A10431W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2015

Friday, 19 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. The Hamiltonian for a 1D harmonic oscillator is

$$\hat{H}_{1D} = \frac{\hat{p}_x^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2 ,$$

where all symbols have their usual meaning. What are its eigenvalues? [3]

The Hamiltonian of a 2D anisotropic harmonic oscillator has the form

$$\hat{H}_{2D} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega^2}{2}\left[\hat{x}^2 + 4\hat{y}^2\right] .$$

Give a general expression for the energy eigenvalues of H_{2D}. What is the energy and degeneracy of the second excited state? [5]

2. A quantum mechanical particle is in a state with wave function

$$\psi(x) = \langle x|\Psi\rangle = \begin{cases} \sqrt{\frac{2k}{\hbar}}\exp\left(-kx/\hbar\right) & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} ,$$

where $k > 0$. What is the probability density for a momentum measurement to give a value equal to q? You may use that

$$\langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(ipx/\hbar) ,$$

$|\Psi\rangle = \Psi \mathbb{I}$

$= \int dp\, \langle p|\Psi\rangle |p\rangle$

where p is the momentum. [6]

$= \int dp \int dx\, \langle p|x\rangle\langle x|\Psi\rangle |p\rangle$

3. Consider a spin-1/2 system with Hamiltonian $\hat{H} = \omega\hat{S}_z$, where ω is a constant. The normalized eigenstates of $\hat{S}_z$ fulfil

$\langle q|\Psi\rangle = \int$

$$\hat{S}_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle .$$

$\langle q|p\rangle = \delta(p-q)$

The system is prepared in the state

$$|\Psi(0)\rangle = a|+\rangle + b|-\rangle , \quad |a|^2 + |b|^2 = 1 ,$$

$H\Psi = E\Psi$ $\quad -\frac{iE}{\hbar} dt$

$i\hbar\frac{\partial\Psi}{\partial t} = E\Psi$

where a and b are constants.

(a) $\hat{S}_z$ is measured. What are the possible results of the measurement, and what are their probabilities? [2]

(b) Consider the time evolution of the system initially prepared in the state $|\Psi(0)\rangle$. Give an explicit expression for $|\Psi(t)\rangle$ in terms of the states $|+\rangle$ and $|-\rangle$. What is the physical significance of the quantity $\langle\Psi(0)|\Psi(t)\rangle$? [3]

(c) What are the eigenstates and eigenvalues of $\hat{S}_x$? [2]

(d) The system is again prepared in $|\Psi(0)\rangle$, but now $\hat{S}_x$ is measured. What are the possible results of the measurement, and what are their probabilities? [3]

(e) What is the expectation value $\langle\Psi(t)|\hat{S}_x|\Psi(t)\rangle$? [2]

$\frac{1}{2}\left(|a+b|^2 + |a-b|^2\right) = |a|^2 + |b|^2$

$a+b$

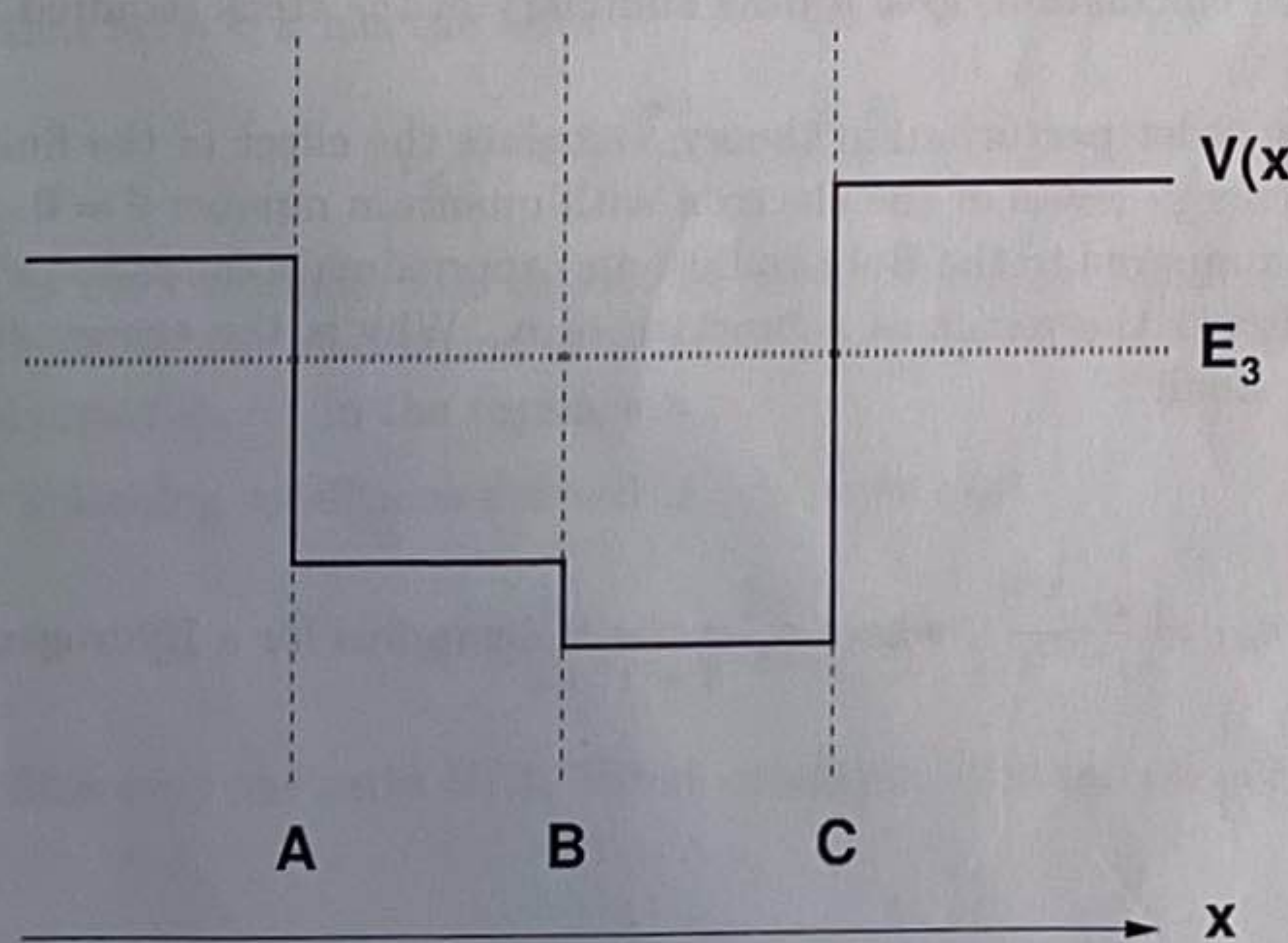
$|a|^2 + |b|^2 - 2ab\, 2|a||b|\cos$

$S_x\left(\frac{1}{2}|+\rangle - \frac{1}{2}|-\rangle\right) =$

4. Two identical *non-interacting* spin-1/2 particles occupy the same one-dimensional simple harmonic oscillator potential. The single-particle wave functions for the ground state and the first excited state are $\psi_g(x)$ and $\psi_e(x)$, respectively, where x is measured along the one dimension. Give expressions for the full wave functions (including the spin part) of the ground and all possible first excited states of the two-particle system. What are the associated total (spin) angular momenta? [6]

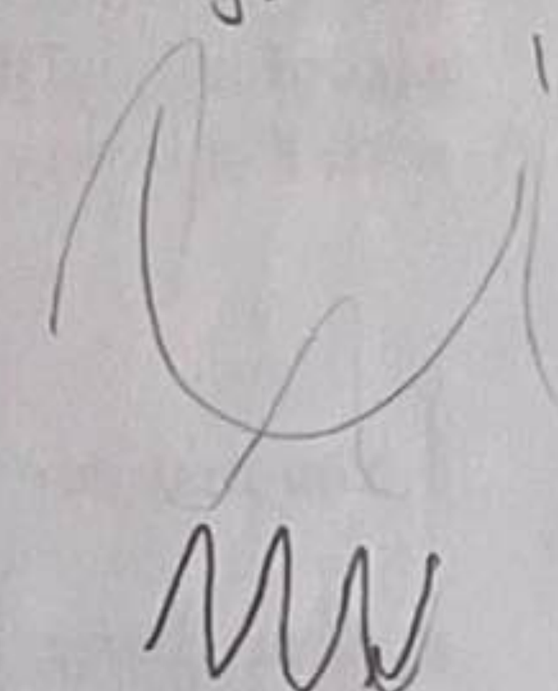
5. Without detailed calculations draw the wave function $\Psi_{\rm inf}(x)$ for the third lowest energy eigenstate of an infinite square well along the x axis. [3]

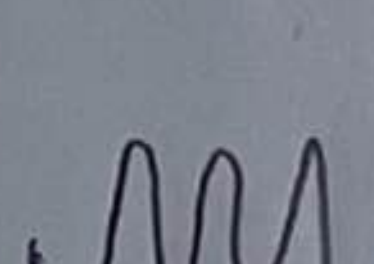
A particle is confined in the 1D potential $V(x)$ shown below. Without detailed calculations, sketch the wave function $\Psi(x)$ of the third lowest energy eigenstate (with energy E_3 indicated by the dashed line). Mark the positions A, B, C on your sketch. Explain the qualitative features of the wave function in the various regions. You may find it useful to take $\Psi_{\rm inf}$ as a starting point of your considerations. [5]

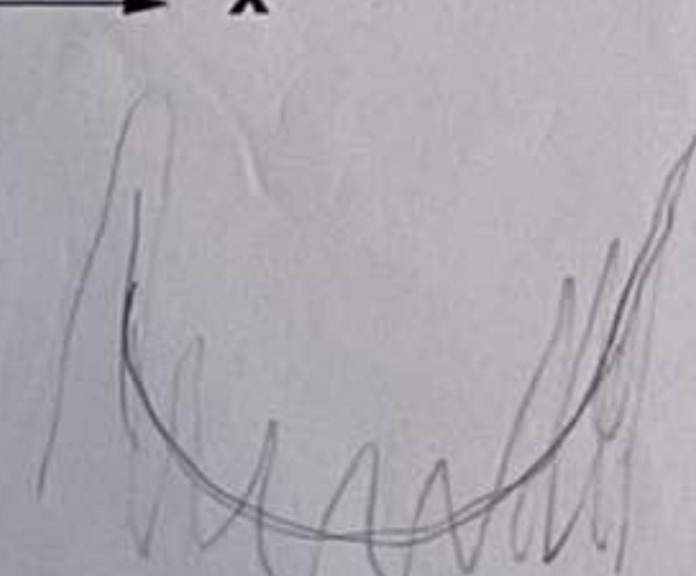


$\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$

1, 2, 3







Section B

6. (a) Give an expression for the energy levels of hydrogen, using n as the principal quantum number. Estimate (ten percent accuracy is sufficient) how these levels would change if the electron were exchanged for a muon. [5]

(b) Given the radial Schrödinger equation for a hydrogen-like atom,

$$-\frac{\hbar^2}{2mr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R_{n,\ell}(r)}{\mathrm{d}r}\right)+\left(-\frac{Ze^2}{4\pi\epsilon_0 r}+\frac{\hbar^2\ell(\ell+1)}{2mr^2}-E_{n,\ell}\right)R_{n,\ell}(r)=0\ ,$$

where all symbols have their usual meaning, show that for very small values of r

$$R_{n,\ell}(r)=c_{n,\ell}\,r^{\ell}\ ,$$

where $c_{n,\ell}$ are constants. [5]

(c) Consider now an atom with a nucleus of charge Z and a single electron. We want to model the nucleus as a sphere of radius R, inside which the protons are uniformly distributed. This can be treated as a perturbation of the usual Hamiltonian for hydrogen by the potential

$$\delta V(r)=\begin{cases}-\frac{Ze^2}{4\pi\epsilon_0 R}\left(1+\frac{R^2-r^2}{2R^2}-\frac{R}{r}\right) & \text{if } r<R\\ 0 & \text{if } r>R\end{cases}.$$

Without detailed calculation, give a brief summary of the steps required to determine the form of $\delta V(r)$.

Using first order perturbation theory, calculate the effect of the finite size of the nucleus on the energy levels of the electron with quantum number $\ell=0$. Assume that R is very small compared to the Bohr radius, and approximate the radial wave functions accordingly. Discuss the result as a function of n. Why is the energy shift for levels with $\ell\neq 0$ very small? [10]

[You may use $c_{n,0}=\frac{2a_Z^{-3/2}}{n^{3/2}}$, where a_Z is the Bohr radius for a Hydrogen-like atom of nuclear charge Z.]

7. A particle is moving in three dimensions in the presence of an attractive potential. Its Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2}{2m} - \lambda_0 \delta(r-a) \ ,$$

where $r = |\mathbf{r}|$, $\delta(x)$ denotes a (one dimensional) delta function, λ_0 and a are constants, and the other symbols have their usual meaning.

(a) Describe the shape of the potential. Argue succinctly that the solutions to the time-independent Schrödinger equations can be written in the form

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r) Y_\ell^m(\theta,\phi) \ ,$$

where $R_{n\ell}$ and Y_ℓ^m have their usual meaning. What is the significance of the parameters m and ℓ? [6]

(b) The Laplacian in spherical polar co-ordinates can be expressed in the form

$$\nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\hat{\mathbf{L}}^2}{\hbar^2 r^2} \ .$$

Write down the radial Schrödinger equation for the functions $R_{n\ell}(r)$. [4]

(c) Assuming that you know the wave function $\psi_{n\ell m}(r,\theta,\phi)$ in the regions $r < a$ and $r > a$, derive "matching conditions" for the wave function and its derivative with respect to r at $r = a$. [3]

(d) For a suitably chosen range of λ_0 there exists a bound state with $\ell = 0$. Show that its wave function for $r < a$ has the form

$$\psi_<(r) = A\frac{\mathrm{e}^{\kappa r}}{r} + B\frac{\mathrm{e}^{-\kappa r}}{r} \ ,$$

where A and B are constants, and relate the parameter κ to the bound state energy. [Hint: you may treat the two terms in $\psi_<(r)$ separately.] Give an expression for the radial wave function $\psi_>(r)$ in the region $r > a$. [4]

(e) Using the matching conditions derived in (c), show that

$$\lambda_0 = \frac{\hbar^2}{2m}\frac{2\kappa}{1 + \mathrm{e}^{-2\kappa a} B/A} .$$

This relation fixes only the ratio B/A. What condition fixes the remaining free parameter? [3]

$$\frac{2m\lambda_0}{\hbar^2} = \frac{2k}{1+e^{-2ka}\frac{B}{A}}$$

8. Rotations of a diatomic molecule such as HCl can be modelled in terms of a rigid rotor. This consists of two point masses m_1 and m_2 at a fixed separation a, that are allowed to rotate around their common centre of mass. In the frame where the centre of mass is stationary, the attitude of the molecule is specified by two angles θ and ϕ. An appropriate Hamiltonian is

$$\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}{2\mu a^2} \ ,$$

where μ is the reduced mass of the two atoms and $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ are the orbital angular momentum operators.

(a) What are the rotational energy levels and corresponding eigenfunctions? What are the degeneracies of the energy levels, and what are the spacings between adjacent levels? [8]

(b) Explain why the pure rotational spectrum of HCl, due to emission or absorption of electric dipole radiation, consists of equally spaced lines. If the spacing is $4.11 \times 10^{-22}\ J$, calculate the separation between the atoms in the molecule (use the mass of the isotope ^{35}Cl). [5]

(c) A diatomic molecule is prepared in a state with wave function

$$\psi(\theta, \phi) = \frac{1}{\sqrt{8\pi}} \left(1 + \sqrt{3} \sin\theta \sin\phi\right) \ .$$

Determine the possible outcomes of a measurement of $\mathbf{L}^2$, their respective probabilities, and give the state of the system after the measurement has been made.
Determine the possible outcomes of a measurement of L_z, their respective probabilities, and give the state of the system after the measurement has been made. [7]

$\left[\text{You may use } Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \ , \ Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \ , \ Y_1^{\pm 1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{\pm i\phi} \ .\right]$

9. Consider a one dimensional square potential well

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases} ,$$

where L is a constant.

(a) What are the energies E_n and normalized wave functions $\psi_n(x)$ of the energy eigenstates of a single particle of mass m moving in presence of the potential $V(x)$? [6]

(b) Consider now two identical spinless particles with mass m moving in presence of the potential $V(x)$. How are the energy eigenvalues and eigenfunctions expressed in terms of the $\psi_n(x)$ and E_n of part (a)? Justify your answer. Give an explicit expression for the ground state. [6]

(c) We now add a perturbation

$$\delta V(x_1, x_2) = -\lambda \cos\left[\frac{\pi(x_1 - x_2)}{2L}\right]$$

to the system, where x_1 and x_2 are the coordinates of the two particles and $\lambda \ll 1$ is a constant. Calculate the first order energy shift for the ground state.

Hint: the integral determining the energy shift can be expressed in the form

$$\int_0^L \mathrm{d}x_1 f(x_1)g(x_1) \int_0^L \mathrm{d}x_2 f(x_2)g(x_2) + \int_0^L \mathrm{d}x_1 h(x_1)g(x_1) \int_0^L \mathrm{d}x_2 h(x_2)g(x_2) ,$$

i.e. in terms of products of independent integrals, where f, g and h are arbitrary functions. [8]

A3 2015 First Attempt.

1. $\hat{H}_{1D} = \frac{\hat{P}_x^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2$

eigenvalues are $E_n = \hbar\omega_0(n+\frac{1}{2})$
where $n = 0, 1, 2, 3, \dots$

$$\hat{H}_{2D} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + 4\hat{y}^2)$$

$$= \left[\frac{\hat{P}_x^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2\right] + \left[\frac{\hat{P}_y^2}{2m} + \frac{m(2\omega)^2}{2}\hat{y}^2\right]$$

$$= \underbrace{\left[\frac{\hat{P}_x^2}{2m} + \frac{m\omega_x^2}{2}\hat{x}^2\right]}_{\hat{H}_x} + \underbrace{\left[\frac{\hat{P}_y^2}{2m} + \frac{m\omega_y^2}{2}\hat{y}^2\right]}_{\hat{H}_y}$$

~~where~~
where $\omega_x = \omega$, $\omega_y = 2\omega$

$\therefore$ product state $\to$ eigenvalue of $\hat{H}_{2D}$
= eigenvalue of $\hat{H}_x$ + eigenvalue of $\hat{H}_y$

$$E_{2D} = E_{(n_x, n_y)} = E^x_{n_x} + E^y_{n_y}$$

$$= \hbar\omega_x(n_x + \tfrac{1}{2}) + \hbar\omega_y(n_y + \tfrac{1}{2})$$

$$= \hbar\omega(n_x + \tfrac{1}{2}) + 2\hbar\omega(n_y + \tfrac{1}{2})$$

$$= \hbar\omega\left(n_x + 2n_y + \tfrac{3}{2}\right)$$

Ground state: $n_x = n_y = 0$ $E_{2D} = \frac{3}{2}\hbar\omega$

First excited state: $n_x = 1, n_y = 0$ $E_{2D} = \frac{5}{2}\hbar\omega$

Second excited state: $(n_x, n_y) = (0, 1)$ or $(2, 0)$

$E_{2D} = \frac{7}{2}\hbar\omega$

degeneracy = 2 ✓

2.

$$|\psi\rangle = \int dp\, |p\rangle\langle p|\psi\rangle = \int dp\, \langle p|\psi\rangle |p\rangle$$

$$= \iint dp\, dx\, \langle p|x\rangle\langle x|\psi\rangle |p\rangle$$

probability amplitude to give q. (let $\hat{P}|q\rangle = q|q\rangle$)

~~then~~ is $\langle q|\psi\rangle$

$$\therefore \langle q|\psi\rangle = \iint dp\, dx\, \langle p|x\rangle\langle x|\psi\rangle \underbrace{\langle q|p\rangle}_{\delta(p-q)}$$

$$= \int_{-\infty}^{\infty} dx\, \langle q|x\rangle\langle x|\psi\rangle$$

$$= \int_0^{\infty} dx \cdot \frac{1}{\sqrt{2\pi\hbar}} \exp(iqx/\hbar) \cdot \sqrt{\frac{2k}{\hbar}} \exp(-kx/\hbar)$$

$$= \sqrt{\frac{2k}{2\pi\hbar^2}} \int_0^{\infty} \exp\left(\frac{iq-k}{\hbar} x\right) dx$$

$$= \sqrt{\frac{k}{\pi\hbar^2}} \cdot \frac{\hbar}{iq-k} \left[\exp\left(\frac{iq-k}{\hbar} x\right)\right]_0^{\infty}$$

$$= \sqrt{\frac{k}{\pi\hbar^2}} \left(\frac{\hbar}{iq-k}\right) \left[\lim_{x\to\infty} \exp\left(\frac{iq-k}{\hbar} x\right) - 1\right]$$

$$\lim_{x\to\infty} \exp\left(\frac{iq-k}{\hbar} x\right) = \Big(\underbrace{\lim_{x\to\infty} e^{\frac{iqx}{\hbar}}}_{|1|^2=1}\Big)\Big(\underbrace{\lim_{x\to\infty} e^{-\frac{kx}{\hbar}}}_{\to 0}\Big) \to 0$$

$$\therefore \langle q|\psi\rangle = \frac{\sqrt{k/\pi}}{iq-k}(-1) = \frac{\sqrt{k/\pi}}{k-iq}$$

probability density to measure q as momentum is

$$P(q) = |\langle q|\psi\rangle|^2 = \frac{k/\pi}{k^2+q^2} = \boxed{\frac{k}{\pi(k^2+q^2)}}$$

3. (a) $|\psi(0)\rangle = a|+\rangle + b|-\rangle$

outcome	$+\frac{\hbar}{2}$	$-\frac{\hbar}{2}$
probability	$|a|^2$	$|b|^2$

(b)

$$|\psi(t)\rangle = a\exp\left(-\frac{iE_+t}{\hbar}\right)|+\rangle + b\exp\left(-\frac{iE_-t}{\hbar}\right)|-\rangle$$

$\because \hat{H} = \omega\hat{S_z}$ $\therefore$ $|+\rangle$, $|-\rangle$ are eigenstates of $\hat{H}$ with eigenvalues

$E_+ = \frac{\hbar}{2}\omega$ and $E_- = -\frac{\hbar\omega}{2}$

$$\therefore |\psi(t)\rangle = a\exp\left(-\frac{i}{2}\omega t\right)|+\rangle + b\exp\left(\frac{i}{2}\omega t\right)|-\rangle$$

$$\langle\psi(0)|\psi(t)\rangle = |a|^2\exp\left(-\frac{i}{2}\omega t\right) + |b|^2\exp\left(+\frac{i}{2}\omega t\right)$$

is the probability ~~to collap~~ amplitude for state $|\psi(t)\rangle$ to collapse to $|\psi(0)\rangle$ when doing a measurement on the spin direction where $|\psi(0)\rangle$ is an eigenstate.

(c) $\hat{S_x}$.

eigenstate	$\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$	$\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$
eigenvalue	$\frac{\hbar}{2}$	$-\frac{\hbar}{2}$

(d)

possible results are $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

~~Proba~~ probabilities :

$$\langle +,x|\psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle+| + \langle-|)(a|+\rangle + b|-\rangle)$$

$$= \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}$$

$$P\left(\frac{\hbar}{2}\right) = \left|\frac{1}{\sqrt{2}}(a+b)\right|^2 = \frac{1}{2}|a+b|^2$$

$$\langle -,x|\psi(0)\rangle = \frac{1}{\sqrt{2}}(\langle+| - \langle-|)(a|+\rangle + b|-\rangle) = \frac{1}{\sqrt{2}}(a-b)$$

$$P\left(-\frac{\hbar}{2}\right) = \left|\frac{1}{\sqrt{2}}(a-b)\right|^2 = \frac{1}{2}|a-b|^2$$

(e) $S_x = \frac{1}{2}(S_+ + S_-)$

$$\langle \psi(t)|S_x|\psi(t)\rangle = \left(a^* e^{\frac{i}{2}\omega t}\langle +| + b^* e^{-\frac{i}{2}\omega t}\langle -|\right)\left(\frac{1}{2}(S_+ + S_-)\right)$$
$$\left(a e^{-\frac{i}{2}\omega t}|+\rangle + b e^{\frac{i}{2}\omega t}|-\rangle\right)$$

$$= \left(\frac{1}{2}a^* b e^{i\omega t} + \frac{1}{2}ab^* e^{-i\omega t}\right)\times\hbar$$

$$= \frac{1}{2}\times 2\,\mathrm{Re}(a^* b e^{i\omega t})\times\hbar$$

$$= \underline{\hbar\,\mathrm{Re}(a^* b e^{i\omega t})}$$

If a, b are real then

$$\langle \psi(t)|S_x|\psi(t)\rangle = \hbar ab\cos(\omega t)$$

$$\left(\begin{array}{l} S_+|+\rangle = 0 \\ S_-|-\rangle = 0 \\ S_+|-\rangle = \hbar|+\rangle \\ S_-|+\rangle = \hbar|-\rangle \end{array}\right)$$

4.

Ground state :

$$\psi_g(x_1, x_2) = \psi_g(x_1)\psi_g(x_2) \otimes \frac{1}{\sqrt{2}}(|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle)$$

~~spin~~ ~~SS~~ $S=0$ $\rightarrow$ Spin angular momentum $= \underline{\underline{0}}$

First excited states :

$\psi_e(x_1, x_2) =$

$$\frac{1}{\sqrt{2}}(\psi_g(x_1)\psi_e(x_2) \mp \psi_e(x_1)\psi_g(x_2)) \otimes |\uparrow_1\rangle|\uparrow_2\rangle$$

or $$\frac{1}{\sqrt{2}}(\psi_g(x_1)\psi_e(x_2) \mp \psi_e(x_1)\psi_g(x_2)) \otimes \frac{1}{\sqrt{2}}(|\uparrow_1\rangle|\downarrow_2\rangle + |\downarrow_1\rangle|\uparrow_2\rangle)$$

or $$\frac{1}{\sqrt{2}}(\psi_g(x_1)\psi_e(x_2) \mp \psi_e(x_1)\psi_g(x_2)) \otimes |\downarrow_1\rangle|\downarrow_2\rangle$$

these 3 states have $S=1$ $\rightarrow$

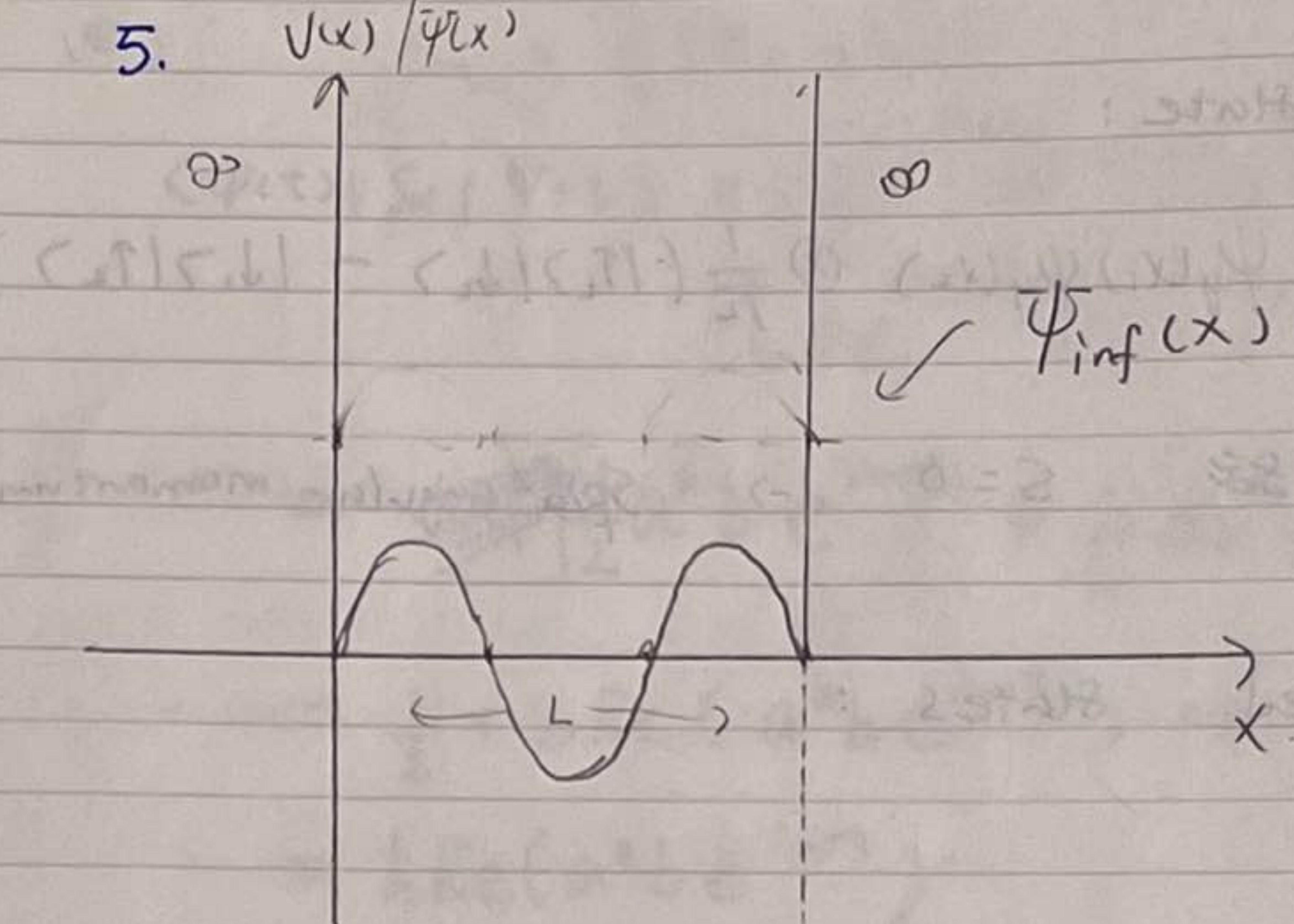
spin angular momentum $= \sqrt{1(1+1)}\hbar = \underline{\underline{\sqrt{2}\hbar}}$

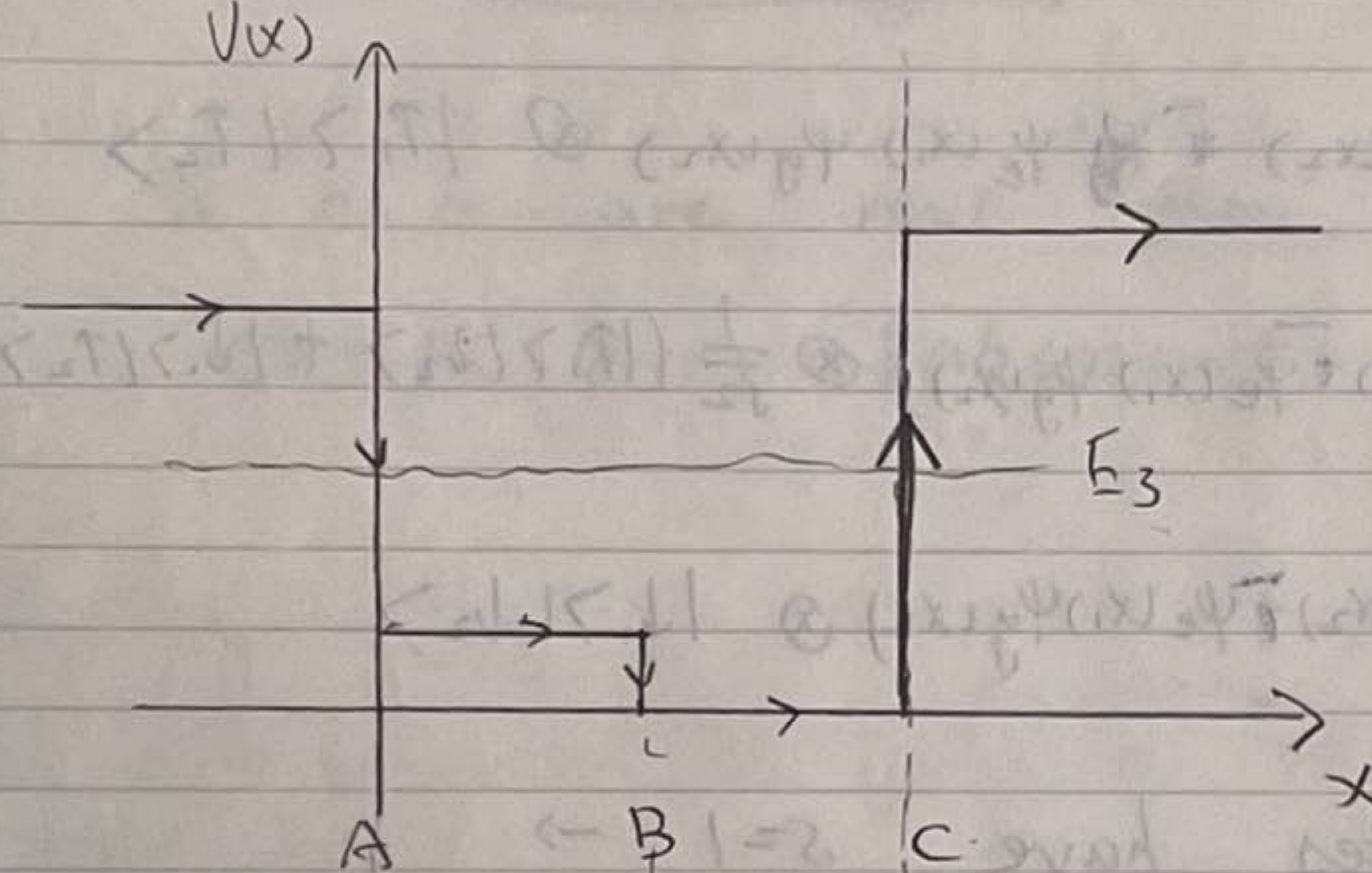
or $$\frac{1}{\sqrt{2}}(\psi_g(x_1)\psi_e(x_2) + \psi_e(x_1)\psi_g(x_2)) \otimes \frac{1}{\sqrt{2}}(|\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle)$$

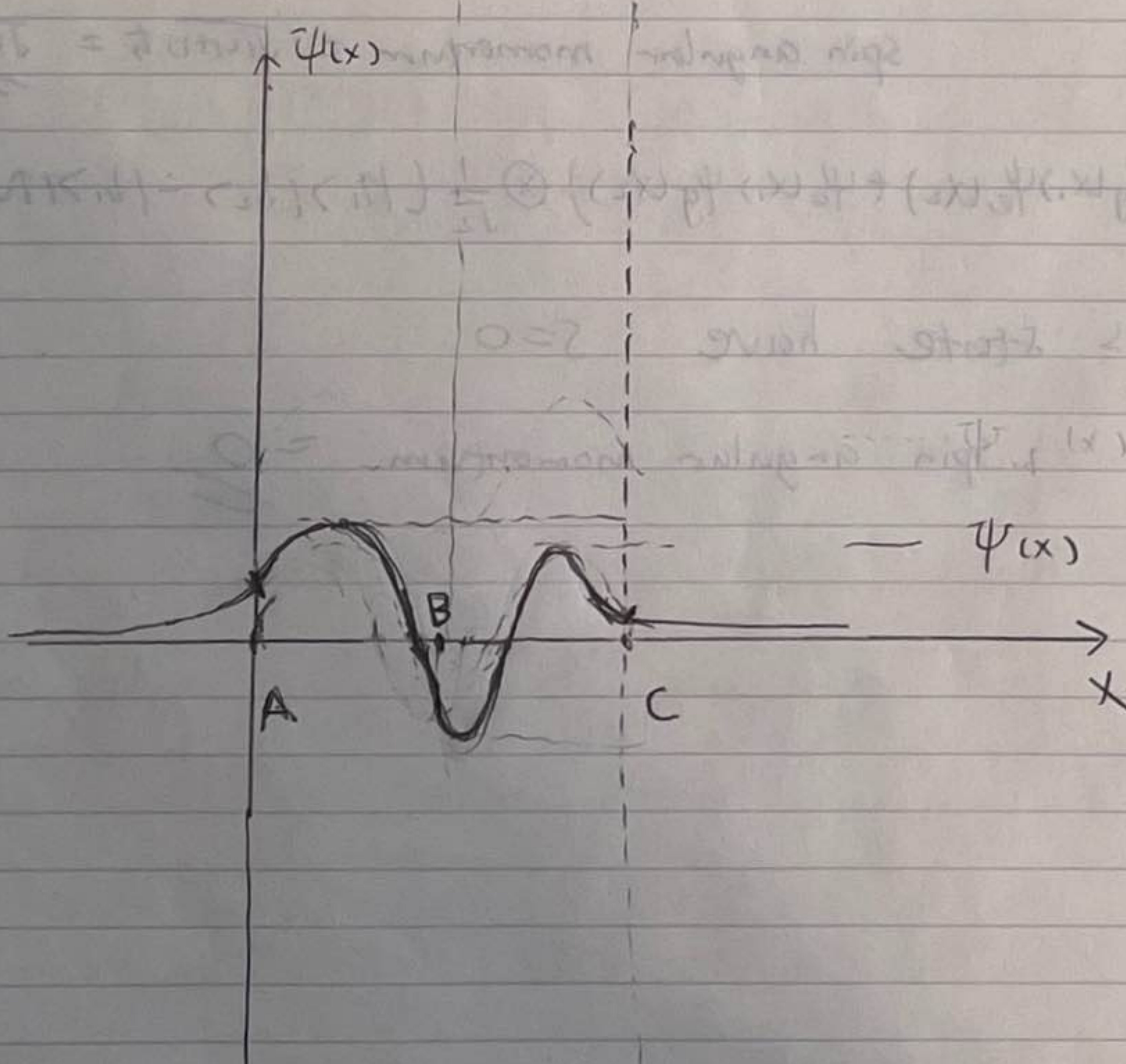
this state have $S=0$

Spin angular momentum $= \underline{\underline{0}}$

5.







6. (a)

$$E = -\frac{Z^2 R}{n^2} \quad (n = 1, 2, 3, \cdots)$$

$$R = \frac{1}{2}\mu\left(\frac{e^2}{4\pi\varepsilon_0\hbar}\right)^2$$

~~other~~ When electron is exchanged for a muon, the ratio in energy is equal to the ratio in reduce mass.

Mass of proton, electron, muon are, in (MeV/c^2):

$m_p = 938.3$, $m_e = 0.5110$, $m_\mu = 105.66$

$$\therefore \mu_{pe} = \frac{(938.3)(0.5110)}{938.3 + 0.5110} = 0.5107$$

$$\mu_{p\mu} = \frac{(938.3)(105.66)}{938.3 + 105.66} = 94.97$$

$\therefore$ ratio in energy for the same "n"

$$\frac{E_\mu}{E_e} \approx \frac{94.97}{0.5107} \approx \underline{186.0}$$

$\therefore$ The energy of each level $\underbrace{\ }_{\text{with muon}}$ will change to 186.0 times ~~the~~ ~~or~~ that with electron.

(b)

$$-\frac{\hbar^2}{2mr^2}\frac{d}{dr}\left(r^2\frac{dR_{n,l}(r)}{dr}\right) + \left(-\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\hbar^2 l(l+1)}{2mr^2} - E_{nl}\right)R_{nl}(r) = 0$$

$$\therefore -\frac{\hbar^2}{2mr^2}\left(2r\frac{dR_{nl}}{dr} + r^2\frac{d^2R_{nl}}{dr^2}\right) + \left(-\frac{Ze^2}{4\pi\varepsilon_0 r} + \frac{\hbar^2 l(l+1)}{2mr^2} - E_{nl}\right)R_{nl} = 0$$

If r is small we ignore $-\frac{Ze^2}{4\pi\varepsilon_0 r}$ and $-E_{nl}$ ~~comp~~ because they are small compare to $\frac{\hbar^2 l(l+1)}{2mr^2}$

$\therefore -\frac{\hbar^2}{2mr^2}\left(r^2R'' + 2rR' - l(l+1)R\right) = 0$

try ~~x~~ $R = Cr^{\alpha}$ then

$r^2R'' + 2rR' - l(l+1)R = 0$

$\rightarrow \cancel{[\alpha(\alpha+1) + 2\alpha - l(l+1)] = 0}$

$[\alpha(\alpha-1) + 2\alpha - l(l+1)] = 0$

$\therefore \alpha(\alpha+1) - l(l+1) = 0$

$\therefore (\alpha - l)(\alpha + l + 1) = 0$

$\therefore \alpha = l$ or $-(l+1)$

$\therefore R = C_{nl}r^{l} + C'_{nl}\frac{1}{r^{l+1}}$ for small r

But for small r, $\frac{1}{r^{l+1}}$ diverges $\therefore$ cannot have this term

$\therefore \underline{R = C_{n,l}r^{l}}$

(c) When $r > R$ potential due to uniformly charged sphere is the same as ~~that~~ that due to a point charge $\Rightarrow \delta V = 0$

When $r < R$ potential inside uniformly charged sphere is ~~potential~~ E-field integrated from ∞ to $r = R$ plus that integrated from $r = R$ to r

Integration from ∞ to R gives $-\frac{Ze^2}{4\pi\varepsilon_0 R}$ $\left(E \sim \frac{1}{r^2}\right)$

The integration from R to r gives

$$-\frac{Ze^2}{4\pi\varepsilon_0 R}\left(\frac{R^2 - r^2}{2R^2}\right)$$

$(E \sim r)$

the sum of these two minus $-\frac{Ze^2}{4\pi\varepsilon_0 r}$, which is the ~~usual~~ potential part of the ~~usual~~ usual Hydrogen Hamiltonian then gives the suggested form of ~~δV~~ $\delta V(r)$

when $l=0$ $R_{n0} = \cancel{C_{n0}}\ C_{n0} r^0 = C_{n0}$
(use this because $R \ll a_z$)

$\therefore$ The first order shift in energy is

$$\Delta E_{n0} = \langle \psi_{n0} | \delta V | \psi_{n0} \rangle = \int dr \cdot r^2 \cdot C_{n0}^2 \cdot \delta V$$

$$= \int_0^R dr \, (C_{n0}^2) \left(\frac{-Ze^2}{4\pi \varepsilon_0 R} \right) \left(r^2 + \frac{1}{2} r^2 - \frac{r^4}{2R^2} - rR \right)$$

$$\int_0^R dr \left(\frac{3}{2} r^2 - \frac{r^4}{2R^2} - rR \right)$$

$$= \left[\frac{1}{2} r^3 - \frac{r^5}{10R^2} - \frac{1}{2} r^2 R \right]_0^R = -\frac{R^5}{10R^2} = \frac{-R^3}{10}$$

$$\therefore \Delta E_{n0} = \frac{4 a_z^{-3}}{n^3} \cdot \frac{-Ze^2}{4\pi \varepsilon_0 R} \cdot -\frac{R^3}{10}$$

$$= \frac{Ze^2 R^2}{10 \pi \varepsilon_0 (a_z n)^3}$$

$\rightarrow E_{n0} \propto \frac{1}{n^3}$

$\therefore$ n large, electron so far away that it does not feel the perturbation potential that only has effect $< R$

$\therefore$ energy increases by ΔE_{n0}.

When $l \neq 0$, $R_{nl} \approx C_{nl} r^l$

$$\underbrace{\sim}_{\text{on the order of}} \left(\frac{r}{a_z} \right)^l R_{n0}$$

$\because R \ll a_z$ and $0 < r < R$ $\therefore \left(\frac{r}{a_z} \right)^l \ll 1$

$$\therefore \int dr \cdot r^2 (R_{nl})^2 \delta V \ll \int dr \, r^2 (R_{n0})^2 \delta V \quad \text{for } l \neq 0$$

$\rightarrow$ energy shift with $l \neq 0$ is very small compare to that with $l = 0$

7.
$$\hat{H} = \frac{\hat{P}^2}{2m} - \lambda_0 \delta(r-a)$$
$$= -\frac{\hbar^2}{2m}\nabla^2 - \lambda_0\delta(r-a)$$

(a) The potential is a spherical delta shell.

~~$\hat{H}$~~ The Laplacian operator in $\hat{H}$ contains $\hat{L}^2$

Y_l^m is the eigenfunction of $\hat{L}^2$ $\therefore$ If we write the solution as $\psi_{nlm}(r,\theta,\phi) = R_{nl}(r) Y_l^m(\theta,\phi)$

~~then~~, ~~we can~~ Since the potential is spherically symmetric, the only angular contribution of $\hat{H}$ is $\hat{L}^2$, we can then ~~rep~~ replace $\hat{L}^2\psi$ by $l(l+1)\hbar^2\psi$ and solve an equation that only depends on r to get $R(r)$ $\therefore$ This form is reasonable.

$m\hbar$ is the eigenvalue of $\hat{L}_z$
$l(l+1)\hbar^2$ is the eigenvalue of $\hat{L}^2$

Both with eigenfunction $Y_l^m(\theta,\phi)$ (spherical harmonics)

(b) $\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\hat{L}^2}{\hbar^2 r^2}$, $\psi = R_{nl}Y_l^m$, $\hat{L}^2\psi = l(l+1)\hbar^2\psi$

$\hat{H}\psi = E_{nl}\psi$

$$\rightarrow \left[-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right) - \lambda_0\delta(r-a)\right] R_{nl} = E_{nl}R_{nl}$$

(c) For wavefunction ψ to properly represent the probability amplitude $\underline{\psi_<(r=a) = \psi_>(r=a)}$ (ψ is continuous at a)

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{l(l+1)}{r^2}R\right) - \lambda_0\delta(r-a)R = ER \quad ①$$

$$\cancel{\int_{a-\varepsilon}^{a+\varepsilon}\frac{d^2R}{dr^2}dr = \left(\frac{dR}{dr}\right)\Big|_{\varepsilon} - \left(\frac{dR}{dr}\right)\Big|_{-\varepsilon}}$$

$$\cancel{\int_{-\varepsilon}^{\varepsilon}\frac{1}{r}\frac{dR}{dr}dr = \int_{r=-\varepsilon}^{r=\varepsilon}\frac{1}{r}dR = \left(\frac{1}{r}R\right)_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon}}$$

$$\rightarrow -\frac{\hbar^2}{2mr^2}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R\right] - \lambda_0\delta(r-a)R = ER$$

$\int \frac{2}{r}\frac{dR}{dr}\,dr = \frac{2}{r}dR + \int R\left(-\frac{2}{r^2}\right)dr \rightarrow 0$

$$\rightarrow -\frac{\hbar^2}{2m}\left[\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)r^2R\right] - \lambda_0\delta(r-a)r^2R = Er^2R \quad ①$$

$$\int_{a-\varepsilon}^{a+\varepsilon}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)dr = \left(r^2\frac{dR}{dr}\right)\Big|_{r=a+\varepsilon} - \left(r^2\frac{dR}{dr}\right)\Big|_{r=a-\varepsilon}$$

as $\cancel{\varepsilon\to 0}$

$$= a^2\left(\frac{dR}{dr}\Big|_{a+\varepsilon} - \frac{dR}{dr}\Big|_{a-\varepsilon}\right)$$

$$\int_{a-\varepsilon}^{a+\varepsilon} r^2R\,dr \rightarrow 0 \quad \text{as } \varepsilon\to 0$$

$$\int_{a-\varepsilon}^{a+\varepsilon} \delta(r-a)r^2R = \int_{-\infty}^{\infty}\delta(r-a)r^2R = a^2R(r=a)$$

$$\int_{a-\varepsilon}^{a+\varepsilon} R\,dr \rightarrow 0 \quad \text{as } \varepsilon\to 0$$

∴ Integrate equation ① we get:

∴ We have $-\frac{\hbar^2a^2}{2m}\left(\frac{dR}{dr}\Big|_{a+\varepsilon} - \frac{dR}{dr}\Big|_{a-\varepsilon}\right) - \lambda_0a^2R(r=a) = 0$

$$\rightarrow \underline{\frac{dR}{dr}\Big|_{a+\varepsilon} - \frac{dR}{dr}\Big|_{a-\varepsilon} = -\frac{2m\lambda_0}{\hbar^2}R(r=a)}$$

multiply by Y_l^m gives

$$\rightarrow \left.\frac{\partial \psi_>}{\partial r}\right|_{a+\varepsilon} - \left.\frac{\partial \psi_<}{\partial r}\right|_{a-\varepsilon} = -\frac{2m\lambda}{\hbar^2}\psi(r=a,\theta,\phi)$$

(d) For $r<a$, $l=0 \rightarrow l(l+1)=0$, $\delta(r-a)=0$

$$\therefore -\frac{\hbar^2}{2m}\left(R'' + \frac{2}{r}R'\right) = ER$$

$$\therefore R'' + \frac{2}{r}R' + \frac{2mE}{\hbar^2}R = 0$$

~~let $k = \sqrt{\frac{2mE}{\hbar^2}}$~~ ~~let $-k^2 = \frac{2mE}{\hbar^2}$, then~~

$\cancel{R'' + \frac{2}{r}R'}$ and for $l=0$, $Y_l^m = Y_0^0 = \frac{1}{\sqrt{4\pi}}$

↳ can be absorbed into the integration constant

$$\psi = RY \propto R$$

For R, try $R = \frac{e^{\alpha r}}{r}$, then $R' = -\frac{1}{r^2}e^{\alpha r} + \alpha\frac{e^{\alpha r}}{r}$

$$R'' = +\frac{2}{r^3}e^{\alpha r} - \alpha\frac{1}{r^2}e^{\alpha r} - \alpha\frac{1}{r^2}e^{\alpha r} + \alpha^2\frac{e^{\alpha r}}{r}$$

$\therefore R'' + \frac{2}{r}R' + \frac{2mE}{\hbar^2}R = 0$ gives

$$\cancel{\frac{2}{r^3}e^{\alpha r}} \cancel{-2\alpha\frac{1}{r^2}e^{\alpha r}} + \alpha^2\frac{e^{\alpha r}}{r} \cancel{-\frac{2}{r^3}e^{\alpha r}} + \cancel{\frac{2\alpha}{r^2}e^{\alpha r}}$$

$$+ \frac{2mE}{\hbar^2}\left(\frac{e^{\alpha r}}{r}\right) = 0$$

Bound state $E<0$, let $k = \sqrt{-\frac{2mE}{\hbar^2}}$ $\rightarrow$ $\frac{2mE}{\hbar^2} = -k^2$

$$\therefore \alpha^2 - k^2 = 0 \rightarrow (\alpha+k)(\alpha-k) = 0$$

$$\rightarrow \alpha = \pm k$$

$\therefore$ General solution is $\psi_<(r) = A\frac{e^{kr}}{r} + B\frac{e^{-kr}}{r}$

$$\psi_<(r) = \frac{1}{r}(Ae^{\varkappa r} + Be^{-\varkappa r})$$

Probability to find the particle in $(r, r+dr]$ is then

$$P(r, dr) = r^2|\psi_<(r)|^2 dr = (Ae^{\varkappa r} + Be^{-\varkappa r})^2 dr$$

when $r \to 0$, $Ae^{\varkappa r} + Be^{-\varkappa r} \to A+B$ doesn't blow up

$\therefore$ No problem.

But if we write ~~$\psi_>(r) = \frac{1}{r}(Ce^{\varkappa r} + Be^{-\varkappa r})$~~

$$\psi_>(r) = \frac{1}{r}(Ce^{\varkappa r} + De^{-\varkappa r})$$

(Because $r<0$ and $r>0$ have the same Hamiltonian)

~~then~~

then $P(r, dr) = |\psi_>(r)|^2 r^2 dr = (Ce^{\varkappa r} + De^{-\varkappa r})^2 dr$

the $Ce^{\varkappa r}$ term blows up when $r \to \infty$

We want bound state so C must $= 0$

$$\therefore \psi_>(r) = D\frac{e^{-\varkappa r}}{r}$$

(e) The Boundary conditions:

① $\psi_<(r=a) = \psi_>(r=a)$

$$\therefore A\frac{e^{\varkappa a}}{a} + \cancel{Be^{\varkappa(-a)}}\; B\frac{e^{-\varkappa a}}{a} = D\frac{e^{-\varkappa a}}{a}$$

$$\therefore Ae^{2\varkappa a} + B = D$$

$$\to \psi_>(r) = (Ae^{2\varkappa a} + B)\frac{e^{-\varkappa r}}{r}$$

② $$\left.\frac{\partial\psi_>}{\partial r}\right|_{r=a} - \left.\frac{\partial\psi_<}{\partial r}\right|_{r=a} = -\frac{2m\lambda_0}{\hbar^2}\psi(a)$$

$$\frac{\partial\psi_>}{\partial r} = (Ae^{2\varkappa a} + B)\left[(-\varkappa - \frac{1}{r})\frac{e^{-\varkappa r}}{r}\right].$$

$$\frac{\partial\psi_<}{\partial r} = A(\varkappa - \frac{1}{r})\frac{e^{\varkappa r}}{r} + B(-\varkappa - \frac{1}{r})\frac{e^{-\varkappa r}}{r}$$

$$\psi(a) = (Ae^{2ka} + B)\frac{e^{-ka}}{a}$$

$$\therefore \cancel{\frac{e^{2ka}}{a}\left(A(e^{2ka})(-k-\frac{1}{a}) - A(k-\frac{1}{a})\right)}$$

$$= \cancel{\frac{e^{2ka}}{a}(Ae^{2ka}+B)(-\frac{2m\lambda_0}{\hbar^2})}$$

$$\therefore \cancel{Ae^{2ka}(-k) - Ae^{2ka}\frac{1}{a} - Ak + \frac{A}{a}}$$

$$\therefore A\frac{e^{ka}}{a}(-k-\frac{1}{a}) - A\frac{e^{ka}}{a}(+k-\frac{1}{a}) = (Ae^{2ka}+B)\frac{e^{-ka}}{a} \times (-\frac{2m\lambda_0}{\hbar^2})$$

$$\therefore A(2k) = \frac{2m\lambda_0}{\hbar^2}(Ae^{2ka}+B)e^{-2ka}$$

$$\therefore 2k = \frac{2m\lambda_0}{\hbar}(1+e^{-2ka}B/A)$$

$$\rightarrow \lambda_0 = \frac{\hbar^2}{2m}\frac{2k}{1+e^{-2ka}B/A}$$

To determine A, B exactly, we need the normalisation condition $\int_0^\infty dr.r^2|\psi(r)|^2 = 1$

$$\rightarrow \int_0^a dr.r^2|\psi_<(r)|^2 + \int_a^\infty dr\, r^2|\psi_>(r)|^2 = 1$$

8.
(a) $\hat{H} = \frac{\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2}{2\mu a^2} = \frac{\hat{L}^2}{2\mu a}$

$\therefore$ energy levels $E_l = \frac{l(l+1)\hbar^2}{2\mu a^2}$ $(l = 0, 1, 2, \dots)$

corresponding eigenfunctions to E_l : $Y_l^m(\theta, \phi)$
spherical harmonics
$m = -l, -l+1, \dots, l-1, l$

Degeneracy of E_l : $2l+1$ 　 $2l+1$ fold degenerate

Spacing between E_{l+1} and E_l is

$$\Delta E_l = E_{l+1} - E_l = \frac{\hbar^2}{2\mu a}\left[(l+1)(l+2) - l(l+1)\right]$$

$$= \frac{\hbar^2}{2\mu a^2}(2l+2) = \frac{\hbar^2(l+1)}{\mu a^2}$$

(b) The selection rule for electric dipole radiation is $\Delta l = \pm 1$

$\therefore$ The allowed transitions are those change the value of l by 1

→ only possible emission and absorption lines are ~~between~~ generated ~~between~~ through transitions between adjacent energy levels.

(1 amu $= 1.66 \times 10^{-27}$ kg)
Cl $= 35$ amu
H $= 1$ amu

The spacing between transition lines is

$$\Delta E = |\Delta E_{l\pm1} - \Delta E_l| = \frac{\hbar^2}{\mu a^2}|(l\pm1-l)| = \frac{\hbar^2}{\mu a^2}$$

$\therefore$ ΔE is independent of l

$\therefore$ pure rotational spectrum has equally spaced lines.

$\mu = \frac{m_{Cl} m_H}{m_H + m_{Cl}} = \frac{35}{36}$ amu $= 1.61 \times 10^{-27}$ kg $\rightarrow \frac{\hbar^2}{\mu a^2} = \Delta E$ $\therefore$ $a = \left(\frac{\hbar^2}{\mu \Delta E}\right)^{\frac{1}{2}} = 1.3 \times 10^{-10}$ m

(c) $\psi(\theta,\phi) = \frac{1}{\sqrt{8\pi}}(1 + \sqrt{3}\sin\theta\sin\phi)$

$= \frac{1}{\sqrt{8\pi}}\left(1 + \sqrt{3}\sin\theta\left(\frac{e^{i\phi} - e^{-i\phi}}{2i}\right)\right)$

$= \frac{1}{\sqrt{8\pi}} + \frac{1}{2i}\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} - \frac{1}{2i}\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{4\pi}}\right) + \frac{i}{2}\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\right) + \frac{i}{2}\left(\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$

$= \left(\frac{1}{\sqrt{2}}\right)Y_0^0 + \left(\frac{i}{2}\right)Y_1^1 + \left(\frac{i}{2}\right)Y_1^{-1}$

$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{2}\right|^2 + \left|\frac{i}{2}\right|^2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 \rightarrow \psi$ is normalised

$\hat{L}^2 \rightarrow$ ~~Measurement probability~~ eigenvalues of Y_m^l is $l(l+1)\hbar^2$

Y_0^0 $Y_0^0 \rightarrow l = 0,\ m = 0$ $l(l+1)\hbar^2 = 0$ $P(l=0) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$Y_1^1 \rightarrow l = +1,\ m = 1$ $l(l+1)\hbar^2 = +2\hbar^2$ ~~$P(l=1) = \left|\frac{i}{2}\right|^2 = \frac{1}{4}$~~ $P\binom{l=1}{m=1} = \frac{1}{4}$

$\frac{1}{\sqrt{2}}(Y_1^1 + Y_1^{-1})$

$Y_1^{-1} \rightarrow l = +1,\ m = -1$ $l(l+1)\hbar^2 = +2\hbar^2$ ~~$P(l=-1) = \left|\frac{i}{2}\right|^2 = \frac{1}{4}$~~ $P\binom{l=1}{m=-1} = \frac{1}{4}$

$\rightarrow P(l=1) = \frac{1}{2}$

$\hat{L}_z \rightarrow$ eigenvalues of Y_m^l is $m\hbar$

Y_0^0 $Y_0^0 \rightarrow m = 0,\ l = 0$ $m\hbar = 0$ $P(m=0) = \frac{1}{2}$

Y_1^{+1} $m = 1,\ l = 1$ $m\hbar = \hbar$ $P(m=1) = \frac{1}{4}$

Y_1^{-1} $m = -1,\ l = 1$ $m\hbar = -\hbar$ $P(m=-1) = \frac{1}{4}$

9. (a)

TISE $\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi = E\psi$

$V(x) = \begin{cases} 0, & 0<x<L \\ \infty, & \text{otherwise} \end{cases}$

$\therefore \frac{\hat{p}^2}{2m}\psi = (E - V(x))\psi$, ~~and~~ when $V(x) = \infty$

We have infinite kinetic energy when ψ is non-zero $\therefore$ $\underline{\psi = 0 \text{ when } x \leq 0, \ x \geq L}$

For $0<x<L$, $V(x)=0$:

TISE $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

$\therefore \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$

If $E<0$ then $\psi = A'e^{+kx} + B'e^{-kx}$ $\quad (k = \sqrt{-\frac{2mE}{\hbar^2}})$

Boundary conditions : at ~~$\psi=0$, $\psi=L$~~ $x=0, L$, $\psi=0$

$\frac{d\psi}{dx}$ can be discontinuous since $\frac{\partial^2\psi}{\partial x^2} \to \infty$

$\psi(0) = 0 \rightarrow A' + B' = 0$

$\psi(L) = 0 \rightarrow A'e^{+kL} + B'e^{-kL} = 0$

$\rightarrow A'(e^{kL} - e^{-kL}) = 0$

$\rightarrow A' \sinh(kL) = 0 \rightarrow k=0 \rightarrow E=0$

contradiction

$\therefore$ If $E=0$

$\psi = A''x + B''$

$\psi(0)=0 \rightarrow B''=0$ $\quad \psi = A''x$

$\psi(L)=0 \rightarrow A''L=0 \rightarrow A''=0 \rightarrow \psi(x)=0$

contradiction

→ We must have $E > 0$. let real number k be ~~$k^2 =$~~ $k = \sqrt{\frac{2mE}{\hbar^2}}$, then

$$\psi'' - k^2\psi = 0 \rightarrow \psi(x) = A\sin(kx) + B\cos(kx)$$

$\psi(0) = 0 \rightarrow$ ~~$B\cos(kL) = 0$~~ $B = 0$, $\psi = A\sin(kx)$

$\psi(L) = 0 \rightarrow A\sin(kL) = 0 \rightarrow kL = n\pi \quad (n = 1, 2, 3, \cdots)$

$\therefore k = \frac{n\pi}{L} \rightarrow \frac{n^2\pi^2}{L^2} = \frac{2mE}{\hbar^2} \rightarrow$

$$\rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$\psi(x) = A\sin\left(\frac{n\pi x}{L}\right)$ for $0 < x < L$

Normalise $\int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 1$

$$A^2\int_0^L dx\, \sin^2\left(\frac{n\pi x}{L}\right) = 1$$

$$\therefore A^2\left[\frac{x}{2} - \frac{L\sin\left(\frac{2\pi n x}{L}\right)}{4\pi n}\right]_0^L = 1$$

$$\therefore \frac{L}{2}A^2 = 1 \rightarrow A = \sqrt{\frac{2}{L}}$$

$$\therefore \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

(b) Spinless particles $\therefore$ only spatial wavefunction
particles are Bosons

Wavefunction is symmetric

two particles in different energy levels:

$$\psi_{n_1,n_2}(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\right)$$

two particles in same energy level:

$$\psi_{n,n}(x_1, x_2) = \psi_n(x_1)\psi_n(x_2)$$

eigenvalue

$$E_{n_1,n_2} = E_{n_1} + E_{n_2} = \underline{\frac{\pi^2\hbar^2}{2mL^2}(n_1^2+n_2^2)}$$

~~(c) perturbation~~ $\cancel{\delta V(x_1,x_2) = -\lambda\cos\left(\frac{\pi(x_1-x_2)}{2L}\right)}$

ground state :

$$\psi_{1,1}(x_1,x_2) = \psi_1(x_1)\psi_1(x_2)$$

$$\cancel{= \sin \frac{2}{L\alpha}}$$

$$\rightarrow \psi_{1,1}(x_1,x_2) = \underline{\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)} \checkmark$$

(c) perturbation $\delta V(x_1,x_2) = -\lambda\cos\left(\frac{\pi(x_1-x_2)}{2L}\right)$

First order shift in ground state energy is

$$\Delta E = \langle 1,1|\delta V|1,1\rangle \quad \cancel{= \langle 1|\langle 1|\delta V|1\rangle|1\rangle}$$

$$\cancel{= \int dx_1 dx_2 \langle 1|x_1\rangle\langle x_1|}$$

$$= \int dx_1 dx_2 \langle 1,1|x_1,x_2\rangle \langle x_1,x_2|\delta V|1,1\rangle$$

$$= \int dx_1 dx_2 \langle 1,1|x_1,x_2\rangle \,\delta V\, \langle x_1,x_2|1,1\rangle$$

$$= \int dx_1 dx_2 |\langle x_1,x_2|1,1\rangle|^2 \delta V$$

$$\cancel{= \int dx_1 dx_2} = \left(\frac{2}{L}\right)^2(-\lambda)\int_0^L dx_1\int_0^L dx_2 \sin^2\left(\frac{\pi x_1}{L}\right)\sin^2\left(\frac{\pi x_2}{L}\right)\cos\left(\frac{\pi(x_1-x_2)}{2L}\right)$$

$$= -\frac{4\lambda}{L^2}\int_0^L dx_1\int_0^L dx_2 \sin^2\left(\frac{\pi x_1}{L}\right)\sin^2\left(\frac{\pi x_2}{L}\right)\left(\cos\left(\frac{\pi x_1}{2L}\right)\cos\left(\frac{\pi x_2}{2L}\right)+\sin\left(\frac{\pi x_1}{2L}\right)\sin\left(\frac{\pi x_2}{2L}\right)\right)$$

$$= -\frac{4\lambda}{L^2}\Bigg[\int_0^L dx_1 \sin^2\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_1}{2L}\right)\int_0^L dx_2 \sin^2\left(\frac{\pi x_2}{L}\right)\cos\left(\frac{\pi x_2}{2L}\right)$$

$$+ \cancel{\int_0^L dx_1 \sin^3\left(\frac{\pi x_1}{2L}\right)} \int_0^L dx_1 \sin^2\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_1}{2L}\right)\int_0^L dx_2 \sin^2\left(\frac{\pi x_2}{L}\right) \times \sin\left(\frac{\pi x_2}{2L}\right)\Bigg]$$

$$\int_0^L dx_1 \sin^2\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_1}{2L}\right) \quad \left\{ \text{let } x = \frac{\pi x_1}{2L} \quad . \; dx = \frac{\pi}{2L} dx_1 \to dx_1 = \frac{2L}{\pi} dx \right.$$

$$x_1 = L \to x = \frac{\pi}{2}$$

$$= \frac{2L}{\pi}\int_0^{\pi/2} \sin^2(2x)\sin(x)\,dx$$

$$= \frac{2L}{\pi}\int_0^{\pi/2} dx\, 2\sin(2x)\cos(2x)\sin(x) = \frac{4L}{2\pi}\int_0^{\pi/2} dx\cos(2x)[\cos(2x-x) - \cos(2x+x)]$$

$$= \frac{2L}{\pi}\int_0^{\pi/2} (\cos(2x)\cos(x) - \cos(2x)\cos(3x))\,dx$$

$$= \frac{2L}{\pi}\int_0^{\pi/2} \cos(3x) + \cos(x) - \cos(5x) - \cos(x)\,dx$$

$$= \frac{2L}{\pi}\left\{\left[\frac{1}{3}\sin(3x)\right]_0^{\pi/2} - \left[\frac{1}{5}\sin(5x)\right]_0^{\pi/2}\right\}$$

$$= \frac{2L}{\pi}\left\{-\frac{1}{3} - \frac{1}{5}\right\} = \frac{2L}{\pi}\left(\frac{8}{15}\right) = \frac{16L}{15\pi}$$

$$\int_0^L dx_1 \sin^2\left(\frac{\pi x_1}{L}\right)\cos\left(\frac{\pi x_1}{2L}\right) = \frac{2L}{\pi}\int_0^{\pi/2}\sin^2(2x)\cos(x)$$

$$= \frac{2L}{\pi}\int_0^{\pi/2}\sin(2x)\sin(2x)\cos(x)$$

$$= \frac{2L}{\pi}\int_0^{\pi/2}\sin(2x)\left(\frac{1}{2}\right)(\sin(3x) + \sin(x))\,dx$$

$$= \frac{L}{\pi}\int_0^{\pi/2}\sin(2x)\sin(3x) + \sin(2x)\sin(x)\,dx$$

$$= \frac{L}{2\pi}\int_0^{\pi/2}\cos(x) - \cos(5x) + \cos(x) - \cos(3x)\,dx$$

$$= \frac{L}{2\pi}\left[2\sin(x)\Big|_0^{\pi/2} - \frac{1}{3}\sin(3x)\Big|_0^{\pi/2} - \frac{1}{5}\sin(5x)\Big|_0^{\pi/2}\right]$$

$$= \frac{L}{2\pi}\left(2 + \frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{L}{2\pi}\left(\frac{32}{15}\right) = \frac{16L}{15\pi}$$

$$\int_0^L dx_1 \, \sin^2\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_1}{2L}\right) = \frac{2L}{\pi}\int_0^{\pi/2} dx \, \sin^2(2x) \sin(x)$$

$$= \frac{2L}{\pi}\int_0^{\pi/2} dx \, \sin(2x) \sin(2x) \sin(x)$$

$$= \frac{2L}{\pi}\int_0^{L/2} dx \, \sin(2x) \, \frac{1}{2}\cdot\left(\cos(x) - \cos(3x)\right)$$

$$= \frac{L}{\pi}\int_0^{\pi/2} dx \, \sin(2x)\cos(x) - \sin(2x)\cos(3x)$$

$$= \frac{L}{2\pi}\int_0^{\pi/2} dx \, \sin(x) + \sin(3x) - (-\sin(x) + \sin(5x))$$

$$= \frac{L}{2\pi}\left[-2\cos(x)\Big|_0^{\pi/2} - \frac{\cos(3x)}{3}\Big|_0^{\pi/2} + \frac{\cos(5x)}{5}\Big|_0^{\pi/2}\right]$$

$$= \frac{L}{2\pi}\left(2 + \frac{1}{3} - \frac{1}{5}\right) = \frac{L}{2\pi}\left(\frac{32}{15}\right) = \frac{16L}{15\pi}$$

$$\therefore \Delta E = -\frac{4\lambda}{L^2}\left[2 \times \left(\frac{16L}{15\pi}\right)^2\right]$$

$$= -\frac{4\lambda}{\cancel{L^2}} \cdot 2 \cdot \frac{16^2 \cancel{L^2}}{15^2\pi^2} = -\frac{8 \times 16^2 \lambda}{15^2\pi^2}$$

$$= -\frac{2048}{225\pi^2}\lambda$$

1. $H = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + 4y^2)$

$E_n = (n + \frac{1}{2})\hbar\omega$

$\psi = \psi_x \psi_y$

$E_{n_x, n_y} = (n_x + \frac{1}{2})\hbar\omega + (n_y + \frac{1}{2})\hbar(2\omega)$

$E_{0,0}\ ,\ E_{1,0}\ \ E_{0,1} = E_{2,0} = \frac{7}{2}\hbar\omega$

~~2.~~ 2. picture taken

3 $\hat{H} = \omega \hat{S_z}$ $\qquad \hat{S_z}|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$

$|\psi(0)\rangle = a|+\rangle + b|-\rangle$

$|a|^2 + |b|^2 = 1$

$|+\rangle \rightarrow \frac{\hbar}{2} \leftarrow P(+) = |a|^2$

$|-\rangle \rightarrow -\frac{\hbar}{2} \leftarrow P(-) = |b|^2$

$|\psi(t)\rangle = \cancel{a\exp}\ a e^{-\frac{iE_+ t}{\hbar}}|+\rangle + b e^{-\frac{iE_- t}{\hbar}}|-\rangle$

$\hat{H}|\pm\rangle = \pm\frac{\hbar\omega}{2}|\pm\rangle$

$\therefore\ |\psi(t)\rangle =$

$\langle\psi(0)|\psi(t)\rangle$ = Probability amplitude for the system to return to state $|\psi(0)\rangle$ after time t.

$\rightarrow$ return amplitude

$|\pm, x\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$ eigenvalues $\pm\frac{\hbar}{2}$

$$\hat{S}_x|\psi(0)\rangle = \hat{S}_x\Big(\frac{a}{\sqrt{2}}(|+,x\rangle + |-,x\rangle)$$

$$+ \frac{b}{\sqrt{2}}(|+,x\rangle - |-,x\rangle)\Big)$$

$$\Rightarrow +\frac{\hbar}{2}, \quad \frac{|a+b|^2}{2} = P(+\tfrac{\hbar}{2})$$

$$\Rightarrow -\frac{\hbar}{2}, \quad \frac{|a-b|^2}{2} = P(-\tfrac{\hbar}{2})$$

$$\langle\psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$= \left(a^* e^{+\frac{i\omega t}{2}}, b^* e^{-\frac{i\omega t}{2}}\right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a e^{-\frac{i\omega t}{2}} \\ b e^{+\frac{i\omega t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(b a^* e^{+i\omega t} + b^* a e^{-i\omega t}\right)$$

$$= \hbar \operatorname{Re}(a b^* e^{-i\omega t})$$

4 $V = \frac{1}{2} m\omega^2 x^2$

Ground state : ~~Ψ_g~~

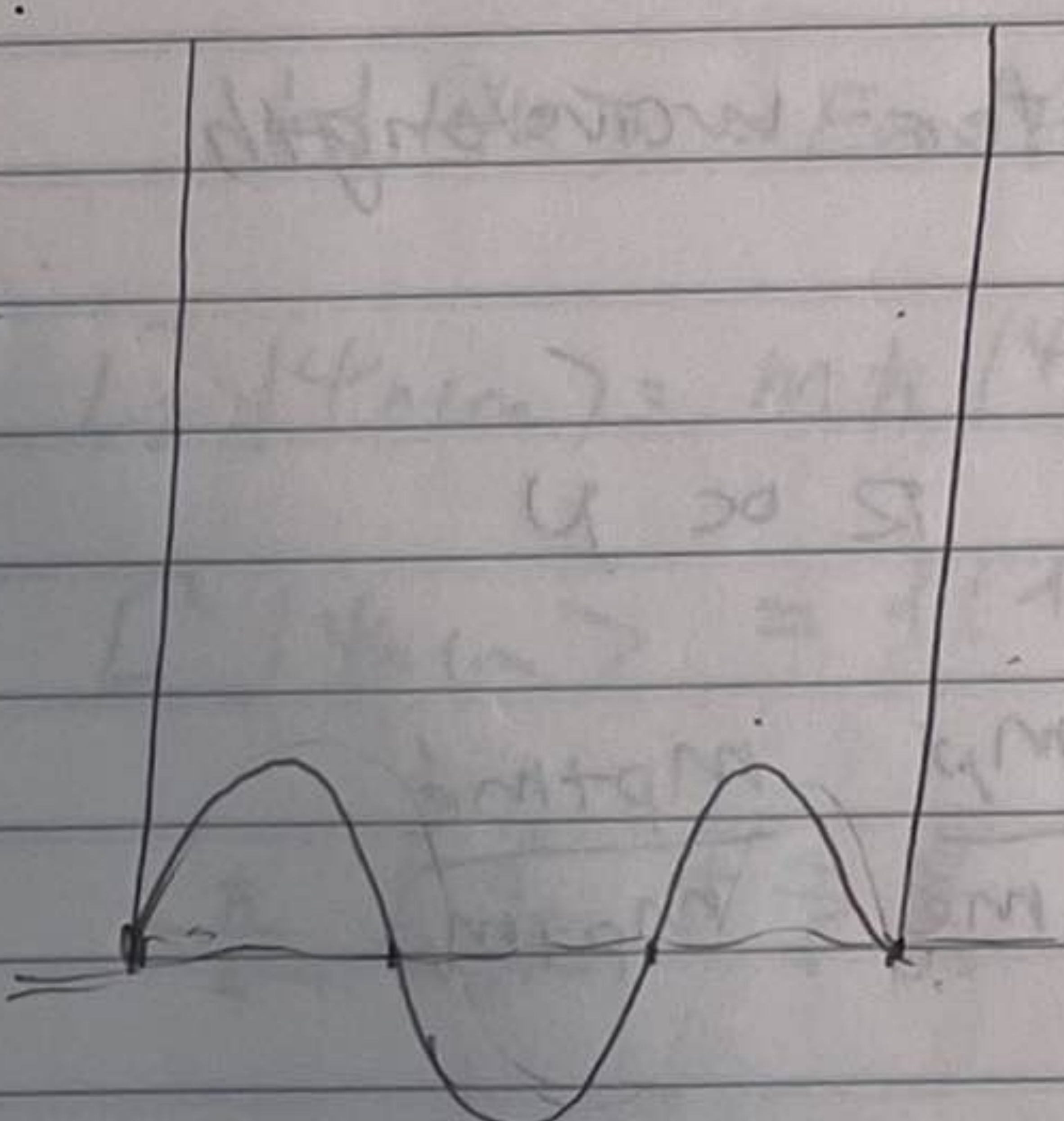
$$\psi_g(x_1)\,\psi_g(x_2)\left(\frac{|+,-\rangle - |-+\rangle}{\sqrt{2}}\right)$$

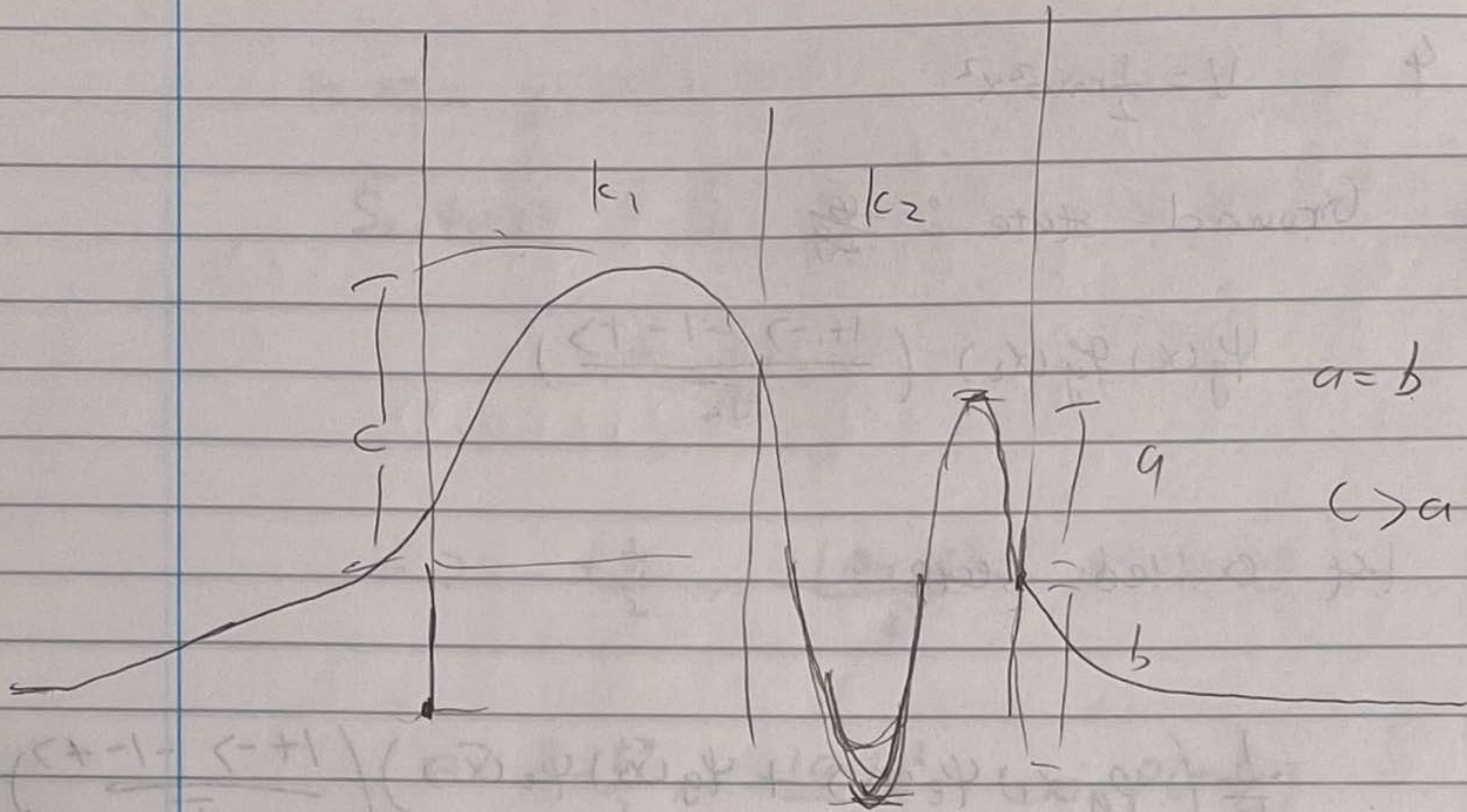
1st excited states :

$$\frac{1}{\sqrt{2}}\left(\psi_g(x_1)\psi_e(x_2) + \psi_g(x_2)\psi_e(x_1)\right)\left(\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}\right)$$

$$\frac{1}{\sqrt{2}}\left(\psi_g(x_1)\psi_e(x_2) - \psi_g(x_2)\psi_e(x_1)\right)\begin{pmatrix} |++\rangle \\ \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \\ |--\rangle \end{pmatrix}$$

5.





$a = b$

$\Leftrightarrow a = b$

$$|\psi_n^{(1)}\rangle = \sum_k \frac{\langle \psi_k^{(0)} | \delta V | \psi_n^{(0)} \rangle}{E_n - E_k} |\psi_k^{(0)}\rangle$$

$k_2 < k_1 \quad \therefore \lambda_2 < \lambda_1$

larger k, shorter wavelength.

$$E_n = -\frac{R}{n^2} \quad R \propto \mu$$

$$\frac{\mu_\mu}{\mu_e} = \frac{\cancel{m}_p m_\mu}{\cancel{m}_p m_e} \frac{m_p + m_e}{m_p + m_\mu}$$

$$m_\mu = \frac{m_p}{9} \simeq 207\ m_e$$

$$\therefore \frac{E_\mu}{E_e} \to 186.$$

6 $\underline{L} = \underline{r} \times \underline{p}$

fix $n \Rightarrow$ fix p

$r \to$ large $\Leftrightarrow$ $\underline{L} \to$ large

$\therefore l$ large $\to$ r large.

$\therefore$ R is small

$\therefore$ l large $\to$ δV has small effect.

$\to$ $\langle \delta V \rangle \to 0$ as l large.

7. $\hat{H} = \frac{\hat{p}^2}{2m} - \lambda_0 \delta(r-a) \longrightarrow$ spherical symmetry.

$\psi_{nlm} = R_{nl}(r) Y_l^m(\theta, \phi)$

$H|\psi_n\rangle = E_n |\psi_n\rangle$

$\hat{L_z} |\psi_{nlm}\rangle = m\hbar |\psi_{nlm}\rangle$

$\hat{L}^2 |\psi_{nlm}\rangle = l(l+1)\hbar^2 |\psi_{nlm}\rangle$

$$-\frac{\hbar}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\right)R_{nl}(r) - \lambda\,\delta(r-a)\,R_{nl}(r)$$

$$= E_n R_{nl}(r)$$

(c) $R_{nl}(a^+) = R_{nl}(a^-)$

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\psi_c = -\frac{2mE}{\hbar^2}\psi_c(r)$$

$\underline{\lambda_0 > 0}$ $\rightarrow$ for bound states.

(a) $E_{\ell m} = \frac{\hbar^2}{2Na^2}\ell(\ell+1)$

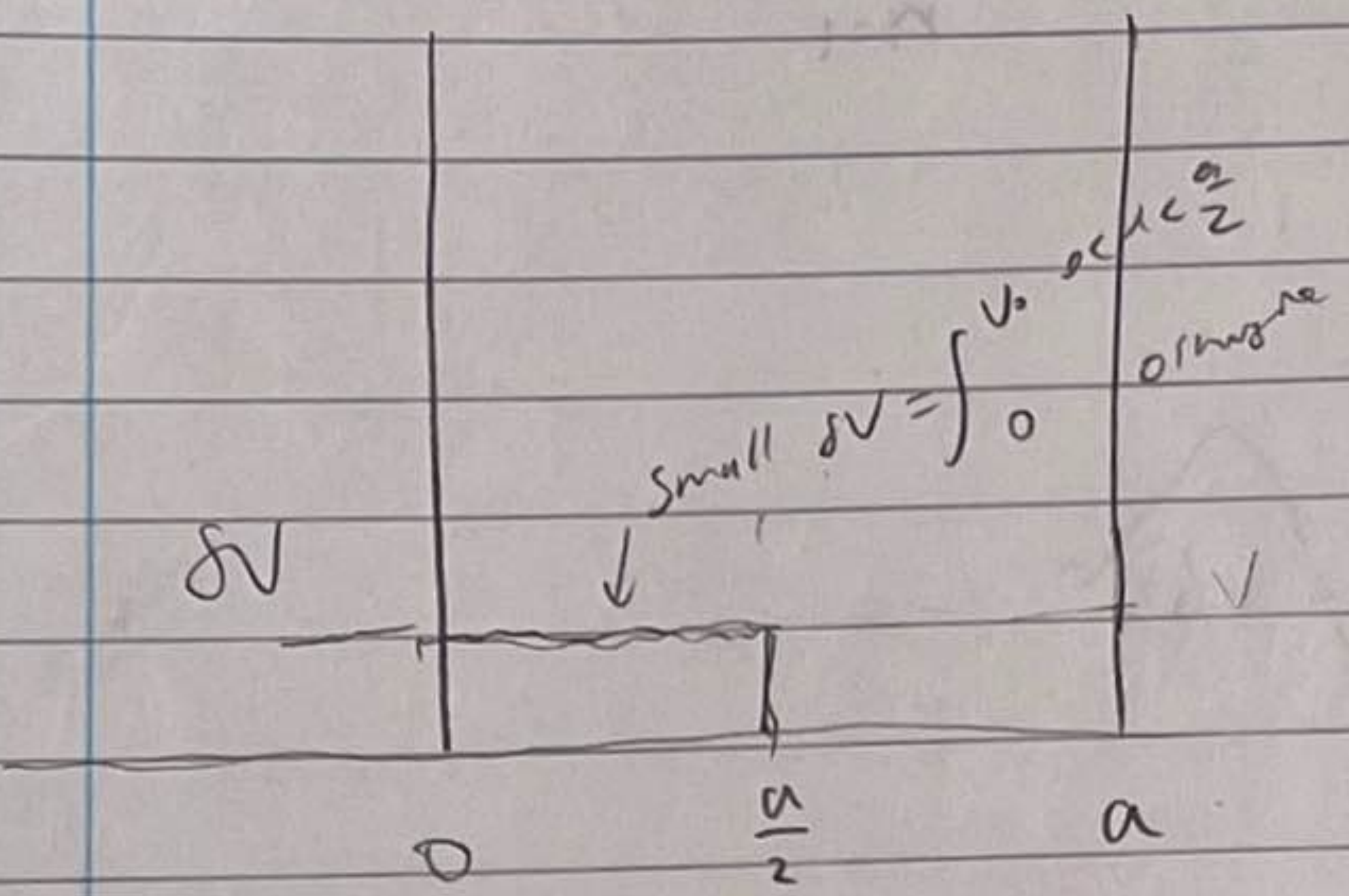
$m-3=$ $\quad (-1)^{\frac{m}{2}+1}$ $\quad m=2n+1$ $\quad n+1$

$\sin(\frac{\pi}{2})=1$

$\sin(\frac{3\pi}{2})=-1$

$2n-3$

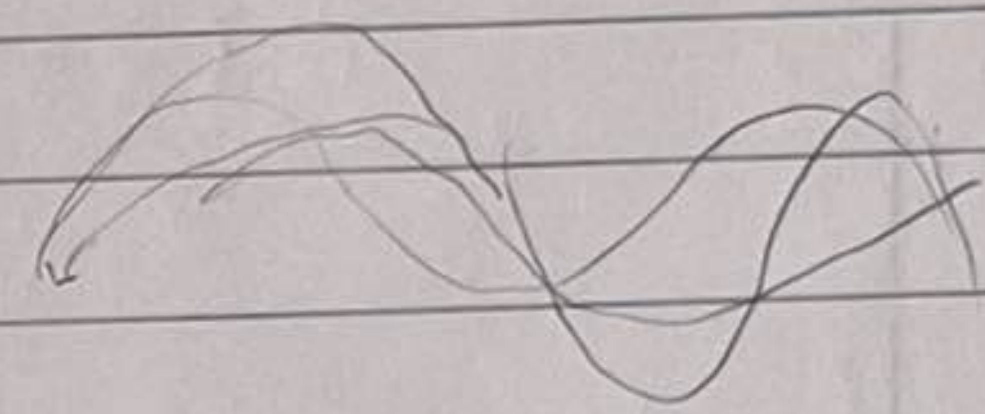
3rd level of Asymmetric Potential well



$$u_1^{(0)}(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$$

$$u_2^{(0)}(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)$$

$$u_3^{(0)}(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)\dots$$



$$E_3^{(0)} = \frac{9\pi^2\hbar^2}{2ma^2} \qquad \delta E_3^{(1)} = \langle 3|\delta V|3\rangle$$

$$\delta|3\rangle^{(1)} = \sum_{m\neq 3} \frac{\langle m|\delta V|3\rangle}{E_3 - E_m}|m\rangle$$

$$\langle m|\delta V|3\rangle = \frac{2V_0}{a}\int_0^{a/2} \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

$$= \frac{2V_0}{a}\left[\frac{a}{2\pi}\right]\left[\frac{\sin\left(\frac{(3-m)\pi x}{a}\right)}{3-m} + \frac{\sin\left(\frac{(3+m)\pi x}{a}\right)}{3+m}\right]_0^{a/2}$$

$$= \frac{\cancel{2}V_0}{\cancel{a}}\cdot\frac{\cancel{a}}{\cancel{2}\pi}\left[\frac{\sin\left(\frac{(m-3)}{2}\pi\right)}{m-3} + \frac{\sin\left(\frac{m+3}{2}\pi\right)}{m+3}\right]$$

$$= \frac{V_0}{\pi}\left[\cancel{\frac{(-1)^n}{2n+3}}\ (-1)^n\left(\frac{1}{2n-3} - \frac{1}{2n+3}\right)\right].$$

$m=2n$

$$= \frac{(-1)^n V_0}{\pi}\left[\frac{1}{2n-3} - \frac{1}{2n+3}\right] = \frac{(-1)^n V_0}{\pi}\frac{6}{4n^2-9}$$

$$E_3 - E_{2n} = \frac{\pi^2\hbar^2}{2ma^2}(9-4n^2)$$

$$\delta|3\rangle = \frac{12 m a^2 \cancel{(-1)^n} V_0}{\pi \hbar^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}|2n\rangle}{(4n^2-9)^2}$$