2663

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2014

Friday, 20 June, 9.30 am – 12.30 pm

ion trap.

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

$x = \frac{n_1}{n_0+n_1} = \frac{2\exp(-\frac{\delta\varepsilon}{k_BT})}{q_e}$ $[A^+A, A]$

$\therefore \cancel{x = \frac{4}{q} - 1}$

$\rightarrow x = \frac{-4+q}{q}$

$\frac{q-4}{q} = x$

$q = \frac{4}{1-x}.$

Section A

$q = 4 + 2e^{-\frac{\delta\varepsilon}{k_BT}}$

1. A system at time $t = 0$ has a state vector $|\psi(0)\rangle = N\left(|E_1\rangle + \mathrm{i}|E_2\rangle\right)$, where $|E_1\rangle$ and $|E_2\rangle$ are normalised energy eigenkets with different eigenvalues E_1 and E_2 ($E_2 > E_1$). Write down the state vector at time t, $|\psi(t)\rangle$. What are the possible values of N for this to be normalised? At what time $T > 0$ is the system first in the state $|\psi(T)\rangle = N'\left[|E_1\rangle + |E_2\rangle\right]$? (The constant N' may differ from N). [5]

$\frac{Q^N}{N!}$

2. A two-state system has a Hamiltonian of the form

$$H = \begin{pmatrix} a+b & c-\mathrm{i}d \\ c+\mathrm{i}d & a-b \end{pmatrix}$$

where a, b, c and d are real numbers. Show that H is Hermitian. Find the eigenvalues of H. Hence show that the only Hamiltonians of this form with degeneracy are multiples of the identity matrix. [5]

ln

Q^N

N

3. Operators $\hat{A}$ and $\hat{B}$ commute: $[\hat{A}, \hat{B}] = 0$. Operator $\hat{A}$ has a complete set of eigenkets

$$\hat{A}|n\rangle = A_n|n\rangle$$

which are non-degenerate. Show that every eigenket of $\hat{A}$ is an eigenket of $\hat{B}$. [3]

Suppose now there is a two-fold degeneracy such that $A_4 = A_5$. Show that in this case $|4\rangle$ is not necessarily an eigenket of $\hat{B}$, but that

$$\hat{B}|4\rangle = \alpha|4\rangle + \beta|5\rangle$$

for some constants α and β. [3]

4. A particle in a (one-dimensional) box can be modelled with a confining potential $V(x)$ which is zero for $0 \leq x \leq a$, and infinite elsewhere. Write down the time-independent Schrödinger equation for the eigenfunctions $u_n(x)$, and state the boundary conditions satisfied by u_n at $x = 0$ and $x = a$. [3]

Solve to find the eigenvalues E_n and normalised eigenfunctions u_n. Hence write down the general solution to the time-dependent Schrödinger equation for a particle confined to the box. [5]

5. The wavefunction for the ground state of a hydrogen-like system with nuclear charge Ze is $\psi(r) = \sqrt{\frac{Z^3}{\pi a^3}}\mathrm{e}^{-Zr/a}$ where a is the Bohr radius and the reduced mass correction has been neglected. Show that this wavefunction is normalised. An atom of tritium (^{3}H) in its ground state decays to singly-ionised helium (^{3}He$^+$) by beta decay, in a process which can be considered instantaneous on the atomic timescale. Calculate the probability that the helium ion is left in the ground state. [7]

6. The Hamiltonian for a hydrogen-like two-particle system is

$$H = \left(\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\right) - \frac{Ze^2}{4\pi\epsilon_0 r}$$

where $\mathbf{p}_1$, $\mathbf{p}_2$ are the two momentum operators, $\mathbf{r}_1$, $\mathbf{r}_2$ are the two position operators, and $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The centre-of-mass operator is $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$. Write down expressions for the momentum operators conjugate to $\mathbf{r}$ and $\mathbf{R}$, $\mathbf{p}$ and $\mathbf{P}$ respectively. Show that $\mathbf{p}_1 = \frac{m_1}{m_1+m_2}\mathbf{P} - \mathbf{p}$. [4]

Separate H into two terms $H = H_{\rm CM} + H_{\rm int}$, each containing only centre-of-mass or internal variables respectively. Give a physical interpretation of the two terms. [5]

∂

Section B

7. The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\hbar\omega\left(a\,a^\dagger + a^\dagger\,a\right)$$

where all the symbols have their usual meanings, and a is the lowering operator

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x + \mathrm{i}\frac{p}{\sqrt{\hbar m\omega}}\right).$$

The action of a on the energy eigenket $|n\rangle$ is $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $n \geq 0$. Find expressions for x and p in terms of a and $a^\dagger$. The oscillator is in the state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|n-1\rangle\,\mathrm{e}^{-\mathrm{i}(n-\frac{1}{2})\omega t} + |n\rangle\,\mathrm{e}^{-\mathrm{i}(n+\frac{1}{2})\omega t}\right).$$

Calculate the expectation values of H, x and p for this state. [9]

Calculate the amplitude of oscillation of a classical oscillator of this frequency and energy $E = \langle\psi(t)|H|\psi(t)\rangle$ and show that it differs from your result for $\langle\psi(t)|x|\psi(t)\rangle$ by a factor independent of n. [4]

Find an expression for the expectation value of x^2 in the state $|\psi(t)\rangle$. Explain briefly why this state has a time-independent value for $\langle\psi(t)|x^2|\psi(t)\rangle$, whereas a superposition of more than two consecutive energy eigenstates would not. [7]

$x_1 \; \frac{\partial}{\partial x_1} \frac{\partial x_1}{\partial}$

两 $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ $\quad \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}\frac{\partial x_1}{\partial x} + \frac{\partial}{\partial y}\frac{\partial y}{\partial}$

内 $\left(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y}, \frac{\partial}{\partial Z}\right)$

$$\frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x_1}$$

$\frac{P_x^2}{2m} + \frac{P_y^2}{m} + \frac{P_z^2}{m} + \frac{1}{2}m\omega^2(\underbrace{x+y+}_{r^2}$

$\Rightarrow E_{n_x n_y n_z} = \hbar\omega(n_x + n_y + n_z + \frac{3}{2})$

$[L_z, z] = 0$

$\therefore L_z z f(r)$
$= z L_z f(r)$
$= \underbrace{\;}_{0} = 0.$

8. A three-dimensional harmonic oscillator with a spherically symmetric potential has a three-fold degenerate first excited level. A possible basis for the degenerate states is provided by the kets $|1,u\rangle$, $|1,v\rangle$ and $|1,w\rangle$ with wavefunctions

$$\langle \mathbf{r}|\,1,u\rangle = x\,f(r) \qquad \langle \mathbf{r}|\,1,v\rangle = y\,f(r) \qquad \langle \mathbf{r}|\,1,w\rangle = z\,f(r)$$

where the explicit form of f is irrelevant. Show that $|1,w\rangle$ is an eigenstate of L_z with eigenvalue zero, so we can give it an alternative designation $|1,w\rangle = |1,0\rangle$. Specify corresponding eigenvalue conditions satisfied by $|1,u\rangle$ and $|1,v\rangle$. Use the operators $L_x \pm \mathrm{i}L_y$ to find the normalised eigenkets of L_z, $|1,\pm1\rangle$, with eigenvalue $\pm\hbar$, in terms of $|1,u\rangle$ and $|1,v\rangle$. Show that there are no states $|1,\pm2\rangle$. [9]

WTF?

The oscillator is subject to a weak perturbation

$$V(\mathbf{r}) = G(2z^2 - x^2 - y^2).$$

Use degenerate perturbation theory in the $\{u,v,w\}$ basis to show that the first-order corrections to the energy are $\Delta E = 2G(K-H)$ and $G(H-K)$ where

$$H = \int x^2y^2f^2(r)\,\mathrm{d}^3r \qquad \text{and} \qquad K = \int x^4f^2(r)\,\mathrm{d}^3r.$$

normalisation of ladder operators

(You may assume that

$$H = \int x^2y^2f^2(r)\,\mathrm{d}^3r = \int y^2z^2f^2(r)\,\mathrm{d}^3r = \int z^2x^2f^2(r)\,\mathrm{d}^3r$$

and similarly

$$K = \int x^4f^2(r)\,\mathrm{d}^3r = \int y^4f^2(r)\,\mathrm{d}^3r = \int z^4f^2(r)\,\mathrm{d}^3r$$

by symmetry.) Assign zeroth-order eigenkets to the two eigenvalues, stating which is still degenerate. [11]

$\langle 1s|-x\rangle$

$\langle 1s| PV + VP |1s\rangle$ $|1s\rangle$ $\int$

$= 0$

Sinc

9. A hydrogen atom in the ground state $|1s\rangle$ is immersed in a radiation field with a weak electric field $\mathbf{E} = \mathcal{E}\hat{\mathbf{z}}\cos\omega t$ where ω is close to the resonant frequency for transitions to the first excited level $n = 2$. This perturbs the hydrogen atom, with interaction Hamiltonian $V(t) = e\mathcal{E}z\cos(\omega t)$. Show that V commutes with the z-component of orbital angular momentum, $[V, L_z] = 0$, and anticommutes with the parity operator $\{V, \mathcal{P}\} = 0$. Hence show that $\langle 1s|V|1s\rangle = 0$, and that the only $n = 2$ excited state with a well-defined value for L_z having non-zero matrix element with $|1s\rangle$ is $|2p, 0\rangle$, with zero eigenvalue of L_z. (Neglect electron spin throughout.) $z|210\rangle$ [6]

At $t = 0$ the atom is in the state $|1s\rangle$, and we write the state at $t > 0$ as

$$|\psi(t)\rangle = c_1\,\mathrm{e}^{-\mathrm{i}\omega_1 t}|1s\rangle + c_2\,\mathrm{e}^{-\mathrm{i}\omega_2 t}|2p, 0\rangle.$$

why no $|3p,0\rangle$ $|4p,0\rangle$ $|5p,0\rangle$

(The constants $\omega_i = E_i/\hbar$, where E_i are the energy eigenvalues for hydrogen.) Show that the equations of motion for the coefficients c_i are

why

$$\begin{aligned}
\mathrm{i}\hbar\frac{\mathrm{d}c_1}{\mathrm{d}t} &= e\mathcal{E}\langle 1s|z|2p, 0\rangle\cos(\omega t)\mathrm{e}^{\mathrm{i}(\omega_1-\omega_2)t}\,c_2\\
\mathrm{i}\hbar\frac{\mathrm{d}c_2}{\mathrm{d}t} &= e\mathcal{E}\langle 2p, 0|z|1s\rangle\cos(\omega t)\mathrm{e}^{\mathrm{i}(\omega_2-\omega_1)t}\,c_1.
\end{aligned}$$

$c_1(0) = 1$

$c_2(0) = 0$

$c_2(\delta t) = c_2(0) + \frac{dc_2}{dt}\Delta t$

$\propto$

Explain briefly why $\frac{\mathrm{d}c_2}{\mathrm{d}t}$ is first-order in $\mathcal{E}$ but $\frac{\mathrm{d}c_1}{\mathrm{d}t}$ is second-order. Hence give first-order equations of motion for c_i. [7]

WTF?

Separate the cosine into two complex exponentials, and explain briefly why one of the two terms in $\frac{\mathrm{d}c_2}{\mathrm{d}t}$ oscillates many times faster than the other. Neglect the fast term and find a first-order approximation for $c_2(t)$. Show that

$$|c_2(t)|^2 = \frac{e^2\mathcal{E}^2|\langle 2p, 0|z|1s\rangle|^2}{\hbar^2\Delta^2}\sin^2\left(\frac{\Delta t}{2}\right)$$

$PV = -VP$

$PV \langle 1s|1s\rangle$ $Px = -xP$

where $\Delta = \omega_2 - \omega_1 - \omega$. Show that in this approximation $|c_1|^2 + |c_2|^2 \geq 1$ for all $t > 0$. Explain why $|c_1|^2 + |c_2|^2$ is expected to remain constant, and hence find a condition on $\mathcal{E}$ and Δ for the first-order approximation to be valid. [7]

$xP = Px$

$\frac{dc_1}{dt} \propto \mathcal{E} c_2$ initial condition

$\frac{dc_2}{dt} \propto \mathcal{E} c_1$ $c_1(0) = 1$

$c_2(0) = 0$

$V_1 = r_1 + ?$

At time $t = \Delta t$ is small

$c_1(\Delta t) = c_1(0) + \frac{dc_1}{dt}\Delta t + \dots \approx 1 + \frac{dc_1}{dt}\Delta t) \approx 1$

$c_2(\Delta t) = c_2(0) + \frac{dc_2}{dt}\Delta t + \dots \approx 0 + \frac{dc_2}{dt}\Delta t + O(\mathcal{E}^2)$

$\therefore \boxed{\frac{dc_2}{dt}} \propto c_1\mathcal{E} \propto \mathcal{E}(1 + \dots) \propto \mathcal{E}$

$\boxed{\frac{dc_1}{dt}} \propto \mathcal{E} c_2 \propto \mathcal{E}(0 + \frac{dc_2}{dt}\Delta t + \dots)$

$\propto \mathcal{E}\frac{dc_2}{dt} \propto \mathcal{E}\cdot\mathcal{E}c_1 \propto \mathcal{E}^2 c_1 \propto \mathcal{E}^2$

$\frac{\partial}{\partial} \quad \frac{\partial}{\partial \phi_1} \cos(\phi_1 - \phi_2) \psi$

$\lambda \langle a, b | Q \psi \rangle = \lambda \langle b, a | \psi \rangle$

$\partial_{12} Q_{12} | \psi \rangle = \lambda Q_{12} | \psi \rangle$

10. The Hamiltonian for helium, neglecting small spin-dependent terms and treating the nucleus as fixed, is

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{e^2}{4\pi\epsilon_0}\left(-2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\right)$$

$\left(\frac{\partial}{\partial r_1} + 2\frac{\partial}{\partial r_2}\right) | r_1 + 2r_2 |$

$\frac{\partial}{\partial r_2} + 2\frac{\partial}{\partial r_1} | r_2 + 2r_1 |$

where the subscripts 1 and 2 identify the two electrons and the distance between the two electrons is:

$$|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2)\right)}.$$

Show that $[L_z, |\mathbf{r}_1 - \mathbf{r}_2|^2] = 0$ where L_z is the z-component of the total orbital angular momentum $\mathbf{L}$:

$|r_1 - r_2| \, L_z, r$

$$L_z = -i\hbar\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}\right).$$

$i\hbar\frac{\partial}{\partial\phi}$

Hence show that $[H, L_z] = 0$ and, assuming that $[H, L_x] = [H, L_y] = 0$, show that $[H, L^2] = 0$. [6]

In the position representation $\langle \mathbf{a}, \mathbf{b} | \psi \rangle$ denotes the amplitude for finding particle 1 at $\mathbf{a}$ and particle 2 at $\mathbf{b}$. Q_{12} is a spatial symmetry operator defined by:

WTF is this?

$$\langle \mathbf{a}, \mathbf{b} | Q_{12} | \psi \rangle = \langle \mathbf{b}, \mathbf{a} | Q_{12} | \psi \rangle.$$

$[L_z, |r_1 - r_2|^2] =$

WTF?

Show that $[Q_{12}, H] = 0$. Explain why H has spatial eigenstates of the form $|E, L, M_L, q\rangle$ where the eigenvalues of H, L^2, L_z and Q_{12} are E, $\hbar^2 L(L+1)$, $\hbar M_L$ and $q = \pm 1$ respectively. The complete ket for the electrons in helium must also include a spin part. Write down possible spin kets which satisfy the symmetry requirements, in terms of the single-electron kets $|\pm\rangle_1$ and $|\pm\rangle_2$ which are eigenstates of the single electron spin operators s_{1z} and s_{2z}. [8]

WTF?

The lowest five eigenvalues of H are given in the following table together with their conventional labels, and quantum numbers for L and q.

Label	$E/\left(e^2/4\pi\epsilon_0 a_0\right)$	L	q
$1s^2$	−2.904	0	+1
$1s2s$	−2.175	0	−1
$1s2s$	−2.145	0	+1
$1s2p$	−2.133	1	−1
$1s2p$	−2.124	1	+1

$\frac{1}{|r_1 - r_2|}$

$\langle a, b | \psi \rangle$

$M_L = 1, 0, -1$

Explain why the assignment of spin parts to these spatial states is constrained by the fact that electrons are fermions. Assign spin parts and hence give degeneracies for these 5 eigenvalues of H. [6]

WTF?

$\langle a, b | Q_{12} | q \rangle$

$= \langle a, b | q \rangle Y_l^m$

$-1 \, Q_{12} \langle b, a | q \rangle$

A3 2014 First Attempt

1. $\because |\psi(0)\rangle = N(|E_1\rangle + i|E_2\rangle)$

$$\therefore |\psi(t)\rangle = N(|E_1\rangle e^{-iE_1t/\hbar} + i|E_2\rangle e^{-iE_2t/\hbar})$$

$$= N(|E_1\rangle e^{-iE_1t/\hbar} + |E_2\rangle e^{i(\frac{\pi}{2} - \frac{E_2t}{\hbar})})$$

$$= Ne^{-iE_1t/\hbar}(|E_1\rangle + |E_2\rangle e^{i(\frac{\pi}{2} + \frac{(E_1-E_2)t}{\hbar})})$$

for $t=T$, $\frac{\pi}{2} + \frac{E_1-E_2}{\hbar}T = \cancel{2\pi}\ 0$ $(\because E_1<E_2)$

$\therefore$ ~~$T = \frac{E_1-E_2}{\hbar}T = \frac{3\pi}{2} \rightarrow T = \frac{3\pi\hbar}{2(E_1-E_2)}$~~

$$\therefore \frac{E_2-E_1}{\hbar}T = \frac{\pi}{2} \quad \therefore T = \frac{\pi\hbar}{2(E_2-E_1)}$$ ✓

2.
$$H = \begin{pmatrix} a+b & c-id \\ c+id & a-b \end{pmatrix}$$

let $|\phi\rangle = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ $|\psi\rangle = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, then

$$\langle\phi|H|\psi\rangle = (x_1^*, y_1^*)\begin{pmatrix} a+b & c-id \\ c+id & a-b \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= (x_1^*, y_1^*)\begin{pmatrix} (a+b)x_2 + (c-id)y_2 \\ (c+id)x_2 + (a-b)y_2 \end{pmatrix}$$

$$= (a+b)x_1^*x_2 + (c-id)x_1^*y_2 + (c+id)y_1^*x_2 + (a-b)y_1^*y_2$$

A3 2014

$$\langle\psi|H|\phi\rangle = (x_2^*, y_2^*)\begin{pmatrix} a+b & c-id \\ c+id & a-b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \;:$$

$$= (a+b)x_1x_2^* + (c-id)x_2^*y_1 + (c+id)x_1y_2^* + (a-b)y_1y_2^*$$

$$\rightarrow \langle\psi|H|\phi\rangle = \langle\phi|H|\psi\rangle^* \quad \rightarrow H \text{ is hermitian}$$

$$H|\psi\rangle = \lambda|\psi\rangle \;:$$

$$\rightarrow \det\begin{pmatrix} a+b-\lambda & c-id \\ c+id & a-b-\lambda \end{pmatrix} = 0$$

$$\rightarrow (a+b-\lambda)(a-b-\lambda) - (c^2+d^2) = 0$$

$$\therefore \lambda^2 - (a+b+a-b)\lambda + (a+b)(a-b) - c^2 - d^2 = 0$$

$$\therefore \lambda^2 - 2a\lambda + a^2 - b^2 - c^2 - d^2 = 0$$

$$\therefore \cancel{\lambda =} \quad \lambda = a \pm \frac{1}{2}\sqrt{4b^2+4c^2+4d^2}$$

$$\rightarrow \lambda = a \pm \sqrt{b^2+c^2+d^2}$$

λ_1 λ_2

$\downarrow$ $\searrow$ $\downarrow$

$|\psi_1\rangle$ $|\psi_2\rangle$ $|\phi\rangle$

3 ?

X

2-D space → only max. 2 Linearly independent vectors. ∵ H is hermitial, so if $\lambda_1 \neq \lambda_2$, then eigenvectors for λ_1 and λ_2 are orthogonal (thus linearly independent) ~~∴ for degenera~~ then neither λ_1 nor λ_2 can have degeneracy. Degeneracy can occur only if $\lambda_1 = \lambda_2$, in which case $b^2+c^2+d^2=0$ → $b=0$ $c=0$ $d=0$

$$\rightarrow H = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = aI$$

QED

3. $[\hat{A},\hat{B}]=0 \rightarrow \hat{A}\hat{B}=\hat{B}\hat{A}$

$\therefore \hat{B}\hat{A}|n\rangle = \hat{A}\hat{B}|n\rangle$, $\hat{B}\hat{A}|n\rangle = \hat{B}A_n|n\rangle = A_n(\hat{B}|n\rangle)$

$\therefore \hat{A}(\hat{B}|n\rangle) = A_n(\hat{B}|n\rangle)$, $\hat{B}|n\rangle$ and $|n\rangle$ are both eigenstates of $\hat{A}$ with same eigenvalue A_n

But A_n is a non-degenerate eigenvalue

$\therefore$ $\hat{B}|n\rangle$ and $|n\rangle$ must be linearly dependent

$\rightarrow$ $\underline{\hat{B}|n\rangle = B_n|n\rangle}$ ✓ → every $|n\rangle$ is an eigenstate of $\hat{B}$

Now if $A_4 = A_5$, then eigenvalue A_4 has 2 linearly independent eigenvectors.

$\therefore$ $\hat{B}|n\rangle$ and $|n\rangle$ can be linearly independent and the above argument fails.

If 2 linearly independent ~~states~~ eigenstates of A_4 are $|4\rangle$ and $|5\rangle$, then since $|4\rangle$, $|5\rangle$, $\hat{B}|4\rangle$, $\hat{B}|5\rangle$ are all eigenstates of $\hat{A}$ with ~~eigenstate~~ eigenvalue A_4, but A_4 only allow for 2-fold degeneracy

$\rightarrow$ $\hat{B}|4\rangle = \alpha|4\rangle + \beta|5\rangle$ $\quad$ $\hat{B}|5\rangle = \alpha'|4\rangle + \beta'|5\rangle$

✓

4. TISE:

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)u_n(x) = E_n u_n(x)$$

$$V(x) = \begin{cases} 0 & 0 \le x \le a \\ \infty & \text{otherwise} \end{cases}$$

for $V(x) = \infty$, $u_n(x) = 0$

for $V(x) = 0$

$$\frac{d^2u_n(x)}{dx^2} + \frac{2mE_n}{\hbar^2}u_n(x) = 0$$

$$k_n = \sqrt{\frac{2mE_n}{\hbar^2}}$$

$$\rightarrow \quad u_n'' + k_n u_n = 0$$

$$\therefore \quad u_n = A\cos(k_n x) + B\sin(k_n x)$$

Boundary conditions

$u_n(x=0) = 0$; $u_n(x=a) = 0$

$\swarrow$ $A = 0$

$\downarrow$ $\sin(k_n a) = 0$

$\rightarrow k_n a = \pi n$

$\rightarrow k_n = \frac{n\pi}{a} \rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ ✓

$$\therefore \quad u_n(x) = B\sin\left(\frac{n\pi x}{a}\right)$$

Normalise $B^2\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = 1$

$$\rightarrow B^2\left(\frac{a}{2}\right) = 1 \quad \rightarrow \quad B = \sqrt{\frac{2}{a}}$$ ✓

$$\rightarrow \quad u_n(x) = \sin^2\left(\frac{n\pi x}{a}\right)\cdot\sqrt{\frac{2}{a}}$$ ✓

General solution: $u(x,t) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{a}}\sin^2\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{in^2\pi^2\hbar t}{2ma^2}\right)\cdot a_n$ ✓

a_n $\swarrow$ arbitrary constants

5. $\int dr \cdot r^2 \cdot \psi^2(r) = \frac{Z^3}{\pi a^3} \int dr \cdot r^2 e^{-2Zr/a}$

$$= \frac{Z^3}{\pi a^3}\left(\frac{a^2}{4Z^2}\right)\left(\frac{a}{2Z}\right)\int_0^\infty \underbrace{du \cdot u^2 e^{-u}}_{2!}$$

$u = \frac{2Z}{a} r$
$r^2 = \frac{a^2}{4Z^2} u^2$
$dr = \frac{a}{2Z} du$

$$= \frac{Z^3 \cdot a^2 \cdot a \cdot Z}{\pi a^3 \cdot 4 \cdot Z^2 \cdot Z \cdot \cancel{Z} \cdot Z} = \frac{1}{4\pi}$$

$$\therefore \langle \psi(r) | \psi(r) \rangle = \int dr d\theta d\phi \, r^2 \sin\theta \, \psi^2(r)$$

$$= \underbrace{\int_0^\infty dr \cdot r^2 \psi(r)}_{\frac{1}{4\pi}} \underbrace{\int_0^\pi d\theta \sin\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 1 \checkmark$$

$\rightarrow$ $\psi(r)$ is normalised.

instantuneous process → sudden approximation.
→ system remains in the eigenstate of H^3 ($|\psi(r, z=1)\rangle$)
→ ~~P~~ Probability amplitude to remain in the ground state. of $^3He^+$ ($|\psi(r, z=2)\rangle$) is :

~~$\langle \psi(r, z=2) | \psi$~~

$\langle \psi(r, z=2) | \psi(r, z=1) \rangle = $ ~~$\sqrt{\frac{8}{\pi a^3}} \cdot \sqrt{\frac{1}{\pi a^3}} \cdot 4\pi \int dr \cdot r^2 e^{-(3/2)}$~~ ✓

$$= \left(\sqrt{\frac{8}{\pi a^3}}\right)\left(\sqrt{\frac{1}{\pi a^3}}\right)(4\pi)\int dr\, r^2 e^{-3r/a}$$

$$= \cancel{\frac{8\sqrt{2}}{a^3}} \left(\frac{8\sqrt{2}}{a^3}\right)\left(\frac{a^2}{9z^2}\right)\left(\frac{a}{3z}\right)\int_0^\infty \underbrace{u^2 e^{-u}\,du}_{2!}$$

$$= \frac{16\sqrt{2}}{27} \checkmark$$

Probability $P = |\langle \psi(r, z=2) | \psi(r, z=1)\rangle|^2$

$$= \left(\frac{16\sqrt{2}}{27}\right)^2 \simeq \underbrace{0.7}\ \checkmark \quad \frac{512}{729}$$

6. $H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m} - \frac{Ze^2}{4\pi\varepsilon_0 r}$

$\hat{\underline{p}} = \cancel{\frac{\partial}{\partial z}} -i\hbar\frac{\partial}{\partial \underline{r}} \qquad \underline{P} = -i\hbar\frac{\partial}{\partial \underline{R}}$

$\left(\underline{r} = \underline{r_2} - \underline{r_1}, \quad \underline{R} = \frac{m_1\underline{r_1} + m_2\underline{r_2}}{m_1+m_2}\right)$

$\underline{p_1} = -i\hbar\frac{\partial}{\partial \underline{r_1}}, \qquad \frac{\partial}{\partial \underline{r_1}} = \frac{\partial \underline{R}}{\partial \underline{r_1}} \cdot \frac{\partial}{\partial \underline{R}} + \frac{\partial \underline{r}}{\partial \underline{r_1}} \cdot \frac{\partial}{\partial \underline{r}}$

Notations : $\frac{\partial}{\partial \underline{r_1}} = \vec{\nabla}_{r_1} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right)^T$

$\frac{\partial \underline{R}}{\partial \underline{r_1}}$ is a tensor, $\frac{\partial \underline{R}}{\partial \underline{r_1}} = \begin{pmatrix} \frac{\partial X}{\partial x_1} & \frac{\partial Y}{\partial x_1} & \frac{\partial Z}{\partial x_1} \\ \frac{\partial X}{\partial y_1} & \frac{\partial Y}{\partial y_1} & \frac{\partial Z}{\partial y_1} \\ \frac{\partial X}{\partial z_1} & \frac{\partial Y}{\partial z_1} & \frac{\partial Z}{\partial z_1} \end{pmatrix}$

Or $\left(\frac{\partial \underline{R}}{\partial \underline{r_1}}\right)_{ij} = \frac{\partial R_j}{\partial r_i}$

in this case $\left(\frac{\partial \underline{R}}{\partial \underline{r_1}}\right)_{ij} = \frac{m_1}{m_1+m_2}\delta_{ij}$

$\left(\frac{\partial \underline{r}}{\partial \underline{r_1}}\right)_{ij} = -\delta_{ij}$

$$\therefore \left(\frac{\partial}{\partial \underline{r}_1}\right)_i = \left(\frac{\partial \underline{R}}{\partial \underline{r}_1}\right)_{ij}\left(\frac{\partial}{\partial \underline{R}}\right)_j + \left(\frac{\partial \underline{r}}{\partial \underline{r}_1}\right)_{ij}\left(\frac{\partial}{\partial \underline{r}}\right)_j$$

$$⑤ = \frac{m_1}{m_1+m_2}\delta_{ij}\left(\frac{\partial}{\partial \underline{R}}\right)_j \cancel{+} - \delta_{ij}\left(\frac{\partial}{\partial \underline{r}}\right)_j$$

$$= \frac{m_1}{m_1+m_2}\left(\frac{\partial}{\partial \underline{R}}\right)_i - \left(\frac{\partial}{\partial \underline{r}}\right)_i$$

$$\rightarrow \cancel{\frac{\partial}{\partial \underline{r}} =} \quad \frac{\partial}{\partial \underline{r}_1} = \frac{m_1}{m_1+m_2}\frac{\partial}{\partial \underline{R}} - \frac{\partial}{\partial \underline{r}}$$

multiply by $-i\hbar$ gives ✓

$$\underline{\hat{p}_1 = \frac{m_1}{m_1+m_2}\underline{\hat{P}} - \underline{\hat{p}}}$$

Similarly $\hat{p}_2 = \frac{m_2}{m_1+m_2}\underline{\hat{P}} + \underline{\hat{p}}$ (obviously $[\underline{P}, \underline{p}] = 0$)

$$\therefore H = \frac{1}{2m_1}\left(\frac{m_1}{m_1+m_2}\underline{P} - \underline{p}\right)^2 + \frac{1}{2m_2}\left(\frac{m_2}{m_1+m_2}\underline{P} + \underline{p}\right)^2 - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$= \frac{1}{2m_1}\left(\frac{m_1}{m_1+m_2}\right)^2 P^2 - \cancel{\frac{2}{2m_1}\frac{m_1}{m_1+m_2}\underline{P}\cdot\underline{p}} + \frac{p^2}{2m_1}$$

$$+ \frac{1}{2m_2}\left(\frac{m_2}{m_1+m_2}\right)^2 P^2 + \cancel{\frac{2}{2m_2}\frac{m_2}{m_1+m_2}\underline{P}\cdot\underline{p}} + \frac{p^2}{2m_2} - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

$$= \underbrace{\left(\frac{P^2}{2(m_1+m_2)}\right)}_{H_{cm}} + \underbrace{\left(\frac{p^2}{2\mu} - \frac{Ze^2}{4\pi\varepsilon_0 r}\right)}_{H_{int}} \quad ✓$$

H_{cm} → kinetic energy of CM

$\mu = \frac{m_1 m_2}{m_1+m_2}$ reduced mass

H_{int} ~~tot~~ ~~tot~~ total energy in the CM frame.

7. $a = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x + i\frac{p}{\sqrt{\hbar m\omega}}\right) = ①$

$a^+ = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,x - i\frac{p}{\sqrt{\hbar m\omega}}\right) = ②$

① + ② → $(a + a^+) = \sqrt{\frac{2m\omega}{\hbar}}\,x$

$\therefore \underline{x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+)}$ ✓

① − ② → $(a - a^+) = 2i\frac{p}{\sqrt{\hbar m\omega}}\cdot\frac{1}{\sqrt{2}}.$

→ $\underline{p = \frac{1}{\cancel{2}i}\sqrt{\frac{\hbar m\omega}{2}}(a - a^+)}$ ✓ $= \cancel{\theta}$ $i\sqrt{\frac{\hbar m\omega}{2}}(a^+ - a)$

$\langle\psi(t)|H|\psi(t)\rangle = \frac{\hbar\omega}{4}\left(\langle n-1|e^{i(n-\frac{1}{2})\omega t} + \langle n|e^{i(n+\frac{1}{2})\omega t}\right)$

$(aa^+ + a^+a)\left(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t}\right.$

$\left.+ |n\rangle e^{-i(n+\frac{1}{2})\omega t}\right)$

$= \frac{\hbar\omega}{4}\left[\underbrace{(2n-1) + (2n+1)}_{4n}\right] = \underline{n\hbar\omega}$ ✓

$\langle\psi(t)|x|\psi(t)\rangle$

$= \frac{1}{2}\cdot\sqrt{\frac{\hbar}{2m\omega}}\left(\langle n-1|e^{i(n-\frac{1}{2})\cancel{\hbar}\omega t} + \langle n|e^{(n+\frac{1}{2})\cancel{\hbar}\omega t}\right)$

$(a + a^+)\left(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t} + |n\rangle e^{-i(n+\frac{1}{2})\omega t}\right).$

$$= \cancel{\tfrac{1}{2}} \sqrt{\frac{\hbar}{8m\omega}} \left(\sqrt{n} e^{-i\omega t} + \sqrt{n} e^{i\omega t}\right)$$

$$= \sqrt{\frac{\hbar}{8m\omega}} \sqrt{n} \cdot 2\cos\omega t = \sqrt{\frac{n\hbar}{2m\omega}} \cos\omega t.$$

✓

$$\langle \psi(t) | P | \psi(t) \rangle$$

$$= \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} \cdot \frac{1}{2} \left(\langle n-1 | e^{i(n-\frac{1}{2})\omega t} + \langle n | e^{i(n+\frac{1}{2})\omega t} \right)$$

$$(a - a^\dagger)\left(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t} + |n\rangle e^{-i(n+\frac{1}{2})\omega t} \right).$$

$$= \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} \cdot \frac{1}{2} \left(\sqrt{n} e^{-i\omega t} - \sqrt{n} e^{+i\omega t}. \right)$$

$$= \frac{1}{i} \sqrt{\frac{\hbar m\omega}{2}} \cdot \frac{1}{2} \cdot 2i \sin(-\omega t) \sqrt{n}.$$

$$= -\sqrt{\frac{n\hbar m\omega}{2}} \sin(\omega t)$$

✓

$$E = \langle \psi(t) | H | \psi(t) \rangle = n\hbar\omega$$

expected to use $\frac{1}{2}m\omega^2 A^2 = E$

$\rightarrow A = \sqrt{\frac{2E}{m\omega^2}}$

then substitute $E = n\hbar\omega$ ~~directly~~ To get the ratio

~~An~~ Classical amplitude A given by

$$\frac{1}{2} m\omega^2 A^2 = n\hbar\omega \rightarrow A^2 = \frac{2n\hbar\omega}{m\omega^2}$$

$$\rightarrow A = \sqrt{\frac{2n\hbar}{m\omega}}$$ ×

use E, $A = \sqrt{\frac{2E}{m\omega^2}}$.

ratio $$\frac{|\langle x \rangle|}{A} = \frac{\sqrt{\frac{n\hbar}{2m\omega}}}{\sqrt{\frac{2n\hbar}{m\omega}}} = \left(\frac{1}{\sqrt{2}}\right)^2$$ independent of n

$$= \frac{1}{2}$$

✓

$$\langle \psi(t)|x^2|\psi(t)\rangle = \frac{1}{2}\cdot\frac{\hbar}{2m\omega}\left(\langle n-1|e^{i(n-\frac{1}{2})\omega t}\right.$$

$$\left.+\langle n|e^{i(n+\frac{1}{2})\omega t}\right)\left(\underline{a^2}+\underline{a^{\dagger 2}}+aa^\dagger+a^\dagger a\right)$$

$$\left(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t}+|n\rangle e^{-i(n+\frac{1}{2})\omega t}\right)$$

$$= \frac{\frac{\hbar}{2m\omega}}{\frac{1}{2}\hbar\omega}\underbrace{\langle\psi(t)|H|\psi(t)\rangle}_{n\hbar\omega}$$

$$= \cancel{\frac{\hbar\cdot\hbar\omega\cdot n\hbar\omega}{\hbar\omega}}\ \frac{\hbar\cdot n\hbar\omega}{m\hbar\omega^2} = \frac{n\hbar}{m\omega}$$ ✓

$$a^\dagger a + aa^\dagger = 2\hat{n}+1$$

$$(\langle n-1| + \langle n| + \langle n+1|)(2\hat{n}+1)(|n-1\rangle + |n\rangle + |n+1\rangle)$$

$$= \text{time independent term}$$

but $$(\langle n-1|e^{i(n-\frac{1}{2})\omega t} + \langle n|^{i(n+\frac{1}{2})\omega t} + \langle n+1|e^{i(n+\frac{3}{2})\omega t})$$

$$(\hat{a}^{\dagger 2}+\hat{a}^2)\,(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t} + |n\rangle e^{-i(n+\frac{1}{2})\omega t}$$

$$+|n+1\rangle e^{-i(n+\frac{3}{2})\omega t})$$

term gives non-zero value if 3 or more consecutive seates are involved. this term is time-dependent. ✓

8. $\because \langle \underline{r}|1,w\rangle = \langle r,\theta,\phi|1,w\rangle = z f(r)$

$\therefore \langle \underline{r}|\hat{L}_z|1,w\rangle = \hat{L}_z\langle \underline{r}|1,w\rangle = \hat{L}_z z f(r) = z\underbrace{\hat{L}_z}_{[\hat{L}_z,\hat{z}]=0} f(r)$

$= z\underbrace{\left(-i\hbar\frac{\partial}{\partial\phi}\right)f(r)}_{=0} = 0$

$\langle \underline{r}|\hat{L}_z|1,w\rangle = 0$, $|1,w\rangle$ is an eigenstate of $\hat{L}_z$ with eigenvalue 0

~~$L_z(x f(r)) =$~~

~~$\langle \underline{r}|\hat{L}_z|1,u\rangle = L_z x f(r)$~~

$\therefore |1,u\rangle$ is the eigenstate of $\hat{L}_x$ with eigenvalue 0

~~$|1,w\rangle$~~

$|1,v\rangle$ is the eigenstate of $\hat{L}_y$ with eigenvalue 0

$\langle \underline{r}|(L_x+iL_y)|1,0\rangle = (L_x+iL_y)z f(r)$ ~~$= z iL_y$~~

$= \underbrace{zL_x f(r)}_{0} + \underbrace{iz L_y f(r)}_{0} + \underbrace{[L_x, z]}_{-i\hbar y} f(r) + i\underbrace{[L_y, z]}_{i\hbar x} f(r)$

$= \hbar(-x-iy)f(r)$

$L_z(L_x+iL_y)|1,0\rangle = L_x \overset{0}{L_z|1,0\rangle} + iL_y\overset{0}{L_z|1,0\rangle}$

$+ \underbrace{[L_z, L_x]}_{+i\hbar L_y}|1,0\rangle + i\underbrace{[L_z, L_y]}_{-i\hbar L_x}|1,0\rangle$

$= \hbar(L_x+iL_y)|1,0\rangle$

∴ $(L_x + iL_y)|1,0\rangle$ is an eigenstate of L_z with eigenvalue $\hbar$

$$\langle \underline{r}|(L_x + iL_y)|1,0\rangle = (-x - iy)f(r)$$

Normalise → $\langle r|1,1\rangle = +\frac{1}{\sqrt{2}}(x + iy)f(r)$

$$\underline{|1,1\rangle = \frac{1}{\sqrt{2}}(|1,u\rangle + i|1,v\rangle)}$$ ✓

$$\langle \underline{r}|L_x - iL_y|1,0\rangle = \cancel{L_x - iL_y}(L_x - iL_y)(zf(r))$$

$$= \underset{\downarrow 0}{zL_xf(r)} - i\underset{\downarrow 0}{zL_yf(r)} + \underbrace{[L_x, z]}_{-i\hbar y}f(r) - i\underbrace{[L_y, z]}_{i\hbar x}f(r)$$

$$= \hbar(x - iy)f(r)$$

$$L_z(L_x - iL_y)|1,0\rangle = L_x\overset{\swarrow ?}{L_z}|1,0\rangle - iL_y\overset{\swarrow 0}{L_z}|1,0\rangle$$

$$+ [\cancel{L_z}L_z, L_x](|1,0\rangle) - i[L_z, L_y]|1,0\rangle$$

$$= i\hbar L_y|1,0\rangle + \hbar L_x|1,0\rangle$$

$$= -\hbar(L_x - iL_y)|1,0\rangle$$

∴ $(L_x - iL_y)|1,0\rangle$ is an eigenstate of L_z with eigenvalue $-\hbar$

Normalise → $\underline{|1,-1\rangle = \frac{1}{\sqrt{2}}(|1,u\rangle - i|1,v\rangle)}$ ✓

$$\hat{L}^2 x_i f(r) = L_j L_j x_i f(r) = \underbrace{L_j x_i L_j f(r)}_{=0} + L_j [\underbrace{L_j, x_i}_{i\hbar\varepsilon_{jik} x_k}] f(r)$$

$$= -i\hbar \varepsilon_{ijk} L_j x_k f(r) = \underbrace{-i\hbar \varepsilon_{ijk} x_k L_j f(r)}_{0} - i\hbar \varepsilon_{ijk} [\underbrace{L_j, x_k}_{i\hbar \varepsilon_{jki} x_i}] f(r)$$

$$= \cancel{-i\hbar\varepsilon_{ijk}} \hbar^2 \underbrace{\varepsilon_{ijk}\varepsilon_{jki}} x_i f(r) = 2\hbar^2 x_i f(r)$$

sum over j, k, not i

$= \cancel{2\hbar}\ 2$

.0

(∵ $|1,1\rangle$, $|1,-1\rangle$, $|1,0\rangle$), $|1,u\rangle$, $|1,v\rangle$ are all eigen~~values~~ states of $\hat{L}^2$ with eigenvalue $2\hbar^2$ → any linear combinations of them is also!

∴ $|1,0\rangle$, $|1,u\rangle$, $|1,v\rangle$ are a set of basis for first energy level.

∴ if $|1,\pm 2\rangle$ exist, then they ~~must~~ must be linear combinations of $|1,0\rangle, |1,u\rangle, |1,v\rangle$

$$\therefore \hat{L}^2 |1,\pm 2\rangle = 2\hbar^2 |1,\pm 2\rangle$$

$$\therefore \langle 1,\pm 2| \hat{L}^2 |1,\pm 2\rangle = 2\hbar^2$$

but $\langle 1,\pm 2| \hat{L}^2 |1,\pm 2\rangle = \langle 1,\pm 2| \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 |1,\pm 2\rangle$

∵ L_i is hermitian ∴ $\langle\psi|\hat{L}_i^2|\psi\rangle = |L_i|\psi\rangle|^2 \geq 0$

$$\therefore \langle 1,\pm 2|\hat{L}^2|1,\pm 2\rangle \geq \langle 1,\pm 2|\hat{L}_z^2|1,\pm 2\rangle$$

$$= |\hat{L}_z|1,\pm 2\rangle|^2 = |\pm 2\hbar|^2 = 4\hbar^2$$

→ $2\hbar^2 \geq 4\hbar^2$ → contradiction

∴ $|1,\pm 2\rangle$ ~~does~~ do not exist.

(eigenvalues of $\hat{L}_z$ can only be integer × ħ)

But if you wish, can do this:

$$L_+ L_+|1,0\rangle = (L_x+iL_y)(L_x+iL_y)\, z f(r)$$

$$= (L_x+iL_y)(xf(r) + \cancel{\hbar}\, iyf(r))$$

$$= -\hbar z \cancel{f(r)} + \hbar z \cancel{f(r)} = 0$$

$$L_- L_- |1,0\rangle = (L_x-iL_y)(xf(r) - iyf(r))$$

$$= \hbar z \cancel{f(r)} - \hbar z \cancel{f(r)} = 0$$

but how to prove that the only way to get $|1,\pm 2\rangle$ is through the application of $L_x \pm iL_y$?

✓

this is enough because eigenstates of spherically symmetric hamiltonians should include spherical harmonics → all rules for spherically harmonics works.

$V(\underline{r}) = G(2z^2 - x^2 - y^2)$ is the perturbation

its matrix in the degenerate subspace $\{u, v, w\}$

is
$$\tilde{V} = \begin{bmatrix} V_{uu} & V_{uv} & V_{uw} \\ V_{vu} & V_{vv} & V_{vw} \\ V_{wu} & V_{wv} & V_{ww} \end{bmatrix}. \quad (V_{ij} = \langle i|V|j\rangle)$$

~~$V_{uu} = V_{vv} = V_{ww} =$~~

If there are odd terms of x, y, z then

$\int_{-\infty}^{\infty}$ vanishes $\rightarrow$ $V_{uv} = V_{vu} = 0$ $\quad V_{uw} = V_{wu} = 0$

$V_{wv} = V_{vw} = 0$

$$V_{uu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f^2(r)\, G(2z^2 - x^2 - y^2)\, dx\,dy\,dz$$

$$= \cancel{\int} G\Big[2\int x^2 z^2 f^2(r)\, d^3r - \int x^4 f^2(r)\, d^3r - \int x^2 y^2 f^2(r)\, d^3r\Big].$$

$$= G(2H - K - H) = G(H - K)$$

Similarly $\quad V_{vv} = G(H - K)$

$$V_{ww} = G\Big[2\int z^4 f^2(r)\, d^3r - \int x^2 z^2 f^2(r)\, d^3r - \int y^2 z^2 f^2(r)\, d^3r\Big]$$

$$= 2G(K - H)$$

$$\therefore \quad \tilde{V} = \begin{bmatrix} G(H-K) & 0 & 0 \\ 0 & G(H-K) & 0 \\ 0 & 0 & 2G(K-H) \end{bmatrix}$$

already diagonal

$$\Delta E = G(H-K) \begin{cases} \nearrow |1,u\rangle \quad \cancel{\text{or } |1,v\rangle} \\ \searrow |1,v\rangle \end{cases}$$

still degenerate ✓

$$\Delta E = 2G(K-H) \rightarrow |1,w\rangle$$ ✓

9. $[L_z, z]\psi = [-i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}), z]\psi$

$= \cancel{-i\hbar x\frac{\partial}{\partial y}}(z\psi) + \cancel{i\hbar y\frac{\partial}{\partial x}}(z\psi) + i\hbar\cancel{z\frac{\partial}{\partial y}}(\psi) - i\hbar\cancel{zy\frac{\partial}{\partial x}}(\psi)$

$= 0 \rightarrow [L_z, z] = 0$ ✓

$[V, L_z] = [e\mathcal{E}z\cos(\omega t), L_z] = -e\mathcal{E}\cos(\omega t)[L_z, z]$

$= 0$ ✓

$\{\hat{z}, \hat{\mathcal{P}}\}\psi = \hat{z}\hat{\mathcal{P}}\psi + \hat{\mathcal{P}}\hat{z}\psi = z\mathcal{P}\psi(x,y,z) + \mathcal{P}(z\psi(x,y,z))$

$= \cancel{z\psi(-x,-y,-z) - z\psi(-x,-y,-z)} = 0$

$\rightarrow \{\hat{z}, \hat{\mathcal{P}}\} = 0$

does parity operator act on l ?

$\therefore \{V, \mathcal{P}\} = \{e\mathcal{E}z\cos(\omega t), \mathcal{P}\}$

$= e\mathcal{E}\cos(\omega t)\{z, \mathcal{P}\} = 0$ ✓

consider $\mathcal{P}|nlm\rangle$, $\langle \underline{x}|nlm\rangle = R_{nl}(r)Y_{lm}(\theta,\phi)$

$\langle \underline{x}|\mathcal{P}|nlm\rangle = \langle -\underline{x}|nlm\rangle = R_{nl}(r)Y_{lm}(\pi-\theta, \phi+\pi)$

$= (-1)^l R_{nl}(r)Y_{lm}(\theta,\phi)$

$\therefore \mathcal{P}|nlm\rangle = (-1)^l|nlm\rangle$

Proof: We know Y_{ll} in general

$$Y_{ll}(\theta,\phi) = \sin^l(\theta) e^{il\phi}$$

$$P Y_{ll}(\theta,\phi) = Y_{ll}(\cancel{\theta}\pi-\theta , \pi+\phi) = \sin^l(\pi-\theta) e^{i\pi l} e^{il\phi}$$

$$= (e^{i\pi})^l \sin^l(\theta) e^{il\phi} = (-1)^l Y_{ll}(\theta,\phi)$$

$$L_i = \varepsilon_{ijk} x_j p_k \qquad P^\dagger L_i P = \varepsilon_{ijk} \underbrace{P^\dagger x_j}_{-x_j P} \underbrace{P P^\dagger}_{1} \underbrace{p_k P}_{-P p_k}$$

$$= \varepsilon_{ijk} x_j p_k = L_i$$

$$\therefore \quad L_i P - P L_i = 0 \quad \to \quad [P, L_i] = 0$$

parity operator commutes with angular momentum (any component)

$\to$ $\hat{L}_- = L_x - iL_y$ commutes with P

$$\therefore \quad P L_- = L_- P \quad \to \quad P L_- Y_{lm} = L_- P Y_{lm} = (-1)^l L_- Y_{lm}$$

$\to$ $P|nlm\rangle = (-1)^l |nlm\rangle$ $\to$ QED.

$$\because \quad PV + VP = 0 \quad \text{i.e.,} \quad P^\dagger = P = P^{-1} \to$$

$$P^\dagger V P = -V$$

$$\therefore \quad \langle 1s | V | 1s \rangle = \langle 100 | V | 100 \rangle = -\langle 100 | P^\dagger V P | 100 \rangle$$

$$= -\left[(-1)^0\right]^2 \langle 100 | V | 100 \rangle = \cancel{\langle 100} - \langle 1s | V | 1s \rangle$$

$$\to \quad \underline{\langle 1s | V | 1s \rangle = 0}$$

~~$\langle 2lm | V | 2lm \rangle = - \langle 2lm | P^{\dagger} V P | 2lm \rangle$~~

$Z \quad \langle 1s | V | 2lm \rangle = \langle 100 | V | 2lm \rangle$

$= -\langle 100 | P^{\dagger} V P | 2lm \rangle = (-1)(-1)^{0}(-1)^{l} \langle 100 | V | 2lm \rangle$

$= (-1)^{l+1} \langle 1s | V | 2lm \rangle$

→ if $l+1$ is even → can be non-zero

if $l+1$ is odd → must be zero

∵ $n=2$ ∴ $l = 1$ or 0 $\quad$ 1 ✓

0 ×

∴ consider $\langle 1s | V | 2P, m \rangle \quad m = -1, 0, 1$

∵ $[\hat{L_z}, \hat{V}] = 0 \quad ∴ L_z (V | 2P, m \rangle) = V L_z | 2P, m \rangle$

$= m (V | 2P, m \rangle)$

∴ $V | 2P, m \rangle$ is an eigenstate of L_z with eigenvalue m.

∵ $|1s\rangle$ is an eigenstate of L_z with eigenvalue 0

∴ if $m \neq 0$, eigenstates of L_z with different eigenvalues are orthogonal

→ $\langle 1s | V | 2P, m \rangle = 0$ for $m \neq 0$

→ only non-zero is $\underline{\langle 1s | V | 2P, 0 \rangle}$

why no $|3p,0\rangle$, $|4p,0\rangle$, ... ?

$$|\psi(t)\rangle = c_1 e^{-i\omega_1 t}|1s\rangle + c_2 e^{-i\omega_2 t}|2p,0\rangle$$

$(E_1 = \omega_1\hbar$, $E_2 = \hbar\omega_2$

TDSE $$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \underset{\downarrow \text{ hydrogen}}{\hat{H}}|\psi(t)\rangle + \underset{\downarrow \text{ perturbation.}}{\hat{V}}|\psi(t)\rangle$$

→

$$i\hbar\left(\frac{dc_1}{dt} - \cancel{i\omega_1 c_1}\right)e^{-i\omega_1 t}|1s\rangle + i\hbar\left(\frac{dc_2}{dt} - \cancel{i\omega_2 c_2}\right)e^{-i\omega_2 t}|2p,0\rangle$$

~~$= c_1(\hbar\omega_1 + V|1s\rangle)e^{-i\omega t} + c_2(\hbar\omega_2 + V|2p,0\rangle)e^{-i\omega_2 t}$~~

$$= c_1(\cancel{\hbar\omega_1} + V)|1s\rangle e^{-i\omega_1 t} + c_2(\cancel{\hbar\omega_2} + V)|2p,0\rangle e^{-i\omega_2 t}$$

$\langle 1s|\cdot$ →

$$e^{-i\omega_1 t}\cdot i\hbar\cdot\frac{dc_1}{dt} = c_1 e^{-i\omega_1 t}\underbrace{\langle 1s|V|1s\rangle}_{0} + c_2 e^{-i\omega_2 t}\langle 1s|V|2p,0\rangle$$

$$\rightarrow i\hbar\frac{dc_1}{dt} = c_2\langle 1s|V|2p,0\rangle e^{i(\omega_1-\omega_2)t}$$

$$\rightarrow i\hbar\frac{dc_1}{dt} = e\mathcal{E}\langle 1s|z|2p,0\rangle\cos(\omega t)e^{i(\omega_1-\omega_2)t}$$

$\langle 2p,0|\cdot$ →

$$e^{-i\omega_2 t}\, i\hbar\frac{dc_2}{dt} = c_1 e^{-i\omega_1 t}\langle 2p,0|V|1s\rangle$$

$$+ c_2 e^{-i\omega_2 t}\langle 2p,0|V|2p,0\rangle$$

$$\langle 2p,0|V|2p,0\rangle = -\langle 2p,0|P^\dagger VP|2p,0\rangle$$

$$= -(-1)^2\langle 2p,0|V|2p,0\rangle \rightarrow \langle 2p,0|V|2p,0\rangle = 0$$

$$\therefore\ i\hbar\frac{dc_2}{dt} = e\mathcal{E}\langle 2p,0|z|1s\rangle\cos(\omega t)e^{i(\omega_1-\omega_2)t}$$

initial condition : $c_1(t=0) = 1$, $c_2(t=0)=0$

$$\frac{dc_1}{dt} \propto \varepsilon c_2 \quad , \quad \frac{dc_2}{dt} \propto \varepsilon c_1$$

~~Ex~~ for $t=\delta t$ is small, we have

$$c_2(t=\Delta t) = \overset{\nearrow 0}{c_2(t=0)} + \frac{dc_2}{dt}\Big|_{t=0}\Delta t + \cdots$$

$$= \frac{dc_2}{dt}\Big|_{t=0} \times \Delta t \propto \varepsilon c_1 \rightarrow \frac{dc_1}{dt} \propto \varepsilon^2 c_1(0)$$

$$c_1(t=\Delta t) = \underset{\underset{1}{\downarrow}}{c_1(t=0)} + \underset{\underset{\propto \varepsilon^2 c_1(t=0)}{\downarrow}}{\frac{dc_1}{dt}\Big|_{t=0}}\Delta t. \approx 1$$

$\propto \varepsilon^2 c_1(t=0) \leftarrow$
$= O(\varepsilon^2)$

$$\therefore \frac{dc_2}{dt}\Big|_{t=\Delta t} \propto \varepsilon c_1(t=\Delta t) \propto \varepsilon c_1(t=0) \propto \varepsilon$$

$$\frac{dc_1}{dt}\Big|_{t=\Delta t} \propto \varepsilon c_2(t=\Delta t) \propto \varepsilon^2 c_1(t=0) \propto \varepsilon^2$$

→ First order equations : (ignore $\frac{dc_1}{dt}$ ∵ it is too small)

$$c_1 = 1$$

$$i\hbar \frac{dc_2}{dt} = e\varepsilon \langle 2p,0|z|1,s\rangle \cos(\omega t)\, e^{i(\omega_2-\omega_1)t}$$

why is that? why faster = can be ignored

$\cos(\omega t)\, e^{i(\omega_2-\omega_1)t}$ $\Delta\omega = \omega_2-\omega_1 > 0$

$= \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})\, e^{i(\omega_2-\omega_1)t}$

$= \frac{1}{2}(e^{i(\omega_2-\omega_1+\omega)t} + e^{i(\omega_2-\omega_1-\omega)t})$

$\downarrow \Delta\omega+\omega$ $\downarrow \Delta\omega-\omega$

$C_2(t)$ is the probability amplitude for trasition ~~in~~ from $|1s\rangle$ into $|2p,0\rangle$

$\because$ ω ~~needs to be~~ is close to $\Delta\omega$, which is the resonant frequency

$\therefore$ $\Delta\omega+\omega >> \Delta\omega-\omega$ $\therefore$ the first term oscillates much faster

$$\Rightarrow \frac{dC_2(t)}{dt} = \frac{e\varepsilon}{2i\hbar}\langle 2p,0|z|1s\rangle e^{i(\omega_2-\omega_1-\omega)t}$$

$\because C_2(t=0)=0$

$$\therefore C_2(t) = \int_0^t \frac{e\varepsilon}{2i\hbar}\langle 2p,0|z|1s\rangle e^{i\underbrace{(\omega_2-\omega_1-\omega)}_{\Delta}t}\, dt.$$

$$= \frac{e\varepsilon\langle 2p,0|z|1s\rangle}{2i\hbar\cdot i\Delta}(e^{i\Delta t}-1)$$

~~$\frac{1}{i^2}(e^{i\Delta t}-1) = e^{i\Delta t}$~~

$\therefore$ ~~$|C_2(t)|^2$~~ $|e^{i\Delta t}-1|^2 = |(\cos(\Delta t)-1)+i\sin\Delta t|^2$

$= \cos^2(\Delta t) - 2\cos(\Delta t) + 1 + \sin^2(\Delta t)$

$= 2-2\cos(\Delta t) = 4\sin^2(\frac{\Delta t}{2})$

$$\therefore |c_2(t)|^2 = \frac{e^2 \varepsilon^2 |\langle 2p,0|z|1s\rangle|^2}{\cancel{4}\hbar^2 \Delta^2} \cancel{\tfrac{1}{4}}\cancel{4} \sin^2\left(\frac{\Delta t}{2}\right)$$

$$\rightarrow |c_2(t)|^2 = \frac{e^2 \varepsilon^2 |\langle 2p,0|z|1s\rangle|^2}{\hbar^2 \Delta^2} \sin^2\left(\frac{\Delta t}{2}\right)$$

$\because \; c_1(t) \simeq 1 \qquad \therefore \; |c_1(t)| = 1$

~~$|c_2(t)|^2 \geq p$~~ $\quad |c_2(t)|^2 \geq 0$

$\therefore \; |c_1(t)|^2 + |c_2(t)|^2 \geq 1$

$|c_1|^2 + |c_2|^2$ is expected to be a constant (= 1) because the probability of being in ~~both states~~ states $|1s\rangle$ ~~and~~ and $|2p,0\rangle$ should sum to 1

So need $|c_2(t)|^2 \leq |c_1(t)|^2 = 1$

$$\therefore \; \frac{e^2 \varepsilon^2 |\langle 2p,0|z|1s\rangle|^2}{\hbar \Delta^2} \ll 1$$

10. $H = \frac{1}{2m}(P_1^2 + P_2^2) + \frac{e^2}{4\pi\varepsilon_0}\left(-2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + \frac{1}{|r_1 - r_2|}\right.$

$$|r_1 - r_2|^2 = r_1^2 + r_2^2 + 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2))$$

$$\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}\right)|r_1 - r_2|^2 \cancel{\phi}\psi$$

$$= \left[r_1^2\frac{\partial\psi}{\partial\phi_1} + r_2^2\frac{\partial\psi}{\partial\phi_1} + 2r_1r_2\cos\theta_1\cos\theta_2\frac{\partial\psi}{\partial\phi_1}\right.$$

$$+ r_1^2\frac{\partial\psi}{\partial\phi_2} + r_2^2\frac{\partial\psi}{\partial\phi_2} + 2r_1r_2\cos\theta_1\cos\theta_2\frac{\partial\psi}{\partial\phi_2}$$

$$+ 2r_1r_2\sin\theta_1\sin\theta_2\underbrace{\left[\frac{\partial}{\partial\phi_1}\cos(\phi_1 - \phi_2)\psi + \frac{\partial}{\partial\phi_2}\cos(\phi_1 - \phi_2)\psi\right]}.$$

$$= \cancel{-\sin(\phi_1 - \phi_2)\psi} + \cos(\phi_1 - \phi_2)\frac{\partial\psi}{\partial\phi_1}$$

$$~~-Si~~ \cancel{+\sin(\phi_1 - \phi_2)\psi} + \cos(\phi_1 - \phi_2)\frac{\partial\psi}{\partial\phi_2}$$

$$= |r_1 - r_2|^2\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}\right)\psi$$

$$\rightarrow \left[\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}, |r_1 - r_2|^2\right] = 0$$

$$\rightarrow [\hat{L_z}, |r_1 - r_2|^2] = 0.$$

~~$\therefore 0 = [L_z, |r_1 - r_2|^2] = |r_1 - r_2|[L_z, |r_1 - r_2|] + [L_z, |r_1 - r_2|]|r_1 - r_2|$~~

$$\therefore [|r_1 - r_2|^2, [L_z, |r_1 - r_2|^2]] = 0$$

let $|r_1 - r_2|^2 = B$ then

$$\therefore [B, [L_z, B]] = 0$$

$$\therefore [L_z, f(B)] = [L_z, B]\frac{df}{dB}$$

consider $f(B) = \frac{1}{\sqrt{B}}$ $\frac{df}{dB} = -\frac{1}{2} B^{-3/2}$

$\therefore$ ~~$\theta = E$~~ $[L_z, f(B)] = \underbrace{[L_z, B]}_{0} \left(-\frac{1}{2} B^{-3/2}\right) = 0$

$\because f(B) = \frac{1}{|r_1 - r_2|}$ $\therefore$ $\underline{[L_z, \frac{1}{|r_1-r_2|}] = 0}$ ①

~~$L_{z_1} =$~~ $\because L_z = -i\hbar\left(\frac{\partial}{\partial\phi_1} + \frac{\partial}{\partial\phi_2}\right)$ does not act on r_1 and r_2

$\therefore$ $\underline{[L_z, (\frac{1}{r_1} + \frac{1}{r_2})] = 0}$ ②

~~$[L_{z_1}, P_{x_1}]\psi = -\hbar^2 [\frac{\partial}{\partial\phi_1}] -\hbar^2 [x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial x}]\psi$~~

~~$= -\hbar^2 \left(x\frac{\partial}{\partial y}\psi - y\frac{\partial}{\partial x} \; x\frac{\partial^2\psi}{\partial y \partial x} - y\frac{\partial^2\psi}{\partial x^2} - \frac{\partial}{\partial x}(x\frac{\partial\psi}{\partial y}) + \frac{\partial}{\partial x}(y\frac{\partial\psi}{\partial x}) \right)$~~

~~$x\frac{\partial^2\psi}{\partial x \partial y} + \frac{\partial\psi}{\partial y}$~~

~~$[L_{z_1}, P_{y_1}]\psi = -\hbar^2 [x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]\psi$~~

~~$= -\hbar^2 (x\partial^2$~~

Lz commutes with spherically symmetric expressions.

$\because [L_i, P_j] = i\hbar\varepsilon_{ijk}P_k$

$\therefore [L_z, P_1^2 + P_2^2] = [L_{z_1} + L_{z_2}, P_{1x}^2 + P_{1y}^2 + P_{1z}^2 + P_{2x}^2 + P_{2y}^2 + P_{2z}^2]$

$= P_{1x}[L_{z_1}, P_{1x}] + [L_{z_1}, P_{1x}]P_{1x}$

$+ P_{1y}[L_{z_1}, P_{1y}] + [L_{z_1}, P_{1y}]P_{1y}$

$+ P_{1z}[L_{z_1}, P_{1z}] + [L_{z_1}, P_{1z}]P_{1z}$

1 and 2 commutes automatically

+ (same for 2)

$$= P_{1x}\,\cancel{i\hbar}\,P_{1y} + P_{1y}\,\cancel{i\hbar}\,P_{1x}$$
$$- P_{1y}\,\cancel{i\hbar}\,P_{1x} - P_{1x}\,\cancel{i\hbar}\,P_{1y}$$
$$+ 0 \quad + 0 = 0 \quad + \quad 0$$

$$= 0$$

$$\therefore \underbrace{[L_z , P_1^2 + P_2^2] = 0}. \quad ③$$

$$\therefore ①, ②, ③ \quad \therefore \quad [L_z , H] = 0$$

$$\therefore \underbrace{[H, L_z] = 0} \quad \text{QED.}$$

Assuming $[H, L_x] = [H, L_y] = 0$

$$\rightarrow [H, L^2] = [H, \underset{\text{sum over } i}{L_iL_i}] = L_i\underbrace{[H, L_i]}_{0} + \underbrace{[H, L_i]}_{0}L_i = 0$$

QED

$$\nearrow \langle \underline{a} . \underline{b} | Q_{12} | \psi \rangle \equiv \langle \underline{b} , \underline{a} | Q_{12} | \psi \rangle \;\times \quad \Rightarrow \langle \underline{a} , \underline{b} | Q_{12} | \psi \rangle = \langle \underline{b} , \underline{a} | \psi \rangle$$

$$\therefore \quad 6 \quad \langle \underline{a} , \underline{b} | Q_{12} H | \psi \rangle = \langle \underline{b} , \underline{a} | Q_{12} H | \psi \rangle$$

a typo

$$\langle \underline{a} , \underline{b} | \cancel{H Q_{12}} | \psi \rangle = \langle \underline{b} , \underline{a} | \cancel{Q_{12}\, H Q_{12}} | \psi \rangle$$

WTF!!!

$\rightarrow \therefore Q_{12} H |\psi\rangle$ and ~~$H Q_{12} |\psi\rangle$~~ ~~are~~ is ~~both~~ symmetric

~~Q_{12} can be represented by the transformation~~

~~$Q_{12} \quad \langle \underline{a} , \underline{b} | \psi \rangle \xrightarrow{Q_{12}} \frac{1}{\sqrt{2}} \langle \underline{a} , \underline{b} | \psi \rangle + \frac{1}{\sqrt{2}} \langle \underline{b} , \underline{a} | \psi \rangle$~~

~~for ~~$\underline{a} \neq \underline{b}$~~ $\langle \underline{a} , \underline{b} | \psi \rangle \neq \langle \underline{b} , \underline{a} | \psi \rangle$~~

~~and $\langle \underline{a}, \underline{b} | \psi \rangle \xrightarrow{Q_{12}} \langle \underline{a}, \underline{b} | \psi \rangle =$~~

~~for $\langle \underline{a}, \underline{b} | \psi \rangle = \langle \underline{b}, \underline{a} | \psi \rangle$~~

$\because$ H is symmetric under exchange of $\underline{r}_1, \underline{r}_2$

$(P_{1,2} = -i\hbar \frac{\partial}{\partial \underline{r}_{1,2}}) \rightarrow \hat{H}(\underline{r}_2, \underline{r}_1) = \hat{H}(\underline{r}_1, \underline{r}_2)$

$\therefore$

consider $\psi(\underline{r}_1, \underline{r}_2)$

~~$Q_{12} \hat{H}(\underline{r}_1, \underline{r}_2) \psi(\underline{r}_1, \underline{r}_2) = \frac{1}{\sqrt{2}} (\hat{H}(\underline{r}_2, \underline{r}_1) \psi(\underline{r}_2, \underline{r}_1) + \hat{H}(\underline{r}_1, \underline{r}_2) \psi(\underline{r}_1, \underline{r}_2))$~~

~~$= \hat{H}(\underline{r}_1, \underline{r}_2) \frac{1}{\sqrt{2}} (\psi(\underline{r}_2, \underline{r}_1) + \psi(\underline{r}_1, \underline{r}_2))$~~ $[\hat{H}, \hat{Q}_{12}] = 0$

~~$= \hat{H} Q_{12} \psi(\underline{r}_1, \underline{r}_2)$~~

$\hat{Q}_{12} \hat{H}(r_1, r_2) \psi(r_1, r_2)$ //

$= \hat{H}(r_2, r_1) \psi(r_2, r_1)$

~~$\rightarrow$ [$\hat{H}$ $[Q_{12}, \hat{H}] = 0$~~ $= \hat{H}(r_1, r_2) \psi(r_2, r_1)$

$= \hat{H}(r_1, r_2) \hat{Q}_{12}(r_1, r_2)$

Similarly $\because$ L^2, L_z, are both symmetric under exchange of $\underline{r}_1$ and $\underline{r}_2$

$\therefore$ $[Q_{12}, \hat{L}^2] = 0$, $[Q_{12}, L_z] = 0$

$\therefore$ H, L^2, L_z, Q_{12} mutually commutes

$\therefore$ $|E, L, M_L, q\rangle$ can be a good quantum state.

(but eigenvalues of Q_{12} are 1 and 0 not 1 and -1, WTF!!!)

possible spin kets for 2 electrons

$S_z=1$ $|+\rangle_1 |+\rangle_2$ ✓

$S_z=0$ $\frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2)$ ✓

$S_z=-1$ $|-\rangle_1 |-\rangle_2$ ✓

} symmetric triplet $S=1$

$S_z=0$ $\frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2)$ ✓ → anti-symmetric singlet $S=0$

Assignment of spin part is constrained by the fact that, since electrons are fermions, they obey pauli-exclusion principle so that the overall wavefunction must be ~~anty~~ antisymmetric

→ spin part must have ~~oppost~~ opposite symmetry to the spatial part.

$1s^2$ $L=0$ $q=+1$ $M_L=0$ $\frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2)$

degeneracy $= \boxed{d=1}$ ✓

$1s2s$ $L=0$ $q=-1$ $M_L=0$ $|+\rangle_1 |+\rangle_2$

$\frac{1}{\sqrt{2}}(|+\rangle_1 |-\rangle_2 + |-\rangle_1 |+\rangle_2)$

$|-\rangle_1 |-\rangle_2$

$\boxed{d=3}$ ✓

1s2s $L=0$ $q=+1$ $M_L=0$ $\therefore \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)$

$\boxed{d=1}$ ✓

1s2p $L=1$ $q=-1$ $M_L=+1, 0, -1$ [redacted]

$\underbrace{M_L=+1,0,-1}_{3}$

$$3\left\{\begin{array}{l} |+\rangle_1|+\rangle_2 \\ \frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 + |-\rangle_1|+\rangle_2) \\ |-\rangle_1|-\rangle_2 \end{array}\right.$$

$\therefore d = 3\times 3 = \boxed{9}$ ✓

[redacted]

[redacted]

1s2p $L=1$ $q=+1$ $M_L=+1, 0, -1$

$\underbrace{M_L=+1,0,-1}_{3}$ $\underbrace{\frac{1}{\sqrt{2}}(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2)}_{1}$

$\boxed{d=3}$ ✓

A3 2014 Tutorial Notes

8. $\langle \underline{r} | 1,i \rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix} f(r)$

$L_z |1,w\rangle = \hat{L}_z \, z \, f(r) = z \, \hat{L}_z f(r) = z \cdot 0 = 0.$

$\hat{L}_x |1,u\rangle = 0$

$\hat{L}_y |1,v\rangle = 0$

$(L_x + iL_y) |l,m\rangle = \hbar\sqrt{l(l+1) - m(m+1)} \, |l, m+1\rangle$

$L_+ |1,0\rangle = \sqrt{2}\,\hbar |1,1\rangle$

$L_x z f = -i\hbar (y\partial_z - z\partial_y) z f$

$= -i\hbar y f$

$L_y z f = +i\hbar x f$

$= -i\hbar |1,v\rangle - \hbar |1,u\rangle$

$(L_x + iL_y) z f = \sqrt{2}\,\hbar |1,1\rangle$

$|1,1\rangle = \frac{-1}{\sqrt{2}} (|1,u\rangle + i |1,v\rangle)$

$|1,-1\rangle = \frac{1}{\sqrt{2}} (|1,u\rangle - i |1,v\rangle)$

$\hat{H} = \hat{T} + \hat{V}$

$\psi \sim R_{nl}(r) \, Y_{lm}(\theta, \varphi)$

$$\langle 1,\xi | V | 1,\xi' \rangle = G(K-H)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$2(K-H)G \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$G(H-K) \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

9. $t=0, \quad c_1 = 1, \quad c_2 = 0$

$$\frac{dc_1}{dt} \sim O(\varepsilon^2)$$

$$\frac{dc_1}{dt} = 0$$

$$\frac{dc_2}{dt} = -\left|\frac{ie\varepsilon}{\hbar}\right| \langle 2p,0 | z | 1s \rangle \underbrace{\cos(\omega t) e^{i(\omega_2-\omega_1)t}}_{\frac{1}{2}\left(e^{i\Delta t} + e^{i(\omega+\omega_2-\omega_1)t}\right)}$$

$$c_2(t) = -\frac{e\varepsilon}{2\hbar\Delta}\left|e^{i\Delta t}-1\right| \langle 2p | \cdots \rangle$$

$$|c_2|^2 = \left(\frac{e\varepsilon}{\hbar\Delta}\right)^2 |\langle \quad \rangle|^2 \cancel{8} \sin\left(\frac{\Delta t}{2}\right)^2$$

$$\Rightarrow |c_2|^2 \ll 1$$

$$|c_2|^2 + |c_1|^2 = |c_2|^2 + 1$$

$$\underline{\left|\frac{\varepsilon}{\Delta}\right|^2 \ll \frac{\hbar}{e|\langle \quad \rangle|^2}}$$

close to the resonant frequency.

$L_z = 0 - i\hbar(\partial\phi_1 + \partial\phi_2)$

$[L_z, |r_1 - r_2|^2]$

$= -i\hbar\,\partial\phi_1(2r_1r_2 \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2))$

$= -i\hbar(\ldots\ldots \sin(\phi_1 - \cancel{\phi_2})) - \sin(\cancel{\phi_1} - \phi_2) - 1)$

$= 0$

if $[A, B] = 0 \Rightarrow [A, f(B)] = 0$

* $[A, f(B)] = [A, B]\frac{df}{dB}$

if $[B, [A, B]] = 0$

$\langle \underline{a}, \underline{b} | Q_{12} | \psi\rangle = \langle \underline{b}, \underline{a} | Q_{12} | \psi\rangle$

$H = Q_{12} \underset{\underline{a}, \underline{b}, \underline{c}}{\sum\!\!\!\!\!\!\int} |b, c\rangle\langle b, c| H |a, b\rangle\langle a, b|.$

$=$

Swap operator

Fucking Typo!

$|\psi'\rangle = P|\psi\rangle$

$-\langle\psi|\underline{V}|\psi\rangle = P\langle\psi|\underline{V}|\psi\rangle$

$= \langle\psi'|\underline{V}|\psi'\rangle$

$= \langle\psi|P^{\dagger}\underline{V}P|\psi\rangle$