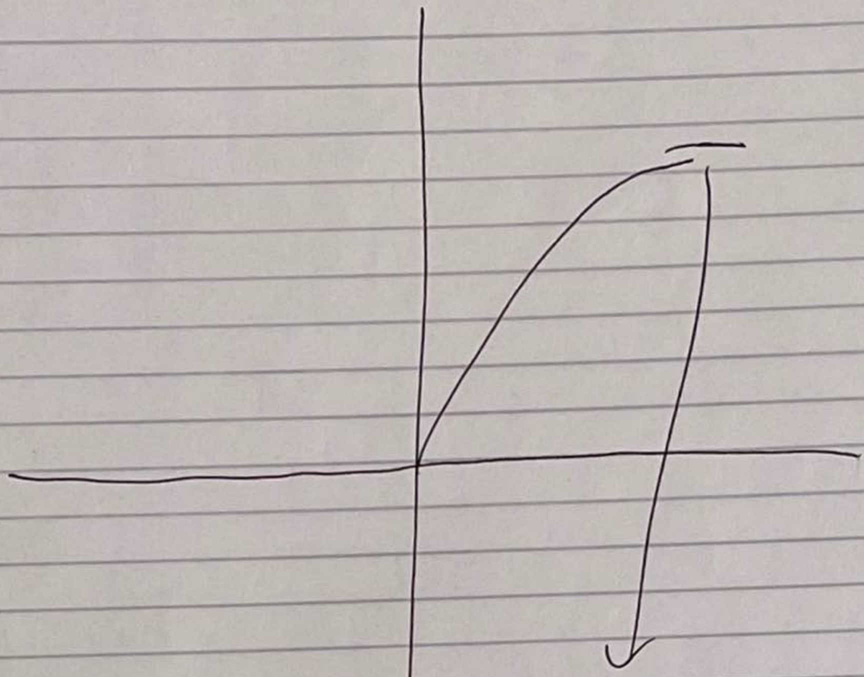
$$W_{no\text{-}gap} = \frac{1}{2} L I^2 = \iiint_{core} \frac{1}{2} B H \, d\tau$$

$$W_{with\,gap} = \frac{1}{2} L I^2 =$$



$$\text{+ve} \rightarrow \int_a^b H\,dB + \int_b^c H\,dB. \leftarrow \text{-ve}$$

$= + \text{Area}_1 - \text{Area}_2$



saturation

point completely

determine ~~t~~ by the total field that the dipoles produce so.

I_{sat} and B_{sat} are symmetrical