2663

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2013

Friday, 14 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. If H is a Hermitian operator, how are $\langle\phi|H|\psi\rangle$ and $\langle\psi|H|\phi\rangle$ related? Show that the eigenvalues of a Hermitian operator are real and explain the significance of this in quantum mechanics. Show that the eigenstates corresponding to distinct eigenvalues are orthogonal. [5]

2. A particle of mass m is in the ground state of the potential $V(x) = 0$ for $0 \leq x < a$ and $V(x) = \infty$ elsewhere. The potential is suddenly changed to $V(x) = 0$ for $0 \leq x < 2a$ and $V(x) = \infty$ elsewhere. What is the probability that the particle remains in the ground state? [6]

3. Suppose, instead, that the potential well in Question 2 is changed slowly from a width a to a width $2a$. Over what time scale must the change occur if the probability that the particle remains in the ground state is to stay close to 1? [5]

4. For a wavefunction $\psi = \psi(\mathbf{r})$ describing particles of mass m, the probability current density is given by

$$\mathbf{j} = \frac{\hbar}{2mi}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right].$$

Show that for the general wavefunction with uniform probability density, $\psi(\mathbf{r}) = A\exp[\mathrm{i}\phi(\mathbf{r})]$, $\mathbf{j}$ is proportional to the gradient of ϕ and find the constant of proportionality. Hence or otherwise, find the wavefunction representing a uniform probability density of one particle per unit volume and a uniform probability current density of $\mathbf{p}/m$. [6]

?

5. What does the operator

$$\int_{-\infty}^{\infty} |x\rangle\langle x|\mathrm{d}x$$

represent? A state $|\psi\rangle$ in the position representation is given by $\langle x|\psi\rangle = a/2$ for $-a \leq x < a$ and $\langle x|\psi\rangle = 0$ elsewhere. Taking

$$\langle x|p\rangle = \frac{1}{\sqrt{h}}\exp\left(\mathrm{i}\frac{px}{\hbar}\right),$$

what is $\langle p|\psi\rangle$? [7]

6. Show that any trial state $|\psi\rangle$ has an expectation energy that is at least as high as the energy of the ground state of the Hamiltonian H. [4]

Using a (normalised) trial wave function of the form $\psi(x) = (a/\pi)^{\frac{1}{4}}\exp(-ax^2/2)$, show that the expectation value of the energy of a particle moving in the potential $V(x) = V_0|x|^3$ is

$$\langle E\rangle(a) = \alpha a + \beta a^{-\frac{3}{2}},$$

and find the constants α and β.

Hence or otherwise find an upper bound for the ground state energy of the particle. [7]

Section B

7. The electron in a hydrogen-like ion has a Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$$

and eigenstates $|n, l, m\rangle$.

(a) Explain the origin of each term in the Hamiltonian, and the meaning of each of the quantum numbers n, l and m. On which quantum number does the energy depend? For $n = 2$, what values may l and m take? [4]

(b) By considering the electric dipole selection rules or otherwise, identify which of the matrix elements $\langle 2, l', m'|z|2, l, m\rangle$ are non-zero. [4]

(c) A small static electric field of strength $\mathcal{E}$ is applied in the z direction. Write down the perturbation Hamiltonian. Calculate its non-zero matrix elements for the basis of states with $n = 2$. [6]

(d) Identify the linear combinations of the $n = 2$ states that diagonalise the perturbation Hamiltonian and calculate the energy shifts. Hence sketch the $n = 2$ energy levels before and after the application of the perturbation. In each case, label the eigenstates and give the magnitude of any energy differences. [6]

[The integral $\int_0^\infty \rho^n e^{-\rho} \mathrm{d}\rho = n!$. The normalised states $|n, l, m\rangle = u_n^l(r)\mathrm{Y}_l^m(\theta, \phi)$, with

$$u_2^0(r) = \frac{2e^{-r/2a_Z}}{(2a_Z)^{3/2}}\left(1 - \frac{r}{2a_Z}\right), \qquad u_2^1(r) = \frac{e^{-r/2a_Z}}{\sqrt{3}(2a_Z)^{3/2}}\frac{r}{a_Z}, \qquad a_Z = \frac{4\pi\epsilon_0\hbar^2}{Zme^2},$$

and $\mathrm{Y}_0^0(\theta, \phi) = \sqrt{\frac{1}{4\pi}}$, $\quad \mathrm{Y}_1^0(\theta, \phi) = \sqrt{\frac{6}{8\pi}}\cos\theta$, $\quad \mathrm{Y}_1^{\pm 1}(\theta, \phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$.]

8. A system of two spin $\frac{1}{2}$ particles, A and B, can be described with a basis representing the projections along z, $\{|\uparrow_A\uparrow_B\rangle, |\uparrow_A\downarrow_B\rangle, |\downarrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle\}$. In this basis, a particular operation on spin A is described by the matrix

$$H_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Find the results of operating on the states $|\uparrow_A\uparrow_B\rangle$ and $|\downarrow_A\uparrow_B\rangle$ with H_A. Measurements of the projections of spins A and B are made on the resulting states. In each case, what is the probability of finding spin A pointing up? What is the probability of finding spin B pointing up? Does the order of the measurements matter for these cases? [8]

Another operation flips the state of spin B if spin A is up and does nothing if spin A is down. Construct the matrix, H_C representing this operation. Find the result, $|\psi\rangle$, of operating on $|\downarrow_A\uparrow_B\rangle$ first with H_A and then with H_C. What are the possible outcomes of measurements of the projections of spin A followed by spin B on state $|\psi\rangle$? What is special about the state $|\psi\rangle$? [9]

For the state $|\psi\rangle$ what is the projection of the total spin along z? Does $|\psi\rangle$ represent a singlet state or a component of a triplet? What do you expect for the outcome of successive measurements of spins A and B along *any* particular axis? [3]

9. Derive an expression for the rate of change of the expectation value of an operator (Ehrenfest's theorem), stating clearly any assumptions that you make. What is a *good quantum number*? What is a *stationary state*? [5]

An apparatus confines a particle of mass m to the (x, y) plane and imposes a potential

$$V(x, y) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2).$$

Write down the energy eigenvalues. For the case $\omega_x = \omega_y + \delta\omega$ where $\delta\omega$ is small compared to both ω_x and ω_y, sketch an energy level diagram showing the lowest six levels and the quantum numbers. [5]

A modification to the apparatus is made so that $\omega_x = \omega_y = \omega$. What new symmetry does the system have? What are the consequences for the energy level diagram? By considering what quantity should be conserved under the new symmetry, identify a new good quantum number and write down the corresponding differential operator. Verify using Ehrenfest's theorem that the expectation value of this operator is time-independent. [7]

Write down the wavefunctions of the lowest three energy eigenstates that are also eigenstates of the new operator. (There is no need to normalise them.) [3]

[You may use without proof the results that for a particle of mass m in a one dimensional potential $V(x) = \frac{1}{2}m\omega^2 x^2$, the energy eigenvalues are $(n + \frac{1}{2})\hbar\omega$ where n is a non-negative integer, and the eigenfunctions for $n = 0$ and $n = 1$ are, respectively, $A_0 \exp(-m\omega x^2/2\hbar)$ and $A_1 x \exp(-m\omega x^2/2\hbar)$.]

L_z , x^2+y^2

$\underbrace{x[L_z, x]}_{i\hbar y} + \underbrace{[L_z, x]}_{i\hbar y}x$

$\underbrace{y[L_z, y]}_{-x}$

10. $\mathcal{X}$ is the operator that exchanges the particles in a two-particle system. By considering wavefunction for two spinless particles $\psi(x_1, x_2)$,

(i) find the eigenvalues of $\mathcal{X}$;

(ii) show that for general operators A and B, if $\mathcal{X}A\mathcal{X} = B$, then $\mathcal{X}A^2\mathcal{X} = B^2$.

Show that $\mathcal{X}$ commutes with the operator for the total kinetic energy, $K = \hat{p}_1^2/2m_1 + \hat{p}_2^2/2m_2$, as long as the masses of the two particles are the same. [Hint: consider $\mathcal{X}K\mathcal{X}$ and use the result that $\mathcal{X}C_1\mathcal{X} = C_2$ where C_1 and C_2 represent any single-particle operator acting on the first and second arguments of the wavefunction respectively.] [6]

The potential energy may contain terms acting on each particle ($V_1(x_1)$ and $V_2(x_2)$) and interaction terms $V_I(x_1, x_2)$. Give conditions on V_1, V_2 and V_I under which $\mathcal{X}$ commutes with the total potential energy (and hence the Hamiltonian, if the masses are the same). [3]

A particle in a particular one-dimensional potential has orthonormal bound states $u(x)$ and $v(x)$ with energies E_u and E_v respectively. A second identical particle, which does not interact with the first, is introduced such that one particle resides in each of the states. Write down the exchange symmetric and antisymmetric two-particle states with total energy $E = E_u + E_v$. Show that when a small interaction potential $V_I(|x_1 - x_2|)$ is turned on, the expectation values for the energies of these states become $E_u + E_v + J_D \pm J_E$. Give integral expressions for J_D and J_E. [7]

For the case where the interaction potential V_I is extremely short-range, and given by $V_I = V_0\delta(x_1 - x_2)$, how are the perturbed energy levels related to the unperturbed energy levels? Explain the result. [4]

To: Michael Barnes

A3 2013 First Attempt Ziyan Li

~~to~~

1. If $\hat{H}$ is hermitian $\langle\phi|\hat{H}|\psi\rangle = \langle\psi|\hat{H}|\phi\rangle^*$

let, $H|\psi\rangle = \lambda|\psi\rangle$ then

$$\cancel{|\lambda|^2\langle\psi|\psi\rangle} = \lambda\langle\psi|\psi\rangle = \langle\psi|(\lambda|\psi\rangle) = \langle\psi|H|\psi\rangle$$

$$= (H|\psi\rangle)^\dagger|\psi\rangle = \lambda^*\langle\psi|\psi\rangle$$

$\because \langle\psi|\psi\rangle \neq 0 \quad \therefore \lambda = \lambda^* \rightarrow$ $\underline{\lambda \text{ is real}}$

$\rightarrow$ eigenvalues are real

~~This~~ Since physical observables are represented by Hermitian operators in Quantum Mechanics, ~~the~~ and the eigenvalues are the results of measurements, the fact that eigenvalues are real means we get real physical measurement results.

If $\hat{H}|\psi_1\rangle = \lambda_1|\psi_1\rangle$ $\hat{H}|\psi_2\rangle = \lambda_2|\psi_2\rangle$

then $\cancel{\lambda_1}\ \cancel{\lambda_2}\ \lambda_2\langle\psi_1|\psi_2\rangle = \langle\psi_1|\lambda_2|\psi_2\rangle$

$$= \langle\psi_1|\hat{H}|\psi_2\rangle = (\hat{H}|\psi_1\rangle)^\dagger|\psi_2\rangle$$

$$= \underbrace{\lambda_1^*\langle\psi_1|\psi_2\rangle = \lambda_1\langle\psi_1|\psi_2\rangle}_{\because \lambda_1 \text{ is real}}$$

$$\rightarrow (\lambda_1 - \lambda_2)\langle\psi_1|\psi_2\rangle = 0$$

$\because \lambda_1 \neq \lambda_2$ (distinct eigenvalues)

$$\therefore \langle\psi_1|\psi_2\rangle = 0$$

2.

Ground state with width a:

$$\langle x | \psi_a \rangle = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)$$

Ground state with width $2a$:

$$\langle x | \psi_{2a} \rangle = \sqrt{\frac{1}{a}} \sin\left(\frac{\pi x}{2a}\right)$$

Sudden expansion the state remains in $|\psi_a\rangle$

$\therefore$ Required Probability Amplitude:

$$\langle \psi_{2a} | \psi_a \rangle = \int dx \, \langle \psi_{2a} | x \rangle \langle x | \psi_a \rangle$$

$$= \int_0^a dx \, \frac{\sqrt{2}}{a} \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{2a}\right)$$

$u = \frac{\pi x}{2a}$

$x = a, \quad u = \frac{\pi}{2}$

$\frac{2a}{\pi} du = dx$

$$= \frac{2}{a} \cdot \frac{\sqrt{2}a}{\pi} \int_0^{\pi/2} du \, \sin(2u) \sin(u)$$

$$= \frac{2}{a} \cdot \frac{\sqrt{2}a}{\pi} \cdot \frac{1}{2} \int_0^{\pi/2} \cos(u) - \cos(3u) \, du$$

$$= \frac{2}{a\pi} \frac{\sqrt{2}}{a\pi} \left[\sin(u) - \frac{1}{3}\sin(3u) \right]_0^{\pi/2} = \frac{4\sqrt{2}}{3\pi}$$

$$1 - \frac{(-1)}{3}$$

$$\cancel{= \frac{4}{a\pi} \frac{2\sqrt{2}}{\pi} = \frac{2\sqrt{2}}{3\pi} \frac{4\sqrt{2}}{3\pi}}$$

Probability:

$$\cancel{P = |\langle \psi_{2a} | \psi_a \rangle|^2 = \boxed{\frac{8}{\pi^2}} =}$$

$$P = |\langle \psi_{2a} | \psi_a \rangle|^2 = \cancel{\frac{8}{9\pi^2}} = \frac{32}{9\pi^2} = 0.36$$

3. For adiabatic ~~prin~~ approximation to apply, the time scale must be much larger than the typical time scale for the expectation of any physical quantity to change.

$\therefore t \gg \Delta t$ where Δt is set by

$$\Delta t \Delta E \geq \frac{\hbar}{2} \quad \rightarrow \quad \Delta t \Delta E \approx \hbar$$

$$\therefore \Delta t \approx \frac{\hbar}{\Delta E}$$

~~groundstat~~ ~~Ground~~ Ground state & states energies of 2a well

For a ~~E~~ Ground state $E_a = \frac{\pi^2\hbar^2}{8ma^2} \cdot 1^2$

~~For 2a~~ First Excited State For 2a : $E_{2a} = \frac{\pi^2\hbar^2}{8ma^2} \cdot 2^2$

$$\Delta E = E_2 - E_1 = \frac{3\pi^2\hbar^2}{8ma^2}$$

$|E_{2a} - E_a|$

$$\therefore \Delta t = \frac{8ma^2}{3\pi^2\hbar}$$

$\therefore$ time scale $t \gg \frac{8ma^2}{3\pi^2\hbar}$ ✓

4. $\underline{j} = \frac{\hbar}{2mi}[\psi^* \nabla\psi - \psi\nabla\psi^*]$

$\psi(\underline{r}) = A\exp[i\phi(\underline{r})]$ $\quad \psi^* = A\exp[-i\phi(\underline{r})]$

$\psi^*\nabla\psi = A\exp(-i\phi(\underline{r})) \cdot i\nabla\phi(\underline{r}) \cdot A\exp[i\phi(\underline{r})] = A^2 i\nabla\phi$

$\psi\nabla\psi^* = A\exp(+i\phi(\underline{r})) \cdot (-i)\nabla\phi(\underline{r}) \cdot A\exp[-i\phi(\underline{r})] = -A^2 i\nabla\phi$

$$\underline{j} = \frac{\hbar}{2mi}[2i][A^2\nabla\phi] = \frac{\hbar A^2}{m}\nabla\phi$$

Consider the plane wave solution ~~$\psi(\underline{r}) = A\exp(\frac{i\underline{p}\cdot\underline{r}}{\hbar}$~~

$$\psi(\underline{r}) = A\exp\left(\frac{i\underline{p}\cdot\underline{r}}{\hbar}\right) =$$

5

A is constant → uniform probability density

$\nabla\left(\frac{\underline{p}\cdot\underline{r}}{\hbar}\right) = \frac{\underline{p}}{\hbar}$

$$\therefore \underline{j} = \frac{\hbar A^2}{m}\cdot\frac{\underline{p}}{\hbar} = \frac{\underline{p}}{m}\cdot A^2 \doteq \frac{\underline{p}}{m} \rightarrow A = 1$$

~~Set A = 1 for normalised wavefunction~~

∴ The required solution is

$$\psi(\underline{r}) = \exp\left(\frac{i\underline{p}\cdot\underline{r}}{\hbar}\right)$$

This satisfies 1 particle per volume

∵ probability density $= |\psi|^2$

probability to find a particle in volume $d\tau$ is

$$P_\tau = |\psi|^2 d\tau = (1)\, d\tau = d\tau$$

5.

$\int_{-\infty}^{\infty} dx \; |x\rangle\langle x| \; dx$ is the identity operator

$$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx \, \langle p|x\rangle\langle x|\psi\rangle \, \cancel{dx}$$

$$= \int_{-a}^{a} dx \left(\frac{1}{\sqrt{\hbar}} \exp\left(\frac{ipx}{\hbar}\right)\right)\left(\frac{a}{2}\right) \qquad h \quad \hbar \qquad WTF!$$

$$= \frac{a}{2\sqrt{\hbar}} \int_{-a}^{a} dx \, \exp\left(\frac{ipx}{\hbar}\right)$$

$$= \frac{a}{2\sqrt{\hbar} \cdot \sqrt{2\pi}} \cdot \frac{\hbar}{ip} \left[\exp\left(\frac{ipa}{\hbar}\right) - \exp\left(-\frac{ipa}{\hbar}\right)\right].$$

$$= \frac{a\sqrt{\hbar}}{p \cdot \sqrt{2\pi} \cdot \cancel{2i}} \cdot \cancel{2i} \sin\left(\frac{pa}{\hbar}\right)$$

$$\therefore \quad = \underbrace{\frac{a}{p}\sqrt{\hbar} \sin\left(\frac{pa}{\hbar}\right)}_{\propto} = \frac{a}{p}\frac{\sqrt{\hbar}}{2\pi}\sin\left(\frac{pa}{\hbar}\right)$$

6. $\hat{H}|E_n\rangle = E_n|E_n\rangle$ be the n^{th} ~~state~~ energy state

$|E_1\rangle$ is ground state. E_1 is its energy ($E_n \geq E_1$)

expand $|\psi\rangle = \sum_n C_n|E_n\rangle$. then

$$\langle\psi|\hat{H}|\psi\rangle = \sum_{m,n} C_m^* C_n \langle E_m|\underbrace{\hat{H}|E_n\rangle}_{E_n|E_n\rangle} = \sum_{m,n} C_m^* C_n E_n \underbrace{\langle E_m|E_n\rangle}_{\delta_{mn}}$$

$$= \sum_n |C_n|^2 E_n \underbrace{\geq}_{E_n \geq E_1} \sum_n |C_n|^2 E_1 \underbrace{=}_{\sum_n |C_n|^2 = 1} E_1$$

$$\therefore \quad \underline{E_1 \leq \langle\psi|\hat{H}|\psi\rangle} \quad \rightarrow \quad QED$$

$\langle x|\psi\rangle = (\frac{a}{\pi})^{\frac{1}{4}} \exp(-ax^2/2)$

$\hat{H} = \frac{\hat{p}^2}{2m} + V_0|x|^3$

$\langle\psi|\hat{H}|\psi\rangle = \frac{1}{2m}\langle\psi|\hat{p}^2|\psi\rangle + V_0 \langle\psi||x|^3|\psi\rangle$

$\langle\psi|\hat{p}^2|\psi\rangle = \cancel{|\hat{p}\psi|^2 =} \int dx\, |\hat{p}\psi|^2$

$= (\frac{a}{\pi})^{\frac{2}{4}} \hbar^2 \int dx \left(\frac{d}{dx} \exp(-ax^2/2)\right)^2$

$= a^2 (\frac{a}{\pi})^{\frac{2}{4}} \hbar^2 \int_{-\infty}^{\infty} dx\, \underbrace{\exp(-ax^2)\, x^2}_{\sqrt{\frac{\pi}{a}}\cdot\frac{1}{2a}}$

$= a^2 \sqrt{\frac{a}{\pi}} \sqrt{\frac{\pi}{a}}\, \hbar^2 \cdot \frac{1}{2a} = a^2\hbar^2 \cdot \frac{1}{2a} = \frac{a\hbar^2}{2}$

$\langle\psi||x|^3|\psi\rangle = 2\sqrt{\frac{a}{\pi}} \int_0^{\infty} \underbrace{x^3 \exp(-ax^2)\, dx}_{= \frac{1}{2a} \times \frac{2}{2a} = \frac{1}{2a^2}}$

even function

$= \sqrt{\frac{a}{\pi}} \frac{1}{a^2} = \frac{1}{\sqrt{\pi a^3}} = \frac{1}{\sqrt{\pi}} a^{-3/2}.$

$\langle E\rangle(a) = \langle\psi|\hat{H}|\psi\rangle = \cancel{\frac{a\hbar^2}{4m}} \frac{\hbar^2}{4m} a + \frac{V_0}{\sqrt{\pi}} a^{-3/2} = \alpha a + \beta a^{-3/2}$

set $0 = \frac{d\langle E\rangle(a)}{da} = \alpha - \frac{3}{2}\beta a^{-\frac{5}{2}}$

$\therefore \frac{3}{2}\beta a^{-\frac{5}{2}} = \alpha \rightarrow \frac{3}{2}\beta = \alpha a^{\frac{5}{2}}$

$\rightarrow \cancel{a = (\frac{3\beta}{2\alpha})^{\frac{5}{2}}}$ $\quad a = (\frac{3\beta}{2\alpha})^{\frac{2}{5}}$

$a_{min} = \left(\frac{6V_0 m}{\sqrt{\pi}\hbar^2}\right)^{2/5}$

$\langle E_{min}\rangle = \langle E\rangle((\frac{3\beta}{2\alpha})^{2/5}) = \alpha(\frac{3\beta}{2\alpha})^{\frac{2}{5}} + \beta(\frac{3\beta}{2\alpha})^{-\frac{3}{5}}$

$= (\frac{3}{2})^{\frac{2}{5}} \alpha^{\frac{3}{5}}\beta^{\frac{2}{5}} + (\frac{3}{2})^{-\frac{3}{5}} \alpha^{\frac{3}{5}}\beta^{\frac{2}{5}} = 1.96\, \alpha^{3/5}\beta^{2/5}$

$\cancel{= 1.96}$

7. (a) $\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r}$

$\uparrow$ kinetic energy $\quad\uparrow$ Coulomb potential

n = principle quantum number $\quad \hat{H}|n l m\rangle = -\frac{Z^2 R}{n^2}|n l m\rangle$

l = anghlar momentum quantum number $\quad \hat{L}^2|n l m\rangle = l(l+1)\hbar^2 |n l m\rangle$

m = magnetic quantum number $\quad \hat{L}_z|n l m\rangle = m\hbar|n l m\rangle$

Energy depends on n

For $n=2$, $l = 0$ or 1

$\downarrow \qquad \downarrow$

$m = 0 \qquad$ ~~m=0,1~~ $m = -1, 0, 1$

(b) Selection rule for z

$\Delta l = \pm 1, \; \Delta m = 0$

$\therefore \underbrace{\langle 210|z|200\rangle}$ and $\underbrace{\langle 200|z|210\rangle}$ are the only 2 non-zero matrix elements.

(c) perturbation $\underbrace{\delta V = +e\mathcal{E}z}$ $\quad$ do we get same result (for $\pm e\mathcal{E}z$?)

$\langle 210|z|200\rangle = \langle 200|z|210\rangle$

$= \int r^2\sin\theta\, dr\, d\theta\, d\phi \frac{2}{(2a_Z)^{3/2}}\left(1-\frac{r}{2a_Z}\right)e^{-r/2a_Z}\frac{1}{2}\sqrt{\frac{1}{\pi}} \times$

$r\cos\theta \times \frac{1}{\sqrt{3}(2a_Z)^{3/2}a_Z} r e^{-r/a_Z\cdot 2}\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$

$= \frac{\cancel{2}}{(2a_Z)^{3/2}}\frac{1}{\cancel{2}}\sqrt{\frac{1}{\pi}}\frac{1}{\sqrt{3}(2a_Z)^{3/2}}\frac{1}{a_Z}\frac{1}{\cancel{2}}\sqrt{\frac{3}{\pi}}\int_0^\infty dr\, r^4\left(1-\frac{r}{2a_Z}\right)e^{-r/a_Z}$

$\times \underbrace{\int_0^\pi d\theta\sin\theta\cos^2\theta}_{2/3} \cdot \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$

$$= \frac{2}{3a_z}\frac{1}{(2a_z)^3}\int_0^\infty dr\, r^4 e^{-r/a_z}$$

$$- \frac{2}{3a_z}\frac{1}{(2a_z)^4}\int_0^\infty dr\, r^5 e^{-r/a_z}$$

$$\int_0^\infty dr\, r^k e^{-r/a_z} \qquad u = \frac{r}{a_z} \quad du = \frac{dr}{a_z}$$

$$dr = a_z du$$

$$= a_z^{k+1}\int_0^\infty \underbrace{du\, u^k e^{-u}}_{k!} = a_z^{k+1} k! \qquad r^k = a_z^k u^k$$

$$\therefore \langle 200|z|210\rangle = \langle 210|z|200\rangle$$

$$= \frac{2}{3}\frac{1}{(2a_z)^3}\frac{1}{a_z} a_z^5 (4!) - \frac{2}{3}\frac{1}{(2a_z)^4} a_z^6 5!$$

$$= \left(\frac{\cancel{2}}{\cancel{3}} \times \frac{1}{\cancel{2}\times\cancel{2}\times\cancel{2}} \times \cancel{4}\times\cancel{3}\times 2\times 1 - \frac{\cancel{2}}{\cancel{3}}\times\frac{1}{2\times\cancel{2}\times\cancel{2}\times\cancel{2}} \times 5\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1\right) a_z$$

$$= (2-5)a_z = \underbrace{-3a_z}$$

(d) let ~~$|x1\rangle$~~ $|1\rangle = |200\rangle$ $\quad |2\rangle = |210\rangle$

$|3\rangle = |211\rangle$ $\quad$ ~~$|4|$~~ $|4\rangle = |21-1\rangle$

~~degenerate~~ let $\underline{|\psi\rangle = \sum_i c_i |i\rangle}$

Matrix of perturbation in degenerate subspace is:

$$\begin{bmatrix} \langle 1|\delta V|1\rangle & \langle 1|\delta V|2\rangle & \langle 1|\delta V|3\rangle & \langle 1|\delta V|4\rangle \\ \langle 2|\delta V|1\rangle & \cdots & & \\ \vdots & & & \\ & & & \langle 4|\delta V|4\rangle \end{bmatrix}$$

$$= e\mathcal{E}\begin{bmatrix} \langle 1|z|1\rangle & \langle 1|z|2\rangle & \cdots & \langle 1|z|4\rangle \\ \langle 2|z|1\rangle & & & \\ \vdots & & \cdots & \langle 4|z|4\rangle \end{bmatrix}$$

$$= e\mathcal{E}\begin{bmatrix} 0 & -3a_z & 0 & 0 \\ -3a_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Only need to diagonalise for non-zero λ

$$\begin{bmatrix} 0 & -3a_z \\ -3a_z & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow \begin{bmatrix} -\lambda & -3a_z \\ -3a_z & -\lambda \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \lambda^2 - (3a_z)^2 = 0 \quad \Rightarrow \quad \lambda = \pm 3a_z$$

$\lambda = -3a_z$:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \underline{\Delta E = -3e\mathcal{E}a_z}$$

~~$\lambda = -3a_z$~~ state $\underline{|\psi\rangle = \frac{1}{\sqrt{2}}(|200\rangle + |210\rangle)}$

$\lambda = +3a_z$:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \underline{\Delta E = +3e\mathcal{E}a_z}$$

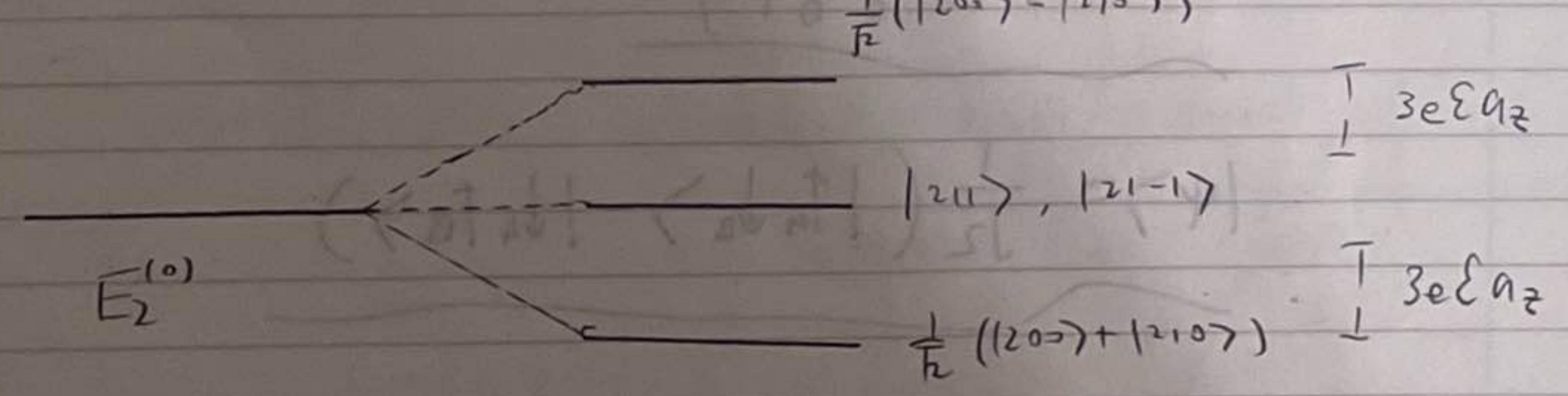
$$\underline{|\phi\rangle = \frac{1}{\sqrt{2}}(|200\rangle - |210\rangle)}$$

or $\lambda = 0$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

∴ eigenstates are still $\underline{|211\rangle}$ or $\underline{|21-1\rangle}$

∴



8.

$$H_A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$|\uparrow_A \uparrow_B\rangle = \cancel{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}} \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\quad$ $\cancel{|\downarrow_A \uparrow_B\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\end{pmatrix}}$ $\quad$ $|\uparrow_A \downarrow_B\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$\therefore \cancel{H_A|\uparrow_A \uparrow_B\rangle = \frac{1}{\sqrt{2}} \frac{1}{2}\begin{pmatrix}2\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}}$

$\cancel{H_A|\downarrow_A \uparrow_B\rangle = \frac{1}{2}}$ $\quad$ $|\downarrow_A \uparrow_B\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\cancel{|\downarrow_A \downarrow_B}$ $\quad$ $|\downarrow_A \downarrow_B\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$$H_A|\uparrow_A \uparrow_B\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_A \uparrow_B\rangle + |\downarrow_A \uparrow_B\rangle\right) \rightarrow P(\uparrow_A) = \frac{1}{2}, \quad P(\uparrow_B) = 1$$

$$H_A|\downarrow_A \uparrow_B\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow_A \uparrow_B\rangle - |\downarrow_A \uparrow_B\rangle\right)$$

$$\rightarrow P(\uparrow_A) = \frac{1}{2}, \quad P(\uparrow_B) = 1$$

They are ~~all~~ both product states $\rightarrow$ order doesn't matter

$$H_C = \cancel{\tfrac{1}{\sqrt{2}}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

($\langle \uparrow\downarrow | H | \uparrow\downarrow \rangle$ is an element ... etc)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_A \downarrow_B\rangle - |\downarrow_A \uparrow_B\rangle\right)$$

~~doesn~~ measuring B doesn't collapse wave function in any way.

measurements:

equal probabilities

spin A $|\uparrow_A\rangle \rightarrow$ Spin B $|\downarrow_B\rangle$

spin A ~~$|\uparrow_B\rangle$~~ $|\downarrow_A\rangle \rightarrow$ Spin B $|\uparrow_B\rangle$

$|\psi\rangle$ is an entangled state. the probability of measuring spin B to be up or down depends on the result of A. order does matter in this case.

$$\hat{S_z} = \hat{S_{Az}} + \hat{S_{Bz}}$$

$$\hat{S_z}|\psi\rangle = (\hat{S_{Az}} + \hat{S_{Bz}})\left(\frac{1}{\sqrt{2}}|\uparrow_A\downarrow_B\rangle - \frac{1}{\sqrt{2}}|\downarrow_A\uparrow_B\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\hbar}{2}|\uparrow_A\downarrow_B\rangle - \frac{\hbar}{2}|\uparrow_A\downarrow_B\rangle + \frac{\hbar}{2}|\downarrow_A\uparrow_B\rangle - \frac{\hbar}{2}|\downarrow_A\uparrow_B\rangle\right)$$

$$= 0\,|\psi\rangle$$

$S_z = 0$ → because A, B always have spin anti-aligned

$|\psi\rangle$ is the spin singlet $|s, s_m\rangle = |0,0\rangle$

No net angular momentum for $|\psi\rangle$

→ spins of A and B should be anti-parallel along any axis.

9. $\hat{H}|\psi\rangle = i\hbar \frac{\partial|\psi\rangle}{\partial t}$ $\langle Q\rangle = \langle\psi|Q|\psi\rangle$

$$\therefore \frac{d\langle Q\rangle}{dt} = \frac{d}{dt}\langle\psi|Q|\psi\rangle = \frac{\partial\langle\psi|}{\partial t}\hat{Q}|\psi\rangle + \langle\psi|\frac{\partial Q}{\partial t}|\psi\rangle$$

$$+ \langle\psi|\hat{Q}|\frac{\partial|\psi\rangle}{\partial t}$$ (assuming $\frac{\partial\hat{Q}}{\partial t} = 0$ / Q = time independent

$$= -\frac{\langle\psi|}{i\hbar}\hat{H}\hat{Q}|\psi\rangle + \langle\psi|\hat{Q}\hat{H}|\frac{|\psi\rangle}{i\hbar}\rangle$$

$$= \frac{1}{i\hbar}\langle\psi|\hat{Q}\hat{H} - \hat{H}\hat{Q}|\psi\rangle$$

$$= \frac{1}{i\hbar}\langle[\hat{Q},\hat{H}]\rangle$$

Stationary states :

eigenstates of operators that commute with Hamiltonian.

Good Quantum number :

eigenvalues of operators that commute with the Hamiltonian.

$$V(x,y) = \frac{1}{2}m(\omega_x^2x^2 + \omega_y^2y^2)$$

$$E_{xy} = (n_x+\tfrac{1}{2})\hbar\omega_x + (n_y+\tfrac{1}{2})\hbar\omega_y$$

let $\omega_x = \omega_y + \delta\omega$, then

$$E_{xy} = (n_x+n_y+1)\hbar\omega_y + (n_x+\tfrac{1}{2})\delta\omega$$

First 6 levels

$(0,0) \rightarrow$ $E_{xy} = \hbar(\omega_y + \frac{1}{2}\delta\omega)$

$(0,1) \rightarrow$ $E_{xy} = \hbar(2\omega_y + \frac{1}{2}\delta\omega)$

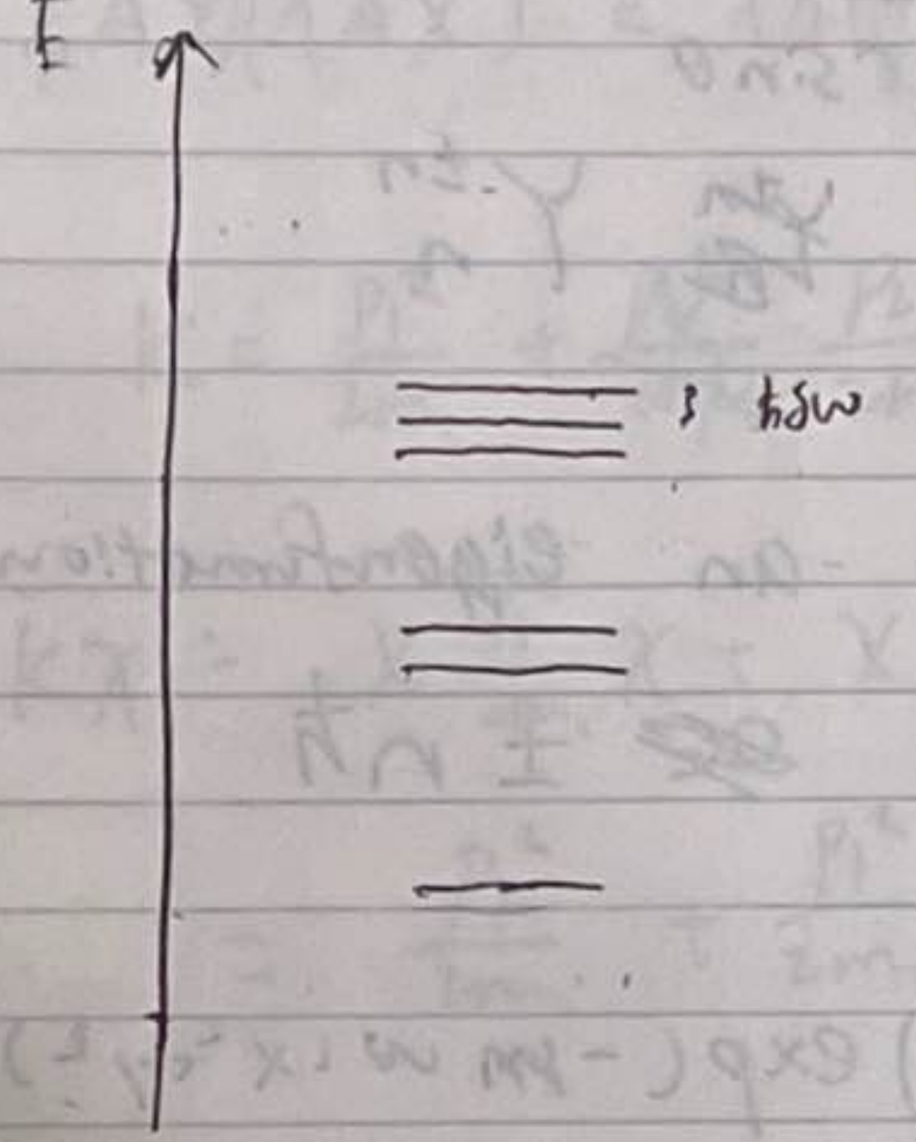
$(1,0) \rightarrow E_{xy} = \hbar(2\omega_y + \frac{3}{2}\delta\omega)$

$(0,2) \rightarrow E_{xy} = \hbar(3\omega_y + \frac{1}{2}\delta\omega)$

$(1,1) \rightarrow E_{xy} = \hbar(3\omega_y + \frac{3}{2}\delta\omega)$

$(2,0) \rightarrow E_{xy} = \hbar(3\omega_y + \frac{5}{2}\delta\omega)$

$\therefore$



If $\omega_x = \omega_y = \omega$, The system now has rotational symmetry in the xy plane.

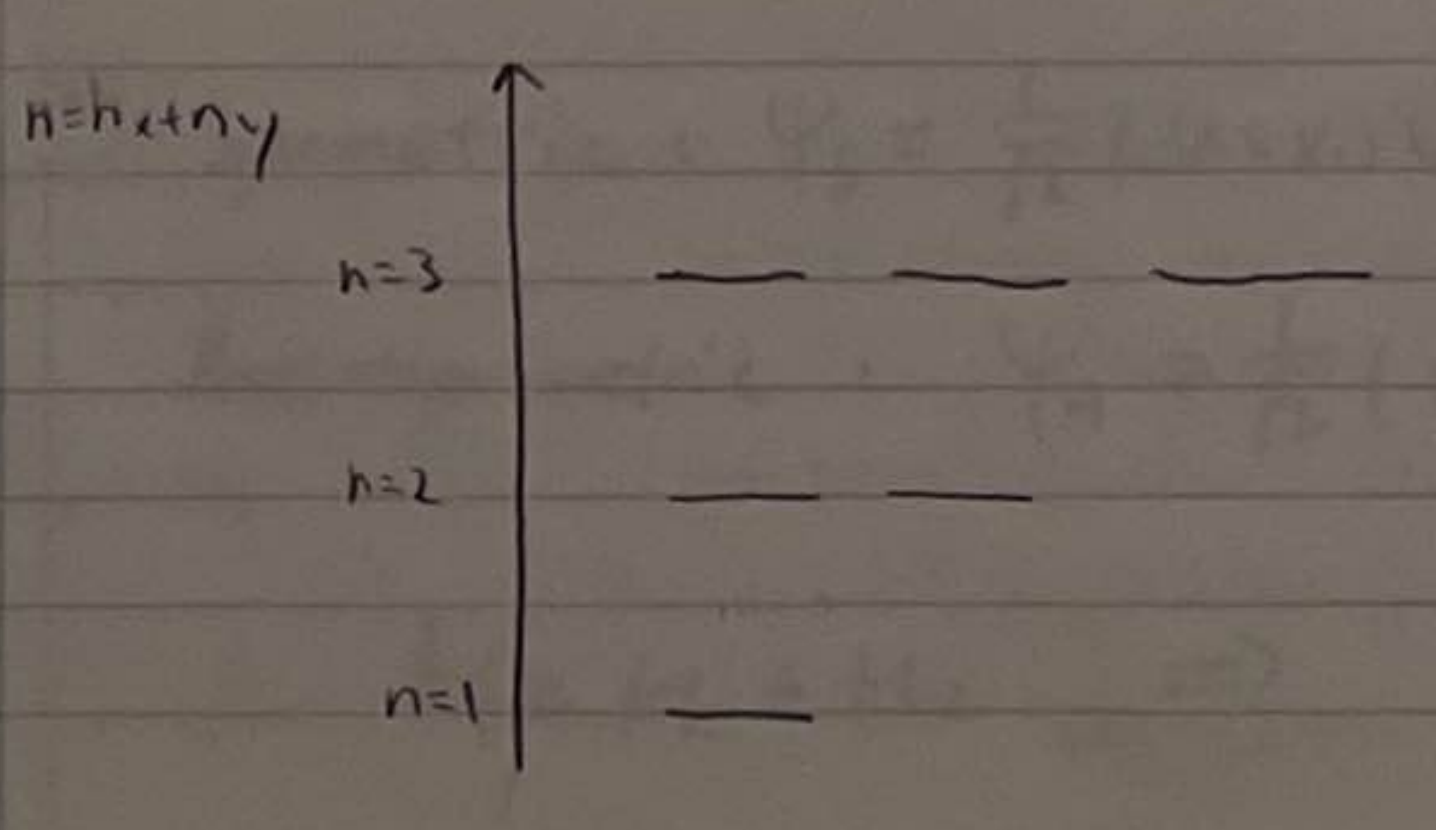
$\rightarrow$ Angular momentum of the z-direction should be conserved.

We expect $[\hat{L}_z, \hat{H}] = 0$

Energy level diagram now has ~~to~~ degeneracy.

A new good quantum number is now $m\hbar$

$\hat{L}_z|\psi\rangle = m\hbar|\psi\rangle$ $\qquad \hat{L}_z = i\hbar\frac{\partial}{\partial\phi} = i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$



$\because [\hat{L}_z, \hat{H}] = 0$

$\therefore \frac{d\langle\hat{L}_z\rangle}{dt} = 0$

lowest 3 energy eigenstates:

$$\psi_{00} = A_0^2 \exp(-m\omega(x^2+y^2)/2\hbar)$$

~~$\psi_{11} = A_0 A_1$~~

ψ_{00} is eigenstate of L_z $\because x^2+y^2 = r^2\sin^2\theta$

$\therefore \psi_{00}$ is independent of ϕ. $\rightarrow$ eigenvalue $= 0$

$$\because (x \pm iy)^n = \Big(\underbrace{(x^2+y^2)^{\frac{1}{2}}}_{r\sin\theta} e^{\pm i\phi}\Big)^n = r^n \sin^n\theta \, e^{\pm i\phi}$$

$$\propto Y_n^{\pm n}$$

$\therefore (x \pm iy)^n$ is an eigenfunction of L_z

with eigenvalue $\pm n\hbar$

$$\therefore \psi_{11} = A_0 A_1 (x+iy) \exp(-m\omega(x^2+y^2)/2\hbar)$$
$$\psi_{1-1} = A_0 A_1 (x-iy) \exp(-m\omega(x^2+y^2)/2\hbar)$$

~~$\psi_{22} = (x+iy)^2 \exp(\ )$~~

~~$\psi_{20} = (x+iy)(x-iy) \exp(\ )$~~

~~$\psi_{2-2} = (x-iy)^2 \exp(\ )$~~

$\vdots$

10. By definition $\chi^2 = I$

(i) If $\chi|\psi\rangle = \lambda|\psi\rangle$, then

$$\chi^2|\psi\rangle = \lambda^2|\psi\rangle = I|\psi\rangle = |\psi\rangle$$

$$\rightarrow \lambda^2 = 1 \Rightarrow \underline{\lambda = \pm 1}$$

(ii) If $\chi A \chi = B$, then

$$B^2 = (\chi A \chi)(\chi A \chi) = \chi A \underbrace{(\chi\chi)}_{I} A \chi = \chi A^2 \chi \rightarrow QED$$

$$K = \frac{P_1^2}{2m_1} + \cancel{\frac{P_2^2}{2m_2}} \frac{P_2^2}{2m_2}.$$

$$\chi K \chi = \chi \frac{P_1^2}{2m_1}\chi + \chi \frac{P_2^2}{2m_2}\chi$$

$$= \frac{P_2^2}{2m_1} + \frac{P_1^2}{2m_2} = K \quad \text{iff} \quad m_1 = m_2$$

$$\chi K \chi = K \Leftrightarrow \underbrace{\chi\chi}_{I} K \chi = \chi K \Leftrightarrow K\chi = \chi K$$

$$\Leftrightarrow [\chi, K] = 0$$

$$\rightarrow \underline{[\chi, K] \Leftrightarrow m_1 = m_2} \rightarrow QED$$

$$\chi(V_1(x_1) + V_2(x_2) + V_I(x_1, x_2))\chi = V_1(x_2) + V_2(x_1) + V_I(x_2, x_1)$$

$$\overset{!}{=} V_1(x_1) + V_2(x_2) + V_I(x_1, x_2)$$

$$\Rightarrow \underline{V_1 = V_2} \quad \text{and} \quad \chi V_I = V_I$$

$$\rightarrow \underline{V_I(x_1, x_2) = V_I(x_2, x_1)}$$

Symmetric: $\psi_S = \frac{1}{\sqrt{2}}(u(x_1)v(x_2) + v(x_1)u(x_2)) = \psi_+$

Anti-symmetric: $\psi_A = \frac{1}{\sqrt{2}}(u(x_1)v(x_2) - v(x_1)u(x_2)) = \psi_-$

$$H = H_1 + H_2 \Rightarrow E = E_u + E_v$$

$$H_I = V_I(|x_1 - x_2|)$$

$$\langle E \rangle = \langle H_0 + H_I \rangle = \langle H_0 \rangle + \langle H_I \rangle$$

$$\langle H_0 \rangle = \langle \psi_{+,-} | H_1 + H_2 | \psi_{+,-} \rangle = E_u + E_v$$

$$\langle H_I \rangle = \langle \psi_{+,-} | V_I(|x_1 - x_2|) | \psi_{+,-} \rangle$$

$$= \frac{1}{2} \int dx_1 \int dx_2 \, V_I \left[u^*(x_1) v^*(x_2) \pm u^*(x_2) v^*(x_1) \right]$$

$$\left[u(x_1) v(x_2) \pm u(x_2) v(x_1) \right]$$

$$= J_D \pm J_E$$

$$\rightarrow \quad J_D = \frac{1}{2} \int dx_1 \int dx_2 \, V_I \left[|u(x_1)|^2 |v(x_2)|^2 + |u(x_2)|^2 |v(x_1)|^2 \right]$$

$$\rightarrow J_E = \frac{1}{2} \int dx_1 \int dx_2 \, V_I \left[u^*(x_1) u(x_2) v^*(x_2) v(x_1) + u^*(x_2) u(x_1) v^*(x_1) v(x_2) \right]$$

If $V_I = V_0 \delta(x_1 - x_2)$

$$\rightarrow J_D = V_0 \int dx \, |u(x)|^2 |v(x)|^2 = J$$

$$J_E = J$$

$$\rightarrow \langle H \rangle = E_u + E_v + J \pm J$$

Anti-symmetric case:

Only if two partides are togather can they have interaction because potential is a δ-function

But for antisymmetric wavefunction the probability that they come toguther is 0

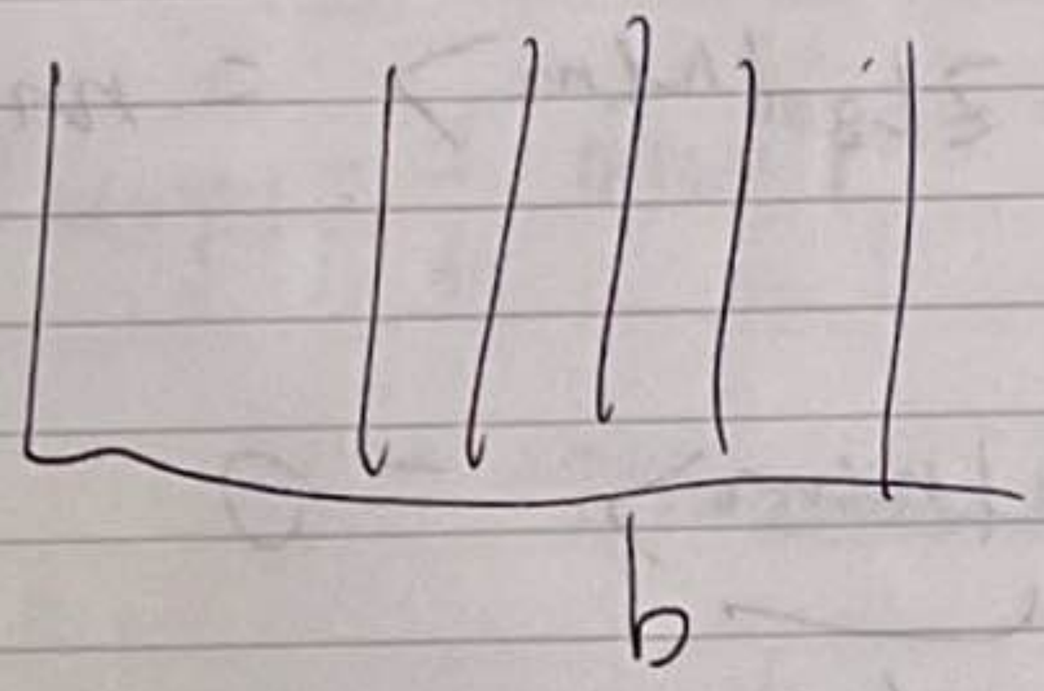
$\therefore$ No interaction energy for ψ_A

A2 2013 ~~Tutorial~~ Tutorial Notes:

$\cancel{Q}$. 1. $\langle\phi|H|\psi\rangle = \langle\psi|H|\phi\rangle^*$

Significance: use them for physical observables
$\to$ has to be real.

3. $\frac{\hbar}{\tau} \ll \Delta E$ $\quad \Delta E = E_2 - E_1 = \frac{\hbar^2\pi^2}{2m(2a)^2}(2^2-1^2)$



$E_{2b} - E_{1b} = \frac{\hbar^2\pi^2}{2mb^2}(2^2-1^2)$

$b_{max} = 2a$

$\Delta E_{min} = (b=2a)$

$\psi(\underline{r}) = Ae^{i\phi(\underline{r})}$

$\underline{j} = \frac{\hbar}{2mi}(A^* e^{-i\phi} Ae^{-i\phi}\, i\nabla\phi$ $\quad \exp(i\underline{k}\cdot\underline{r})$

$+ Ae^{i\phi}A^* e^{-i\phi}\, i\nabla\phi$ $\quad \int_0^1\int_0^1\int_0^1 d^3x\, A^2 = 1$

$\downarrow$

$|\psi|^2$.

$= \cancel{\frac{\hbar^2|A|^2}{2m}}$ or $\boxed{\frac{\hbar|A|^2}{m}}$

~~Unit~~ uniform prob density $|A|^2 = 1$?

uniform $\underline{j} \to$ ~~i~~ $\nabla\phi$ = const

$\to \phi(\underline{r}) = \underline{k}\cdot\underline{r}$

$\nabla\phi = \underline{k}$

$\langle 2, l', m' | z | 2, 1, m \rangle$

$\downarrow$

$r\cos\theta$

$$\int dr\, r^3 \int d\theta \sin\theta \int d\phi \cos\theta\, Y_{l'}^{m'} Y_l^m$$

If $l' = l$, $m' = m$, the product is even.

$[L_z, z] = 0$

$L_z(z|nlm\rangle) = zL_z|nlm\rangle = m\hbar(z|nlm\rangle)$

$\therefore \langle n'l'm' | z | nlm \rangle = 0$

Orthogonal if $m \neq m'$

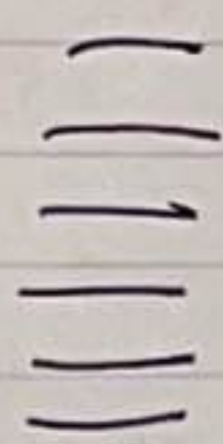
$-e\mathcal{E}\hat{z} = -\nabla \delta H$

$= -\nabla(e\mathcal{E}z)$

$\rightarrow \delta H = +e\mathcal{E}z$

9. $E_{n_x,n_y} = (n_x+\frac{1}{2})\hbar(\omega_y+\delta\omega)$

$+(n_y+\frac{1}{2})\hbar(\omega_y)$



$V(x,y) = \frac{1}{2} m\omega^2 r^2$

→ rotartional symmetry

→ degeneracies

$E_{00} = \hbar\omega \quad E_{10} = E_{01} = 2\hbar\omega$

$E_{20} = E_{02} = E_{11} = 3\hbar\omega$

10. $XAX = B$

$\rightarrow B^2 = XAXXAX = XAX^2AX = XA^2X$

$XKX = X\frac{p_1^2}{2m_1}X + X\frac{p_2^2}{2m_2}X$

$= \cancel{X}\frac{p_2^2}{2m_1}\cancel{X} + \cancel{X}\frac{p_1^2}{2m_2}\cancel{X}$

if $m_1 = m_2$

$= K$

$\rightarrow \cancel{XK = KX} \quad KX = XK \rightarrow [X,K] = 0$

$\chi V \psi_{12}$ $\qquad \psi_{12} = \psi(x_1, x_2)$

$\chi V \psi_{12} \Leftrightarrow V \chi \psi_{12}$

$\chi^2 V \psi_{12} = \chi V \chi \psi_{12}$

$\chi V_{12} \psi_{12}$ $\qquad V_{12} \chi \psi_{12}$

$= V_{21} \psi_{12}$ $\qquad = V_{12} \psi_{21}$

$\rightarrow \quad V_1(x_2) + V_2(x_1) + V_I(x_2, x_1)$

$= V_1(x_1) + V_2(x_2) + V_I(x_1, x_2)$

$\rightarrow \quad V_1 = V_2 \qquad V_I = V_I(|x_1 - x_2|)$?