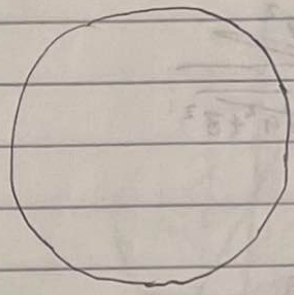
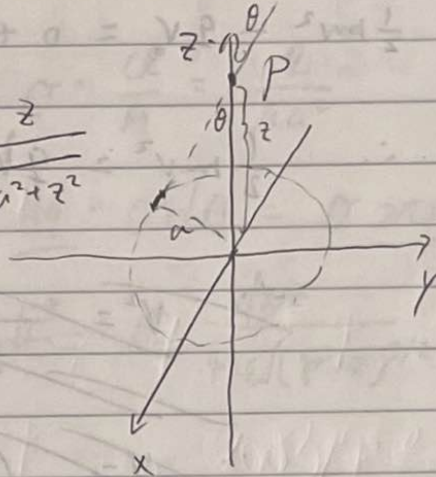
EM Problem Set 0 Ziyan Li

1)



$$\cos\theta = \frac{z}{\sqrt{a^2+z^2}}$$

$$r^2 = a^2 + z^2$$



$\vec{E} = \int d\vec{E}$, by symmetry components along x and y cancel, so

$$\vec{E} = \left(\int dE\right) \hat{e_z} \cos\theta = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{e_z}\cos\theta.$$

$$= \frac{\hat{e_z}}{4\pi\varepsilon_0}\int \frac{dq}{a^2+z^2}\frac{z}{\sqrt{a^2+z^2}}$$

$$= \boxed{\frac{qz\,\hat{e_z}}{4\pi\varepsilon_0(a^2+z^2)^{3/2}}}$$

(b) Potential at P

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r} = \boxed{\frac{q}{4\pi\varepsilon_0(a^2+z^2)^{1/2}}}$$

(c) at infinity $T + V = 0 + 0$

at $z=0$ $T + V = \frac{1}{2}mv^2 - qV_0$

$$\therefore \ \frac{1}{2}mv^2 - qV_0 = 0 + 0$$

$$\therefore \ \frac{1}{2}mv^2 = qV_0 = \cancel{\frac{q^2}{4\pi\varepsilon_0\sqrt{a^2+z^2}}}$$

$$\cancel{\therefore \ v^2 = \frac{q^2}{2\pi\varepsilon_0 m\sqrt{a^2+z^2}}}$$

$$\cancel{\therefore \ v = \left(\frac{q^2}{2\pi\varepsilon_0 m\sqrt{a^2+z^2}}\right)^{1/2}}$$

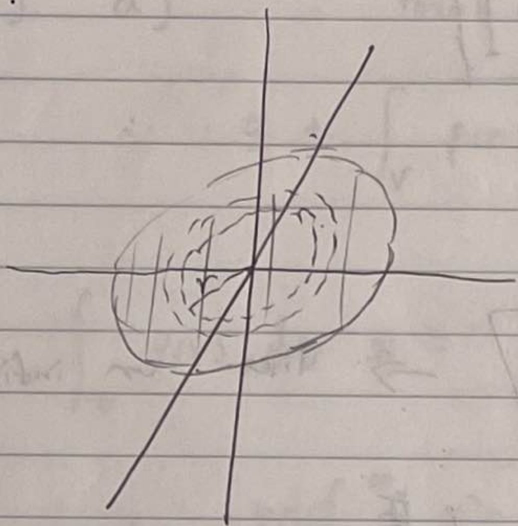
$$V_0 = V(z=0) = \frac{q^2}{4\pi\varepsilon_0 a}$$

$$\therefore \ \frac{1}{2}mv^2 = \frac{q^2}{4\pi\varepsilon_0 a}$$

$$\therefore \ v^2 = \frac{q^2}{2\pi\varepsilon_0 ma}$$

$$\therefore \ \boxed{v = \frac{q}{\sqrt{2\pi\varepsilon_0 ma}}}$$

2.



$$\sigma = \frac{Q}{A} = \frac{q}{\pi a^2}$$

$$dq = \sigma\, dA = \sigma\, 2\pi r\, dr$$

$$d\vec{E} = \hat{e_z} \frac{(dq)z}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}$$

$$\vec{E} = \int d\vec{E} = \hat{e_z} \int \frac{z\, dq}{4\pi\varepsilon_0 (r^2+z^2)^{3/2}}$$

$$= \cancel{\frac{\hat{e_z} z}{4\pi\varepsilon_0}} \frac{\hat{e_z} z}{4\pi\varepsilon_0} \int_{r=0}^{r=a} \frac{dq}{(r^2+z^2)^{3/2}}$$

$$= \frac{\hat{e_z} z}{4\pi\varepsilon_0} \int_0^a \frac{\sigma\, 2\pi r\, dr}{(r^2+z^2)^{3/2}}$$

$$= \frac{\hat{e_z} z\sigma}{2\varepsilon_0} \int_0^a \frac{r\, dr}{(r^2+z^2)^{3/2}}$$

$$= \frac{\hat{e_z} z\sigma}{2\varepsilon_0} \left[-\frac{1}{\sqrt{r^2+z^2}} \right]_0^a$$

$$= \frac{\hat{e_z} z\sigma}{2\varepsilon_0} \left[\frac{1}{|z|} - \frac{1}{\sqrt{a^2+z^2}} \right].$$

$$= \frac{\sigma}{2\varepsilon_0} \left[\frac{z}{|z|} - \frac{z}{\sqrt{a^2+z^2}} \right].$$

$$= \boxed{\frac{\hat{e}_z q}{2\pi\varepsilon_0 a^2}\left[\frac{z}{|z|} - \frac{z}{\sqrt{a^2+z^2}}\right]}$$

when $z = 0$

$$\vec{E} = \boxed{\hat{e}_z \frac{\sigma}{2\varepsilon_0}} \rightarrow \text{like an infinite plate}$$

when $z \gg a$,

$$\vec{E} = \frac{\hat{e}_z q}{2\pi\varepsilon_0 a^2}\left[1 - z(a^2+z^2)^{-\frac{1}{2}}\right].$$

$$= \frac{\hat{e}_z q}{2\pi\varepsilon_0 a^2}\left(1 - \left(1+\frac{a^2}{z^2}\right)^{-\frac{1}{2}}\right].$$

$$= \frac{\hat{e}_z q}{2\pi\varepsilon_0 a^2}\left(1 - \left(1 - \frac{a^2}{2z^2}\right)\right)$$

$$= \frac{\hat{e}_z q}{2\pi\varepsilon_0 a^2}\left(\frac{a^2}{2z^2}\right).$$

$$= \boxed{\frac{\hat{e}_z q}{4\pi\varepsilon_0 z^2}} \rightarrow \text{like a point charge}$$

$\therefore$ Consistent with expectations.

3) a) $\because$ total charge of atom is 0

$$\therefore \quad q + \int_V \rho(r)\, dV = 0$$

$$\int_V \rho(r)\, dV = \int_0^\infty -Ce^{-2r/a_0}\, 4\pi r^2\, dr$$

$$= -4\pi C \int_0^\infty r^2 e^{-2r/a_0}\, dr$$

Note:

$$\underline{\int r^2 e^{kr}\, dr} = \frac{1}{k}\int r^2\, d(e^{kr})$$

$$= \frac{1}{k}\left[r^2 e^{kr} - \int e^{kr}\, d(r^2) \right]$$

$$= \frac{1}{k}\left[r^2 e^{kr} - 2\int r e^{kr}\, dr \right]$$

$$= \frac{1}{k}\left[r^2 e^{kr} - \frac{2}{k}\int r\, d(e^{kr}) \right]$$

$$= \frac{1}{k}\left[r^2 e^{kr} - \frac{2}{k}\left(re^{kr} - \int e^{kr}\, dr \right) \right]$$

$$= \frac{1}{k}\left[r^2 e^{kr} - \frac{2}{k}\left(re^{kr} - \frac{1}{k}e^{kr} \right) \right]$$

$$= \frac{1}{k} r^2 e^{kr} - \frac{2}{k^2} r e^{kr} + \frac{2}{k^3} e^{kr}$$

$$= \underline{\frac{e^{kr}}{k^3}\left(k^2 r^2 - 2kr + 2 \right)}$$

In the above case

$$k = -\frac{2}{a_0}$$

$$\therefore \int_V \rho \, d\tau = -4\pi C \int_0^\infty r^2 e^{(-\frac{2}{a_0})r} \, dr$$

$$= -4\pi C \left[\frac{e^{(-\frac{2}{a_0})r}}{(-\frac{2}{a_0})^3} \left(\left(-\frac{2}{a_0}\right)^2 r^2 - 2\left(-\frac{2}{a_0}\right) r + 2 \right) \right]_0^\infty$$

$$= \cancel{+} 4\pi C \left\{ \frac{a_0^3 e^{-\frac{2r}{a_0}}}{8} \left(\frac{4}{a_0^2} r^2 + \frac{4}{a_0} r + 2 \right) \right]_0^\infty$$

$$= \boxed{\pi C \left\{ a_0^3 e^{-2r/a_0} \left(2\frac{r^2}{a_0^2} + 2\frac{r}{a_0} + 1 \right) \right]_0^\infty}$$

$$= \pi C \left[\left\{ a_0^3 \cancel{\left(} \lim_{r \to \infty} e^{-2r/a_0} r^2 \frac{2}{a_0} + \lim_{r \to \infty} e^{-2r/a_0} 2\frac{r}{a_0} \right. \right.$$

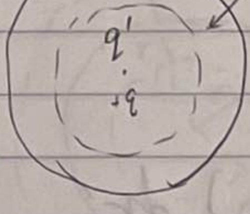
$$\left. + \lim_{r \to \infty} e^{-2r/a_0} \right) - (a_0^3)\left(0 + 0 + \cancel{a_0 e^{-2}} \right.$$

$$\left. + a_0^3 \cancel{e^{0}} \, 0 \right].$$

$$= -\pi C a_0^3$$

$$\therefore \quad q - \pi C a_0^3 = 0 \quad \Rightarrow \quad \boxed{C = \frac{q}{\pi a_0^3}}$$

b)



Gaussian Surface

~~Gaus~~ Integral form of Gauss's Law :

$$\oiint_{\partial V} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV$$

$$\therefore \quad (4\pi r^2) E = \frac{1}{\varepsilon_0}(q + q') \quad \text{where}$$

$$q' = \int_V \rho(r) \, dV = \int_{r'=0}^{r'=r} \rho(r') \, dV'$$

$$= \pi C a_0^3 \left\{ e^{-2r/a_0} \left(2\frac{r^2}{a_0^2} + 2\frac{r}{a_0} + 1 \right) \right\}_0^r$$

$$= \pi C a_0^3 \left\{ e^{-2r/a_0} \left(2\frac{r^2}{a_0^2} + \frac{2r}{a_0} + 1 \right) - 1 \right\}$$

$$= q \left\{ e^{-2r/a_0} \left(2\frac{r^2}{a_0^2} + \frac{2r}{a_0} + 1 \right) - 1 \right\}$$

$$\therefore \quad (4\pi \varepsilon_0 r^2) E = q e^{-2r/a_0} \left(2\frac{r^2}{a_0^2} + \frac{2r}{a_0} + 1 \right)$$

$$\therefore \quad \boxed{\vec{E} = \frac{q e^{-2r/a_0}}{4\pi \varepsilon_0} \left(\frac{2}{a_0^2} + \frac{2}{a_0 r} + \frac{1}{r^2} \right) \hat{e}_r}$$

c)

$$V(r) = -\int_{\infty}^{r} E(r')\,dr'$$

$$= -\frac{q}{4\pi\varepsilon_0}\int_{\infty}^{r} e^{-\frac{2r'}{a_0}}\left(\frac{1}{r'^2} + \frac{2}{a_0 r'} + \frac{2}{a_0^2}\right) dr'$$

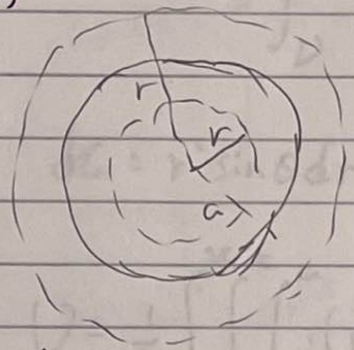
$$= -\frac{q}{4\pi\varepsilon_0}\left[\int_{\infty}^{r} \frac{dr'\,e^{(-\frac{2}{a_0})r'}}{r'}\left(\frac{1}{(\frac{2}{a_0})r'} + 1\right)\left(\frac{2}{a_0}\right) + \int_{\infty}^{r}\frac{1}{a_0}e^{-\frac{2r'}{a_0}}\left(\frac{2}{a_0}\right)dr'\right]$$

$$= -\frac{q}{4\pi\varepsilon_0}\left(\cancel{\left(\frac{2}{a_0}\right)}\left[-\frac{e^{-\frac{2r'}{a_0}}}{\cancel{(\frac{2}{a_0})}r'}\right]_{\infty}^{r} + -\frac{e^{-\frac{2r'}{a_0}}}{a_0}\Bigg|_{\infty}^{r}\right).$$

$$= \boxed{\frac{q}{4\pi\varepsilon_0}\left(e^{-2r/a_0}\right)\left(\frac{1}{r} + \frac{1}{a_0}\right)}$$

4)

a)



$\cancel{f(r) =} \ \rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi a^3} = \frac{3Q}{4\pi a^3}$

Gauss's Law:

$$\oint_{\partial V} \vec{E}\cdot d\vec{S} = \frac{Q_V}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_V \rho \, dV$$

E(r) :

Inside sphere :

$$Q_V = \frac{r^3}{a^3} Q$$

$$\therefore \cancel{4\pi\varepsilon_0} \ (4\pi r^2) E = \frac{1}{\varepsilon_0} \frac{r^3}{R^3} Q$$

$$\therefore \cancel{\boxed{E = \frac{Qr}{4\pi\varepsilon_0 R^3}}} \quad \boxed{E = \frac{Qr}{4\pi\varepsilon_0 a^3}}$$

Outside sphere :

$$Q_V = Q$$

$$\therefore (4\pi r^2) E = \frac{1}{\varepsilon_0} Q$$

$$\therefore \boxed{E = \frac{Q}{4\pi\varepsilon_0 r^2}}$$

V(r) :

Inside sphere

$$V(r) = -\int_{\infty}^{r} E(r) \, dr$$

$$= \int_r^{\infty} \frac{Q}{4\pi\varepsilon_0 r^2}\, dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r}\right)\Bigg|_r^{\infty}$$

$$= \boxed{\frac{Q}{4\pi\varepsilon_0 r}}$$

Inside sphere

$$V(r) = \int_r^{\infty} E(r)\, dr$$

$$= \int_a^{\infty} \frac{Q}{4\pi\varepsilon_0 r^2}\, dr + \int_r^{a} \frac{Qr}{4\pi\varepsilon_0 a^3}$$

$$= \frac{Q}{4\pi\varepsilon_0 a} + \frac{Q}{4\pi\varepsilon_0 a^3}\left(\frac{r^2}{2}\right)\Bigg|_r^{a}$$

$$= \frac{Q}{4\pi\varepsilon_0 a^3}\left(a^2 + \frac{a^2}{2} - \frac{r^2}{2}\right)$$

$$= \boxed{\frac{Q}{4\pi\varepsilon_0 a^3}\left(\frac{3}{2}a^2 - \frac{1}{2}r^2\right).}$$

b) $$U = \frac{1}{2}\int_V \rho V \, d\tau$$

$$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \left(\rho \frac{Q}{4\pi\varepsilon_0 a^3}\right)\left(\frac{3}{2}a^2 - \frac{1}{2}r^2\right) r^2 \sin\theta \, \overset{dr}{\downarrow} d\theta \, d\phi \, \cancel{d\tau}$$

$$= \frac{\rho Q}{8\pi\varepsilon_0 a^3} \cancel{\int d\phi} (2\pi) \int_0^{\pi} \sin\theta \, d\theta \int_0^{a} \left(\frac{3}{2}a^2 r^2 - \frac{1}{2}r^4\right) dr$$

$$= \left(\frac{\rho Q}{8\pi\varepsilon_0 a^3}\right)(2\pi)(2) \left[\frac{1}{2}a^2 r^3 - \frac{1}{10} r^5\right]_0^a$$

$$= \left(\frac{3Q}{4\pi a^3}\right)\left(\frac{Q}{\cancel{8}\pi\varepsilon_0 a^3}\right)\cancel{(2\pi)}\cancel{(2)}\left(\frac{\cancel{2}}{5}a^5\right)$$

$$= \boxed{\frac{3Q^2}{20\pi\varepsilon_0 a}}$$

c)

$$U = \int_{\text{space}} \frac{\varepsilon_0 E^2}{2}\, d\tau = U_1 + U_2$$

$$U_1 = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 r^2 \sin\theta\, \overset{dr}{d\theta}\, d\phi$$

$$= \cancel{(2\pi)}(2)\left(\frac{1}{2}\cancel{\varepsilon_0}\right)\left(\frac{Q^2}{\underset{4\pi}{\cancel{16\pi^2}}\,\varepsilon_0}\right) \underbrace{\int_a^{\infty} \frac{1}{r^2}\, dr}_{= -\frac{1}{r}\Big|_0^{\infty} = \frac{1}{a}}$$

$$= \frac{Q^2}{8\pi\varepsilon_0 a}$$

$$U_2 = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{1}{2}\varepsilon_0 \left(\frac{Q^2 r^2}{16\pi^2\varepsilon_0^2 a^6}\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \cancel{(2\pi)(2)}\left(\frac{1}{2}\right)(\cancel{\varepsilon_0})\left(\frac{Q^2}{\cancel{(4\pi)}(4\pi\varepsilon_0)\cancel{\varepsilon_0} a^6}\right)\int_0^a r^4\, dr$$

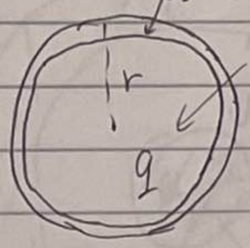
$$= \frac{Q^2}{8\pi\varepsilon_0 a^6}\left(\frac{a^5}{5}\right) = \frac{Q^2}{40\pi\varepsilon_0 a}$$

$$U = U_1 + U_2 = \left(\frac{1}{8} + \frac{1}{40}\right)\frac{Q^2}{\pi\varepsilon_0 a}$$

$$= \boxed{\frac{3Q^2}{20\pi\varepsilon_0 a}}$$

d)

$dq = \rho\, dv = 4\pi r^2 \rho\, dr \qquad \rho = \frac{3Q}{4\pi a^3}$



$q = \frac{r^3}{a^3} Q$

$$U = \int_V dU = \int_V \frac{1}{4\pi\varepsilon_0}\frac{q\,dq}{r}$$

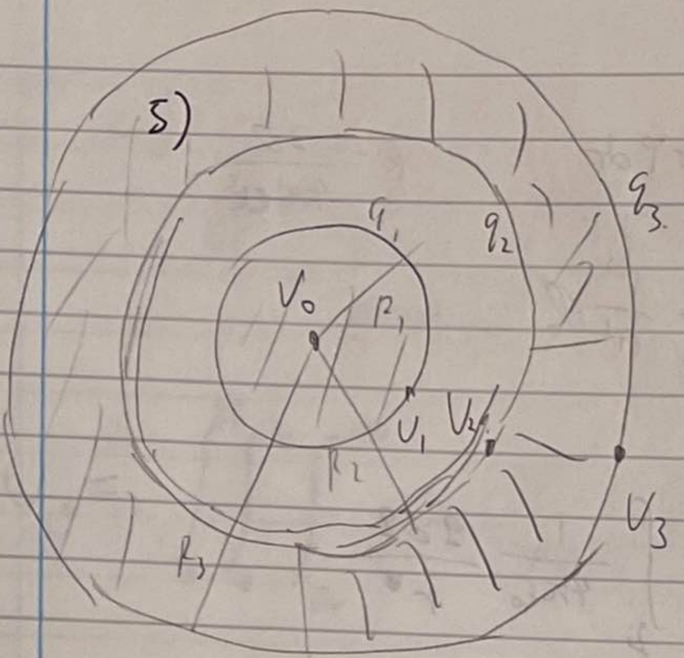
$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\left(\frac{r^3}{a^3} Q \rho\, 4\pi r^2 dr\right)}{r}$$

$$= \left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{3Q}{4\pi a^3}\right)\left(\frac{Q}{a^3}\right)\cancel{(4\pi)}\int_0^a r^4\, dr$$

$\frac{1}{5} a^5$

$$= \boxed{\frac{3Q^2}{20\pi\varepsilon_0 a}}$$

5)



$V_\infty = 0$

a) By ~~Gauss~~ Gauss's Law and properties of Conductors

$$\boxed{\begin{aligned} q_1 &= q \\ q_2 &= -q \\ q_3 &= q \end{aligned}}$$

$$\boxed{\begin{aligned} \sigma_1 &= \frac{q}{4\pi R_1^2} \\ \sigma_2 &= \frac{-q}{4\pi R_2^2} \\ \sigma_3 &= \frac{q}{4\pi R_3^2} \end{aligned}}$$

b) By Gauss's Law

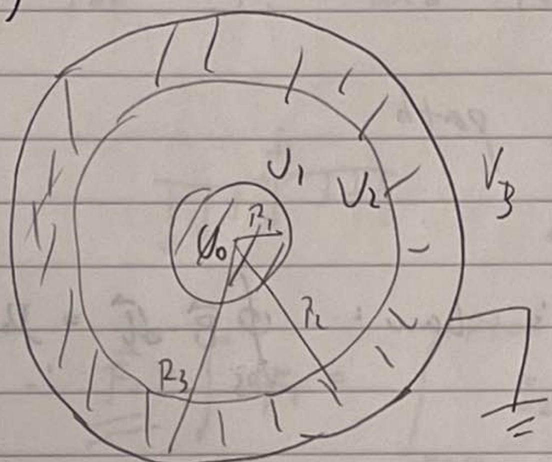
$V_\infty = 0 \rightarrow V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_3}$

$V_2 = V_3 = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_3}$

$V_1 = V_2 + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$, $V_0 = V_1$

$$\therefore \boxed{V_0 = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3}\right)}$$

c)



V_∞

Now that $q_3 = 0$ and $V_3 = 0$

$\therefore$ a) $\boxed{\begin{aligned} q_1 &= q \\ q_2 &= -q \\ q_3 &= 0 \end{aligned}}$ $\boxed{\begin{aligned} \sigma_1 &= \frac{q}{4\pi R_1^2} \\ \sigma_2 &= \frac{-q}{4\pi R_2^2} \\ \sigma_3 &= 0 \end{aligned}}$

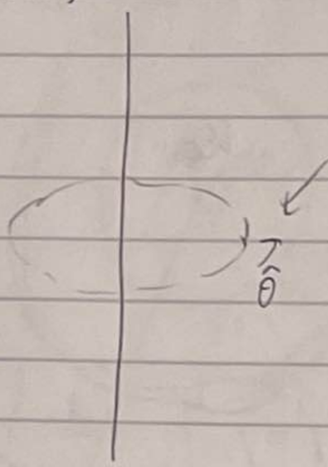
b) $V_3 = 0$ $V_2 = V_3 = 0$

$$V_1 = V_2 + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad . \quad V_0 = V_1$$

$$\therefore \boxed{V_0 = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$

6)

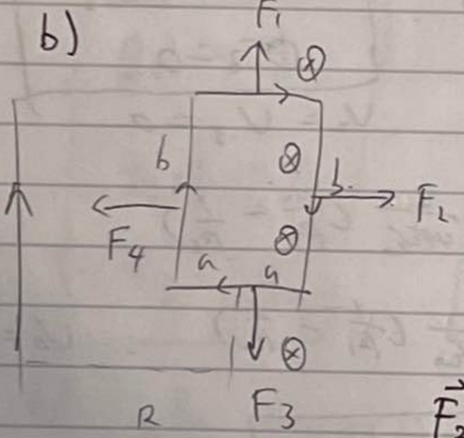


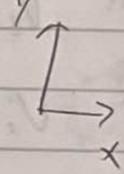
Ampere's Law : $\oint_{\partial S} \vec{B} \cdot d\vec{l} = \mu_0 I_s$

a) $B(2\pi r) = \mu_0 I$

$\therefore \boxed{\vec{B} = \frac{\hat{e}_\theta \mu_0 I_1}{2\pi r}}$

b)





$\cancel{\vec{F}_1 = \hat{e}_y}$

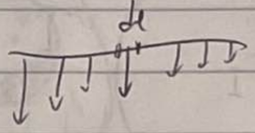
$$\vec{F}_2 = I_2 B l \, \hat{e}_x$$

$$= I_2 \left(\frac{\mu_0 I_1}{2\pi (R+a)} \right) b \, \hat{e}_x$$

$$= \boxed{\frac{\mu_0 I_1 I_2 b}{2\pi (R+a)} \hat{e}_x}$$

$$\vec{F}_4 = \boxed{\frac{-\mu_0 I_1 I_2 b}{2\pi (R-a)} \hat{e}_x}$$

For $\vec{F_1}$ and $\vec{F_3}$



$$\cancel{d\vec{F} = I_2 \vec{B}(l)}$$

$$d\vec{F_1} = I_2 d\vec{l} \times \vec{B}(l) = I_2 B(l) dl \, \hat{e_y}$$

$$\therefore \underline{\underline{\vec{F_1}}} = \int dF_1 = \int_{R-a}^{R+a} I_2 \frac{\mu_0 I_1}{2\pi r} dr$$

$$= \boxed{\frac{\mu_0 I_1 I_2}{2\pi} \ln\left(\frac{R+a}{R-a}\right) \hat{e_y}}$$

$$\underline{\underline{\vec{F_3}}} = -\vec{F_1} = \boxed{-\frac{\mu_0 I_1 I_2}{2\pi} \ln\left(\frac{R+a}{R-a}\right) \hat{e_y}}$$

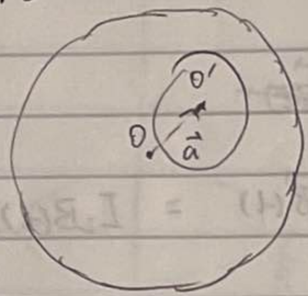
c) Net force on the loop

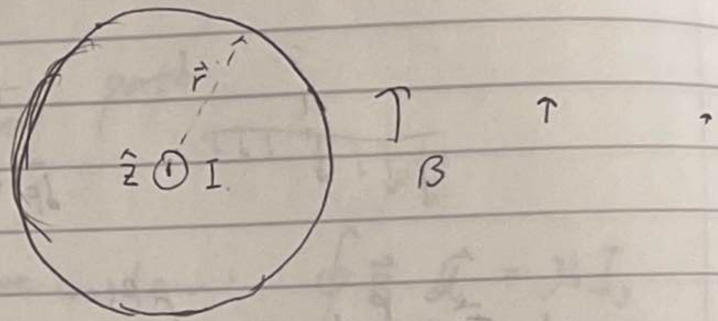
$$\vec{F}_{net} = \frac{-\mu_0 I_1 I_2 b}{2\pi} \left(\frac{1}{R-a} - \frac{1}{R+a}\right) \hat{e_x}$$

$$= \frac{-\mu_0 I_1 I_2 ab}{\pi (R^2 - a^2)}$$

7)

Solid cylinder





a) Ampere's Law gives

$\cancel{\hat{z}\times\vec{B} = \mu_0\vec{J}}$ $\quad \oint_{\partial S} \vec{B}\cdot d\vec{l} = \mu_0 I_s$

$I_s = \pi r^2 J \qquad \therefore \; (2\pi r) B = \pi r^2 J \mu_0$

$\therefore \; B = (\mu_0 J/2)\, r$

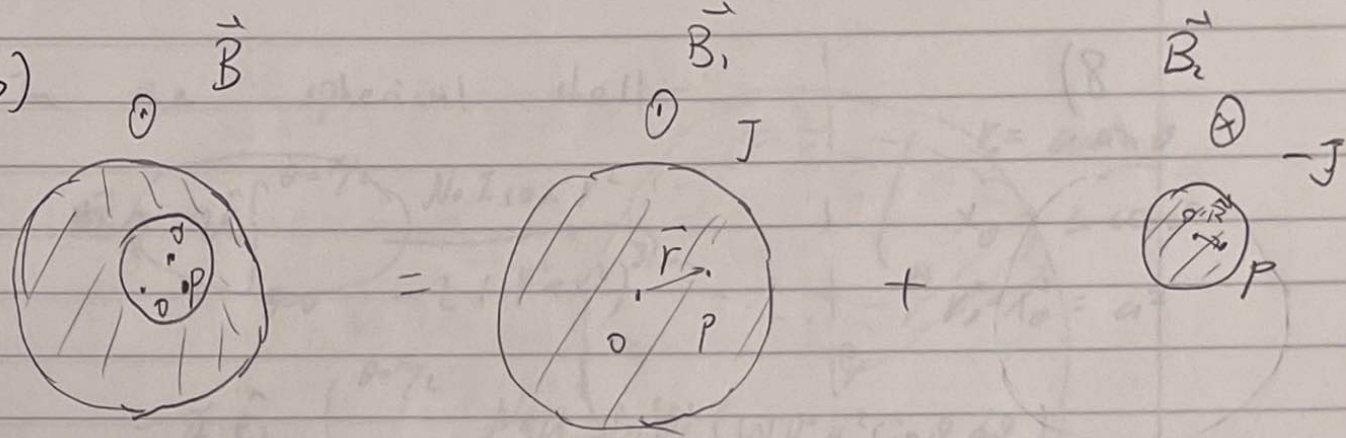
$\vec{B}$ is along the direction $\hat{z}\times\hat{r}$ by right hand rule

$\therefore \; \cancel{\vec{B} = (\mu_0 J/2)\, r \; (\hat{r}\times}$

$\cancel{\vec{B} = (\mu_0 J/2)(\hat{z}\times\hat{r})}$

$\vec{B} = (\mu_0 J/2)\, r\, (\hat{z}\times\hat{r})$

$\because \; \vec{r} = r\hat{r} \qquad \therefore \; \underline{\vec{B} = (\mu_0 J/2)(\hat{z}\times\vec{r})}$

b) 

Point of interest is P

$\vec{B} = \vec{B}_1 + \vec{B}_2$

Using the conclusion from a)

$\vec{B}_1 = (\mu_0 J/2)\, \hat{z} \times \vec{r}$

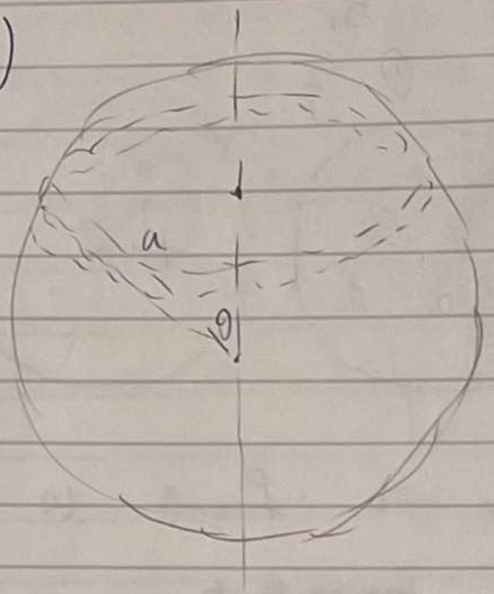
$\vec{B}_2 = -(\mu_0 J/2)\, \hat{z} \times \vec{R}$

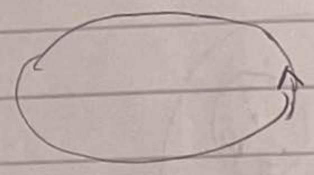
$\vec{B} = \vec{B}_1 + \vec{B}_2 = (\mu_0 J/2)\, \hat{z} \times (\vec{r} - \vec{R})$

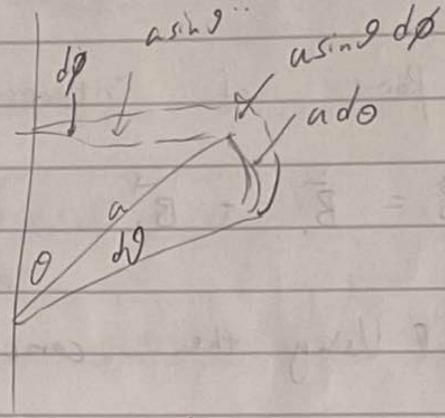
$\because \vec{r} - \vec{R} = \vec{a}$ by definition

$\therefore \underline{\vec{B} = (\mu_0 J/2)\, \hat{z} \times \vec{a}}$ which is uniform in both magnitude and direction.

8)







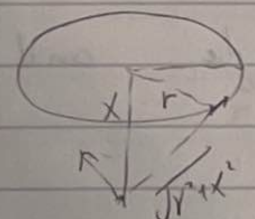
$$\cancel{dI} \quad dI = \cancel{\frac{dq}{dt}} = \cancel{\sigma a^2 \sin\theta \, d\theta \, d\phi}$$

$$dI_{(\theta)} = \frac{dq}{dt} = \frac{\sigma dA}{dt} = \cancel{\sigma a^2 \sin\theta \, d\theta \, d\phi} \quad \frac{2\pi \sigma a^2 \sin\theta \, d\theta}{2\pi/\omega}$$

$$= \underline{\omega \sigma a^2 \sin\theta \, d\theta}$$

consider a ring of radius r then the field at P, $\vec{B}_P$, is given by the Biot-Savart Law:

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\hat{l} \times \hat{r}}{r^2}$$



$$\vec{B}_P = \int d\vec{B}_P = \hat{e}_z \int d\vec{B} \cos\theta$$

$$= \frac{r\hat{e}_z}{\sqrt{r^2+x^2}} \frac{\mu_0 I}{4\pi(r^2+x^2)} (2\pi r)$$

$$= \underline{\frac{\mu_0 I r^2}{2(r^2+x^2)^{3/2}} \hat{e}_z}$$

In the spherical shell

$$\vec{B} = 2\hat{e}_z \int_{\theta=0}^{\theta=\pi/2} \frac{\mu_0 I(\theta)\, r^2}{2(r^2+x^2)^{3/2}}$$

$$r_\theta = a\sin\theta$$
$$x_\theta = a\cos\theta$$
$$r_\theta^2 + x_\theta^2 = a^2$$

$$= \cancel{2}\,\hat{e}_z \int_{\theta=0}^{\theta=\pi/2} \frac{\mu_0 (a^2\sin^2\theta)(\omega\sigma a^2 \sin\theta\, d\theta)}{\cancel{2}\, a^3}.$$

$$= \hat{e}_z\, \mu_0 a\omega\sigma \int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$\int \sin^3\theta\, d\theta = \int \sin\theta(1-\cos^2\theta)\, d\theta = \int \sin\theta\, d\theta - \int \cos^2\theta \sin\theta\, d\theta$$

$$= -\cos\theta + \int \cos^2\theta\, d(\cos\theta) = -\cos\theta + \frac{\cos^3\theta}{3}$$

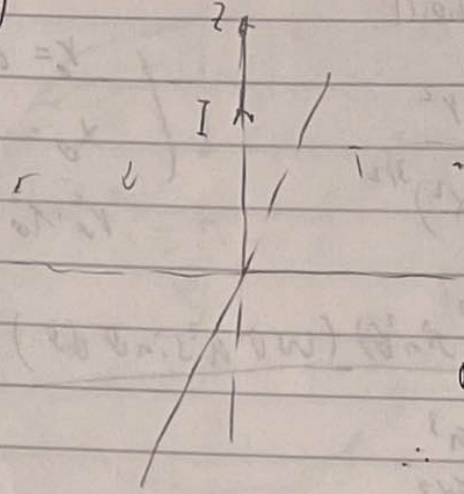
$$\int_0^{\pi/2} \sin^3\theta\, d\theta = -\cos\pi/2 + \frac{\cos^3\pi/2}{3} + \cos 0 - \frac{\cos^3 0}{3}$$

$$= \frac{2}{3}$$

$\therefore$

$$\cancel{\vec{B} = \frac{\hat{e}_z 2 \mu_0 a}{3}} \quad \vec{B} = \frac{2}{3}\hat{e}_z \mu_0 a\omega\sigma$$

9)



Ampere's Law :

$$\oint_{\partial s} \vec{B} \cdot d\vec{l} = \mu_0 I_s$$

$$B(2\pi r) = \mu_0 I$$

$$\therefore \quad B = \frac{\mu_0 I}{2\pi r}.$$

$\hat{B}$ is along the direction $\hat{z} \times \hat{r}$

$$\therefore \quad \vec{B} = \frac{\mu_0 I (\hat{z} \times \vec{r})}{2\pi r^2} = \frac{\mu_0 I}{2\pi r}\hat{\theta} \quad \boxed{\vec{r} = (r, \theta, \cancel{\theta} z)}$$

~~Force on particle F~~

a) Power $P = \frac{dE}{dt} = \frac{dT}{dt}$ (T = kinetic energy).

$$= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\frac{d\underline{v}}{dt}\cdot\underline{v} = \vec{F}\cdot\vec{v} = (q\vec{v}\times\vec{B})\cdot\vec{v} = 0$$

$\therefore$ T is constant.

b)

$$\cancel{\vec{F}} \quad \underline{F} = q(\underline{v} \times \underline{B})$$

$$= q\underline{v} \times \frac{\mu_0 I}{2\pi r^2}\hat{z}\times\underline{r}$$

$(\underline{v} = \dot{\underline{r}})$

$$= \frac{q\mu_0 I}{2\pi r^2}(\dot{\underline{r}} \times (\hat{z} \times \underline{r})) = \underline{\frac{q\mu_0 I}{2\pi r}\dot{\underline{r}} \times \hat{\underline{\theta}}}$$

c) $m\frac{dv}{dt} = \underline{F}$

$$\therefore \quad \cancel{\text{but}} \quad \underline{m\ddot{\underline{r}} = \frac{q\mu_0 I}{2\pi r^2}(\dot{\underline{r}} \times (\hat{z} \times \underline{r}))}$$

$$\cancel{\ddot{\underline{r}} = \frac{q\mu_0 i}{2\pi m}\left(\frac{1}{r^2}\right)(\dot{\underline{r}} \times (\hat{z} \times r))}$$

$$m\ddot{\underline{r}} = \frac{q\mu_0 i}{2\pi r}\dot{\underline{r}} \times \hat{\underline{\theta}}$$

$$\therefore \quad \ddot{\underline{r}} = \frac{q\mu_0 I}{2\pi m}\frac{1}{r}\dot{\underline{r}} \times \hat{\underline{\theta}}$$

$$\cancel{\dot{\underline{r}} = (\dot{x}, \dot{y}, \dot{z})} \qquad \underline{r} = (\dot{r}, \dot{\theta}, \dot{z})$$

$$\dot{\underline{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

$$\dot{\underline{r}} \times \hat{\underline{\theta}} = \dot{r}\hat{\underline{r}} \times \hat{\underline{\theta}} + r\dot{\theta}\,\hat{\underline{\theta}} \times \hat{\underline{\theta}} + \dot{z}\hat{\underline{z}} \times \hat{\underline{\theta}}$$

$$= \dot{r}\hat{z} + 0 + \dot{z}(-\hat{r})$$

$$= -\dot{z}\hat{r} + \dot{r}\hat{z}$$

$$\therefore \quad \ddot{\underline{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} + \ddot{z}\hat{z}$$

$$= \left(\frac{q\mu_0 i}{2\pi m}\right)\left(\frac{1}{r}\right)(-\dot{z}\hat{r} + \dot{r}\hat{z})$$

$$\therefore \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\mu_0 I q}{2\pi m}\frac{\dot{z}}{r} \qquad (1)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \qquad (2)$$

$$\ddot{z} = \frac{q\mu_0 i}{2\pi m}\frac{\dot{r}}{r} \qquad (3)$$

d) $\cancel{③ \Rightarrow}$ $\dot{z} = V_z$ $\ddot{z} = 0$

③ $\Rightarrow$ $\dot{r} = 0$ $\Rightarrow$ $r = r_0$ $\Rightarrow$ $\ddot{r} = 0$

$\therefore$ ① $\Rightarrow$ $\dot{\theta}^2 = \dfrac{\mu_0 I q}{2\pi m r_0^2} V_z$

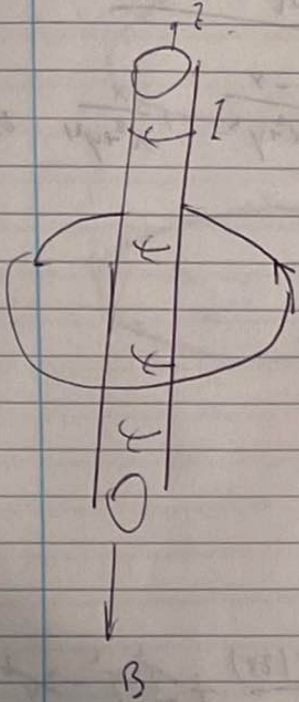
$\because$ $\ddot{\theta} = 0$ $\dot{r} = 0$ $\Rightarrow$ ② is satisfied

The particle rotates ~~above~~ about z - axis with angular velocity

$$w = \dot{\theta} = \left(\frac{\mu_0 I q}{2\pi m r_0^2} V_z \right)^{1/2}.$$

10)

a)



By Lenz' Law, the induced current flows counterclockwise.

b) $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$

$\cancel{\Phi = \int \vec{B}\cdot d\vec{A}}$

$\Phi = \int \vec{B}\cdot d\vec{S}$

For $r < a$

$\Phi = \int \vec{B}\cdot d\vec{S} = \mu_0 n I (\pi r^2) = \pi \mu_0 n I r^2$

$\because I = kt \quad \therefore \quad \Phi = \pi \mu_0 n r^2 k t \qquad \frac{d\Phi}{dt} = \pi \mu_0 n r^2 k$

$\therefore (E)(2\pi r) = \pi \mu_0 n r^2 k$

$\boxed{E = \frac{1}{2}\mu_0 n r k}$ $\qquad \vec{E} = \frac{1}{2}\mu_0 n r k\, \hat{\theta}$

For $r > a$ $\qquad \Phi = \pi \mu_0 n a^2 k t$

$E(2\pi r) = \pi \mu_0 n a^2 k$

$\therefore \boxed{E = \frac{\mu_0 n a^2 k}{2r}} \qquad \vec{E} = \frac{\mu_0 n a^2 k}{2r}\,\hat{\theta}$

Orientation agrees with what is given by ~~Lent~~ Lenz' Law

c) For $r < a$ $\qquad -\frac{\partial B}{\partial t} = +\mu_0 n k \hat{z}$

$\vec{\nabla}\times\vec{E} = \mu_0 n k \left(\vec{\nabla}\times\left(\frac{1}{2} r\hat{\theta}\right)\right)$

$= \frac{1}{2}\mu_0 n k \left(\vec{\nabla}\times(-y, x, 0)\right)$

$= \frac{1}{2}\mu_0 n k\,(0, 0, 2)$

$= \mu_0 n k \hat{z}$

$\hat{\theta} = \left(-\frac{y}{r}, \frac{x}{r}, 0\right)$

$r = x^2 + y^2$

$r > a$

$$\vec{E} = \frac{\mu_0 n a^2 k}{2r}\hat{\theta} \qquad \frac{\partial \vec{B}}{\partial t} = \vec{0}.$$

$$\vec{\nabla} \times \vec{E} = \alpha \quad \vec{\nabla} \times \left(\frac{\hat{\theta}}{r}\right) = \vec{\nabla} \times \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$$

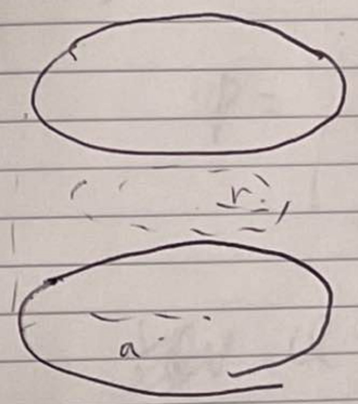
$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= i(0) + j(0) + k\left(\frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right)$$

$$= (0, 0, 0).$$

$$\therefore \vec{\nabla} \times \vec{E} = \vec{0}$$

11. a)



$Q(t) = It$ on a plate.

$\cancel{\sigma = \frac{Q}{S}}$ $\sigma = \frac{Q}{A} = \frac{It}{\pi a^2}$.

$\because$ $w \ll a$ we treat plates as infinite

$\therefore$ By Gauss's Law

$$E = \frac{\sigma}{\varepsilon_0}$$

$$\therefore E = \frac{It}{\pi \varepsilon_0 a^2}.$$

b) $\vec{\nabla} \times \vec{B} = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$

$$\therefore \int (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{S}$$

Stokes Theorem $\Big\{$ $\int \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \frac{\partial \Phi_E}{\partial t}$.

$$\therefore B(2\pi r) = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \left(\frac{It}{\pi \varepsilon_0 a^2} (\pi r^2) \right)$$

$$= \boxed{\frac{\mu_0 I r}{2\pi a^2}.}$$

Energy density $u = \frac{1}{2} \varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$.

$$= \cancel{\frac{I^2t^2 + \dots}{2 \dots a^2}} \; \frac{1}{2} \varepsilon_0 \frac{I^2 t^2}{\pi^2 \varepsilon_0^2 a^4} + \frac{1}{2\mu_0} \frac{\mu_0^2 I^2 r^2}{4\pi^2 a^4}.$$

$$= \frac{I^2 t^2}{2 \varepsilon_0 \pi^2 a^4} + \frac{\mu_0 I^2 r^2}{8\pi^2 a^4}.$$

$$\frac{\partial u}{\partial t} = \frac{I^2 t}{\varepsilon_0 \pi^2 a^4}$$

$$\vec{E} = \frac{It}{\pi \varepsilon_0 a^2}\,\hat{z}$$

$$\vec{B} = \frac{\mu_0 I r}{2\pi a^2}\,\hat{\theta}$$

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{I^2 t r}{2\varepsilon_0 \pi^2 a^4}\underbrace{\hat{z}\times\hat{\theta}}_{-\hat{r}}$$

$$\cancel{\therefore \vec{S} \propto \nabla}$$

$$\underline{-\vec{\nabla}\cdot\vec{S}} = \frac{I^2 t}{2\varepsilon_0 \pi^2 a^4}\,\vec{\nabla}\cdot(r\,\hat{r}) = \frac{I^2 t}{2\varepsilon_0 \pi^2 a^4}\,\vec{\nabla}\cdot(\vec{r})$$

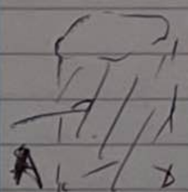
$$= \frac{I^2 t}{2\varepsilon_0 \pi^2 a^4}\,\vec{\nabla}\cdot(x, y, 0).$$

$$= \frac{I^2 t}{2\varepsilon_0 \pi^2 a^4}\,(2) = \frac{I^2 t}{\varepsilon_0 \pi^2 a^4} = \underline{\frac{\partial u}{\partial t}}.$$

d) total power $P = \frac{dE}{dt} = \frac{d}{dt}\int u\,dv$

$$= \int \frac{\partial u}{\partial t}\,dv = \left(\frac{I^2 t}{\varepsilon_0 \pi^2 a^4}\right)(\pi b^2 w)$$

$$= \frac{\pi I^2 b^2 w}{\varepsilon_0 \pi^2 a^4}\,t = \boxed{\frac{I^2 b^2 w t}{\varepsilon_0 \pi a^4}}$$



Poynting vector points ~~dt~~ perpendicularly into the ~~surface~~ cylinderical surface A

$$\int \vec{S}\cdot d\vec{A} = \frac{I^2 t b}{2\varepsilon_0 \pi^2 a^4}\,(2\pi b)(w) = \boxed{\frac{I^2 b^2 w t}{\varepsilon_0 \pi a^4}}$$

e) When $b = a$.

$$p = \boxed{\frac{I^2 w t}{\varepsilon_0 \pi a^2}}$$

$\cancel{\frac{1}{2}QV}$ $Q = It$ $V = Ew = \frac{Itw}{\pi \varepsilon_0 a^2}$

$$\frac{d}{dt}\left(\frac{1}{2}QV\right) = \frac{d}{dt}\left(\frac{1}{2}\frac{I^2 t^2 w}{\pi \varepsilon_0 a^2}\right).$$

$$= \boxed{\frac{I^2 w t}{\varepsilon_0 \pi a^2}}$$

$$\therefore P = \frac{d}{dt}\left(\frac{1}{2}QV\right).$$