To : Caroline Terquem

Electromagnetism 1

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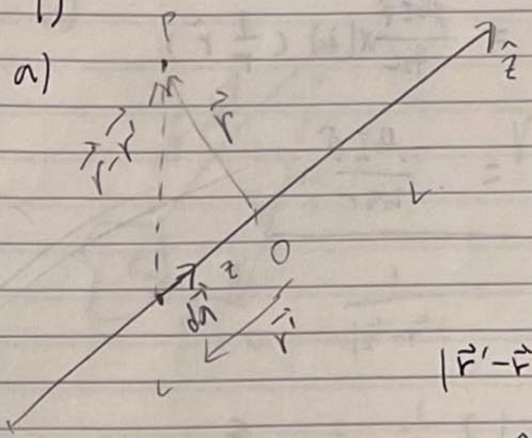
Excellent ! Only questions 4c-d need to be redone

1)

a)



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_P \frac{I(\vec{r}')}{|\vec{r}'-\vec{r}|}\, d\vec{\lambda}'$$

$|\vec{r}'-\vec{r}| = |\vec{r}-\vec{r}'| = \sqrt{z^2+r^2}$, $I(\vec{r}') = I$, $d\vec{\lambda}' = \hat{z}\,dz$

$$\therefore \quad \vec{A}(\vec{r}) = \frac{\mu_0 \hat{z} I}{4\pi}\int_{z=-L}^{z=L} \frac{dz}{\sqrt{z^2+r^2}}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi}\left[\ln\left(z+\sqrt{z^2+r^2}\right)\right]_{-L}^{L}$$

$$= \frac{\mu_0 I \hat{z}}{4\pi}\left[\ln\left(L+\sqrt{L^2+r^2}\right) - \ln\left(-L+\sqrt{L^2+r^2}\right)\right]$$

$$= \frac{\mu_0 I \hat{z}}{4\pi}\ln\left(\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right) \checkmark$$

b) Assuming $L \gg r$, $\sqrt{L^2+r^2} = L\sqrt{1+\frac{r^2}{L^2}} = L\left(1+\frac{r^2}{2L^2}+O(r^4)\right)$

$$\approx \cancel{L\left(1+\frac{r^2}{2L^2}\right)}\ L\left(1+\frac{r^2}{2L^2}\right) = L+\frac{r^2}{2L}$$

$$\therefore \quad \vec{A}(\vec{r}) = \frac{\mu_0 I \hat{z}}{4\pi}\ln\left(\frac{2L}{\cancel{L}+\frac{r^2}{2L}-\cancel{L}}\right) = \frac{\mu_0 I \hat{z}}{4\pi}\ln\left(\frac{4L}{r^2}\right)$$

→ should be L^2

$\vec{\nabla}(\phi\vec{F}) = \phi(\vec{\nabla}\times\vec{F}) + \vec{\nabla}\phi\times\vec{F}$, in this case $\vec{F} = \frac{\mu_0 I \hat{z}}{4\pi}$,

$\phi = \ln\left(\frac{4L}{r^2}\right)$ $\quad \vec{\nabla}\times\vec{F} = \frac{\mu_0 I}{4\pi}(\vec{\nabla}\times\hat{z}) = 0$

$$\therefore \quad \vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = \vec{\nabla}\left(\ln\left(\frac{4L}{r^2}\right)\right)\times\frac{\mu_0 I \hat{z}}{4\pi}$$

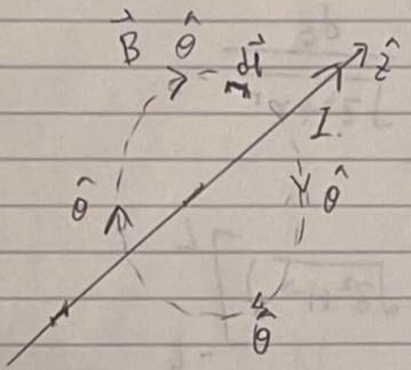
$$= \frac{\mu_0 I \hat{z}}{4\pi}\times\vec{\nabla}\left(\ln\left(\frac{r^2}{4L}\right)\right)$$

$$= \frac{\mu_0 I \hat{z}}{4\pi}\times\vec{\nabla}\left(\ln(r^2)\right)$$

$$= \frac{\mu_0 I \hat{z}}{4\pi} \times 2 \vec{\nabla}(\ln(r)) = \frac{\mu_0 I \hat{z}}{4\pi} \times (2)\left(\frac{1}{r}\hat{r}\right)$$

$$= \frac{\mu_0 I}{2\pi r}(\hat{z} \times \hat{r}) = \frac{\mu_0 I \hat{\theta}}{2\pi r}$$ ✓

Using Ampere's Law :



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$\therefore B(2\pi r) = \mu_0 I$$

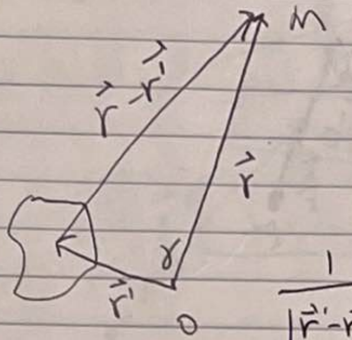
$$\therefore B = \frac{\mu_0 I}{2\pi r}$$

By right-hand rule $\vec{B}$ is along direction $\hat{\theta}$

$$\therefore \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$ ✓

Consistent.

2)



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\vec{r}')}{|\vec{r}'-\vec{r}|}\,d\tau'$$

$$\frac{1}{|\vec{r}'-\vec{r}|} = |\vec{r}-\vec{r}'|^{-1} = \left(r^2 - 2rr'\cos\gamma + r'^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left(1 - 2\cos\gamma\left(\frac{r'}{r}\right) + \left(\frac{r'}{r}\right)^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left[1 - \left(2\cos\gamma\left(\frac{r'}{r}\right) - \left(\frac{r'}{r}\right)^2\right)\right]^{-\frac{1}{2}}$$

$\because$ $(1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$, let $\frac{r'}{r} = x$
(for $x \ll 1$)

$$\therefore \frac{1}{|\vec{r}'-\vec{r}|} \approx \frac{1}{r}\Big(1 + \frac{1}{2}(2x\cos\gamma - x^2) + \frac{3}{8}(2x\cos\gamma - x^2)^2 + \frac{5}{16}(2x\cos\gamma - x^2)^3\Big)$$

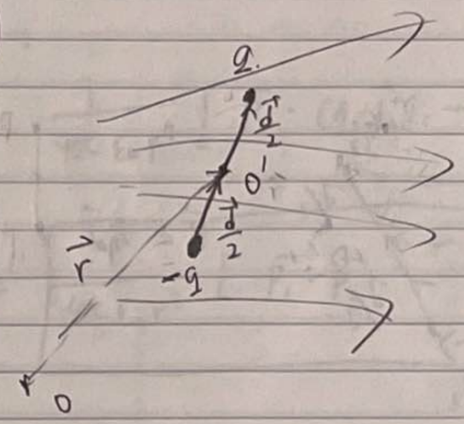
$$= \frac{1}{r}\Big(\underbrace{1}_{P_0(\cos\gamma)} + x\underbrace{(\cos\gamma)}_{P_1(\cos\gamma)} + x^2\underbrace{\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right)}_{P_2(\cos\gamma)} + x^3\underbrace{\left(\frac{5}{2}\cos^3\gamma - \frac{3}{2}\cos\gamma\right)}_{P_3(\cos\gamma)} + O(x^4)\Big)$$

$$= \frac{1}{r}\sum_{n=0}^{3} x^n P_n(\cos\gamma) = \frac{1}{r}\sum_{n=0}^{3}\left(\frac{r'}{r}\right)^n P_n(\cos\gamma) + O\left(\left(\frac{r'}{r}\right)^4\right)$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho(\vec{r}')}{|\vec{r}'-\vec{r}|}\,d\tau'$$

$$= \cancel{\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{3}} = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{3}\frac{1}{r^{n+1}}\iiint_V (r')^n P_n(\cos\gamma)\rho(\vec{r}')\,d\tau' + O(r^4)$$

3) a)



$\vec{E}(\vec{r})$

$V(r)$ $\qquad$ $\vec{E}(\vec{r}) = -\vec{\nabla} V(r)$

$\vec{p} = q\vec{d}$

Energy of dipole :

$$U = qV(\vec{r} + \frac{\vec{d}}{2}) + (-q)\, V(\vec{r} - \frac{\vec{d}}{2})$$

$$= q\left[V(\vec{r} + \frac{\vec{d}}{2}) - V(\vec{r} - \frac{\vec{d}}{2}) \right]$$

$$= q\left[\left(V(\vec{r} + \frac{\vec{d}}{2}) - V(\vec{r}) \right) + \left(V(\vec{r}) - V(\vec{r} - \frac{\vec{d}}{2}) \right) \right].$$

$$\cancel{= 2q\, D_{\frac{\vec{d}}{2}}} \qquad \cancel{= 2q\, (D_{\vec{d}} V)(\frac{\vec{d}}{2})}$$

~~where $D_{\vec{d}} V$ is the directional~~

we know that : $\quad dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = \vec{\nabla} V \cdot d\vec{r}$

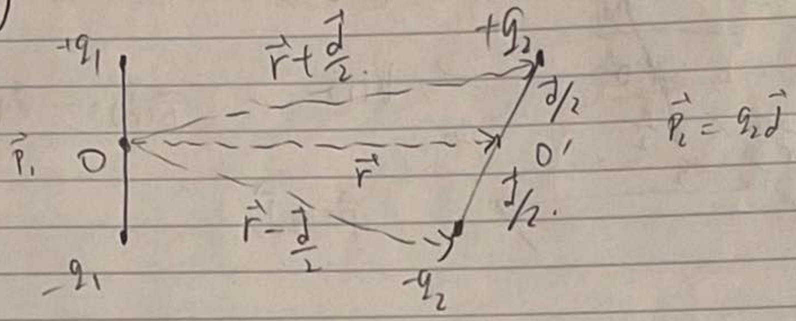
and in this case $\quad d\vec{r} = \frac{\vec{d}}{2}$

$$\therefore \quad dV = V(\vec{r} + \frac{\vec{d}}{2}) - V(\vec{r}) = V(\vec{r}) - V(\vec{r} - \frac{\vec{d}}{2}) = \vec{\nabla} V \cdot \frac{\vec{d}}{2}$$

$$\therefore \quad U = q\left[2 \times \vec{\nabla} V \cdot \frac{\vec{d}}{2} \right] = q(\vec{\nabla} V \cdot \vec{d}) \Rightarrow$$

$$= \vec{p} \cdot \vec{\nabla} V = -\vec{p} \cdot \vec{E} \checkmark$$

b)



Dipole potential $V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}_1\cdot\vec{r}'}{r'^3}$, energy $U = qV$

$$\therefore \quad U_{int} = \frac{q_2}{4\pi\varepsilon_0}\left(\frac{\vec{p}_1\cdot(\vec{r}+\frac{\vec{d}}{2})}{|\vec{r}+\frac{\vec{d}}{2}|^3} - \frac{\vec{p}_1\cdot(\vec{r}-\frac{\vec{d}}{2})}{|\vec{r}-\frac{\vec{d}}{2}|^3}\right)$$

More simply: $U_{int} = -\vec{p}_2\cdot\vec{E}_1$ with $\vec{E}_1 = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}_1\cdot\hat{r})\hat{r} - \vec{p}_1\right]$

$$\left|\vec{r}+\frac{\vec{d}}{2}\right|^{-3} = \left(r^2 + \vec{r}\cdot\vec{d} + \frac{d^2}{4}\right)^{-3/2}$$

$$= \frac{1}{r^3}\left(1 + \frac{\vec{r}\cdot\vec{d}}{r^2} + \frac{d^2}{4r^2}\right)^{-3/2} \underbrace{\simeq}_{d \ll r} \frac{1}{r^3}\left(1+\frac{\vec{r}\cdot\vec{d}}{r^2}\right)^{-3/2}$$

$$\simeq \frac{1}{r^3}\left(1 - \frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{r^2}\right)$$

similarly $\left|\vec{r}-\frac{\vec{d}}{2}\right|^{-3} = \frac{1}{r^3}\left(1+\frac{3}{2}\frac{\vec{r}\cdot\vec{d}}{r^2}\right)$

$$\therefore \boxed{U_{int}} = \frac{q_2}{4\pi\varepsilon_0 r^3}\Big[\left(\vec{p}_1\cdot\vec{r} + \tfrac{1}{2}\vec{p}_1\cdot\vec{d}\right)\left(1 - \tfrac{3}{2}\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right)$$

$$- \left(\vec{p}_1\cdot\vec{r} - \tfrac{1}{2}\vec{p}_1\cdot\vec{d}\right)\left(1 + \tfrac{3}{2}\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right)\Big]$$

$$= \frac{q_2}{4\pi\varepsilon_0 r^3}\Big[\cancel{\vec{p}_1\cdot\vec{r}} + \tfrac{1}{2}\vec{p}_1\cdot\vec{d} - \tfrac{3}{2}(\vec{p}_1\cdot\vec{r})\left(\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right) - \cancel{\tfrac{3}{4}(\vec{p}_1\cdot\vec{d})\left(\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right)}$$

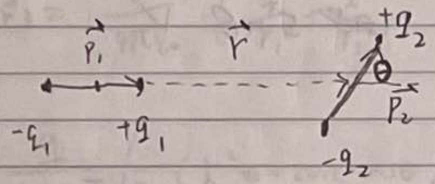
$$- \cancel{\vec{p}_1\cdot\vec{r}} + \tfrac{1}{2}\vec{p}_1\cdot\vec{d} - \tfrac{3}{2}(\vec{p}_1\cdot\vec{r})\left(\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right) + \cancel{\tfrac{3}{4}(\vec{p}_1\cdot\vec{d})\left(\tfrac{\vec{r}\cdot\vec{d}}{r^2}\right)}$$

$$= \frac{q_2}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1\cdot\vec{d} - 3(\vec{p}_1\cdot\vec{r})\left(\frac{\vec{r}\cdot\vec{d}}{r^2}\right)\right].$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1 \cdot (q_2\vec{d}) - 3\left(\vec{p}_1 \cdot \frac{\vec{r}}{r}\right)\left(q_2\vec{d} \cdot \frac{\vec{r}}{r}\right) \right]$$

$$= \boxed{\frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r}) \right].}$$

c) (i) when $\vec{p}_1$ is parallel to $\vec{r}$

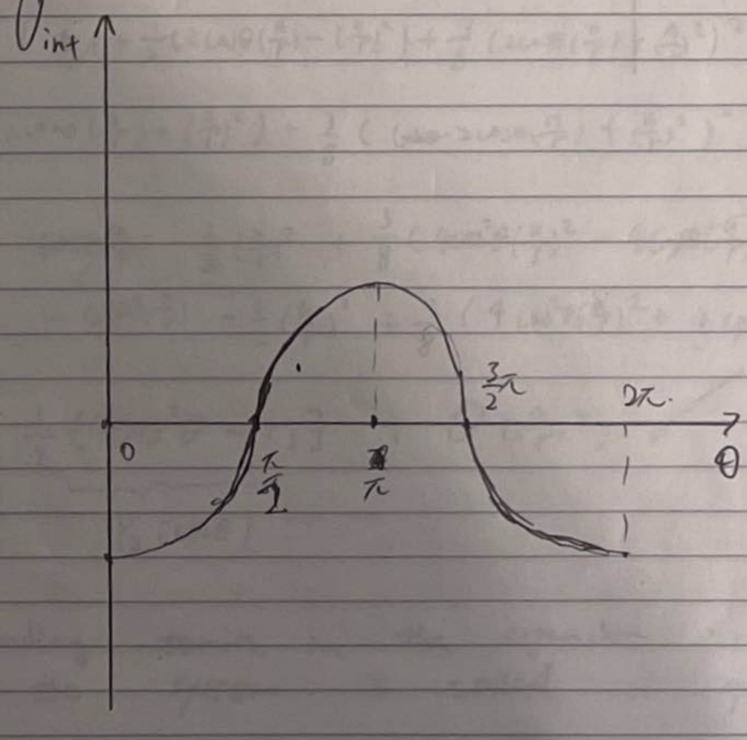


let $p_1 = |\vec{p}_1|$, then $\vec{p}_1 = p_1\hat{r}$ and $\vec{p}_1 \cdot \hat{r} = p_1 \hat{r} \cdot \hat{r} = p_1$

$$\therefore \quad U_{int} = \frac{1}{4\pi\varepsilon_0 r^3}\left[p_1 (\vec{p}_2 \cdot \hat{r}) - 3p_1 (\vec{p}_2 \cdot \hat{r}) \right].$$

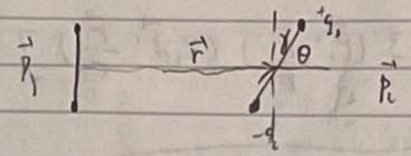
$$= -\frac{p_1}{2\pi\varepsilon_0 r^3}(\vec{p}_2 \cdot \hat{r}) = -\frac{p_1 p_2}{2\pi\varepsilon_0 r^3}\cos\theta$$

$$\therefore \quad U_{int}(\theta) \propto -\cos\theta \qquad \theta \in [0, 2\pi).$$



OK

(ii) when $\vec{p}_1$ is perpendicular to ~~$\vec{p}$~~ $\vec{r}$



$\gamma + \theta = 90°$

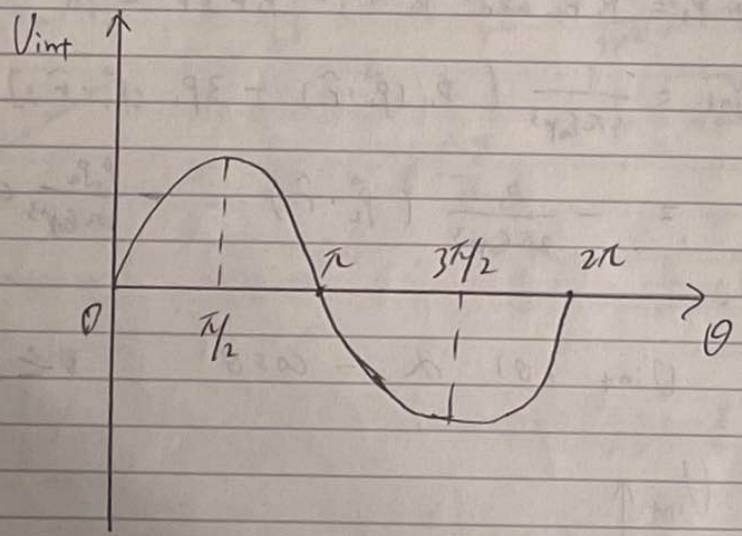
$$U_{int} = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1\cdot\vec{p}_2 - 3\underbrace{(\vec{p}_1\cdot\hat{r})}_{0}(\vec{p}_2\cdot\hat{r})\right]$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p}_1\cdot\vec{p}_2\right]$$

$$= \cancel{\frac{p_1p_2}{4\pi\varepsilon_0 r^3}\cos\theta} \quad \frac{p_1p_2}{4\pi\varepsilon_0 r^3}\cos\gamma$$

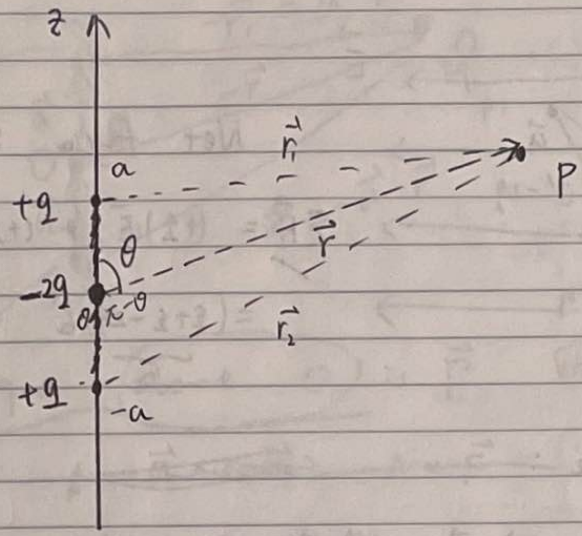
$$= \frac{p_1p_2}{4\pi\varepsilon_0 r^3} \cancel{\cos} \sin\theta \quad ✓$$

$\therefore U_{int} \propto \sin\theta \quad , \quad \theta \in [0, 2\pi]$



OR

4)



a)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{q}{r_2} - \frac{2q}{r}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left((r^2 - 2ar\cos\theta + a^2)^{-1/2} + (r^2 + 2ar\cos\theta + a^2)^{-1/2} - \frac{2}{r}\right).$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1 - 2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right)^{-1/2} + \frac{1}{r}\left(1 + 2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right)^{-1/2} - \frac{2}{r}\right)$$

$$= \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - \left(2\cos\theta\left(\frac{a}{r}\right) - \left(\frac{a}{r}\right)^2\right)\right)^{-1/2} + \left(1 + \left(2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right)\right)^{-1/2} - 2\right]$$

use the multipole expansion without rederiving it

$$(1+x)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \cdots \qquad (1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

$$\therefore V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\Big[\cancel{1} + \frac{1}{2}\left(2\cos\theta\left(\frac{a}{r}\right) - \left(\frac{a}{r}\right)^2\right) + \frac{3}{8}\left(2\cos\theta\left(\frac{a}{r}\right) - \left(\frac{a}{r}\right)^2\right)^2 + \cancel{1}$$

$$-\frac{1}{2}\left(2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right) + \frac{3}{8}\left(2\cos\theta\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^2\right)^2 \cancel{-2}.\Big]$$

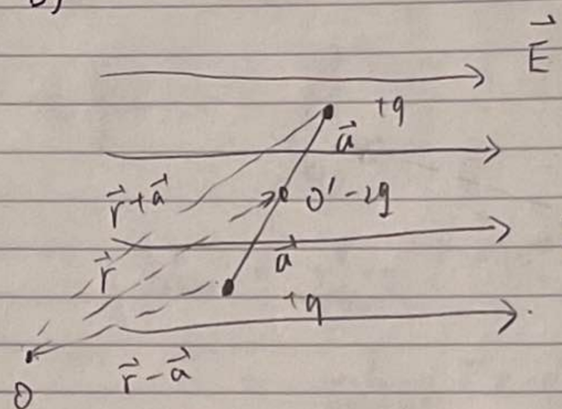
$$= \frac{q}{4\pi\varepsilon_0 r}\Big[\cancel{\cos\theta\left(\frac{a}{r}\right)} - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{3}{8}\left(4\cos^2\theta\left(\frac{a}{r}\right)^2 - 4\cos\theta\left(\frac{a}{r}\right)^3 + \left(\frac{a}{r}\right)^4\right).$$

$$-\cancel{\cos\theta\left(\frac{a}{r}\right)} - \frac{1}{2}\left(\frac{a}{r}\right)^2 + \frac{3}{8}\left(4\cos^2\theta\left(\frac{a}{r}\right)^2 + 4\cos\theta\left(\frac{a}{r}\right)^3 + \left(\frac{a}{r}\right)^4\right)$$

$$= \frac{2qa^2}{4\pi\varepsilon_0 r^3}\Big[\underbrace{\frac{1}{2}(3\cos^2\theta - 1)}_{P_2(\cos\theta)}\Big] + O\left(\left(\frac{a}{r}\right)^4\right) \checkmark$$

The leading term in the expansion is the quadruple term so the system is called a quadruple. ✓

b)



Net force :

$$\sum \vec{F} = (+q)(\vec{E}) + (+q)(\vec{E}) + (-2q)(\vec{E})$$

$$= \underbrace{(q+q-2q)}_{0}\vec{E} = 0$$

Net Torque :

$$\sum \vec{\tau} = \cancel{(\vec{r}+\vec{a})\times(q\vec{E})} \sum_i \vec{r}_i \times \vec{F}_i$$

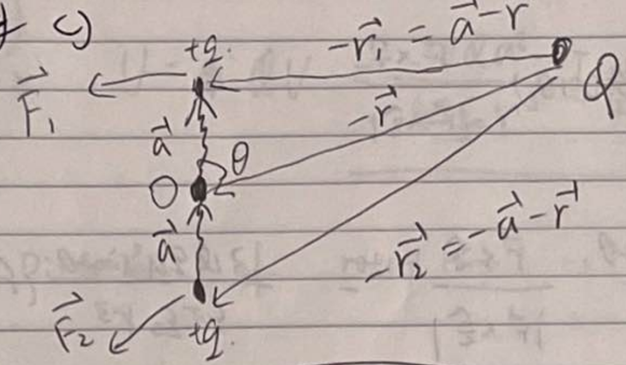
$$= (\vec{r}+\vec{a}) \times q\vec{E} + (\vec{r}-\vec{a}) \times q\vec{E} + \vec{r} \times (-2q)\vec{E}$$

$$= q\vec{r}\times E + \cancel{q\,\vec{a}\times\vec{E}} + q\vec{r}\times\vec{E} - \cancel{aq}\ \cancel{q\vec{a}\times\vec{E}} - 2q\vec{r}\times\vec{E}$$

$$= 0$$

$\therefore$ No translational force or couple on the quadrupole in a uniform $\vec{E}$ field.

c)



Torque about O is $\vec{\Gamma}$ This is not what you were asked to calculate.

$$\cancel{\vec{\Gamma} = \vec{a} \times (-\vec{r}_1)} \quad A \times \vec{F}_1$$

$$\vec{\Gamma} = \vec{a} \times \vec{F}_1 + (-\vec{a}) \times \vec{F}_2$$

$$= \vec{a} \times \frac{qQ}{4\pi\varepsilon_0 r_1^3}(\vec{a} - \vec{r}) - \vec{a} \times \frac{qQ}{4\pi\varepsilon_0 r_2^3}(-\vec{a} - \vec{r})$$

$$= \frac{qQ}{4\pi\varepsilon_0 r_1^3}(\underbrace{\vec{a} \times \vec{a}}_{0} - \vec{a} \times \vec{r}) - \frac{qQ}{4\pi\varepsilon_0 r_2^3}(\underbrace{\vec{a} \times -\vec{a}}_{0} - \vec{a} \times \vec{r})$$

$$= (\vec{r} \times \vec{a})\left(\frac{qQ}{4\pi\varepsilon_0}\right)\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)$$

$$= \underbrace{\frac{\vec{r} \times \vec{a}}{|\vec{r} \times \vec{a}|} ar\sin\theta \left(\frac{qQ}{4\pi\varepsilon_0}\right)}_{\vec{K}}\left[(r^2 - 2ar\cos\theta + a^2)^{-3/2} - (r^2 + 2ar\cos\theta + a^2)^{-3/2}\right]$$

$$= \cancel{\vec{K}\,[\;1\;(} \quad \frac{\vec{K}}{r^3}\left[\left(1 - 2\frac{a}{r}\cos\theta\right)^{-3/2} - \left(1 + 2\frac{a}{r}\cos\theta\right)^{-3/2}\right]$$

$\because r \gg a \therefore \left(\frac{a}{r}\right)^2$ can be neglected

$$= \frac{\vec{K}}{r^3}\left[1 + 3\frac{a}{r}\cos\theta - 1 + 3\frac{a}{r}\cos\theta\right].$$

$$= \frac{3\vec{K}}{r^3}\left(\frac{1}{r}\right)(2\cos\theta)(a)$$

$$= \frac{\vec{r} \times \hat{z}}{|\vec{r} \times \hat{z}|} a\cancel{r}\sin\theta\left(\frac{qQ}{4\pi\varepsilon_0}\right)\left(\frac{3}{r^3}\right)\left(\frac{1}{\cancel{r}}\right)(2\cos\theta)(a)$$

$$\frac{\vec{r} \times \vec{a}}{|\vec{r} \times \vec{a}|} = \frac{\vec{r} \times \hat{z}}{|\vec{r} \times \hat{z}|}$$

$$= \frac{3Qqa^2}{4\pi\varepsilon_0 r^3}(2\sin\theta\cos\theta)\,\frac{\vec{r}\times\hat{z}}{|\vec{r}\times\hat{z}|}$$

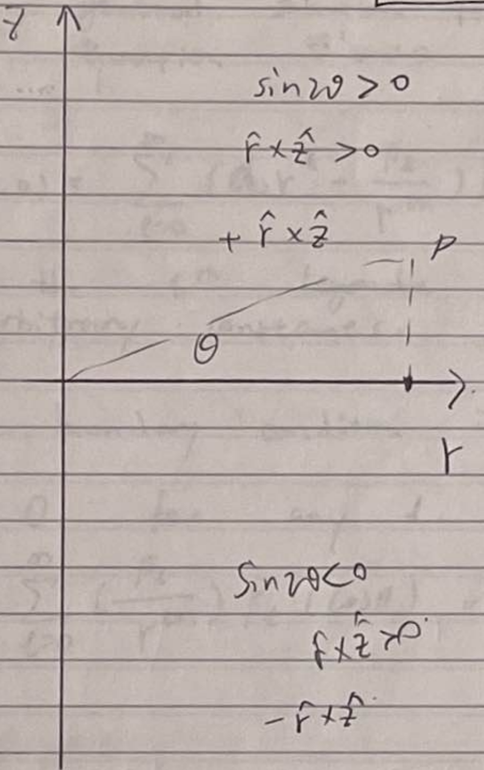
$$= \frac{3Qq\,a^2\sin 2\theta}{4\pi\varepsilon_0 r^3}\,\frac{\vec{r}\times\hat{z}}{|\vec{r}\times\hat{z}|} = \frac{3Qqa^2\sin 2\theta}{4\pi\varepsilon_0 r^3}(\hat{r}\times\hat{z})$$

$\therefore\ \Gamma = \dfrac{3Qqa^2\sin 2\theta}{4\pi\varepsilon_0 r^3}$, direction along $\dfrac{\sin 2\theta}{|\sin 2\theta|}(\hat{r}\times\hat{z})$

d)

$$U = \cancel{q}\, QV = \frac{2qQa^2}{4\pi\varepsilon_0 r^3}\left[\frac{1}{2}\left(3\cos^2\theta - 1\right)\right] ✓$$

Direction of rotation is $\boxed{\frac{\sin 2\theta}{|\sin 2\theta|}\hat{r}\times\hat{z}}$ $\cancel{\frac{\sin 2\theta \sin\theta}{|\sin 2\theta \sin\theta|}}$

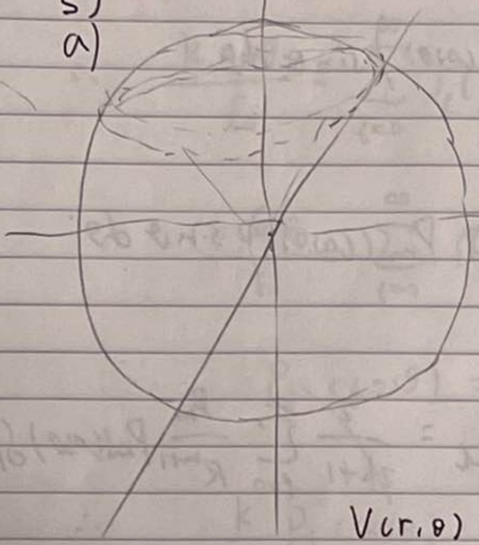


∴ when $0<\theta<\pi/2$, counterclockwise rotation

when $\frac{\pi}{2}<\theta<\pi$, clockwise rotation

when $\theta = 0,\ \pi/2,$ or π, no rotation.

5)
a)



By symmetry $V(r, \theta, \phi) = V(r, \theta)$

V is independent of ϕ

By separation of variables the general solution to the Laplace's equation $\vec{\nabla}^2 V = 0$ is

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \frac{\beta_l}{r^{l+1}}\right) P_l(\cos\theta)$$

where P_l is the l^{th} Legendre polynomial and α_l, β_l are arbitrary constants.

when $r > R$:

The boundary condition is $V(\infty, \theta) = 0$

$\therefore \alpha_l = 0$ for any l.

$$V_o(r, \theta) = \sum_{l=0}^{\infty} \left(\frac{\beta_l}{r^{l+1}}\right) P_l(\cos\theta)$$ ✓

When $r < R$:

The boundary condition is $V(0, \theta)$ is finite

$\therefore \beta_l = 0$ for any l

$$\therefore V_i(r, \theta) = \sum_{l=0}^{\infty} (\alpha_l r^l) P_l(\cos\theta)$$ ✓

Boundary Condition: at $r = R$, the function $V(r, \theta)$ is continuous, i.e. $V_o(R, \theta) = V_i(R, \theta)$

$$\therefore \sum_{l=0}^{\infty} \alpha_l R^l P_l(\cos\theta) = \sum_{m=0}^{\infty} \frac{\beta_m}{R^{m+1}} P_m(\cos\theta)$$

Using the orthogonality of Legendre Polynomials

$$\int_0^{\pi} P_k(\cos\theta) P_l(\cos\theta) \sin\theta \, d\theta = \frac{2\delta_{kl}}{2k+1}$$

$$\therefore \int_0^{\pi} P_k(\cos\theta)\left(\sum_{l=0}^{\infty} \alpha_l R^l P_l(\cos\theta)\right)\sin\theta\, d\theta$$

$$= \int_0^{\pi} P_k(\cos\theta)\left(\sum_{m=0}^{\infty} \frac{\beta_m}{R^{m+1}} P_m(\cos\theta)\right)\sin\theta\, d\theta$$

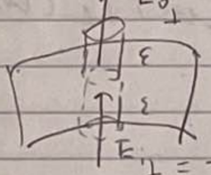
$$\Rightarrow \cancel{\frac{2}{2k+1}} \sum_{l=0}^{\infty} \alpha_l R^l P_l(\cos\theta)\,\delta_{kl} = \cancel{\frac{2}{2k+1}} \sum_{l=0}^{\infty} \frac{\beta_m}{R^{m+1}} P_m(\cos\theta)\,\delta_{km}$$

$$\Rightarrow \alpha_k R^k \cancel{P_k(\cos\theta)} = \frac{\beta_k}{R^{k+1}} \cancel{P_k(\cos\theta)}$$

$$\Rightarrow \underline{\beta_k = \alpha_k R^{2k+1}} \checkmark \text{ for any } k$$

Boundary condition:

$\vec{E}_{0\perp} = -\frac{\partial V_0}{\partial r}$



$\vec{E}_{i\perp} = -\frac{\partial V_i}{\partial r}$

Gauss's law:

$$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

$$\therefore (\vec{E}_{0\perp} - \vec{E}_{i\perp})S = \frac{\sigma S}{\varepsilon_0}$$

$$\therefore \vec{E}_{0\perp} - \vec{E}_{i\perp} = \frac{\sigma}{\varepsilon_0}\hat{r} \Rightarrow -\frac{\partial V_0}{\partial r} + \frac{\partial V_i}{\partial r}\bigg|_{r=R} = \frac{\sigma}{\varepsilon_0} = \frac{k\cos\theta}{\varepsilon_0} \checkmark$$

$$\therefore \frac{k\cos\theta}{\varepsilon_0} = \frac{\partial V_i}{\partial r} - \frac{\partial V_0}{\partial r}\bigg|_{r=R}$$

$$= \frac{\partial}{\partial r}\left(\sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos\theta)\right)\bigg|_{r=R} - \frac{\partial}{\partial r}\left(\sum_{l=0}^{\infty}\left(\frac{\beta_l}{r^{l+1}}\right)P_l(\cos\theta)\right)\bigg|_{r=R}$$

$$= \sum_{l=0}^{\infty}\left[l\alpha_l r^{l-1} P_l(\cos\theta) + (l+1)\frac{\beta_l}{r^{l+2}} P_l(\cos\theta)\right]_{r=R}$$

$$\cancel{= \sum_{l=0}^{\infty}\left[l r^{l-1} + (l+1)\frac{1}{r^{l+2}}\right]}$$

$$= \sum_{l=0}^{\infty} P_l(\cos\theta)\left[l\alpha_l r^{l-1} + (l+1)\frac{\alpha_l R^{2l+1}}{r^{l+2}}\right]_{r=R}.$$

$$\therefore \quad \frac{k\cos\theta}{\varepsilon_0} = \sum_{l=0}^{\infty} P_l(\cos\theta)\,\alpha_l\,[\,lR^{l-1} + (l+1)R^{l-1}]$$

$$\therefore \quad \frac{k\cos\theta}{\varepsilon_0} = \sum_{l=0}^{\infty} \alpha_l\, P_l(\cos\theta)\,[\,(2l+1)R^{l-1}]$$

$$\because \quad P_1(\cos\theta) = \cos\theta$$

$$\therefore \quad \frac{k}{\varepsilon_0} P_1(\cos\theta) = \sum_{l=0}^{\infty} \alpha_l[(2l+1)R^{l-1}]\,P_l(\cos\theta)$$

$$\frac{k}{\varepsilon_0}\int_0^{\pi} P_m(\cos\theta)\,P_1(\cos\theta)\sin\theta\,d\theta = \int_0^{\pi}\left(\sum_{l=0}^{\infty} \alpha_l(2l+1)R^{l-1} P_l(\cos\theta)\right)P_m(\cos\theta)\sin\theta\,d\theta$$

$$\Rightarrow \quad \frac{k}{\varepsilon_0}\frac{2}{2(1)+1}\delta_{1m} = \frac{2}{2m+1}(\alpha_m(2m+1)R^{m-1})$$

$\therefore$ ~~$\alpha_m = 0$~~ $\alpha_m \neq 0$ iff $m = 1$, in which case

~~$\frac{k}{\varepsilon_0} = \alpha_1(3)(R^0) \Rightarrow \alpha_1 = \frac{k}{3\varepsilon_0 R^2} \Rightarrow \alpha_1 = \frac{k}{3\varepsilon_0}$~~

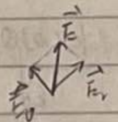
$$\therefore \quad \frac{k}{\varepsilon_0} = \alpha_1(3)(R^0) = 3\alpha_1 \quad \Rightarrow \quad \alpha_1 = \frac{k}{3\varepsilon_0} \quad \Rightarrow \beta_1 = \frac{kR^3}{3\varepsilon_0}$$

$$\therefore \quad V_i(r,\theta) = \frac{kr\cos\theta}{3\varepsilon_0}, \quad V_o(r,\theta) = \frac{kR^3\cos\theta}{3\varepsilon_0 r^2}$$

$$\therefore \quad V(r,\theta) = \begin{cases} \dfrac{kr\cos\theta}{3\varepsilon_0} & r \le R \;\checkmark \\[2ex] \dfrac{kR^3\cos\theta}{3\varepsilon_0 r^2} & r > R \;\checkmark \end{cases}$$

b) $\vec{E} = -\vec{\nabla} V = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}$

inside sphere : $V(r, \theta, \phi) = \frac{k}{3\varepsilon_0} r\cos\theta$



$$E_r = -\frac{\partial V}{\partial r} = -\frac{k\cos\theta}{3\varepsilon_0}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{k\sin\theta}{3\varepsilon_0}$$

$\therefore$ in fact $\vec{E} = -\frac{k}{3\varepsilon_0}\hat{z}$ = constant

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

inside sphere. Yes

Outside sphere : $V(r, \theta, \phi) = \frac{kR^3}{3\varepsilon_0}\frac{\cos\theta}{r^2}$

$$E_r = -\frac{\partial V}{\partial r} = \frac{2kR^3}{3\varepsilon_0}\frac{\cos\theta}{r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{kR^3}{3\varepsilon_0}\frac{\sin\theta}{r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0$$

c) The dipole moment of a charge distribution

is $\vec{p} = \iiint_V \vec{r}'\rho(\vec{r}')\, d\tau' = \iiint_V \vec{r}'\rho(\vec{r}')\, \underset{\uparrow \atop \varepsilon \to 0}{dr}\, d\theta\, d\phi\, \underset{\uparrow \atop = R^2 \text{ as } \varepsilon \to 0}{r'^2 \sin\theta}$

$$= \iiint_V \vec{r}'\, \underbrace{\rho(\vec{r}')\varepsilon}_{\sigma(\vec{r}')}\, \underbrace{R^2 \sin\theta\, d\theta\, d\phi}_{dS'} = \cancel{\oint\!\!\oint_\Sigma \vec{r}'\sigma\, dA}\ \cancel{\oint\!\!\oint_\Sigma \vec{r}'\sigma\, ds}$$

$$= \oint\!\!\oint_\Sigma \vec{r}'\, \sigma(\vec{r}')\, dS'$$

By symmetry $\vec{p}$ points to the $+\hat{z}$ direction

$$\therefore \vec{p} = \hat{z} \oint\!\!\oint_\Sigma \vec{r}' \cdot \hat{z}\, \sigma(\vec{r}')\, dS' = \hat{z} \oint\!\!\oint_\Sigma R\cos\theta\, k\cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$$

$$= \hat{z} \int_0^{2\pi}\!\!\int_0^{\pi} R^3 k \cos^2\theta \sin\theta\, d\theta\, d\phi$$

$$= \underbrace{2\pi R^3 k \hat{z} \int_{-1}^{1}}_{} \underbrace{u^2\, du}_{\frac{u^3}{3}\Big|_{-1}^{1}} = \boxed{\frac{4\pi}{3} R^3 k \hat{z}}$$

$u = \cos\theta$

~~$du = \sin$~~

$-du = \sin\theta\, d\theta$

$u(\theta = 0) = 1$

$u(\theta = \pi) = -1$

$\therefore$ dipole potential

$$V_1 = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}, \quad \vec{E}_1 = -\vec{\nabla} V_1 = \frac{1}{4\pi\varepsilon_0 r^3}[3\vec{p}\cdot\hat{r}\hat{r} - \vec{p}]$$

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\frac{4\pi}{3} R^3 k\cos\theta\right)\hat{r} - \frac{4\pi}{3} R^3 k\hat{z}\right]$$

$\leftarrow \hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$

$$= \frac{R^3}{\varepsilon_0 r^3}\left(k\cos\theta\hat{r} - \frac{1}{3}k\cos\theta\hat{r} + \frac{1}{3}k\sin\theta\hat{\theta}\right)$$

$$= \frac{2kR^3\cos\theta\,\hat{r}}{3\varepsilon_0 r^3} + \frac{kR^3\sin\theta\,\hat{\theta}}{3\varepsilon_0 r^3}$$, which is

consistent with the $\vec{E}$ outside sphere calculated in (b)

As a result, this charge distribution is a pure dipole, all higher multipoles are 0!

The polarization $\vec{k} = \frac{\vec{p}}{V} = \frac{\cancel{\frac{4\pi}{3}R^3} k\hat{z}}{\cancel{\frac{4\pi}{3}R^3}} = k\hat{z}$

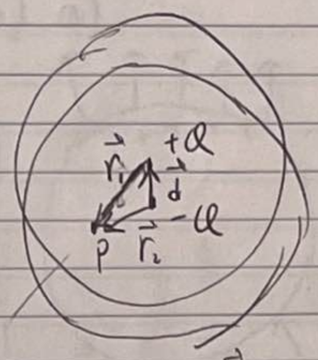
Field inside sphere:

$$\cancel{V_i} \quad E_i = -\frac{k\cos\theta}{3\epsilon_0}\hat{r} + \frac{k\sin\theta}{3\epsilon_0}\hat{\theta}$$

$$= -\frac{k}{3\epsilon_0}\underbrace{(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}_{\hat{z}} = -\frac{k\hat{z}}{3\epsilon_0}$$

$$= -\frac{\vec{k}}{3\epsilon_0}$$

d)



$\vec{r_2} = \vec{r_1} + \vec{d}$

Gauss's Law ~~~~ Outside a sphere

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

~~Outside sphere~~

$$E_0 = \frac{Q}{4\pi\varepsilon_0 r^2}$$ which is the same as a point charge

$\therefore$ Outside sphere the system can be modelled by a ~~phys~~ physical dipole of charge Q and distance vector $\vec{d}$

$$\therefore \quad V_1 = \frac{(Qd)\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{(\frac{4\pi}{3}R^3 k)\cos\theta}{4\pi\varepsilon_0 r^2}$$

result of previous question

$$\therefore \quad \vec{p} = Q\vec{d} = \frac{4\pi}{3}R^3 k\hat{z} \checkmark$$ is the required relation.

Gauss's Law inside sphere!

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q}{\varepsilon_0} \quad \therefore \quad E(4\pi r^2) = \frac{Qr^3}{R^3\varepsilon_0}$$

$$\therefore \quad E = \frac{Qr}{4\pi\varepsilon_0 R^3} \Rightarrow E(\vec{r}) = \frac{Q\vec{r}}{4\pi\varepsilon_0 R^3}$$

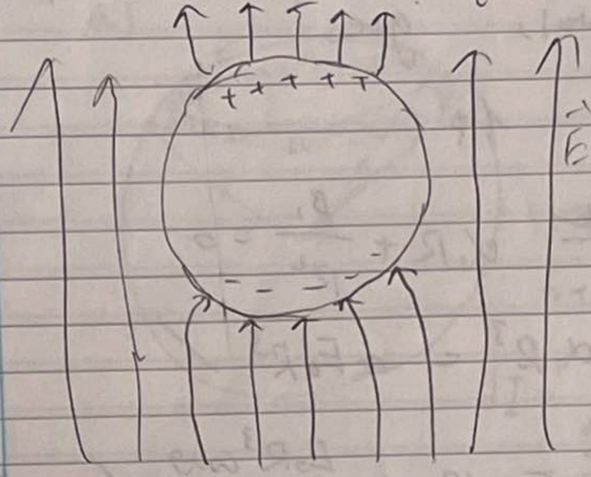
Superposition of 2 spheres.

$$\Rightarrow \quad \vec{E_i} = \frac{Q\vec{r_1}}{4\pi\varepsilon_0 R^3} + \frac{(-Q)\vec{r_2}}{4\pi\varepsilon_0 R^3} = \frac{Q}{4\pi\varepsilon_0 R^3}\underbrace{(\vec{r_1} - \vec{r_2})}_{-\vec{d}}$$

$$= \frac{-Q\vec{d}}{4\pi\varepsilon_0 R^3} = \frac{-\frac{4\pi}{3}R^3 k\hat{z}}{4\pi\varepsilon_0 R^3} = -\frac{k\hat{z}}{3\varepsilon_0} \checkmark$$

same as previous question.

6) a)



The constant ~~electricalk~~ electric field $\vec{E}$ pushes positive charges to the upper half of the sphere and negative charges to the lower half of the sphere

The ~~redistru~~ redistribution of charges ~~causes~~ produces ~~its own elect~~ additional electric fields that superpose on $\vec{E}$ to give the overall new electric field.

b) $\vec{\nabla}^2 V = 0$ outside the sphere and has azimuthal symmetry

$$\therefore \quad V(r,\theta) = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \frac{\beta_l}{r^{l+1}}\right) P_l(\cos\theta)$$

Boundary Condition : $V(R, \theta) = 0$ ①

$$\Rightarrow \quad 0 = \sum_{l=0}^{\infty} \left(\alpha_l R^l + \frac{\beta_l}{R^{l+1}}\right) P_l(\cos\theta)$$

Boundary Condition : $V(\infty, \theta) = -E_0 z + \cancel{c}$ because the external field is uniform ($\vec{E} = E_0 \hat{z}$)

$$\therefore \quad -E_0 z = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \frac{\beta_l}{r^{l+1}}\right) P_l(\cos(\theta)) \Big|_{r\to\infty}$$

$$\Rightarrow \quad -E_0 r\cos\theta = \sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos\theta)$$

$$\Rightarrow \quad -E_0 r P_1(\cos\theta) = \sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos\theta)$$

By the same argument as 5 (a)

~~α~~ $\alpha_l = 0$ iff $l \neq 1$, and

$-E_0 r P_1(\cos\theta) = \alpha_1 r P_1(\cos\theta) \Rightarrow \alpha_1 = -E_0$ ✓

~~Also $\beta_l = 0$ iff $l \neq 1$~~ and

From ①, the linear independence of Legendre polynomials gives

$$\beta_l = 0 \quad \text{for} \quad l \neq 1$$

and $\cancel{\alpha_l R + \frac{\beta_l}{}}$ $\quad \alpha_1 R + \frac{\beta_1}{R^2} = 0$

$$\therefore \beta_1 = -\alpha_1 R^3 = E_0 R^3 \checkmark$$

$$\therefore V(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^3 \cos\theta}{r^2} \checkmark$$

(c) The radial component of overall electric field at $r=R$ is

$$\vec{E}_\perp = \vec{E}'_r = -\frac{\partial V}{\partial r}\Big|_{r=R} = E_0\cos\theta + \frac{2E_0 R^3 \cos\theta}{r^3}\Big|_{r=R}$$

$$= 3E_0\cos\theta$$

On the surface of a conductor

$$E_\perp = \frac{\sigma}{\varepsilon_0} \quad \therefore \sigma = \varepsilon_0 E_\perp \quad \therefore \boxed{\sigma = 3E_0\varepsilon_0\cos\theta} \checkmark$$

(d) Compare to 5 (b), $\sigma = k\cos\theta = 3E_0\varepsilon_0\cos\theta$

$\therefore k = 3E_0\varepsilon_0$

electric field produced by σ is $\underbrace{\vec{E}_\sigma}_{}$ (inside sphere) $= -\frac{k\hat{z}}{3\varepsilon_0} = -\frac{3E_0\varepsilon_0\hat{z}}{3\varepsilon_0} = \underline{-E_0\hat{z}}$

Total $\vec{E}$-field inside sphere is

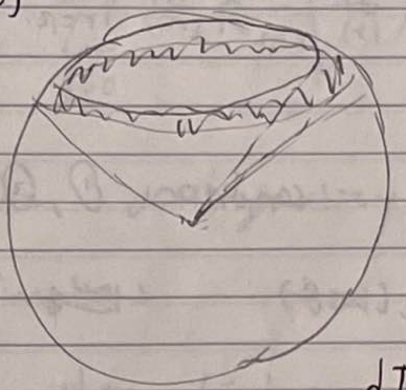
$$\vec{E}' = \vec{E} + \vec{E}_\sigma = E_0\hat{z} - E_0\hat{z} = \underline{0} \quad \text{yes}$$

This is ~~electric~~ electrostatic shielding (Faraday's Cage)

No field line can go from a conductor to itself, so the electric field must be $\underline{0}$ inside a hollow conductor if there is no charge in the cavity.

7)

a)



Area of ring

$$dA = \int_0^{2\pi} d\phi \; R^2 \sin\theta \, d\theta = 2\pi R^2 \sin\theta \, d\theta$$

total charge $dQ = \sigma \, dA = 2\pi\sigma R^2 \sin\theta \, d\theta$

$$dI = \frac{dQ}{\Delta t} = \frac{2\pi\sigma R^2 \sin\theta \, d\theta}{2\pi/w} = w\sigma R^2 \sin\theta \, d\theta$$

$$\therefore \underline{K(\vec{r})} = \frac{dI}{dl'} = \frac{w\sigma R^2 \sin\theta \, d\theta}{R \, d\theta} = \boxed{w\sigma R \sin\theta}$$

b) $\because \vec{\nabla} \times \vec{B} = \mu_0 \vec{j}$ for static current.

and ~~when~~ in free space where current density $\vec{j} = \underline{0}$ (Inside and outside the sphere, not on the sphere),

we have $\vec{\nabla} \times \vec{B} = \underline{0}$.

$\because \vec{\nabla} \times \vec{\nabla} \psi = \underline{0}$ is an vector identity

$\therefore$ we can ~~writt~~ write $\vec{B}$ as $\vec{B} = \vec{\nabla} \psi$

$\because \vec{\nabla} \cdot \vec{B} = 0 \quad \therefore \vec{\nabla} \cdot (\vec{\nabla} \psi) = 0 \quad \therefore \vec{\nabla}^2 \psi = 0$

$\therefore \psi$ satisfies Laplace's Equation.

Boundary conditions of ψ:

1. $\psi(r, \theta, \phi) = \psi(r, \theta)$ (Azimuthal symmetry)
2. $\psi(\infty, \theta) = 0$
3. $\psi(0, \theta)$ is finite.
4. $\vec{B}_o = \vec{B}_{out}$ $\quad \vec{B}_i = \vec{B}_{in}$. then

$$B_o^{\parallel} - B_i^{\parallel} = \mu_0 K = \mu_0 w\sigma R \sin\theta$$

$$\therefore \left.\frac{1}{r}\frac{\partial \psi_o}{\partial \theta}\right|_{r=R} - \left.\frac{1}{r}\frac{\partial \psi_i}{\partial \theta}\right|_{r=R} = \mu_0 w \sigma R \sin\theta$$

<u>yes</u>

(5) ~~Bo~~ $B_o^{\perp} = B_i^{\perp} \Rightarrow \left.\frac{\partial \psi_o}{\partial r}\right|_{r=R} = \left.\frac{\partial \psi_i}{\partial r}\right|_{r=R.}$ yes

c) Based on Boundary conditions (1), (2) and (3)

$$\psi_o(r,\theta) = \sum_{l=0}^{\infty} \left(\frac{\beta_l}{r^{l+1}}\right) P_l(\cos\theta) \qquad r > R.$$

$$\psi_i(r,\theta) = \sum_{l=0}^{\infty} (\alpha_l r^l) P_l(\cos\theta) \qquad r < R.$$

ψ cannot be defined and thus is discontinuous at $r=R$. since $\vec{\nabla} \times \vec{B} \neq 0$

Excellent!

$$(5) \Rightarrow \sum_{l=0}^{\infty} l\alpha_l R^{l-1} P_l(\cos\theta) = \sum_{l=0}^{\infty} -(l+1)\frac{\beta_l}{R^{l+2}} P_l(\cos\theta)$$

$$\Rightarrow l\alpha_l R^{l-1} = -(l+1)\frac{\beta_l}{R^{l+2}}.$$

$$\cancel{\therefore \beta_l = -\frac{l}{l+1} R^{2l+2} \alpha_l}$$

$$\therefore \beta_l = -\frac{l}{l+1} R^{2l+1} \alpha_l.$$

$$(4) \Rightarrow \mu_0 \omega \sigma R \sin\theta = \frac{1}{R}\left(\frac{\partial \psi_o}{\partial \theta} - \frac{\partial \psi_i}{\partial \theta}\right)$$

$$\Rightarrow \mu_0 \omega \sigma R^2 \sin\theta = \frac{\partial}{\partial \theta}(\psi_o - \psi_i)$$

$$\Rightarrow \frac{\partial}{\partial \theta}(\mu_0 \omega \sigma R^2 \cos\theta) = \frac{\partial}{\partial \theta}(\psi_i - \psi_o)$$

choose the integration constant to be 0 and integrate we get

$$\psi_i - \psi_o = \mu_0 \omega \sigma R^2 \cos\theta$$ yes

$$\therefore \sum_{l=0}^{\infty}\left(\alpha_l R^l - \frac{\beta_l}{R^{l+1}}\right)P_l(\cos\theta) = N_0 \omega \sigma R^2 P_1(\cos\theta)$$

The orthogonality of Legendre polynomial implies that $\alpha_l = 0$, $\beta_l = 0$ for $l \neq 1$

when $l = 1$,

$$\cancel{\alpha_1 R = \frac{\beta_1}{R^2} \Rightarrow \beta_1 =}$$

$$\cancel{\alpha_1 R_1 = \frac{\beta_1}{R^2}}$$

$$\alpha_1 R - \frac{\beta_1}{R^2} = N_0 \omega \sigma R^2 \qquad \because \beta_1 = -\frac{1}{2}R^3\alpha_1$$

$$\therefore \alpha_1 R + \cancel{\tfrac{1}{2}\beta_1}\ \frac{1}{2}\alpha_1 \frac{R^3}{R^2} R = N_0 \omega \sigma R^2$$

$$\therefore \cancel{\alpha}\ \frac{3}{2}\alpha_1 = N_0 \omega \sigma R$$

$$\therefore \alpha_1 = \frac{2}{3} N_0 \omega \sigma R$$

$$\beta_1 = -\frac{1}{3} N_0 \omega \sigma R^4$$

$$\therefore \psi = \begin{cases} \frac{2}{3} N_0 \omega \sigma R r \cos\theta & r < R \\ -\frac{1}{3} N_0 \omega \sigma \frac{R^4}{r^2}\cos\theta & r > R \end{cases}$$

yes

This is the solution because given the full boundary conditions Laplace's Equation only has one solution (by uniqueness theorem) ✓

d) Inside sphere:

$$B_r = \frac{\partial \psi}{\partial r} = \frac{2}{3}\mu_0 \omega \sigma R \cos\theta$$

$$B_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\frac{2}{3}\mu_0 \omega \sigma \sin\theta$$

$$B_\phi = 0$$

$$\vec{B}_i = \frac{2}{3}\mu_0 \omega \sigma R \,(\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{\hat{z}}) = \underline{\frac{2}{3}\mu_0 \omega \sigma R\,\hat{z}}$$

Outside sphere:

$$B_r = \frac{\partial \psi}{\partial r} = \frac{2}{3}\mu_0 \omega \sigma \frac{R^4}{r^3}\cos\theta$$

$$B_\theta = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{3}\mu_0 \omega \sigma \frac{R^4}{r^3}\sin\theta$$

$$B_\phi = 0$$

e)

$$d\vec{m} = (dI)\,\vec{a}(\theta) = (\omega\sigma R^2 \sin\theta\, d\theta)(\pi R^2 \sin^2\theta)\,\hat{z}$$

$$\vec{m} = \int d\vec{m} = \int_0^{\pi} \hat{z}\pi\omega\sigma R^4 \sin^3\theta\, d\theta$$

$$= \pi\sigma R^4 \vec{\omega} \int_0^{\pi} \sin^3\theta\, d\theta$$

$$\int_0^{\pi} \sin^3\theta\, d\theta = \int_{\theta=0}^{\pi=\theta} (1-\cos^2\theta)(-d\cos\theta)$$

$$= \int_{\theta=0}^{\theta=\pi} (\cos^2\theta - 1)\, d\cos\theta$$

$$= \int_{-1}^{1} (1-\cos^2\theta)\, d(\cos\theta) \quad \text{let } u = \cos\theta$$

$$= \cancel{\left[\cos\theta - \frac{\cos^3\theta}{3}\right]_{-1}^{1}}$$

$$= \left[u - \frac{u^3}{3}\right]_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$$

$$\therefore \quad \vec{m} = \frac{4\pi}{3}\sigma R^4 \vec{\omega} = \underline{\frac{4\pi}{3}\sigma R^4 \omega\, \hat{z}}$$

f) B- field due to the dipole is

$$\vec{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}\right]$$

$$= \frac{\mu_0}{4\pi r^3}\left(3\cdot\frac{4\pi}{3}\sigma R^4\omega\cos\theta\,\hat{r} - \frac{4\pi}{3}\sigma R^4\omega(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\right)$$

$$= \underline{\frac{2}{3}\mu_0\omega\sigma\frac{R^4}{r^3}\cos\theta\,\hat{r} + \frac{1}{3}\mu_0\omega\sigma\frac{R^4}{r^3}\sin\theta\,\hat{\theta}}$$

which is exactly the same as the total $\vec{B}$ field produced by the sphere outside the sphere

$\therefore$ This is a pure magnetic dipole;
all higher multipoles are 0! ✓

magnetisation $\vec{k}_m = \frac{\vec{m}}{V} = \frac{\cancel{\frac{4\pi}{3}R^3}(\sigma\omega R\hat{z})}{\cancel{\frac{4\pi}{3}R^3}} = \sigma\omega R\hat{z}$

magnetic field inside the sphere

$\vec{B}_i = \frac{2}{3}\mu_0(\omega\sigma R\hat{z}) = \frac{2}{3}\mu_0\vec{k}_m$ ✓

8) a) Laplace's equation in cylindrical coordinates with no z dependence is given by

$$\vec{\nabla}^2 V = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

~~let~~ To separate variables, let $V(r,\theta) = F(r)\,G(\theta)$

then

$$\cancel{\frac{G}{r}\frac{\partial}{\partial r}} \quad \frac{G}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \frac{F}{r^2}\frac{d^2G}{d\theta^2} = 0.$$

$$\therefore \underbrace{\frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{dr}\right)}_{\text{only depend on } r} + \underbrace{\frac{1}{G}\frac{d^2G}{d\theta^2}}_{\text{only depend on } \theta} = 0$$

they are constants.

Cases:

① $$\frac{1}{G}\frac{d^2G}{d\theta^2} = 0 \Rightarrow \frac{d^2G}{d\theta^2} = 0 \Rightarrow G = \underbrace{A\theta + B}_{G \text{ must be periodical in } 2\pi \ \therefore A=0} = B$$

$$\frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = 0 \Rightarrow r\frac{dF}{dr} = C$$

$$\Rightarrow dF = \frac{C}{r}\,dr \Rightarrow F = C\ln r + D$$

② $$\frac{1}{G}\frac{d^2G}{d\theta^2} = -k^2 \Rightarrow \frac{d^2G}{d\theta^2} + k^2 G = 0$$

$$\Rightarrow G(\theta) = \cancel{A\sin(k\theta) + B\cos} \; A\cos(k\theta) + B\sin(k\theta)$$

$$\frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = k^2$$

$$\therefore \frac{r}{F}\left(r\frac{d^2F}{dr^2} + \frac{dF}{dr}\right) = k^2.$$

$$\therefore r^2 \frac{d^2F}{dr^2} + r\frac{dF}{dr} - k^2 F = 0$$

let $F(r) = \cancel{\sum_{l=0}^{\infty} \alpha_l r^l} \sum_{l=-\infty}^{\infty} \alpha_l r^l$ ✓

substitute this into the equation we have

$$\sum_{n=1}^{\infty} \left(l(l-1)\alpha_l + l\alpha_l - k^2\alpha_l \right) r^l = 0$$

$\therefore \because$ ~~poly~~ r^l's are linearly independent

$$\therefore \quad \alpha_l (l^2 - k^2) = 0$$

$\therefore \quad l = k$ or $-k$. $\quad \because l$ is integer

$\therefore$ k must also be integers

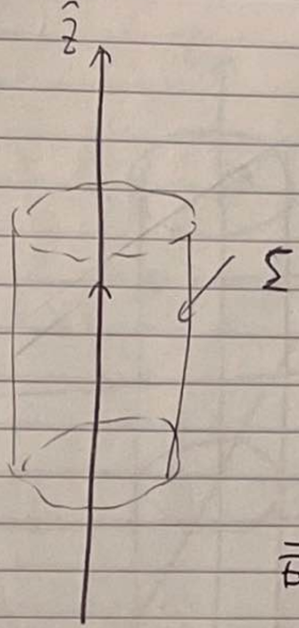
$\therefore$ for $k = l \qquad F(r) = \alpha_l r^l$ is the solution

$\therefore$ The general solution is

$$\cancel{V(r,\theta) = C\ln r + D + \sum_{l=1}^{\infty} r^l (\alpha_l \cos(l\theta) + \beta_l \sin(l\theta))}$$

$$\cancel{+ \sum_{l=-\infty}^{-1} r^l (\alpha_l \cos(l\theta)) + \beta}$$

$$V(r,\theta) = C\ln r + \sum_{l=-\infty}^{\infty} r^l (\alpha_l \cos(l\theta) + \beta_l \sin(l\theta)) \checkmark$$

b) 

Gauss's Law: $\oiint_{\Sigma} \vec{E}\cdot d\vec{S} = \frac{Q}{\varepsilon_0}$

$\therefore E(2\pi r \cancel{\ell}) = \frac{\lambda \ell}{\varepsilon_0}$

$\therefore$ $\cancel{E = \frac{\lambda}{2\pi\varepsilon_0 \ell}}$ $E = \frac{\lambda}{2\pi\varepsilon_0 r}$

$\vec{E} = \frac{\lambda \hat{r}}{2\pi\varepsilon_0 r}$

$\because \vec{E} = -\vec{\nabla}V \quad \therefore \quad \frac{\lambda}{2\pi\varepsilon_0 r} = -\underbrace{\frac{\partial V}{\partial r}}_{\text{no } \theta \text{ or } \phi \text{ dependence by symmetry}} = -\frac{dV}{dr}$

$\therefore \boxed{V = -\frac{\lambda}{2\pi\varepsilon_0}\ln r}$ ✓, which is of course included by the general solution.

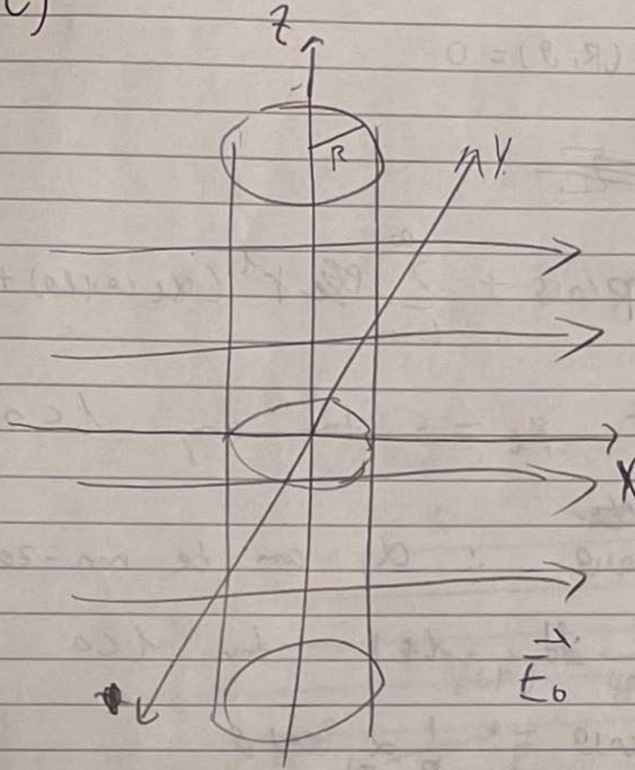
check:

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(-\frac{\lambda}{2\pi\varepsilon_0}\ln r\right)\right) = \frac{\partial}{\partial r}\left(r\left(-\frac{\lambda}{2\pi\varepsilon_0}\right)\frac{1}{r}\right)$$

$$= \frac{\partial}{\partial r}\left(-\frac{\lambda}{2\pi\varepsilon_0}\right) = 0$$

$\therefore$ It is a solution to $\vec{\nabla}^2 V = 0$

c)



General solution

$$V(r,\theta) = C\ln r + \sum_{l=-\infty}^{\infty} r^l (\alpha_l \cos l\theta + \beta_l \sin l\theta)$$

Boundary conditions:

1 $\quad V(\infty, \theta) = -E_0 x = -E_0 r\cos\theta$ ✓

$$\therefore \ -E_0 r\cos\theta = C\ln r + \sum_{l=-\infty}^{\infty} r^l (\alpha_l \cos(l\theta) + \beta_l \sin(l\theta))$$

~~By linear~~ ~~By orthogonality of sin and cos functions~~

$\therefore$ <1> $\beta_l = 0$ for any $l \geq 0$

<2> $l \geq 0$ and $\alpha_l = 0$ iff $l \neq 1$

<3> $C = 0$

$-E_0 r\cos\theta = \alpha_1 r\cos\theta \quad \therefore \alpha_1 = -E_0$ ✓

②

$\cancel{V}$ $V(R,\vartheta) = 0$

$\therefore$ ~~$0 = E$~~

$$0 = -E_0 R\cos\vartheta + \sum_{l=-\infty}^{\infty} r^l (\alpha_l \cos(l\theta) + \beta_l \sin(l\theta))$$

$\therefore$ ~~$E_0 r\cos\vartheta$~~ $\beta_l = 0$ for any $l<0$

$\because \cos(-\theta) = \cos\vartheta$ $\therefore$ α_1 can be non-zero

$\alpha_l = 0$ iff $l \neq 1$ for $l<0$

$$\therefore \quad E_0 R\cos\vartheta = \frac{1}{R}\alpha_{-1}\cos\theta$$

$$\therefore \quad \alpha_{-1} = E_0 R^2$$

$\therefore$ Overall solution is

$$V(r,\theta) = -E_0 r\cos\theta + \frac{E_0 R^2 \cos\vartheta}{r}$$

radial component of ~~the~~ electric field at $r=R$ is

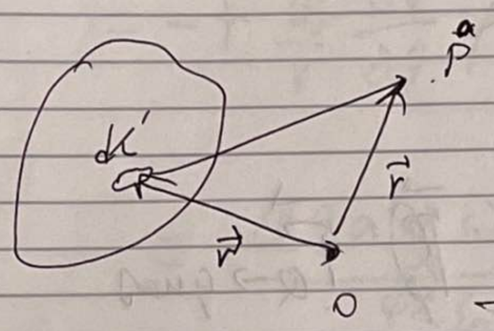
$$E_r\Big|_{r=R} = -\frac{\partial V}{\partial r}\Big|_{r=R} = E_0\cos\vartheta + E_0\cos\vartheta \frac{R^2}{R^2} = 2E_0\cos\vartheta$$

$\therefore$ surface charge density σ is

$$\sigma = \varepsilon_0 E_r = \boxed{2E_0\varepsilon_0\cos\vartheta}$$

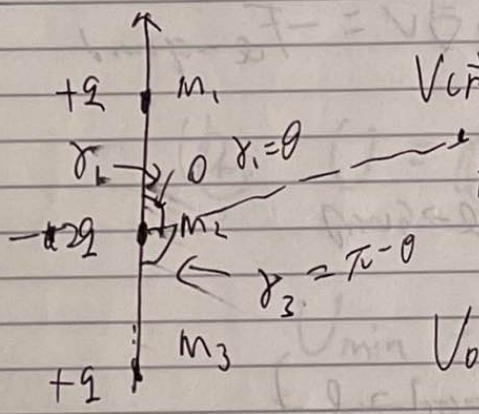
Problem 4

multipole expansion:



$$\cancel{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\iiint_V (r')^l \rho(\vec{r}')P_l(\cos\gamma)\,d\tau'$$



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\sum_{i=1,2,3}(Om_i)^l q_i P_l(\cos\gamma_i)$$

If $l=1$

$$V_0(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r}\underbrace{\sum_{i=1,2,3} q_i \overset{P_0}{(1)}}_{0} = 0$$

$l=1 \Rightarrow$

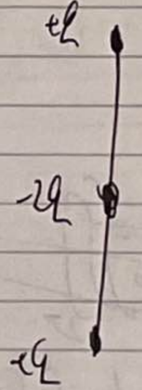
$$V_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^2}\underbrace{\left[Om_1(+q)\cos\theta + (+q)\cos(\pi-\theta)\right]}_{0}$$

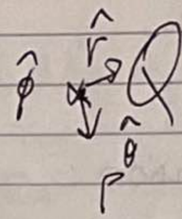
$$= 0$$

$l=2 \Rightarrow$

$$V_2(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r^3}\left[Om_1^2(+q)\frac{1}{2}3\cos^2\theta - 1 + 0 + Om_3^2(+q)\frac{1}{2}(3\cos^2\theta - 1)\right]$$

$$= \frac{qa^2}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$





we want $\vec{\tau}_{Q\to quad}$

$$\vec{F}_{quad\to Q} = -Q\vec{\nabla}V = -\vec{F}_{Q\to quad}$$

$$\vec{\tau}_{Q\to quad} = \vec{PO}\times\vec{F}_{Q\to quad}$$

$$= \vec{PO}\times\left(-\vec{F}_{quad\to Q}\right)$$

$$= \vec{OP}\times\vec{F}_{quad\to Q}$$

$$= \vec{\tau}_{quad\to Q}.$$

$$\vec{P} = r\hat{r} \times (-Q\vec{\nabla}V)$$

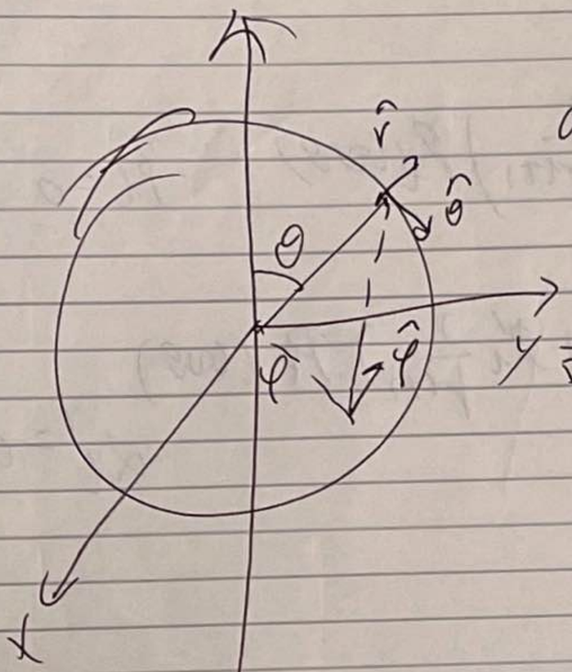
$$= -r\frac{Q}{r}\frac{\partial V}{\partial \theta}\hat{\varphi}$$

$$= \frac{Qqa^2 3\sin 2\theta}{4\pi\varepsilon_0 r^3}\hat{\varphi}$$

Lem. $U = QV$

$U_{min} = \cancel{}\ \theta = \pi/2$ stable position.

$U_{max} = \cancel{}\ 0, \pi$.

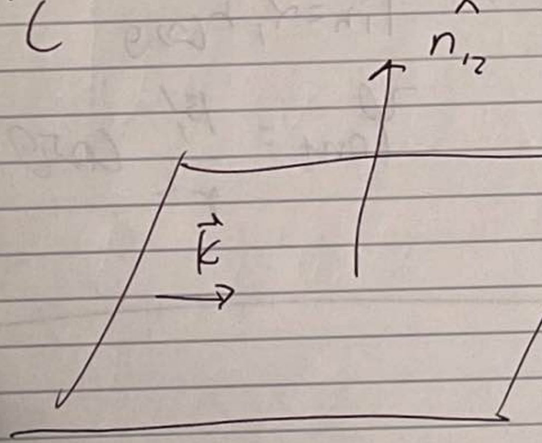


at $r = R$ $\quad \vec{\nabla} \times \vec{B} = 0$

$$\vec{B} = \vec{\nabla} \psi$$

$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla}^2 \psi = 0$$

B. C



$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$$

$$B_2^{\perp} = B_1^{\perp}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{B}_2^{\parallel} - \vec{B}_1^{\parallel} = \mu_0 \vec{K} \times \hat{n}_{12}$$

$$B_r^{out}\Big|_{r=R} = B_r^{in}\Big|_{r=R} \Rightarrow \frac{\partial \psi_{out}}{\partial r}\Big|_{r=R} = \frac{\partial \psi_{out}}{\partial r}\Big|_{r=R}$$

$$\vec{B}_{out}^{\parallel}\Big|_{r=R} - B_{in}^{\parallel}\Big|_{r=R} = \mu_0 K \underbrace{\hat{\varphi} \times \hat{r}}_{\theta} \cancel{= -\frac{1}{r}\frac{\partial \psi_{out}}{\partial \theta}}$$

$$\Rightarrow \overbrace{\frac{1}{r}\frac{\partial \psi_{out}}{\partial \theta}}^{B_\theta^{out}}\Big|_{r=R} - \overbrace{\frac{1}{r}\frac{\partial \psi_{in}}{\partial \theta}}^{B_\theta^{in}}\Big|_{r=R} = \mu_0 K$$

$$\psi_{in} = \sum_l \left(\alpha_l r^l + \beta_l \frac{1}{r^{l+1}}\right) P_l(\cos\theta) \qquad \beta_l = 0$$

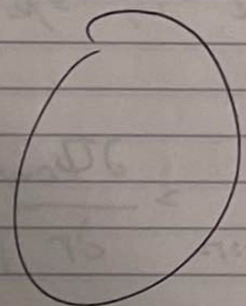
$$\psi_{out} = \sum_l \left(\alpha_l' r^l + \beta_l' \frac{1}{r^{l+1}}\right) P_l(\cos\theta)$$

$$\alpha_l' = 0$$

Only $l=1$ $\Rightarrow$ $\psi_{in} = \alpha_1 r \cos\theta$

$$\psi_{out} = \frac{\beta_1'}{r^2}\cos\theta$$

P.5



$$E_2^{\perp} - E_1^{\perp} = \frac{\sigma}{\epsilon_0} = \frac{k\cos\theta}{\epsilon_0}$$

$V(R^+) = V(R^-)$ or $E_2^{\parallel} = E_1^{\parallel}$