2663

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A3: QUANTUM PHYSICS

TRINITY TERM 2011

Friday, 24 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. For a quantum mechanical operator Q, what do the *expectation value* $\langle Q\rangle$ and the *dispersion* $D(Q) = \sqrt{\langle (Q - \langle Q\rangle)^2\rangle}$ represent?

If a system is in a eigenstate of Q with eigenvalue q, evaluate $\langle Q\rangle$ and $D(Q)$ for the system. [4]

2. What is the *parity* operator in quantum mechanics and what are the parity eigenvalues of the eigenstates under its action?

If ψ_1 and ψ_2 are *degenerate* eigenstates of a quantum-mechanical Hamiltonian, what does this imply about them?

Show that the non-degenerate eigenstates of a Hamiltonian which is symmetric under space inversion of the coordinates have definite parity. Comment on the degenerate case. [5]

3. How is the *probability current density* related to the static probability density?

Considering a particle moving in one dimension in a potential $V(x)$, derive an expression for the probability current density j, starting from the time-dependent Schrödinger equation. [6]

4. By considering the time-independent Schrödinger equation, comment on the general form of the wavefunction ψ in regions where $E < V$ and in regions where $E > V$, taking V to be a constant. If a particle of energy E travels from the region $x < 0$, for which $V = 0$, into the region $x > 0$, for which $V = V_0 > E$, what is the probability that the particle will be reflected? What is the probability to observe the particle at distance $x = d$ $(d > 0)$, relative to the probability to observe it at $x = 0$? [7]

5. At time $t = 0$, a free particle of mass m moving in one dimension is described by the normalised wavefunction

$$\psi(x) = \left(\frac{2}{\pi a^2}\right)^{1/4} \exp\left(-\frac{x^2}{a^2} + ikx\right).$$

Obtain the expectation values of the position and momentum of the particle and estimate the size of the region in which it is located. Show that the probability current density can be written as $(\hbar k/m)|\psi(x)|^2$. [9]

6. For a two-state system, the time-independent Schrödinger equation, $H|\psi\rangle = E|\psi\rangle$, can be written as an eigenvalue equation for the energy E:

$$\begin{pmatrix} E_1 + \lambda & \lambda \\ \lambda & E_2 + \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix}.$$

What are the eigenvalues of H when $E_1 = E_2$ and what are the associated eigenstates of the system?

Comment on how your answer relates to first-order perturbation theory. [9]

57/100

Section B

7. Hydrogen-like wavefunctions have the form

$$\psi_{n\ell m} = R_{n\ell}(r) Y_{\ell m}(\theta, \phi).$$

Which property of the Coulomb interaction allows for the above separation of variables? What do the symbols n, ℓ and m represent? For a given n, what values can ℓ and m take? [5]

The ground-state wavefunction is given by

$$R_{10}(r) = \left(\frac{Z}{a_0}\right)^{3/2} 2\,\mathrm{e}^{-Zr/a_0} \qquad \text{and} \qquad Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}\,,$$

where $a_0 = 4\pi\epsilon_0\hbar^2/\mu e^2$. What do μ and Z represent? What is the significance of a_0? Sketch the probability density for the electron as a function of r. [5]

A tritium atom, ^{3}H is in its ground state when the nucleus undergoes a beta decay and becomes ^{3}He. Using the 'sudden approximation' calculate the probability that this helium ion is in the $1s$ state. [10]

8. J_x, J_y and J_z, are the angular momentum operators. Evaluate the commutators $[J_z, J_+]$, $[J_z, J_-]$, $[J^2, J_+]$, and $[J^2, J_-]$, where $J^2 = J_x^2 + J_y^2 + J_z^2$ and $J_\pm = J_x \pm iJ_y$. [3]

If $|j, m\rangle$ is an eigenket of J^2, J_z with eigenvalues $j(j+1), m$ show that $J_+|j, m\rangle$ and $J_-|j, m\rangle$ are also eigenkets and give the corresponding eigenvalues. [3]

Derive matrices representing J_x, J_y, J_z for $j = 1$. [8]

The matrix for the angular momentum in a direction at 45° to the z axis in the xz plane is:

$$J_{45^\circ} = \frac{1}{\sqrt{2}}(J_x + J_z).$$

Find the eigenvector of this operator corresponding to the eigenvalue zero. [3]

A spin-1 particle is prepared in a state with magnetic quantum number $m = 0$ with respect to a reference axis at 45° to the z axis in the xz plane. It then passes through a filter which only transmits particles in the $m = 1$ state with respect to the z direction. What is the probability that the particle is still in the $m = 0$ state with respect to the original 45° axis? [3]

9. A particle is confined in an infinitely deep square well described by a potential $V(x)$ which is zero for $-\frac{1}{2}a \leq x \leq \frac{1}{2}a$. Use the Schrödinger equation to find the energies available to the particle and sketch the wavefunctions of the three states of lowest energy. [8]

Explain how the wavefunction of the ground state is changed if the well is perturbed by a small added potential V':

(a) $V' = V_1 x \quad (V_1 > 0,\ V_1 a \ll E_0$, the ground-state energy),

(b) $V' = V_2 x^2 \quad (V_2 > 0,\ V_2 a^2 \ll E_0)$. [5]

The well is perturbed by the addition to $V(x)$ of a potential $V' = V_3 \sin(\pi x/a)$. Show that in perturbation theory to first order, only the first excited-state wavefunction then mixes with the ground-state wavefunction and evaluate its relative contribution. [7]

10. The Hamiltonian for the interaction between an electron and an uniform magnetic field $\vec{B}$ is

$$H = \frac{e\hbar}{m_{\mathrm{e}}}\vec{B}\cdot\vec{S},$$

where $\vec{S}$ is the electron spin with components:

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The magnetic field (of magnitude B) is along the z-axis. Show that the wavefunction for the electron takes the form

$$\psi(t) = \begin{pmatrix} a(0)\exp\left(-\frac{1}{2}\mathrm{i}\,\omega_L t\right) \\ b(0)\exp\left(\frac{1}{2}\mathrm{i}\,\omega_L t\right) \end{pmatrix}, \text{ where } \omega_L = eB/m_{\mathrm{e}}$$

and specify the energy eigenvalues. [6]

A magnetic field of magnitude $B' \ll B$, which rotates in the xy plane at angular velocity ω, is applied to the system in the lower eigenstate. What is the probability that it will be found in the higher energy eigenstate at time t? [14]

A3 2011 First Attempt

1.

$\langle Q \rangle$ is the average ~~of~~ ~~val~~ value obtained when the observable Q is measured successively.

$D(Q)$ is the standard deviation from the mean $\langle Q \rangle$; ~~to~~ it measure the ~~ne~~ uncertainty of Q.

If $Q|q\rangle = q|q\rangle$. then

$$\langle Q \rangle = \langle q|Q|q\rangle = q\langle q|q\rangle = q$$

$$\cancel{D} \quad \langle Q^2 \rangle = \langle q|Q^2|q\rangle = q^2$$

$$D(Q) = \sqrt{\langle (Q-\langle Q\rangle)^2 \rangle} = \sqrt{\langle Q^2 - 2Q\langle Q\rangle + \langle Q\rangle^2 \rangle}$$

$$= \sqrt{\langle Q^2\rangle - \langle Q\rangle^2} = \sqrt{q^2 - q^2} = 0$$

2. The parity operator reverses the ~~spacial~~ coordinates of the state.

$$\underline{P\langle \underline{x}|\psi\rangle = \langle -\underline{x}|\psi\rangle}$$

$\therefore$ $P^2 \langle \underline{x}|\psi\rangle = \langle \underline{x}|\psi\rangle$

if $|\psi\rangle$ is an eigenstate then this implies $\lambda^2 = 1$ $\rightarrow$ $\underline{\lambda = \pm 1 \text{ are eigenvalues}}$

$\rightarrow$ ~~ψ_1 and ψ_2~~ If ψ_1 and ψ_2 are degenerate eigenstates of the same eigenvalue,

then any linear combination of them is also an eigenstate of the same eigenvalue

-1

→ symmetric Hamiltonian $[P, H] = 0$

∴ states of H are non-degenerate

∴ eigenstates of H must be eigenstates of P

∴ eigenstates of H have definite parity.

→ Degenerate case: eigenstates of H with degenerate eigenvalues does not necessarily have definite parity, but we can construct linear ~~constit~~ combinations of them that have.

need extra notes

-2

3. $\cancel{j}\ \frac{\partial |\psi|^2}{\partial t} = -\nabla \cdot \underline{J} = -\frac{\partial J}{\partial x}$ (in 1-D)

1-D: $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

TDSE $\quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi \quad$ ①

taking complex conjugate: $-i\hbar\frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{d\psi^*}{dx^2} + V\psi^* \quad$ ②

ψ^*① $-$ $\cancel{\psi^*}\psi$② $\Rightarrow i\hbar\left(\psi^*\frac{\partial \psi}{\partial t} + \psi\frac{\partial \psi^*}{\partial t}\right)$

$= -\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right)$

$$\therefore\ i\hbar \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m}\left(\left(\underbrace{\psi^* \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}}_{\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x}\right)}\right) - \left(\underbrace{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x} + \psi \frac{\partial^2 \psi^*}{\partial x^2}}_{\frac{\partial}{\partial x}\left(\psi \frac{\partial \psi^*}{\partial x}\right)}\right)\right)$$

$$\rightarrow \quad \frac{\partial |\psi|^2}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)$$

$$\therefore\ J = \frac{i\hbar}{2m}\left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x}\right)$$

4.

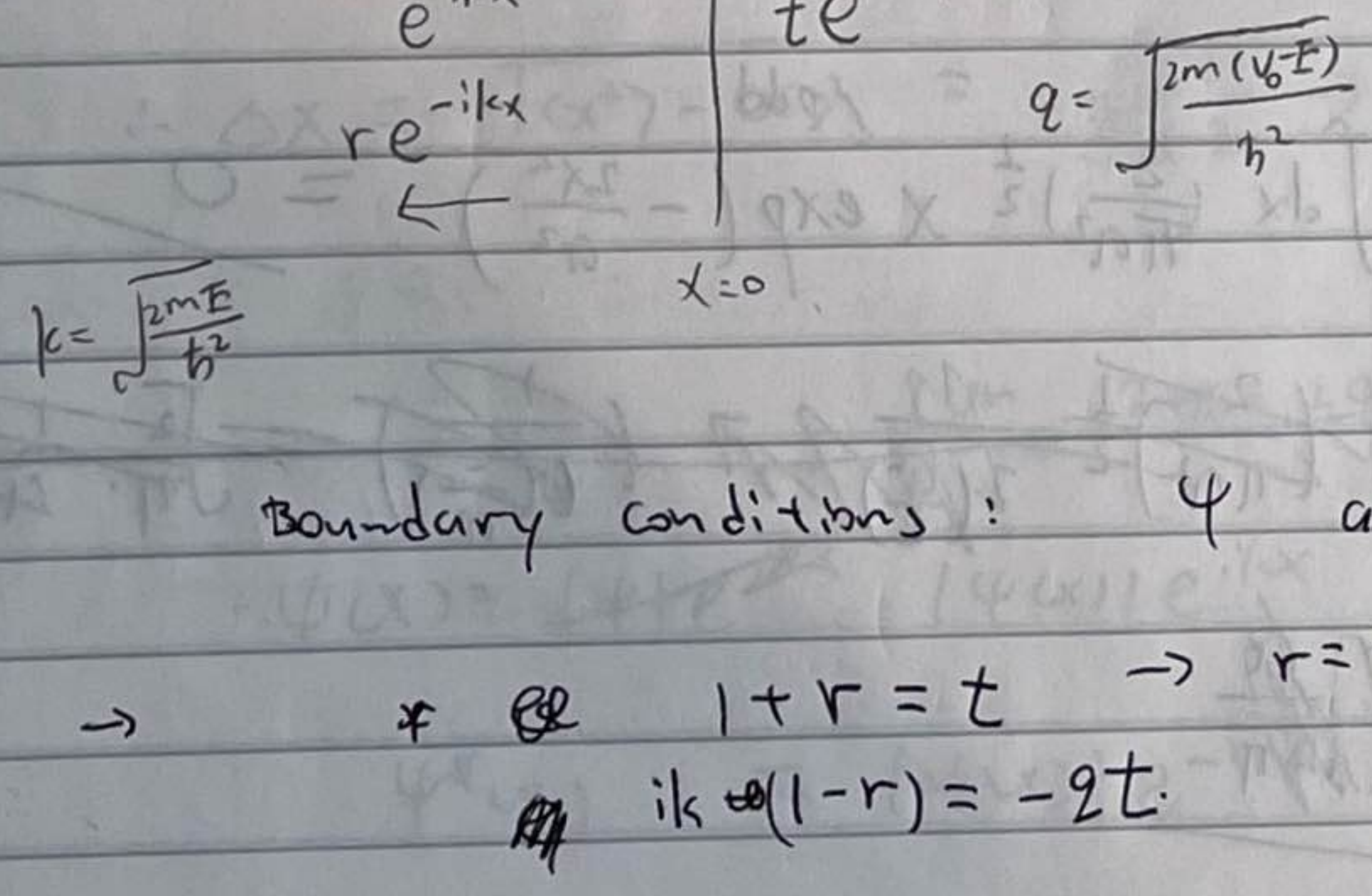
$E < V \qquad \psi = Ae^{qx}$ or Be^{-qx} if ~~$q > 0$~~

$\qquad\qquad\qquad\downarrow \qquad\qquad\qquad \downarrow$

$\qquad\qquad\quad x < 0 \qquad\qquad x \geq 0 \qquad \left(q = \sqrt{\frac{2m(V-E)}{\hbar^2}}\right)$

$E > V \qquad \psi = Ce^{ikx} + De^{-ikx}$

$\left(k = \sqrt{\frac{2m(E-V)}{\hbar^2}}\right).$



Boundary conditions: ψ and $\frac{d\psi}{dx}$ continuous

$\rightarrow \quad 1 + r = t \quad \rightarrow \quad r = t - 1$

$\qquad ik(1 - r) = -qt.$

$\therefore$ ~~$ik(2-t)=-$~~ $ik(2-t) = -\kappa t$

$\therefore$ $2ik = (ik - \kappa)t$

$\rightarrow$ $t = \dfrac{2ik}{ik-\kappa}$

transmission current $J_t = \dfrac{i\hbar}{2m}\Big(te^{-\kappa x}\dfrac{\partial}{\partial x}(te^{-\kappa x})$

$- \cancel{t^{-\kappa}}\, te^{-\kappa x}\dfrac{\partial}{\partial x}(te^{-\kappa x})\Big) = 0$

$\therefore$ No transmitted probability.

$\rightarrow$ total reflection $\rightarrow$ prob $= 1$

relative probability to be observed at d.

$$P(d) = \frac{(e^{-\kappa d})^2}{(1)^2} = e^{-2\kappa d} \quad \left(\kappa = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}\right)$$

5. $\psi(x) = \cancel{e}\left(\dfrac{2}{\pi a^2}\right)^{1/4}\exp\left(-\dfrac{x^2}{a^2}\right)\exp(ikx)$

$\psi^*(x) = \left(\dfrac{2}{\pi a^2}\right)^{1/4}\exp\left(-\dfrac{x^2}{a^2}\right)\exp(-ikx)$

$\therefore \langle x\rangle = \int dx \left(\dfrac{2}{\pi a^2}\right)^{\frac{1}{2}} \underbrace{x}_{\text{odd}} \exp\left(-\dfrac{2x^2}{a^2}\right) = 0$

$= \cancel{\left(\dfrac{2}{\pi a^2}\right)^{\frac{1}{2}} \dfrac{1}{2\left(\frac{a^2}{2}\right)} \left(\dfrac{1}{\left(\frac{2}{a^2}\right)}\right) = \sqrt{\dfrac{2}{\pi}}\cdot\dfrac{1}{a}\cdot\dfrac{a^2}{2}}$

$= \cancel{a\dfrac{\sqrt{2}}{2\sqrt{\pi}}}$

$\cancel{\langle p \rangle = -\left(\frac{2}{\pi a^2}\right)^{1/2} i\hbar \int dx}$

$\cancel{P\psi = p\psi}$

$$P\psi(x) = -i\hbar\frac{\partial}{\partial x}\left[\left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{a^2} + ikx\right)\right].$$

$$= -i\hbar\left(\frac{2}{\pi a^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{a^2} + ikx\right)\left(-\frac{2x}{a^2} + ik\right)$$

odd term gives 0 contribution

$$\therefore \langle p \rangle = \int dx\ \psi^* p \psi = \hbar k \underbrace{\left(\frac{2}{\pi a^2}\right)^{\frac{1}{2}} \int dx\ \exp\left(-\frac{2x^2}{a^2}\right)}_{1} = \underline{\hbar k}.$$

$$\langle x^2 \rangle = \int dx\ x^2 |\psi|^2 = \left(\frac{2}{\pi a^2}\right)^{\frac{1}{2}} \underbrace{\int dx\ x^2 e\left(-\frac{2x^2}{a^2}\right)}_{\frac{\sqrt{\pi/(2/a^2)}}{2 \times 2/a^2}}.$$

careless mistake

-3

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{a} \cdot \frac{a^2}{4} \sqrt{\frac{\pi}{2}} \cdot a = \frac{a^2}{4}$$

$$\therefore \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{2} \rightarrow \text{size where it is located.}$$

(or $\cancel{2\Delta x}$ $2\Delta x = a$.).

$\psi(x) = \cancel{|\psi| e^{ikx}}\ |\psi(x)| e^{ikx}$

$\psi^*(x) = |\psi(x)| e^{-ikx}$

$$J = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{i\hbar}{2m}\left(\cancel{|\psi(x)|e^{ikx}\frac{\partial|\psi|}{\partial x}e^{-ikx}} + |\psi(x)|^2\frac{\partial}{\partial x}(e^{-ikx})\right.$$

$$\left. - \cancel{|\psi(x)|e^{-ikx}\frac{\partial|\psi|}{\partial x}e^{ikx}} - |\psi(x)|^2\frac{\partial}{\partial x}(e^{ikx})\right)$$

$$= \frac{i\hbar}{2m}(-2ik)|\psi(x)|^2$$

$$= \frac{\hbar k}{m}|\psi(x)|^2$$

6. $$\begin{pmatrix} E_1+\lambda & \lambda \\ \lambda & E_2+\lambda \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = E\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\rightarrow \det\begin{pmatrix} E_1+\lambda-E & \lambda \\ \lambda & E_2+\lambda-E \end{pmatrix} = 0$$

when $E_1 = E_2 \rightarrow (E_1+\lambda-E)^2 - \lambda^2 = 0$

$$\therefore (E_1-E)(E_1-E+2\lambda) = 0$$

$\therefore$ $\underline{E = E_1}$ or $\underline{E' = E_1 + 2\lambda}$

$\swarrow$ $a+b=0 \rightarrow a=-b$ $\qquad$ $\downarrow$ $-a+b=0 \rightarrow a=b$

$\underline{|\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ $\qquad$ $\underline{|\phi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

→ E_1 and $E_1 + 2\lambda$ are the splitting of originally degenerate eigenvalue E_1, they are first order approximation to energy

→ $|\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $|\phi'\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ are the corresponding zeroth order eigenstates of E and E'.

$|\phi\rangle$ is antisymmetric → particles not togather ∴ E doesn't change

-5 $|\phi\rangle'$ is symmetric → particles togather

part A $\frac{27}{40}$ E increases.

7.

→ The Coulamb interaction is spherically symmetric.

n → energy

l → total orbital angular momentum

m → z-component of orbital angular momentum

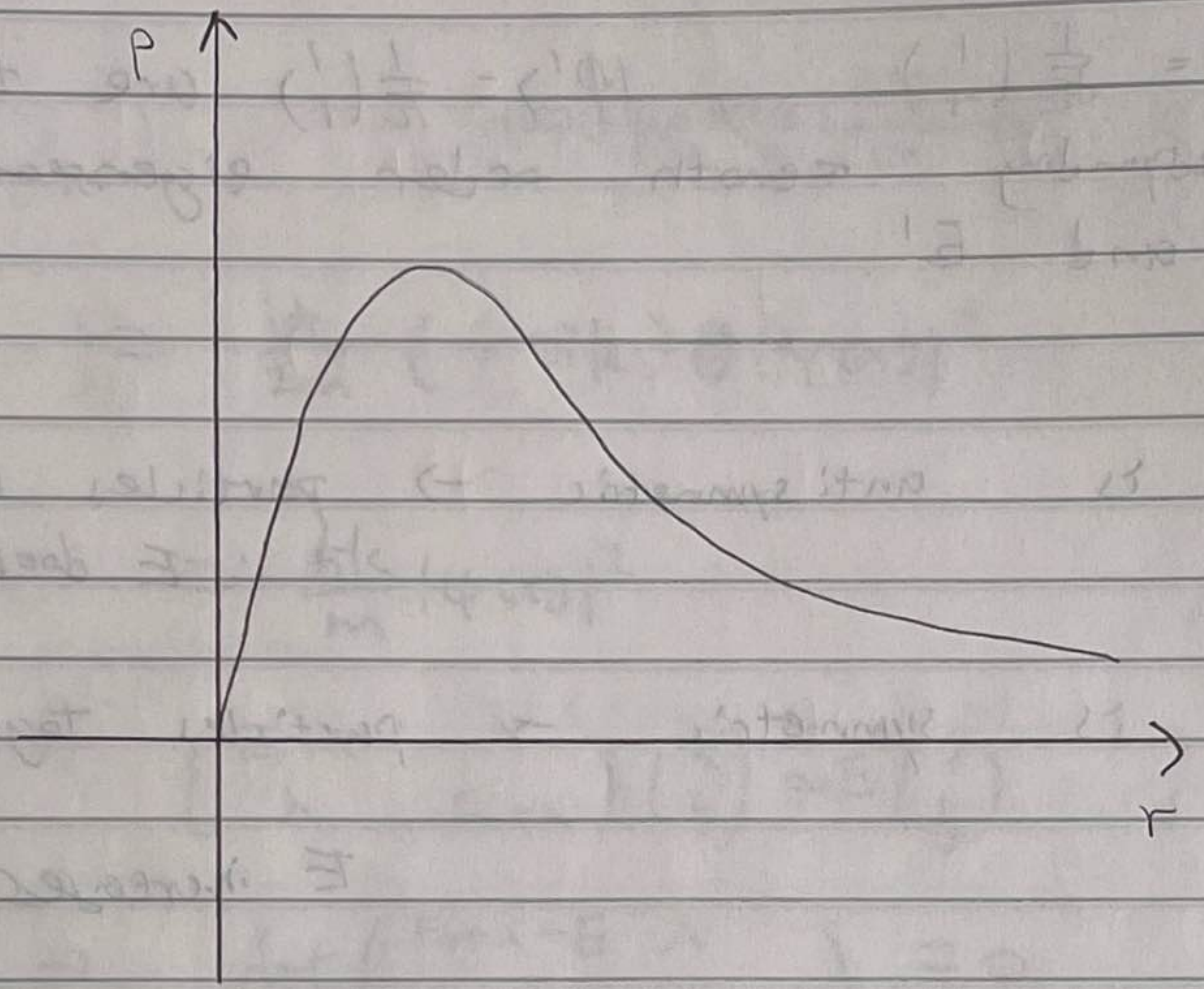
n → $l = 0, 1, 2, \cdots, n-1$. $m = -l, -l+1, \cdots, l-1, l$.

μ → reduced mass Z → nuclear charge

→ a_0 is the most probable radius of ground state wavefunction that the particle can be found.

Probability density with respect to dr is

$P \propto r^2 e^{-Zr/a_0}$ →



Beta decay : ${}^3_1H \to {}^0_{-1}e + {}^3_2He^+$

$\downarrow$ $Z=1$ $\downarrow$ $Z=2$

both hydrogen like atoms

∵ sudden approximation ∴ state is $|1s, Z=1\rangle$ after decay

→ the required amplitude :

$$\langle 1s, Z=2 | 1s, Z=1\rangle = \int dr \cdot r^2 \cdot \left(\frac{1}{a_0}\right)^{3/2}\left(\frac{2}{a_0}\right)^{3/2} \cdot 4 e^{-2r/a_0} \times e^{-r/a_0}$$

$u = \frac{3r}{a_0}$

$r = \frac{a_0 u}{3}$

$$= \left(\frac{1}{a_0}\right)^{3/2}\left(\frac{2}{a_0}\right)^{3/2} \cdot 4 \cdot \int dr \cdot r^2 e^{-3r/a_0}$$

$$= \left(\frac{1}{a_0}\right)^{3/2}\left(\frac{2}{a_0}\right)^{3/2} 4 \left(\frac{a_0}{3}\right)^3 \int \underbrace{du \cdot u^2 e^{-u}}_{2! = 2}$$

$$= \frac{2\sqrt{2} \times 4 \times 2}{3^3} = \cancel{\frac{8\sqrt{2}}{27}} \quad \frac{16\sqrt{2}}{27}$$

Probability $\quad P = |\langle 1s, 2 | 1s, 1\rangle|^2 = \underline{\frac{512}{729}} \approx 0.7$

8. $\quad J_+ = J_x + iJ_y \qquad J_- = J_x - iJ_y$

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

$\rightarrow [J_z, J_+] = \underbrace{[J_z, J_x]}_{i\hbar J_y} + i\underbrace{[J_z, J_y]}_{i\hbar(-J_x)} = i\hbar(J_x + iJ_y) = \underline{\hbar J_+}$

$\rightarrow [J_z, J_-] = \underbrace{[J_z, J_x]}_{i\hbar J_y} - i\underbrace{[J_z, J_y]}_{-i\hbar J_x} = i\hbar(-J_x + iJ_y) = \underline{-\hbar J_-}$

$\because [J^2, J_i] = 0 \qquad (i = x, y, z)$

$\rightarrow \therefore [J^2, J_+] = [J^2, J_-] = \underline{0}$

$\rightarrow J_z(J_+|j,m\rangle) = J_+ J_z|j,m\rangle + \underbrace{[J_z, J_+]}_{\hbar J_+}|j,m\rangle$

$= m\hbar(J_+|j,m\rangle) + \hbar J_+|j,m\rangle$

$= (m+1)\hbar\,(J_+|j,m\rangle) \rightarrow$ QED

↑ eigenvalue

$$\rightarrow \quad J^2 J_+ |j,m\rangle = J J_+ J |j,m\rangle = J_+ J^2 |j,m\rangle$$

$$= \underbrace{j(j+1)\hbar^2}_{\text{eigenvalue}} (J_+ |j,m\rangle) \quad \rightarrow \text{QED}$$

$$\rightarrow \quad J_z (J_- |j,m\rangle) = J_- \underbrace{J_z |j,m\rangle}_{m\hbar} + \underbrace{[J_z, J_-]}_{-\hbar J_-} |j,m\rangle$$

$$= \cancel{(m+1)\hbar |j\rangle} \; \underbrace{(m-1)\hbar}_{\text{eigenvalue}} (J_- |j,m\rangle) \quad \rightarrow \text{QED}$$

$$\rightarrow \quad J^2 J_- |j,m\rangle = J_- J^2 |j,m\rangle = \underbrace{j(j+1)\hbar^2}_{\text{eigenvalue}} (J_- |j,m\rangle) \rightarrow \text{QED}$$

$j=1 \qquad m = -1, 0, 1$

let $\underline{|1,1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}} = |1\rangle \quad \underline{|1,0\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}} = |2\rangle \quad \underline{|1,-1\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}} = |3\rangle$

then ~~≡~~ trivially $\underline{J_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}}$

$$J_x = \frac{1}{2}(J_+ + J_-)$$

$\langle 1|J_x|1\rangle = \frac{1}{2}\langle 1,1|J_+ + J_-|1,1\rangle = 0$

similarly $\langle 2|J_x|2\rangle = \langle 3|J_x|3\rangle = 0$ (states orthogonal)

$\langle 1|J_x|2\rangle = \frac{1}{2}\langle 1,1|J_+ + J_-|1,0\rangle = \frac{1}{2}\langle 1,1|J_+|1,0\rangle$

$J_+|1,0\rangle = \sqrt{2}\,|1,1\rangle \quad \therefore \langle 1|J_x|2\rangle = \langle 2|J_x|1\rangle = \frac{\sqrt{2}}{2}$

$\cancel{\langle 1|J_x}\quad \langle 1|J_x|3\rangle = \frac{1}{2}\langle 1,1|\cancel{J_x} + J_-|1,-1\rangle = 0 = \langle 3|J_x|1\rangle$

$\langle 2|J_x|3\rangle = \frac{1}{2}\langle 1,0|J_+ + J_-|1,-1\rangle$

$= \frac{1}{2}\langle 1,0|J_+|1,-1\rangle = \cancel{\sqrt{\ldots}}$

$= \frac{1}{2}\sqrt{(1)(1+1)-(-1)(0)} = \cancel{\sqrt{2}} \; \frac{\sqrt{2}}{2} = \langle 3|J_x|2\rangle$

$$\therefore \quad \underline{J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}$$

$J_y = \frac{1}{2i}(J_+ - J_-)$

$\therefore$ similarly $\langle 1|J_y|1\rangle = \langle 2|J_y|2\rangle = \langle 3|J_y|3\rangle = 0$

$\cancel{\langle 1}\quad \langle 1|J_y|2\rangle = \frac{1}{2i}\langle 1,1|J_+ - J_-|1,0\rangle = \frac{\sqrt{2}}{2i} \; \cancel{= \langle 2|J_y}$

$\langle 2|J_y|1\rangle = -\frac{\sqrt{2}}{2i}$

$\langle 1|J_y|3\rangle = \langle 3|J_y|1\rangle = 0$

$\langle 2|J_y|3\rangle = \frac{1}{2i}\langle 1,0|J_+ - J_-|1,-1\rangle = \frac{\sqrt{2}}{2i} = -\frac{i}{\sqrt{2}}$

$\langle 3|J_y|2\rangle = -\frac{\sqrt{2}}{2i} = \frac{i}{\sqrt{2}}$

$$\rightarrow \quad \underline{J_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}}$$

$$J_{45} = \frac{1}{\sqrt{2}}(J_x + J_z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

For the eigenvalue $m=0$ let $|\phi\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. then

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left.\begin{array}{ll} \frac{1}{\sqrt{2}}a + \frac{1}{2}b = 0 & \rightarrow b = -\sqrt{2}a \\ \frac{1}{2}a + \frac{1}{2}c = 0 & \rightarrow c = -a \\ \frac{1}{2}b - \frac{1}{\sqrt{2}}c = 0 & \rightarrow b = \sqrt{2}c \end{array}\right\} \Rightarrow |\phi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ -1 \end{pmatrix}$$

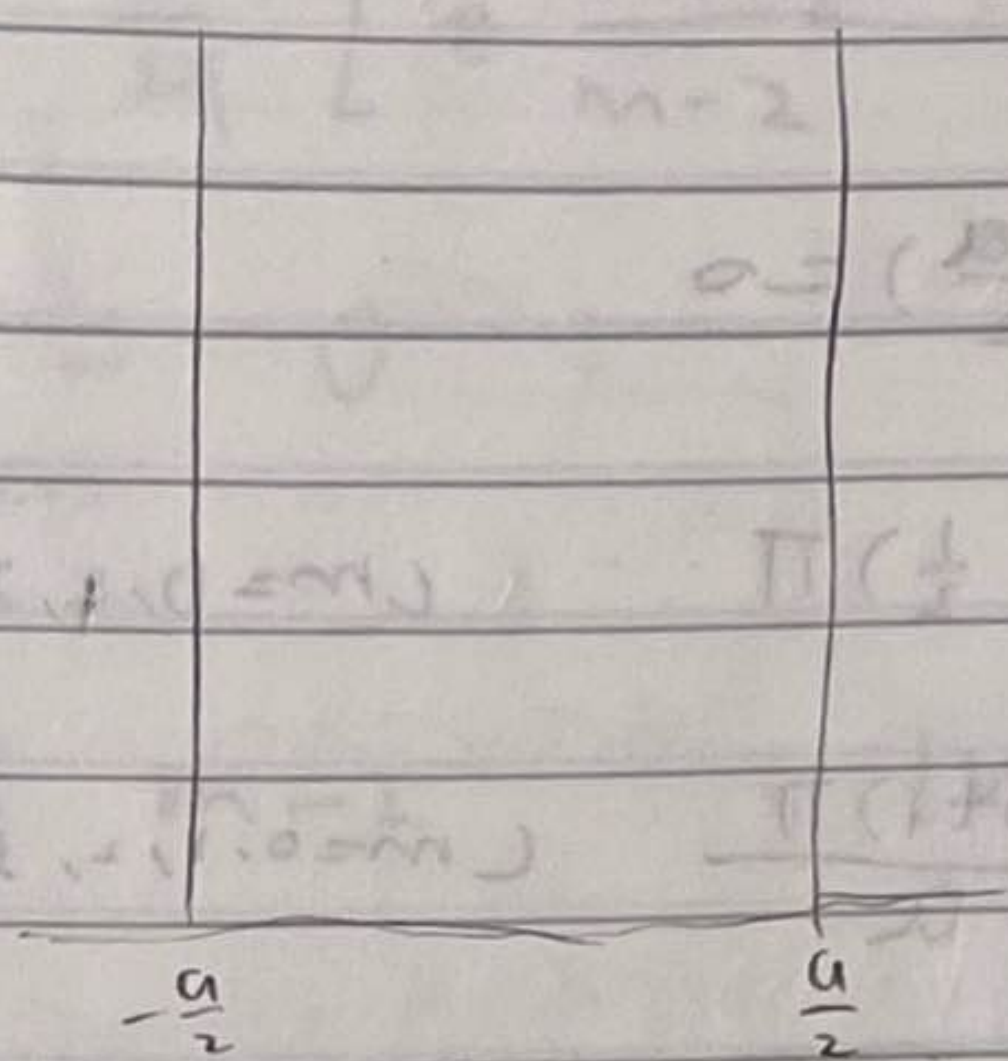
$$\rightarrow |\phi\rangle = \frac{1}{2}|1,1\rangle - \frac{\sqrt{2}}{2}|1,0\rangle - \frac{1}{2}|1,-1\rangle$$

After passing the filter the state ~~is~~ of particle is $|1,1\rangle$

→ required amplitude :

$$\langle\phi|1,1\rangle = \frac{1}{2} \quad \rightarrow \quad \text{probability } P = |\langle\phi|1,1\rangle|^2 = \frac{1}{4}$$

9.



TISE $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

$$V(x) = \begin{cases} 0 & -\frac{1}{2}a \le x \le \frac{a}{2} \\ \infty & \text{otherwise.} \end{cases}$$

$\therefore \psi = 0$ for $x < -\frac{1}{2}a$ and $x > \frac{a}{2}$

For $-\frac{1}{2}a \le x \le \frac{1}{2}a \qquad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$

let $\quad k^2 = \frac{2mE}{\hbar^2} \quad \rightarrow \quad \psi'' + k^2\psi = 0$

$\therefore \psi = A\cos(kx) + B\sin(kx)$

Boundary Conditions : $\psi(-\frac{a}{2}) = \psi(\frac{a}{2}) = 0$

$\therefore \rightarrow \quad 0 = A\cos(-\frac{ka}{2}) + B\sin(-\frac{ka}{2})$

$\qquad = A\cos(\frac{ka}{2}) - B\sin(\frac{ka}{2}) \qquad (1)$

$\rightarrow \quad 0 = A\cos(\frac{ka}{2}) + B\sin(\frac{ka}{2}) \qquad (2)$

$(1) + (2) \rightarrow \quad 2A\cos(\frac{ka}{2}) = 0 \qquad (3)$

$(1) - (2) \rightarrow \quad 2B\sin(\frac{ka}{2}) = 0 \qquad (4)$

$\cos(\frac{ka}{2})$ and $\sin(\frac{ka}{2})$ cannot $= 0$ for same a

, and if $A = B = 0$ then solution is meaningless.

$\therefore$ Either $\quad A = 0, \quad \sin(\frac{ka}{2}) = 0$

$\rightarrow \quad \frac{ka}{2} = n\pi \qquad (n = 1, 2, 3, \dots)$

$\therefore \quad k = \frac{2n\pi}{a} \quad (n = 1, 2, 3, \cdots)$

OR $\quad B = 0 \quad \cos\left(\frac{ka}{2}\right) = 0$

$\rightarrow \quad \frac{ka}{2} = \left(m + \frac{1}{2}\right)\pi \quad (m = 0, 1, 2, 3, \cdots)$

$\therefore \quad k = \frac{(2m+1)\pi}{a} \quad (m = 0, 1, 2, 3, \cdots)$

Overall : $\quad k = \frac{n\pi}{a} \quad (n = 1, 2, 3, \cdots) \rightarrow E_n = \frac{n^2\hbar^2\pi^2}{2ma^2}$

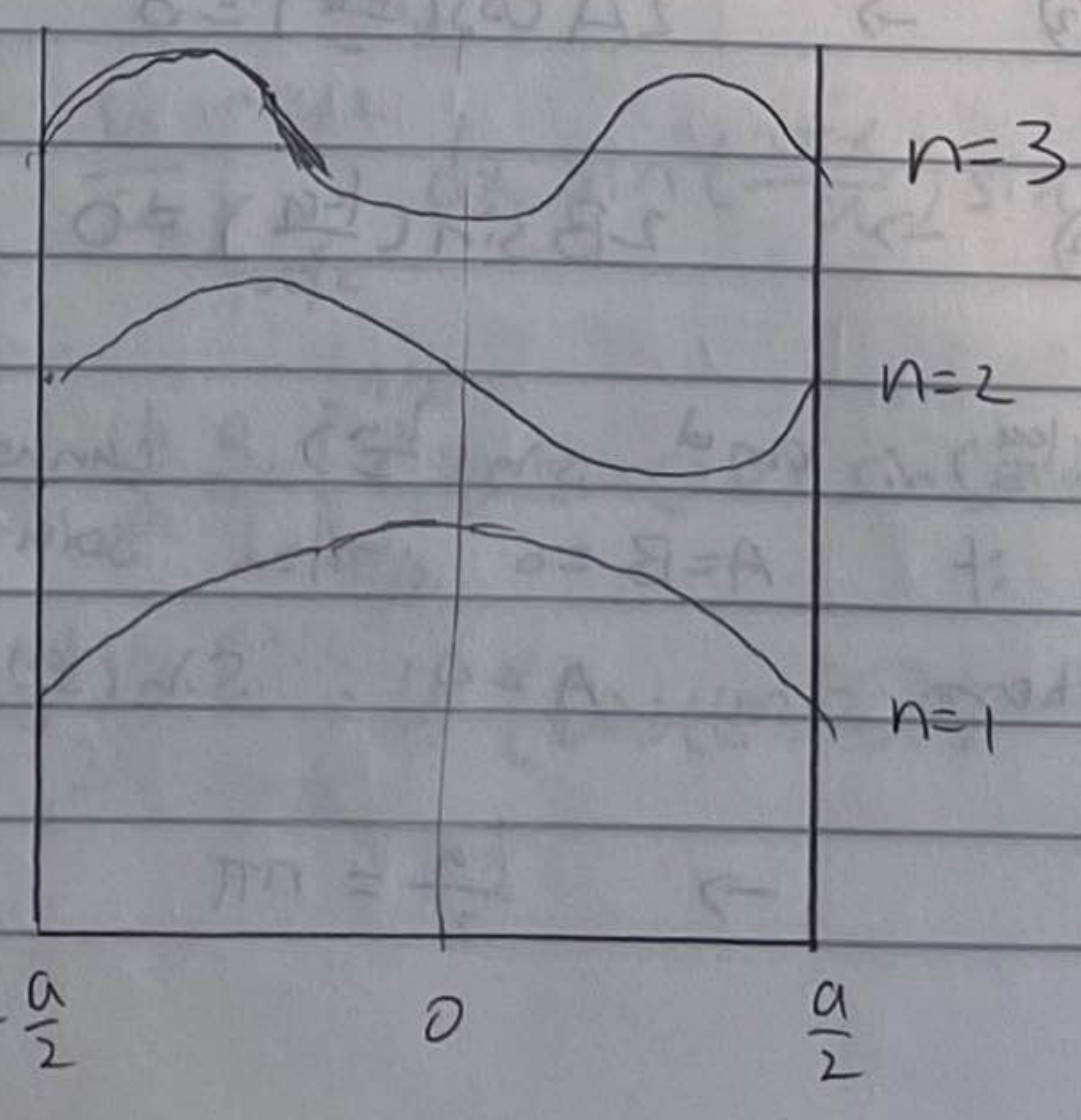
$\psi_n = B\sin\left(\frac{n\pi x}{a}\right)$ if n even

$\psi_n = A\cos\left(\frac{n\pi x}{a}\right)$ if n odd

Normalising $\rightarrow \quad A^2\int \sin^2\left(\frac{n\pi x}{a}\right)dx = B^2\int \cos^2\left(\frac{n\pi x}{a}\right)dx = 1$

$\rightarrow \quad A = B = \sqrt{\frac{2}{a}}$

$$\therefore \quad \psi_n = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) & \text{if } n \text{ odd} \\ \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & \text{if } n \text{ even} \end{cases}$$



$$= \frac{V_3}{2\pi}\left[\cancel{\theta}\ \frac{\sin((m-2)\theta)}{m-2} - \frac{\sin((m+2)\theta)}{m+2}\right]_{-\pi/2}^{\pi/2}.$$

$\therefore$ $= 0$

m even

If $m=2$ $\quad \langle 1|V'|m\rangle = \frac{V_3}{2\pi}\int_{-a/2}^{a/2} d\theta \sin^2(2\theta)$

$$\cancel{= \frac{V_3}{2\pi}\int_{-a/2}^{a/2} \sin(2\theta)\cos(2\theta)\, d(2\theta)}$$

$$\cancel{d\sin(2\theta)}$$

$$\cancel{= \frac{V_3}{2\pi}\sin}$$

$$= \frac{V_3}{2\pi}\int_{-\pi/2}^{\pi/2} d\theta \left[\frac{1}{2} - \frac{1}{2}\cos(4\theta)\right].$$

$$= \cancel{\frac{V_3 a}{4\pi}}\ \frac{V_3}{2}$$

relative contribution

$$= \frac{V_3}{2}\left(\frac{1}{E_1 - E_2}\right) = \frac{V_3}{2}\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^{-1}$$

$$= \frac{V_3 ma^2}{3\pi^2\hbar^2}$$

$\therefore$ only $m=2$ (first excited state) mixes with the ground state.

10. $\because$ B along z $\quad \therefore$ $H = \frac{e\hbar}{m_e} B S_z = \omega_L \hbar S_z$

$S_z|+\rangle = \frac{1}{2}|+\rangle \qquad S_z|-\rangle = -\frac{1}{2}|-\rangle \qquad |+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$\therefore$ $H|+\rangle = E_+|+\rangle = \underline{\frac{1}{2}\hbar\omega_L}|+\rangle \qquad H|-\rangle = \underline{-\frac{1}{2}\hbar\omega_L}|-\rangle = E_-|-\rangle$

$\therefore$ $\cancel{|\psi(t)\rangle =}$ $|\psi(0)\rangle = a(0)|+\rangle + b(0)|-\rangle$

$\therefore$ $|\psi(t)\rangle = a(0)e^{-\frac{iE_+t}{\hbar}}|+\rangle + b(0)e^{-\frac{iE_-t}{\hbar}}|-\rangle$

(a) $V' = V_1 x$

First order $\cancel{|\delta\psi_1^{(1)}\rangle = \sum_{m\neq 1} \frac{\langle 1|V'|m\rangle}{E_1 - E_m}|m\rangle}$

$$|\delta\psi_1^{(1)}\rangle = \sum_{m\neq 1} \frac{\langle m|V'|1\rangle}{E_1 - E_m}|m\rangle$$

$$\cancel{\langle m|V'|1\rangle = \frac{2V_1}{a}\int_{-a/2}^{a/2} dx\, x \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)} = 0$$

~~because integrand is odd~~ ~~(for even m)~~

$$\cancel{\text{and } \langle m|V'|1\rangle - \frac{2V_1}{a}\int_{-a/2}^{a/2} dx\, x \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)}$$

For odd m

$$\langle m|V'|1\rangle = \frac{2V_1}{a}\int_{a/2}^{a/2} dx\, x \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)$$

$= 0$ b/c integrand odd

For even m

$$\langle m|V'|1\rangle = \frac{2V_1}{a}\int_{-a/2}^{a/2} dx\, x \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

$$= \frac{2V_1}{a}\frac{a}{2\pi^2}\left[\frac{\pi x(m-1)\sin\left(\frac{\pi x(m-1)}{a}\right) + a\cos\left(\frac{\pi x(m-1)}{a}\right)}{(m-1)^2} + \frac{\pi x(m+1)\sin\left(\frac{\pi x(m+1)}{a}\right) + a\cos\left(\frac{\pi x(m+1)}{a}\right)}{(m+1)^2}\right]_{-a/2}^{a/2}$$

→ only states with ~~n≠2~~ m even ~~(odd state~~ (odd parity) mixes with the ground state.

(b) $V' = V_2 x^2$

~~∴~~ if m is even

$$\langle 1|V'|m\rangle = \frac{2V_2}{a}\int_{-a/2}^{a/2} dx\, x^2 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

$= 0$ ∵ integrand odd

∴ only states with m odd (even parity) mixes with the ground state.

~~∴~~ → If $V' = V_3 \sin\left(\frac{\pi x}{a}\right)$, then if m odd.

$$\langle 1|V'|m\rangle = \frac{2V_3}{a}\int_{-a/2}^{a/2} dx\, \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{m\pi x}{a}\right)$$

$= 0$ ∵ integrand odd

For m even:

$$\langle 1|V'|m\rangle = \frac{2V_3}{a}\int_{-a/2}^{a/2} dx\, \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

$$= \frac{V_3}{a}\int_{-a/2}^{a/2} dx\, \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

$\theta = \frac{\pi}{a}x$

$x = \frac{a\theta}{\pi}$

$$= \frac{V_3}{a}\frac{a}{\pi}\int_{-\pi/2}^{\pi/2} d\theta\, \sin(2\theta)\sin(m\theta)$$

→ if $2 \neq m$ ($m > 2$)

$$= \frac{V_3}{2\pi}\int_{-\pi/2}^{\pi/2} d\theta\, [\cos((m-2)\theta) - \cos((m+2)\theta)]$$

$$= \frac{V_3}{2\pi}\left[\cancel{\theta}\ \frac{\sin((m-2)\theta)}{m-2} - \frac{\sin((m+2)\theta)}{m+2}\right]_{-\pi/2}^{\pi/2}.$$

$\therefore = 0$
m even

If $m=2$ $\quad \langle 1|V'|m\rangle = \frac{V_3}{2\pi}\int_{-a/2}^{a/2} d\theta \sin^2(2\theta)$

$$\cancel{= \frac{V_3}{2\pi}\int_{-a/2}^{a/2} \sin(2\theta)\cos(2\theta)\, d(2\theta)}$$

$\cancel{d\sin(2\theta)}$

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$$= \cancel{\frac{V_3 a}{4\pi}}\ \frac{V_3}{2}$$

relative contribution $= \frac{V_3}{2}\left(\frac{1}{E_1 - E_2}\right) = \frac{V_3}{2}\left(\frac{3\pi^2\hbar^2}{2ma^2}\right)^{-1} = \frac{V_3 m a^2}{3\pi^2\hbar^2}$

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$\therefore \cancel{|\psi(t)\rangle =}\ |\psi(0)\rangle = a(0)|+\rangle + b(0)|-\rangle$

$\therefore |\psi(t)\rangle = a(0)e^{-\frac{iE_+t}{\hbar}}|+\rangle + b(0)e^{-\frac{iE_-t}{\hbar}}|-\rangle$

$\therefore \quad \underline{|\psi(t)\rangle = a(0)\, e^{-\frac{1}{2} i \omega_L t} |+\rangle + b(0)\, e^{\frac{1}{2} i \omega_L t} |-\rangle}$

energy eigenvalues $\quad \underline{E_+ = \frac{1}{2}\hbar\omega_L} \quad \underline{E_- = -\frac{1}{2}\hbar\omega_L}$

let the time dependent perturbation be.

$$V = \cancel{B' \sin}\ \frac{e\hbar}{m_e} B' \left[(\cos\omega t) S_x + (\sin\omega t) S_y \right].$$

initial condition : start from lower energy state $\therefore \quad a(0) = 0 \quad b(0) = 1$

Time dependent perturbation theory :

$$i\hbar \dot{a} e^{-\frac{1}{2} i\omega_L t} = a e^{-\frac{1}{2} i \omega_L t} \langle + | V | + \rangle$$
$$+ b e^{\frac{1}{2} i \omega_L t} \langle + | V | - \rangle \quad ①$$

$$i\hbar \dot{b} e^{+\frac{1}{2} i \omega_L t} = a e^{-\frac{1}{2} i \omega_L t} \langle - | V | + \rangle$$
$$+ b e^{\frac{1}{2} i \omega_L t} \langle - | V | - \rangle \quad ②$$

$$\langle + | V | + \rangle = \frac{e\hbar B'}{m_e} \left[\cos\omega t\, (0) + \sin\omega t\, (\cancel{0}) \right] = \cancel{\frac{e\hbar B'}{2m_e} \sin\omega t}\ \underline{0}$$

$$\langle + | V | - \rangle = \frac{e\hbar B'}{m_e} \left[\cos\omega t \left(\tfrac{1}{2}\right) + \sin(\omega t) \left(-\tfrac{i}{2}\right) \right]$$
$$= \underline{\frac{e\hbar B'}{2m_e} e^{-i\omega t}}.$$

$$\langle - | V | + \rangle = \underline{\frac{e\hbar B'}{2m_e} e^{i\omega t}}. \qquad \underline{\langle - | V | - \rangle = 0}$$

$\therefore \quad i\hbar \dot{a} e^{-\frac{1}{2} i\omega_L t} = b e^{\frac{1}{2} i\omega_L t} \left(\frac{e\hbar B'}{2m_e}\right) e^{-i\omega t}$

$\rightarrow \quad \cancel{i\hbar \dot{a} = b} \quad i\hbar \dot{a} = \left(\frac{e\hbar B'}{2m_e}\right) b\, e^{i(\omega_L - \omega)t}$ ③

$i\hbar \dot{b} e^{\frac{1}{2} i\omega_L t} = a e^{-\frac{1}{2} i\omega_L t} \left(\frac{e\hbar B'}{2m_e}\right) e^{i\omega t}.$

$\rightarrow \quad i\hbar \dot{b} = \cancel{a} \left(\frac{e\hbar B'}{2m_e}\right) a\, e^{i(\omega - \omega_L)t}$ ④

$\therefore \quad a = \frac{2im_e}{eB'} \dot{b} e^{i(\omega_L - \omega)t}$

$\rightarrow \quad \dot{a} = \frac{2im_e}{eB'} \left[\ddot{b} e^{i(\omega_L - \omega)t} + i(\omega_L - \omega) \dot{b} e^{i(\omega_L - \omega)t}\right]$

$\because \quad \dot{a} = \left(\frac{eB'}{2im_e}\right) b\, e^{i(\omega_L - \omega)t}$

$\therefore \quad \left(\frac{eB'}{2im_e}\right) b = \left(\frac{2im_e}{eB'}\right)\left(\ddot{b} + i(\omega_L - \omega)\dot{b}\right)$

$\therefore \quad \ddot{b} + i\underbrace{(\omega_L - \omega)}_{\Delta\omega}\dot{b} + \underbrace{\left(\frac{eB'}{2m_e}\right)^2}_{m^2} b = 0$

try $\quad b(t) = e^{ict}$

$\therefore \quad -c^2 + c\Delta\omega + m^2 = 0$

$\therefore \quad c^2 - c\Delta\omega - m^2 = 0 \quad \rightarrow \quad c = \frac{\Delta\omega \pm \sqrt{\Delta\omega^2 + 4m^2}}{2}$

$i\hbar \dot{a} e^{i(\omega - \omega_L)t} =$