2662

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

A2: ELECTROMAGNETISM AND OPTICS

TRINITY TERM 2014

Saturday, 21 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

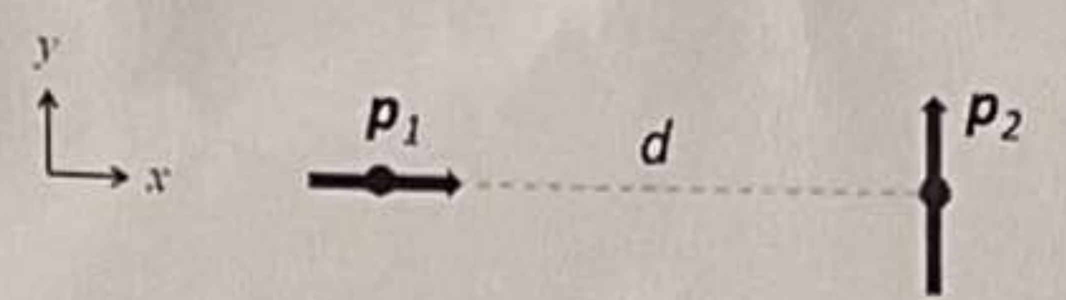
Section A

1. What is the relationship between **B** and and the vector potential **A**? Demonstrate, using selected Maxwell's equations, under what circumstances it is correct to represent the **E**-field as the gradient of a scalar potential ϕ, and when **A** must also be included. Consider an infinite straight wire carrying a steady current I. For the region outside the wire write down the **B**-field and a valid expression for **A**. [5]

2. Show that an electric dipole of moment **p** placed in an electric field **E** experiences a torque $\mathbf{T} = \mathbf{p} \times \mathbf{E}$ around its centre. [2]

Two perfect electric dipoles of moment $\mathbf{p}_1$ and $\mathbf{p}_2$ are placed a distance d apart, with one aligned at right angles to the other.

$\frac{\vec d}{2} \times q\vec E + (-q\frac{\vec d}{2}) \times (-q\vec E)$



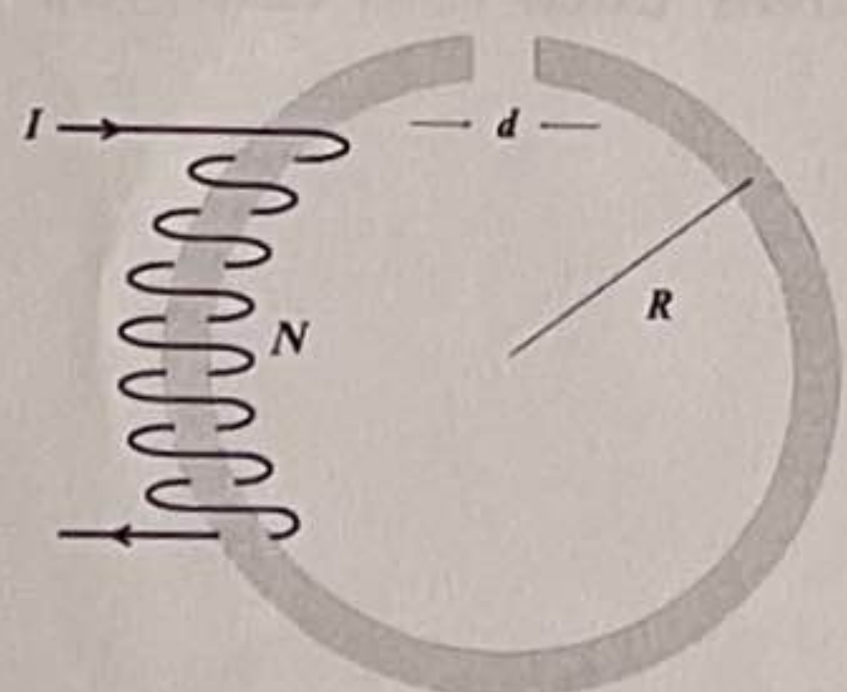
$-q$ =

Determine the field each dipole experiences, and the torque on each dipole evaluated around its own centre. Why are these torques not equal and opposite? [5]

[You may assume that the potential at a position **r** from a dipole of moment **p** is given by $V = \mathbf{p}.\mathbf{r}/4\pi\epsilon_0 r^3$.]

3. Derive the condition that relates the **B**-fields in two media separated by a boundary. [2]

A circular ring of radius R is formed of soft-iron of relative permeability μ_r, with a small air gap of extent d.



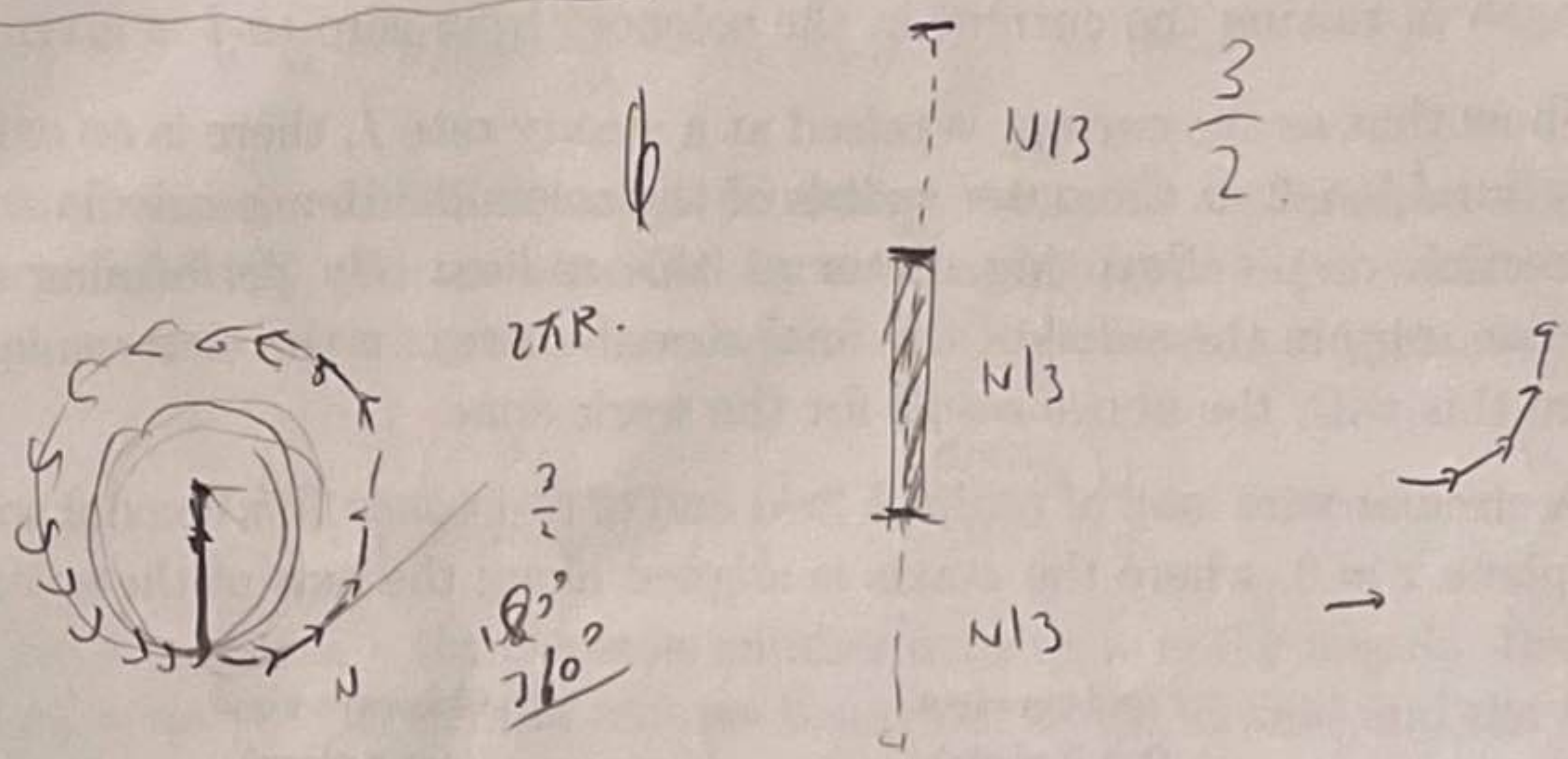
$\cos\theta$ $\sin\theta$

$-\sin\theta$

Around the ring is a coil of N turns of wire through which a current I passes. Calculate B_g, the flux density in the gap. [4]

A ferromagnet of the same geometry (without the coil) is prepared with large magnetisation of magnitude M, directed around the ring. What is B_g in this case? Sketch the B and H field lines inside the magnet material and in the gap. [2]

4. A plane diffraction grating consists of N very narrow, equally spaced slits. The centremost $N/3$ slits are obscured by an opaque strip. Using a phasor diagram, show that the angular distance from the principal maximum to the nearest minimum is $3/4$ of that displayed by the unobscured grating. [6]



5. Sketch the arrangement of optical components employed in a Michelson interferometer. State how the components can be configured to give circular fringes and straight fringes and how these are produced. How can the interferometer be used to measure the wavelength of a light source? [8]

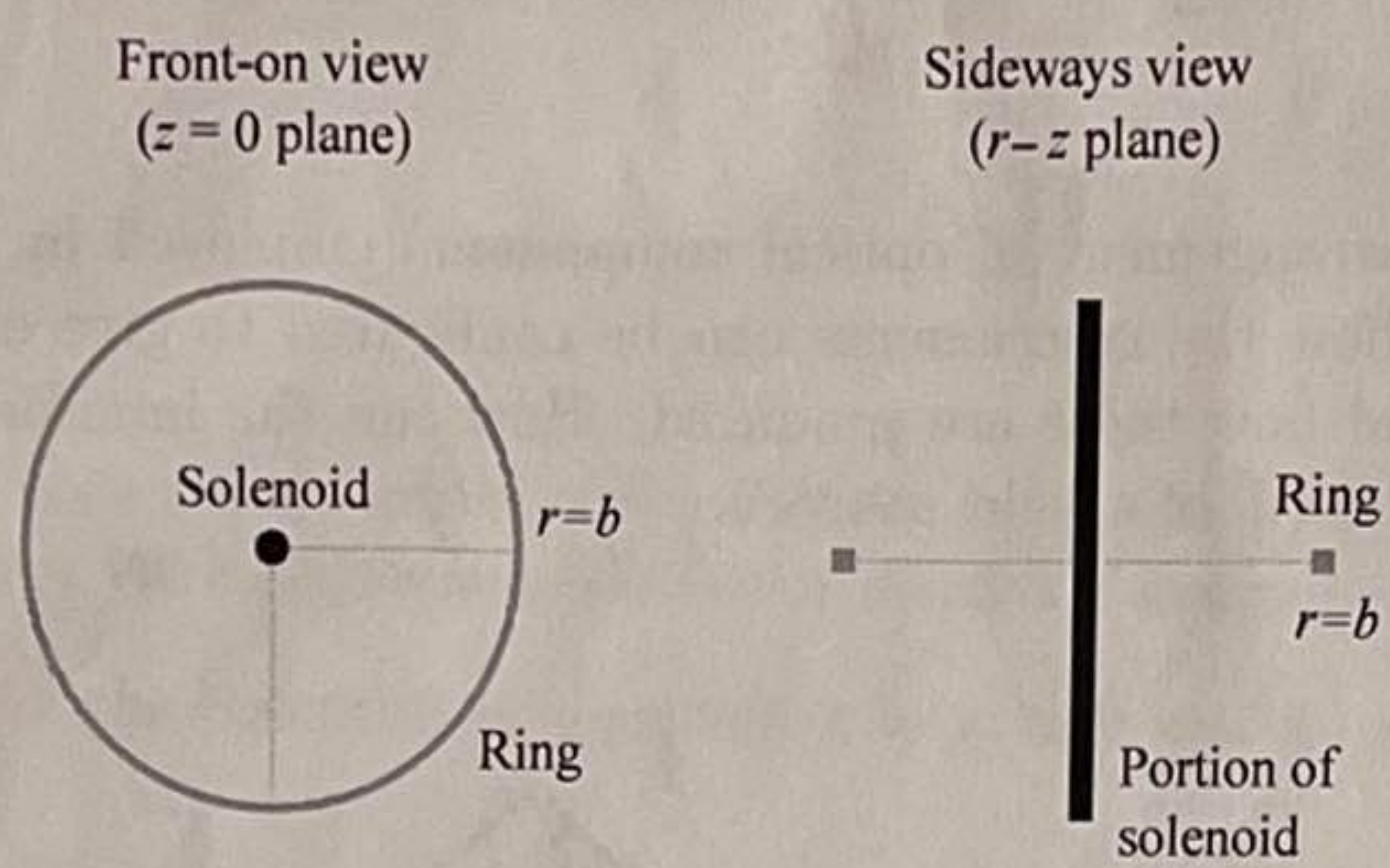
6. Define what is meant by plane polarized, circularly polarized, elliptically polarized and unpolarized light. State how the phase angle of elliptically polarized light can be changed to any other through an appropriate combination of quarter-wave and half-wave plates. (A derivation of the properties of the plates is not required.) [6]

Section B

7. Derive an expression for the self-inductance per unit length, L, of an infinite air-filled solenid of n turns per unit length, and radius a. Show that the work done per unit length in raising the current in the solenoid from zero to I is given by $\frac{1}{2}LI^2$. [4]

Show that as the current is raised at a steady rate $\dot{I}$, there is an azimuthal electric field $E_\phi = n\dot{I}\mu_0 a/2$ at the outer radius of the solenoid. Hence calculate the magnitude and direction of the Poynting vector at this radius. By performing an appropriate integration, obtain the value of the final stored energy in the electromagnetic field and compare this with the above result for the work done. [6]

A circular wire loop of radius $b \gg a$ and of resistance R is coaxial with the solenoid in the plane $z = 0$, where the z-axis is aligned along the axis of the solenoid.



The current in the solenoid is now decreased gradually at a rate α. Calculate I_l, the current induced in the loop. By considering the magnetic field arising from I_l, show that the Poynting vector evaluated just outside the outer radius of the solenoid and at a longitudinal distance z away from the loop has magnitude

$$S = \frac{\mu_0 n a b^2 I_l \alpha}{4(b^2+z^2)^{3/2}}.$$

Sketch the lines of Poynting flux during this period for the region $r > a$ as viewed in the r-z plane. [7]

Show that the total power obtained by evaluating this Poynting flux over the surface area of the solenoid is equal to the rate of energy dissipated in the resistive ring. [3]

[You may use $\int \frac{\mathrm{d}x}{(c+x^2)^{3/2}} = \frac{x}{c(c+x^2)^{1/2}}$.]

?

physical ~~sig~~ significance?

8. An electromagnetic wave propagates through a homogeneous, isotropic medium of conductivity σ, relative permittivity ϵ_r and relative permeability μ_r. Use Maxwell's equations to show that the the electric field $\mathbf{E}$ obeys the following relation

$$\nabla^2 \mathbf{E} - \sigma \mu_0 \mu_r \frac{\partial \mathbf{E}}{\partial t} - \epsilon_0 \epsilon_r \mu_0 \mu_r \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$ [4]

Consider a plasma in which electrons of charge $-e$ move under the influence of a sinusoidal electric field $\mathbf{E}$ and are retarded by a damping force $-\xi \mathbf{v}$, where $\mathbf{v}$ is the velocity of the electrons. Show that the conductivity of this plasma is given by

$$\sigma = \frac{\sigma_0}{1 + (\sigma_0 m \omega / n e^2)^2} \left[1 - \mathrm{i} \left(\frac{\sigma_0 m \omega}{n e^2} \right) \right],$$

where m is the electron mass, n the electron number density, ω is the angular frequency of the field and $\sigma_0 = n e^2 / \xi$. Sketch the relative behaviour of the $\mathbf{E}$ field and the current density $\mathbf{J}$ as a function of time when $\xi = m\omega$. [6]

For a low density plasma damping effects may be neglected. For such a plasma derive the dispersion relation

$$c^2 k^2 = \omega^2 - \omega_p^2$$

where $\omega_p = (n e^2 / m \epsilon_0)^{1/2}$ is the plasma frequency. Obtain an expression for the group velocity of waves in the plasma, and state for what range of ω a wave can propagate through the plasma. [6]

Radio waves are detected from a pulsar which is 530 light years distant. The arrival time of the pulses differs by 5 s if they are detected in a narrow band centred around frequencies of 400 MHz compared with a band centred on 200 MHz. Explain this observation and calculate the mean electron density in the space between earth and the pulsar. [4]

cos θ + sin θ

tan θ

tan θ

$J = ne \frac{d\vec{x}}{dt}$

$\omega_p^2 = \frac{ne^2}{m\epsilon_0}$

9. A plane diffraction grating consists of N very narrow slits, equally spaced by a distance d, illuminated by monochromatic light of wavenumber k at normal incidence. Show that the intensity of radiation, $I(\theta)$, diffracted by an angle θ is given by

$$I(\theta) = \frac{I_0}{N^2}\frac{\sin^2(Nud/2)}{\sin^2(ud/2)}$$

where $u = k\sin\theta$ and I_0 is the intensity of the principal maxima.
Determine how the intensity of the principal maxima, I_0, varies with the number of slits N when d is held constant.
Sketch the diffracted intensity $I(\theta)$ as a function of $\sin\theta$. [8]

State and prove the convolution theorem.
How is the convolution theorem useful for solving difficult diffraction problems? [7]

By making appropriate use of the convolution theorem, derive and sketch the intensity of the diffraction pattern created by the above grating when the slits have finite width a. [5]

10. Explain, with the aid of a suitable sketch, the physics of a Fabry–Pérot etalon. [3]

Show that the intensity of the fringe pattern varies as:

$$I_t(\delta) = \frac{I_0}{1 + (4F^2/\pi^2)\sin^2(\delta/2)},$$

where $F = \pi\sqrt{R}/(1-R)$ is the finesse of the etalon, and you should provide suitable definitions of δ, R and I_0. Sketch the variation of I_t with δ in the limits of high and low values of R.

In an appropriate limit, show that the finesse can be interpreted as the ratio of the distance between adjacent peaks and the peak half-width. [10]

A monochromatic beam of light passes through an etalon at normal incidence with an optical path length of 1 mm. By gradually tilting the plate, a periodic change in intensity is observed from 100% to 1% of the peak transmitted beam intensity. The angle corresponding to the 10th peak is 0.07 radians. Estimate the wavelength of the light and the resolving power of the etalon. [7]

$1 - 2\sin^2(\delta/2) = \cos(\delta)$

$4\pi\bar{\nu}d\cos\theta = 2\pi(p-10)$

$4\pi\bar{\nu}d = 2\pi p$

$4\pi\bar{\nu}d =$

To: Caroline Terquem

A2 2014 First Attempt Ziyan Li

1. $\underline{B} = \nabla \times \underline{A}$ ✓

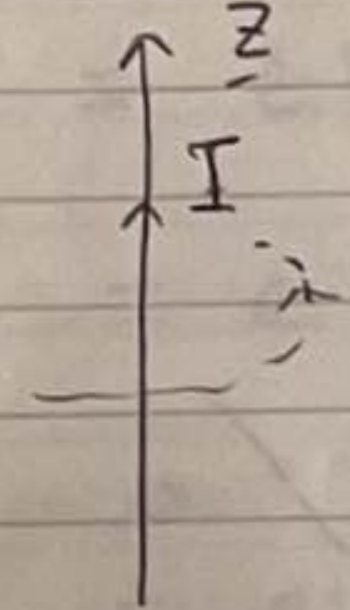
$\because \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ $\therefore$ when $\underline{B}$ is constant in time,

$\nabla \times \underline{E} = 0 \rightarrow$ we then can express $\underline{E} = -\nabla\phi$ ✓

If $\underline{B}$ varies with time

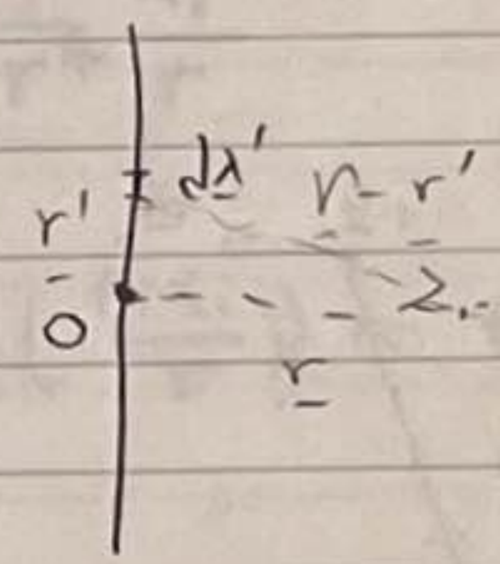
$$\frac{\partial \underline{B}}{\partial t} = \nabla \times \frac{\partial \underline{A}}{\partial t} \rightarrow \nabla \times \underline{E} = -\nabla \times \frac{\partial \underline{A}}{\partial t}$$

$$\therefore \underline{E} = -\nabla\phi - \frac{\partial \underline{A}}{\partial t}$$ ✓



$\oint \underline{B} \cdot d\underline{l} = \mu_0 I$ (Ampere's Law)

$\therefore \underline{B} = \frac{\mu_0 I}{2\pi r} \underline{\hat{\theta}}$ $(r = \sqrt{x^2+y^2})$



$$\underline{A} = \frac{\mu_0}{4\pi}\int \frac{I(\underline{r}')}{|\underline{r}'-\underline{r}|} d\lambda'$$

$$= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} \frac{I}{\sqrt{z^2+r^2}} dz$$

$$= \frac{\mu_0 I}{4\pi}\hat{\underline{z}}\int_{-\infty}^{\infty} \frac{dz}{\sqrt{z^2+r^2}}$$

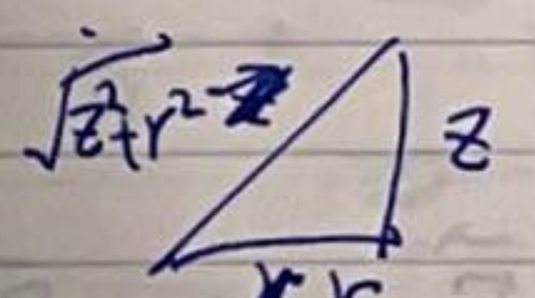
let $z = \cancel{r\sec\theta}\ r\tan\theta$

$dz = r\sec^2\theta\ \cancel{\tan\theta}\ d\theta$

$$\cancel{=\frac{\mu_0 I}{4\pi}} \int \frac{dz}{\sqrt{z^2+r^2}} = \int \frac{r\sec^2\theta\ \cancel{\tan\theta}\ d\theta}{r\sqrt{1+\tan^2\theta}} = \cancel{\int \sec\theta\, d\theta}$$

$\cancel{\tan\theta} \rightarrow \sec\theta$ $\int \sec\theta\, d\theta$

$$= \ln(\sec\theta + \tan\theta)$$



$\rightarrow \sec\theta = \frac{1}{r}\sqrt{z^2+r^2}$

$$= \ln\left(\frac{1}{r}(\sqrt{z^2+r^2} + z)\right) + C$$

$$= \ln(\sqrt{z^2+r^2} + z) - \ln r + C = \ln(\sqrt{z^2+r^2} + z) + C$$

$$\therefore \underline{A} = \cancel{\frac{\mu_0 I}{4\pi}} \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty} \frac{dz}{\sqrt{z^2+r^2}}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^{\infty} \frac{dz}{\sqrt{z^2+r^2}} = \frac{\mu_0 I}{2\pi} \ln\left(\sqrt{z^2+r^2}+z\right)\Big|_0^{L\to\infty}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left(\cancel{\sqrt{z^2+L^2}+}\sqrt{r^2+L^2}+L\right) - \ln(r)\right]$$

$$\approx \frac{\mu_0 I}{2\pi}\left(\ln(2L) - \ln(r)\right)\hat{\underline{z}}$$

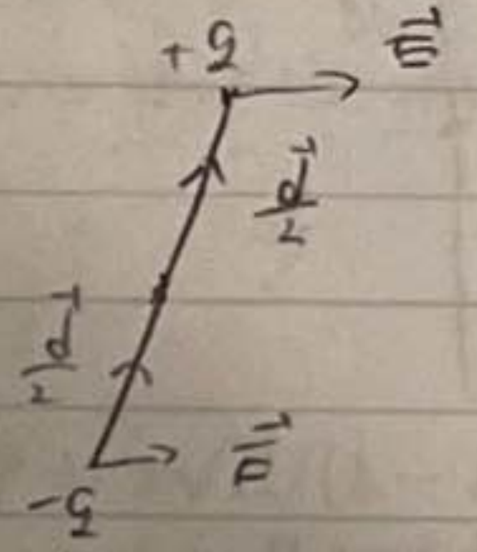
(as $L\to\infty$)

$\therefore$ An valid $\underline{A}$ is

$$\underline{A} = -\frac{\mu_0 I}{2\pi}\ln(r)\,\hat{\underline{z}}$$

✓

2.



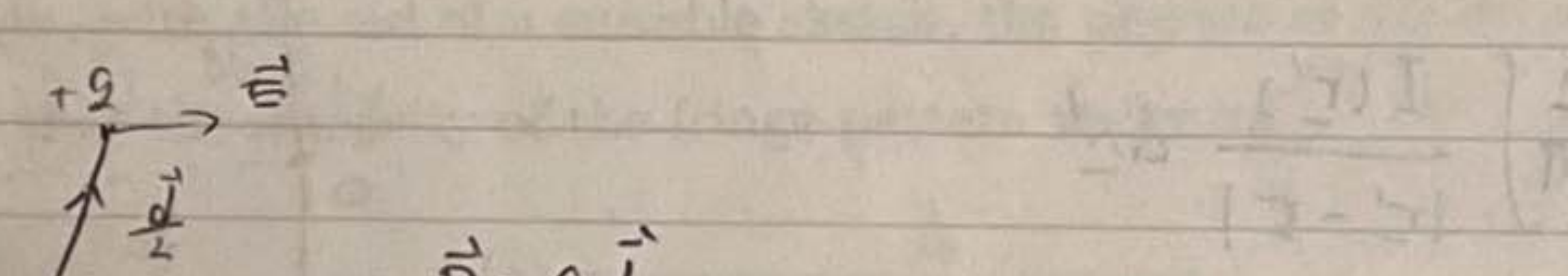
$\vec{p} = q\vec{d}$

Torque $\cancel{\vec{\tau} = \sum \vec{r}\times\vec{E}}$ $\vec{T} = \sum \vec{r}\times\vec{F}$ $\cancel{(\vec{F} = q\vec{E})}$

$$= \frac{\vec{d}}{2}\times(q\vec{E}) + \left(-\frac{\vec{d}}{2}\right)\times(-q\vec{E})$$

$$= (q\vec{d})\times\vec{E} = \vec{p}\times\vec{E}$$

non-uniform field? expand ∇V.



$$\vec{T}_{2\to1} = \vec{p_1}\times\vec{E}_{2\to1}$$

$$\vec{T}_{1\to2} = \vec{p_2}\times\vec{E}_{1\to2}$$

Potential due to dipole:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

electric field $\vec{E} = -\vec{\nabla}V = -\frac{1}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\hat{r}}{r^2}\right)$

$$\vec{\nabla}\cdot\left(\vec{p}\cdot\frac{\hat{r}}{r^2}\right) = \vec{p}\times\underbrace{\left(\vec{\nabla}\times\frac{\hat{r}}{r^2}\right)}_{=0} + (\vec{p}\cdot\vec{\nabla})\frac{\hat{r}}{r^2}$$

$$+ \underbrace{\frac{\hat{r}}{r^2}\times(\vec{\nabla}\times\vec{p})}_{=0} + \underbrace{\left(\frac{\hat{r}}{r^2}\cdot\vec{\nabla}\right)\vec{p}}_{=0} \quad \because \vec{p} \text{ is constant}$$

$$= (\vec{p}\cdot\vec{\nabla})\frac{\hat{r}}{r^2}$$

$$= e_i p_j \partial_j\left(\frac{x_i}{r^3}\right) \qquad \left(\hat{r} = \frac{e_i x_i}{r}\right)$$

$$= e_i p_j \frac{1}{r^3}\underbrace{\partial_j x_i}_{\delta_{ij}} + e_i p_j x_i \partial_j\left(\frac{1}{r^3}\right)$$

$$= e_i p_i \frac{1}{r^3} + e_i x_i p_j (-3)\frac{1}{r^4}\frac{x_j}{r}$$

$$= \cancel{e_i p_i \frac{1}{r^3}} - \frac{1}{r^3}\left(3p_j\left(\frac{x_j}{r}\right)\left(\frac{e_i x_i}{r}\right) - e_i p_i\right)$$

$$= -\frac{1}{r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

$$\therefore \vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right)$$

$(\vec{r}_{1\to2} = \vec{d},\ \vec{r}_{2\to1} = -\vec{d})$

$$\therefore \vec{E}_{2\to1} = \frac{1}{4\pi\varepsilon_0 d^3}\left(3(\vec{p}_2\cdot(-\hat{d}))(-\hat{d}) - \vec{p}_2\right)$$

$$\vec{p}_2 \perp \vec{d} \quad \therefore \vec{p}_2\cdot(-\hat{d}) = 0$$

$$\therefore \vec{E}_{2\to1} = \frac{-\vec{p}_2}{4\pi\varepsilon_0 d^3}$$

$\vec{p}_1 = p_1\hat{x}$ $\quad$ $\vec{p}_2 = p_2\hat{y}$

$$\vec{T}_{2\to1} = \vec{p}_1\times\vec{E}_{2\to1} = -\frac{\vec{p}_1\times\vec{p}_2}{4\pi\varepsilon_0 d^3}$$

$$= -\frac{p_1 p_2}{4\pi\varepsilon_0 d^3}\hat{z}$$

$$\vec{E}_{1\to2} = \frac{1}{4\pi\varepsilon_0 d^3}\left(3(\vec{p}_1\cdot\hat{d})\hat{d} - \vec{p}_1\right)$$

$(\hat{d} = \hat{x})$

$\vec{p}_1 = p_1\hat{x}$

$\vec{p}_1\cdot\hat{d} = p_1$

$\vec{p}_1 = p_1\hat{d}$

$$= \frac{1}{4\pi\varepsilon_0 d^3}\left(3p_1\hat{d} - p_1\hat{d}\right)$$

$$= \frac{2p_1\hat{d}}{4\pi\varepsilon_0 d^3} = \frac{2p_1\hat{x}}{4\pi\varepsilon_0 d^3} = \frac{2\vec{p}_1}{4\pi\varepsilon_0 d^3}$$

$$\vec{T}_{1\to2} = \vec{p}_2\times\vec{E}_{1\to2} = \frac{2\vec{p}_2\times\vec{p}_1}{4\pi\varepsilon_0 d^3} = -\frac{2\vec{p}_1\times\vec{p}_2}{4\pi\varepsilon_0 d^3}$$

$$= \underline{-\frac{2p_1p_2}{4\pi\varepsilon_0 d^3}\hat{z}}$$

$\vec{T}_{1\to2} \neq \vec{T}_{2\to1}$ because the torques are measured from the centres of ~~each~~ two individual dipoles rather than from a common point.

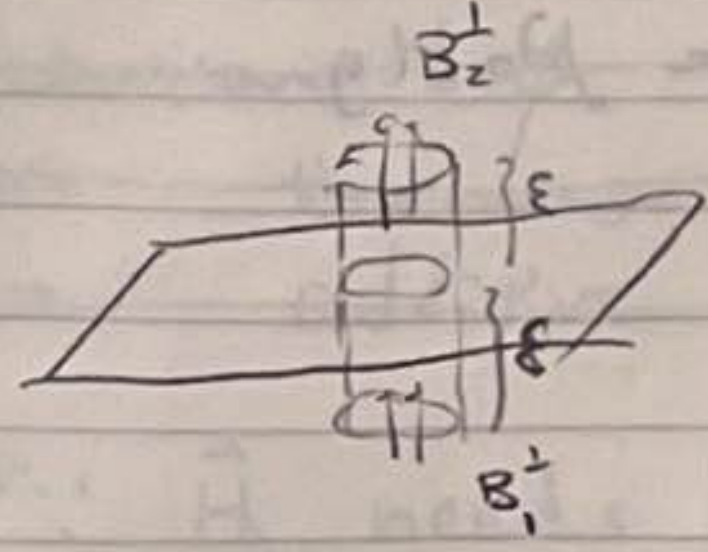
→ $\vec{T}_{1\to2}$ ~~measured~~ is with respect to centre of $\vec{p}_2$

→ $\vec{T}_{2\to1}$ is with respect to centre of $\vec{p}_1$

They are not related by ~~New~~ Newton's 3rd Law because they are not measured from a common point.

∴ They don't ~~have~~ have to be equal.

3.

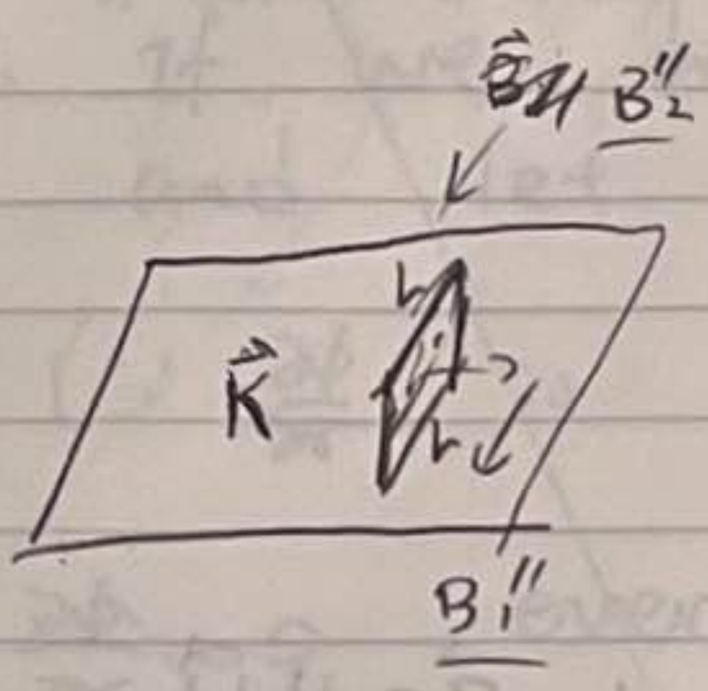


$$\oiint \vec{B}\cdot d\vec{S} = 0$$

i.e. As $\varepsilon \to 0$

$$B_2^{\perp} S - B_1^{\perp} S = 0 \to \underline{B_2^{\perp} = B_1^{\perp}}$$

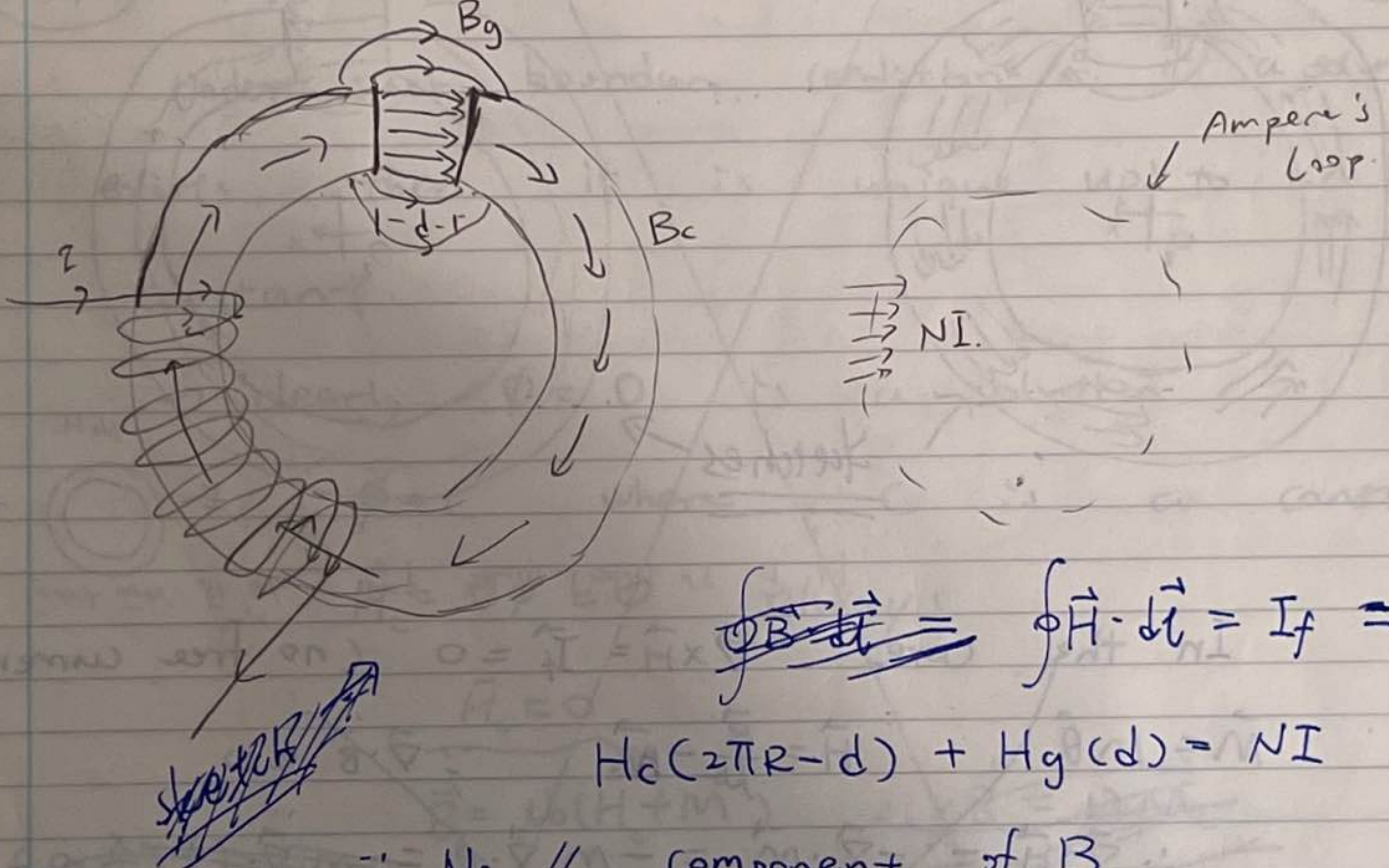
✓



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

$$\Rightarrow B_2^{\parallel} L - B_1^{\parallel} L = \mu_0 K L$$

$$\to \underline{B_2^{\parallel} - B_1^{\parallel} = \vec{K} \times \hat{n}_{12}}$$



$$\cancel{\oint \vec{B}\cdot d\vec{l}} = \oint \vec{H}\cdot d\vec{l} = I_f = NI$$

$$H_c(2\pi R - d) + H_g(d) = NI$$

$\because$ No // component of B

and $B_g^{\perp} = B_c^{\perp}$ $\therefore$ $B_g = B_c$

& $H_g = \dfrac{B_g}{\mu_0}$ $H_c = \dfrac{B_c}{\mu_r \mu_0}$ $\therefore$ $H_c = \dfrac{H_g}{\mu_r}$

$$\therefore \frac{H_g}{\mu_r}(2\pi R - d) + H_g(d) = NI$$

$$\therefore H_g = \frac{\mu_r N I}{(2\pi R - d) + \mu_r d} = \cancel{\frac{\mu_r NI}{2\pi R + (\mu_r - d)}}$$

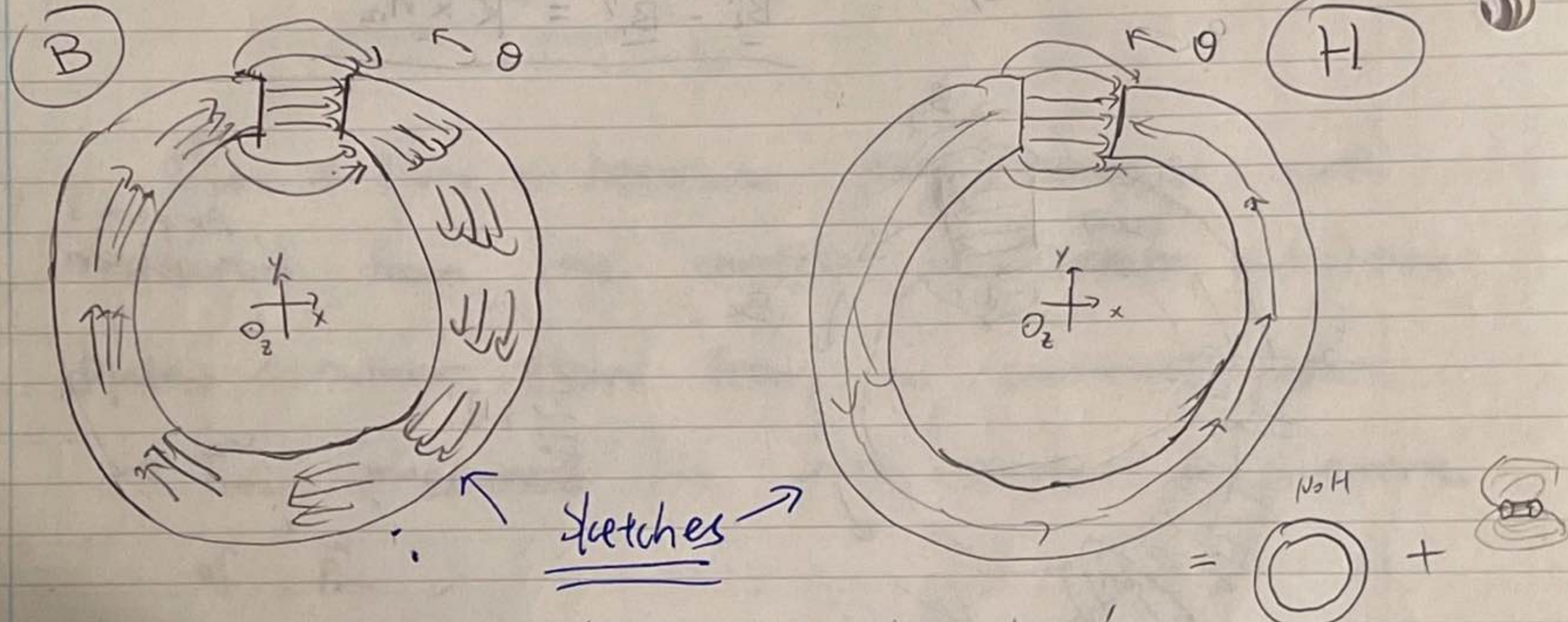
$$H_g = \frac{\mu_r N I}{2\pi R + (\mu_r - 1) d} \qquad B_g = \mu_0 H_g$$

$$B_g = \frac{\mu_0 \mu_r N I}{2\pi R + (\mu_r - 1) d}$$

Ferromagnetism :

In gap ~~B~~ : $\vec{M} = \frac{\vec{B}_c}{\mu_0} - \vec{H}_c$

In air : $\vec{M} = 0 \quad \therefore \vec{B}_g = \mu_0 \vec{H}_g$



Sketches

H, B No superposition, but ! $\vec{B} = \mu_0 \vec{M} + \mu_0 \vec{H}$

In the core $\vec{\nabla} \times \vec{H} = \vec{J}_f = 0$ (no free current)

$\vec{M} = M\hat{\theta} \qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \because \vec{\nabla} \cdot \vec{B} = 0$

$\therefore \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = -M \vec{\nabla} \cdot \hat{\theta} = \cancel{-M\vec{\nabla} \cdot (\frac{y}{r}, \frac{x}{r}, 0)}$

$\cancel{-M\vec{\nabla} \cdot (-\frac{x}{r}, \frac{y}{r}, 0)} = -M \vec{\nabla} \cdot (-\frac{y}{r}, \frac{x}{r}, 0)$

$= 0$

For finite energy we want $\hat{H}$ vanish at ∞

$\because \vec{\nabla} \times \vec{H} = 0$ let $\vec{H} = \vec{\nabla}\phi$ without loss of generality

$\vec{\nabla} \cdot \vec{H} = 0 \quad \therefore \nabla^2 \phi = 0 \rightarrow$ Laplace equation.

By uniqueness theorem, if we can find a

~~Solution ϕ that satisfy the Boundary condition then this solution is unique up to an additive constant.~~

$\because$ $\vec{H}$ needs to vanish at infinity ~~tho~~
(or in this case the boundary of the toriod, since $H_{//}$ is continuous, and $\vec{H}$ in vacuum is 0 → $\vec{H}$ on the boundary of toroid is 0 → this is the boundary condition)

$\therefore$ If we draw a surface with radius R and let $R \to \infty$ then at Surface $\frac{\partial \phi}{\partial r} \to 0$

($\because$ $\frac{\partial \phi}{\partial r}$ is the radial component of $\vec{H}$ ($\vec{H} = \vec{\nabla}\phi$))

$\frac{\partial \phi}{\partial r} = 0$ everywhere on the Boundary is a type of Neumann boundary condition of Laplace's equation

Under such boundary condition, if a solution is exists, then it is unique up to an additive constant.

→ clearly $\phi = 0$ is a solution

→ $\phi = C$ where C is a constant

$\therefore$ $\vec{H}_c = \vec{\nabla}\phi = 0$ always

$\therefore$ $\vec{H}_c = 0$

$\because$ $\vec{B}_c = \mu_0(\vec{H}_c + \vec{M})$ $\therefore$ ~~$\vec{B}_c = \mu_0 \vec{M}_c$~~

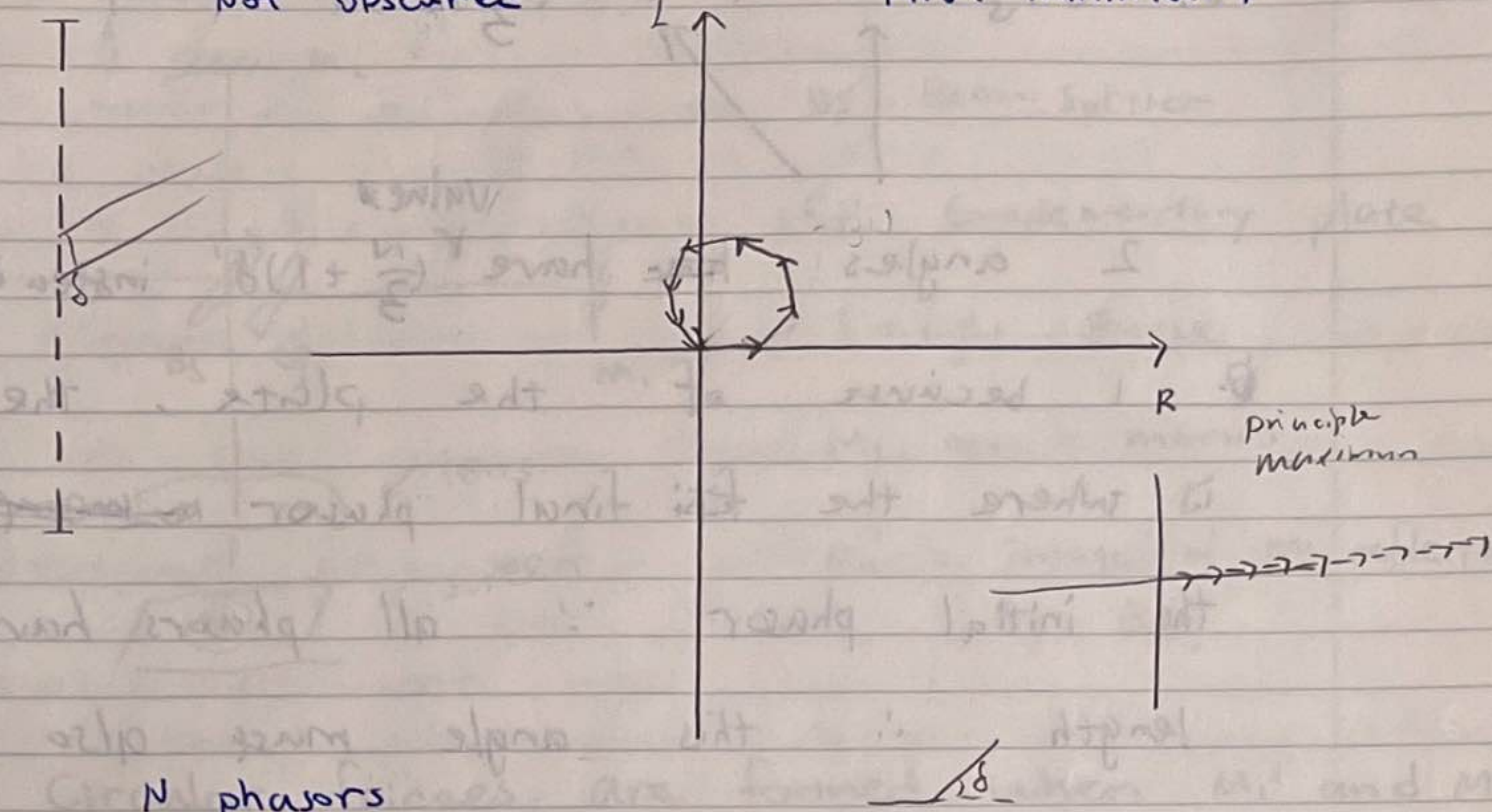
$\therefore$ $\vec{B}_c = \mu_0 \vec{M}$

$\because$ $\vec{B}_c = \vec{B}_g$ (boundary condition derived ~~ear~~ earlier)

$\therefore$ $\vec{B}_g = \mu_0 \vec{M}$ (assuming $\vec{B}_g$ is perpendicular to the surface of cross-section $\because$ $d \ll R$)

$\because$ $\vec{B}_g = \mu_0 \vec{H}_g$ $\therefore$ $\vec{H}_g = \vec{M}$

4. Not obscured First minimum

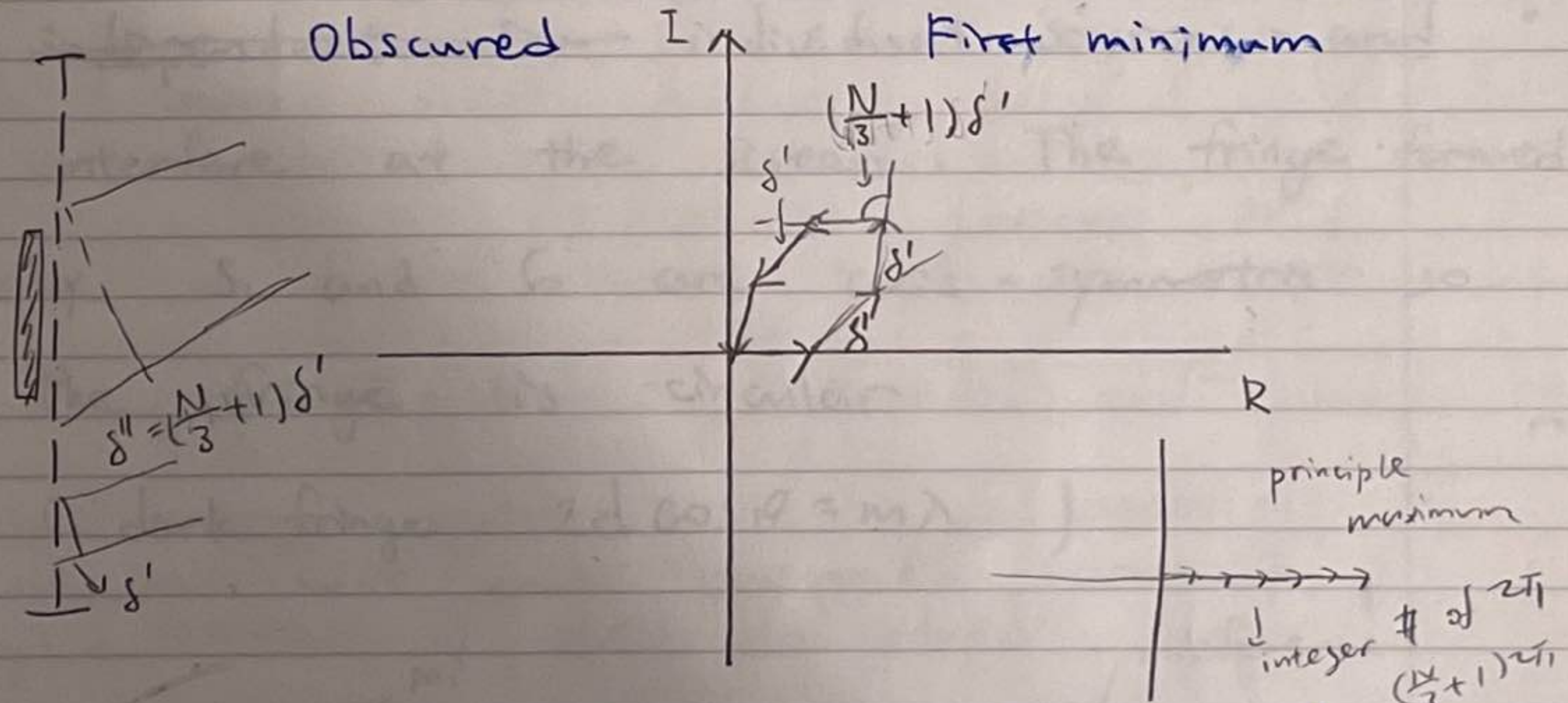


N phasors

Deviation angle $= \delta$ for all phasors

$N\delta = 2\pi$ $\cancel{9\delta = 2\pi}$ $\rightarrow$ $\cancel{\delta = \frac{2\pi}{9}}$ $\delta = \frac{2\pi}{N}$

Obscured First minimum



Consider the two interfering beams adjacent to the opaque ~~plt~~ plate

the path difference is $\frac{N}{3}+1$ times ~~the~~ that between two adjacent interfering beams,

$\therefore$ $\cancel{\delta'(\frac{2}{3}N) + 2(\frac{N}{3}+1)\delta' = 2\pi}$ $\delta = kd'\sin\theta$

$\therefore$ $\cancel{\delta''(\frac{4}{3}N) = 2\pi}$ $\rightarrow \delta'' = kd''\sin\theta = k(\frac{N}{3}+1)d'\sin\theta$

$= (\frac{N}{3}+1)\delta'$

$\therefore \quad \delta'(\tfrac{2}{3}N-2) + 2\times(\tfrac{N}{3}+1)\delta' = 2\pi$

2 angles ~~have~~ have values $(\frac{N}{3}+1)\delta'$ instead of δ',

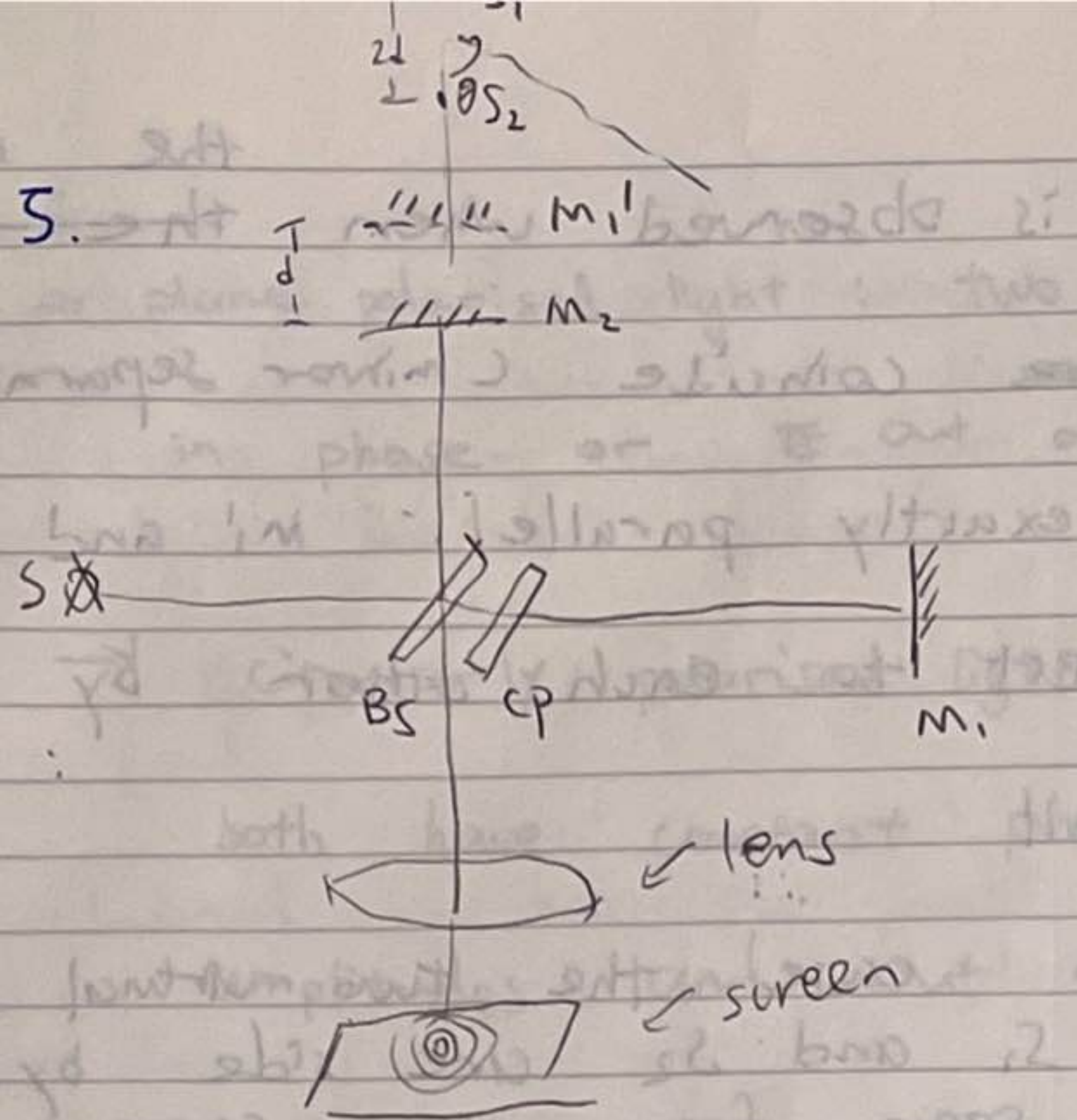
1 because of the plate, the other is where the ~~fini~~ final phasor ~~to meet thea~~ meets the initial phasor $\because$ all phasors have the same length $\therefore$ this angle must also be $(\frac{N}{3}+1)\delta'$

$\rightarrow \quad (\tfrac{4}{3}N)\,\delta' = 2\pi \quad \rightarrow \quad \delta' = \frac{3\pi}{2N} = \frac{3}{4}\left(\frac{2\pi}{N}\right)$

$= \frac{3}{4}\delta$

$\rightarrow$ QED

5.



BS = Beam Splitter

CP = Complementary plate

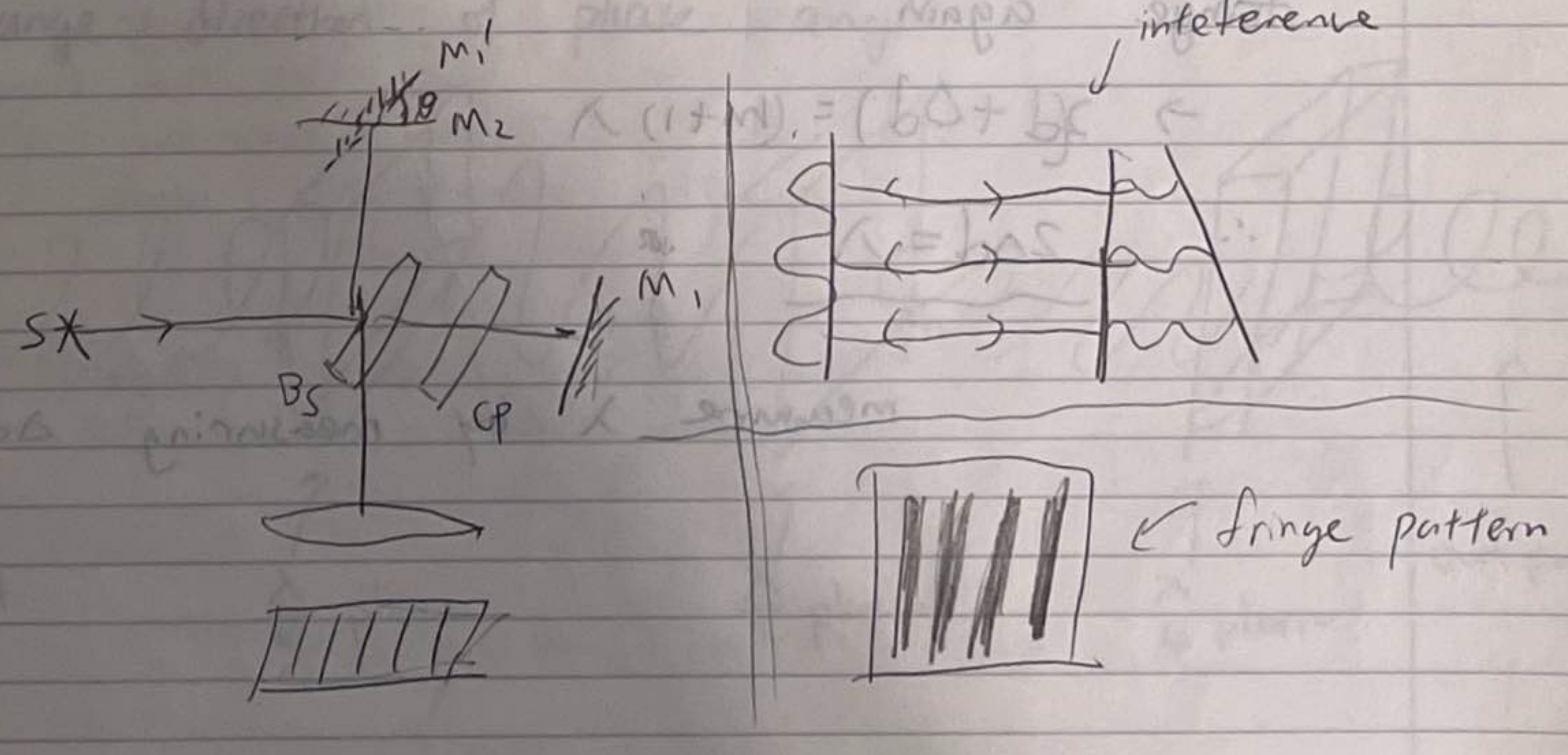
S = light source

M_1, M_2 = mirrors

M_1' = image of M_1 reflected by BS.

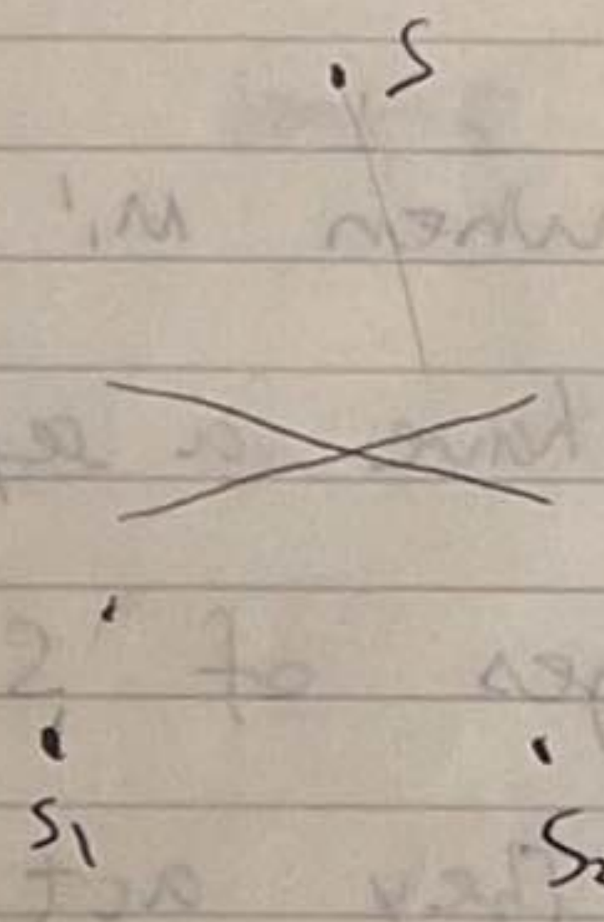
→ Circular fringes are formed when M_1' and M_2 are exactly parallel and have a separation d. S_1 and S_2 are images of S formed by M_1' and M_2 respectively. They act as ~~independent sou~~ individual sources and interfere at the screen. The fringe formed by S_1 and S_2 are axis-symmetric so the fringe is circular.

(dark fringe $2d\cos\theta = m\lambda$)



→ Linear fringes is observed when the centre ~~the two~~ M_1' and M_2 ~~are~~ coincide (mirror separation = 0) but are not exactly parallel. M_1' and M_2 are tilted with respect to each other by an angle θ that is small

~~If we~~ In this case the two virtual sources ~~are~~ S_1 and S_2 are side by side as seen from the screen so the interference patter is linear just like Young's slits.



In the circular fringe case, consider the central fringe

→ $2d = m\lambda$

increase d so that fringe appears and then ~~discrease~~ disappears such that we have a dark fringe again

→ $2(d + \Delta d) = (m+1)\lambda$

$\therefore$ $\underline{2\Delta d = \lambda}$

measure λ by measuring Δd

6.

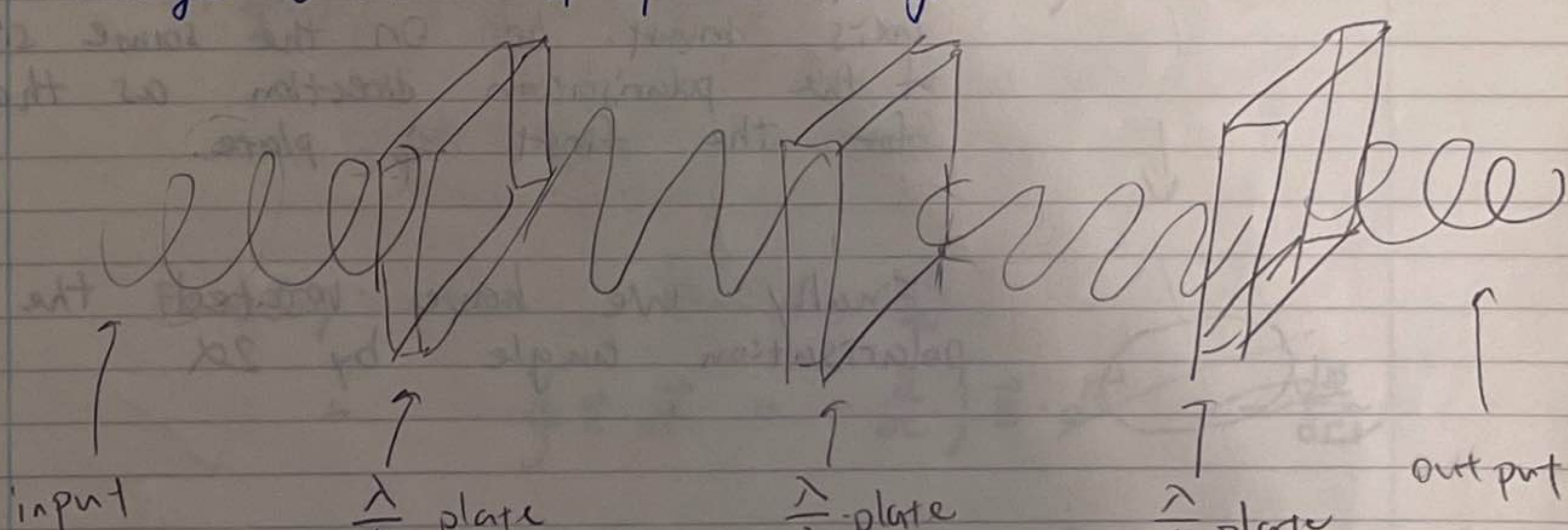
→ plane polarised light : two orthogonal components are in phase or ~~π~~ out of phase by $\pm\pi$.

→ circularly polarised light : two orthogonal components both have constant phase factor. They have equal amplitudes and out of phase by $\pm\frac{\pi}{2}$

→ Elliptically polarised light : two orthogonal components both have constant phase factor. They can either have unequal amplitudes , in which case phase difference $\delta \neq 0, \pm\pi$, or they can have equal amplitudes , in which case $\delta \neq 0, \pm\pi, \pm\frac{\pi}{2}$

→ Unpolarised light : the components of light have no definite phase relationship. ~~Polarisats~~ Instantaneous polarisation direction is random and varies with time

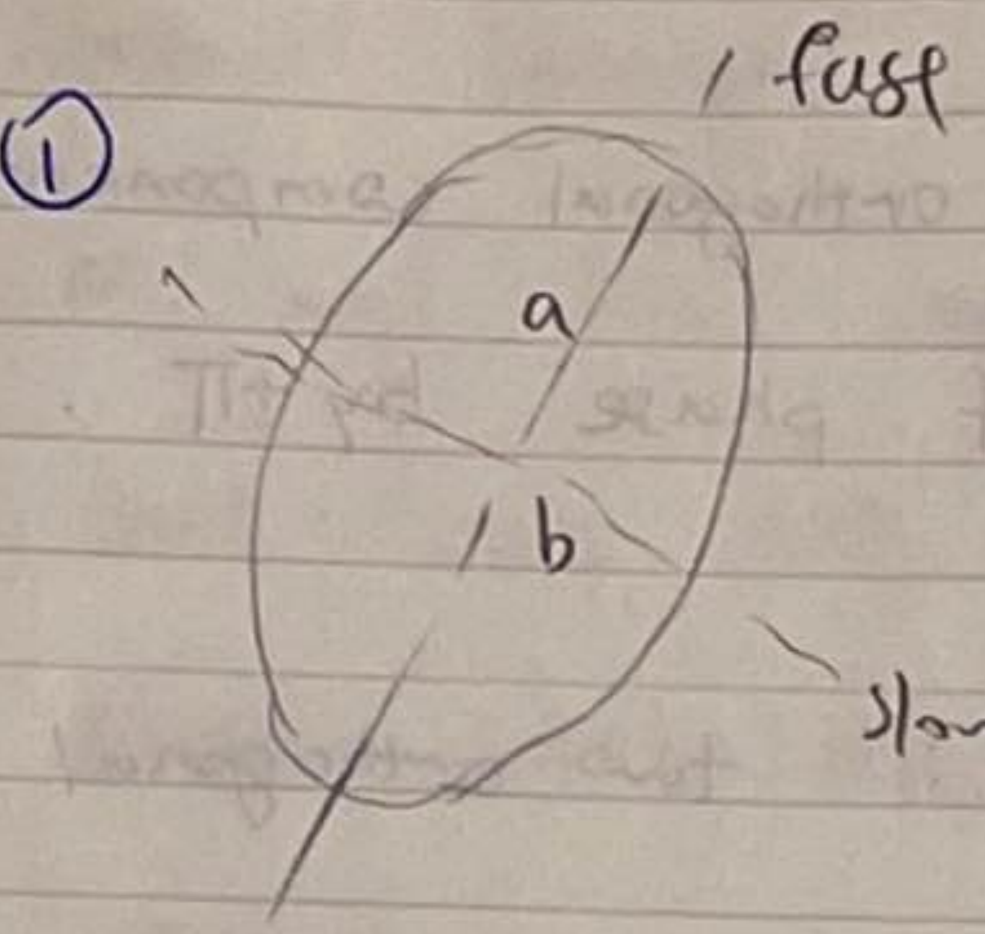
Change direction of phase angle :



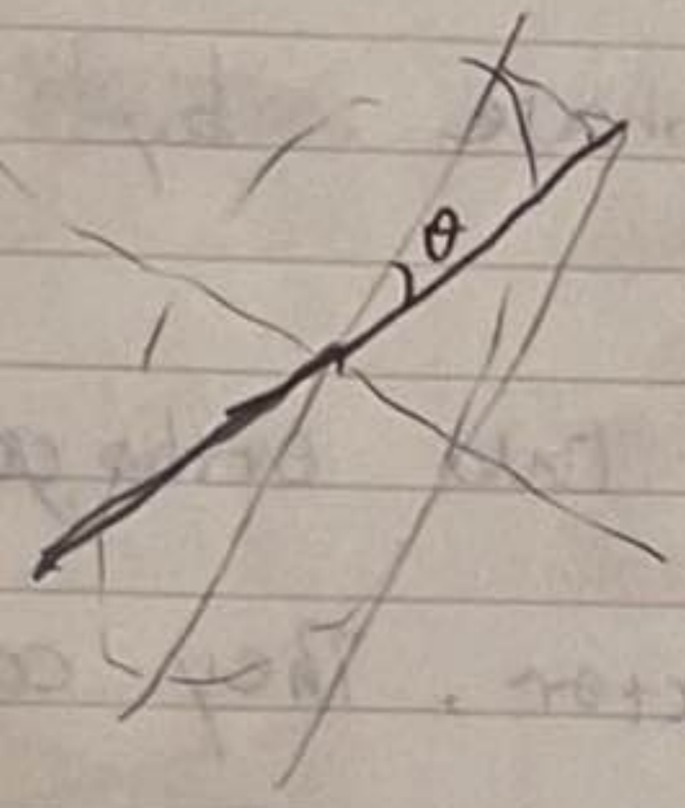
of the

↑ $\frac{\lambda}{4}$

①



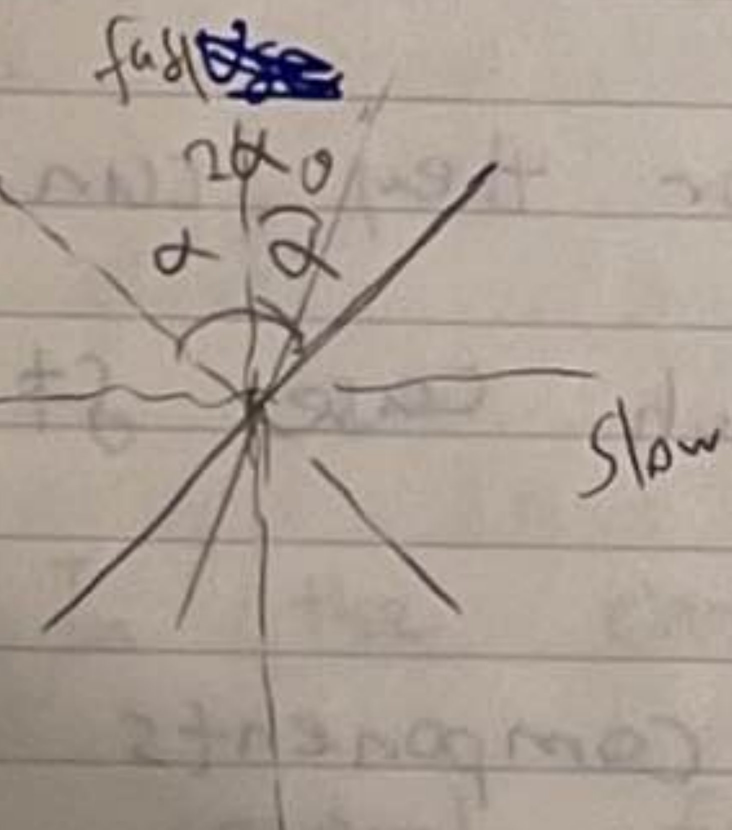
Orient the fast and slow axes such that they coincide with the major and minor axis of the elliptically polarised light



Then we get a linearly polarised light

Angle θ with respect to fast axis is $\tan\theta = \frac{b}{a}$

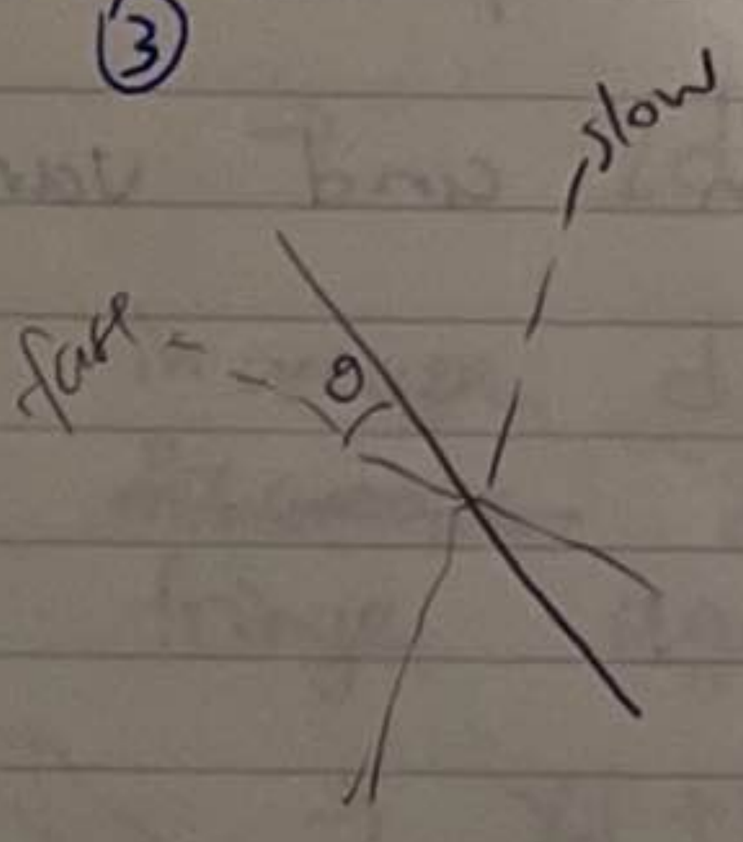
②



Orient the fast axis of $\frac{\lambda}{2}$ plate angle α w.r.t the linearly polarised light

Polarisation is rotated by 2α

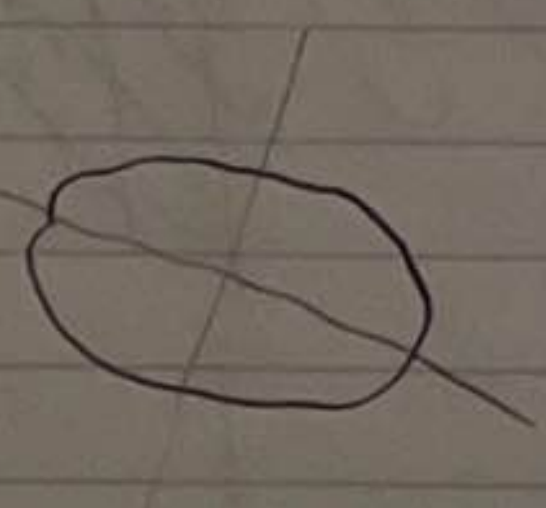
③



Now orient the second $\frac{\lambda}{4}$ plate such that its fast axis makes an angle θ w.r.t the polarisation direction (Angle 2α w.r.t the first $\frac{\lambda}{4}$ plate)

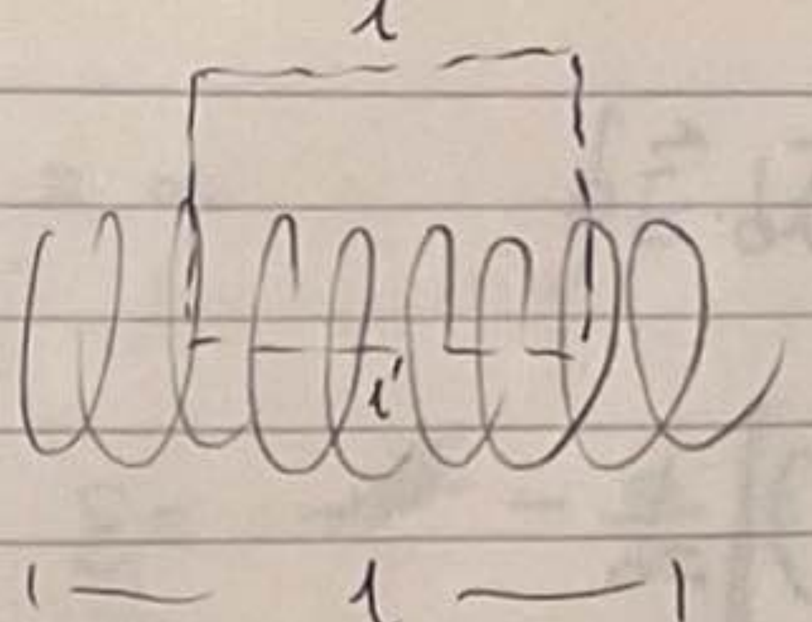
Note that in this case the fast axis must be on the same side of the polarisation direction as that for the first $\frac{\lambda}{4}$ plate.

⇓



Finally we have rotated the polarisation angle by 2α

7.



$n = \frac{N}{l}$

Ampere's Law $\quad \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$

$\rightarrow \quad B l' = \mu_0 n l' I \quad \rightarrow \quad B = \mu_0 n I$

Total flux for length l is

$$\Phi = (n l \pi a^2)(\mu_0 n I) = \mu_0 \pi a^2 n^2 I l$$

inductance per unit length:

$$L = \frac{\Phi}{I l} = \underline{\mu_0 \pi a^2 n^2} \checkmark$$

Energy density $\quad u_B = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{1}{2\mu_0}\mu_0^2 n^2 I^2 = \frac{1}{2}\mu_0 n^2 I^2$

Total energy in length l $\quad E = u_B V = u_B S l = \frac{1}{2}\mu_0 n^2 I^2 (\pi a^2) l$

Work done per unit length

$$W = \frac{E}{l} = \underline{\frac{1}{2}\mu_0 \pi a^2 n^2 I^2}$$

$\therefore \frac{1}{2} L I^2 = \frac{1}{2}\mu_0 \pi a^2 n^2 I^2$

$\therefore \quad \underline{W = \frac{1}{2} L I^2} \rightarrow$ QED

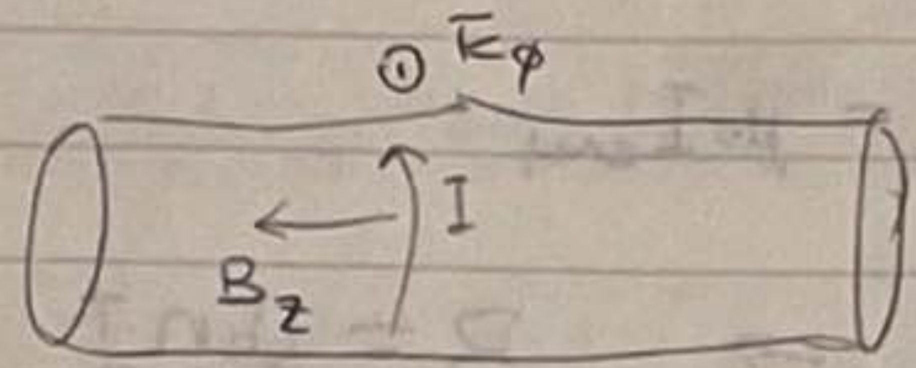
Faraday's Law $\quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

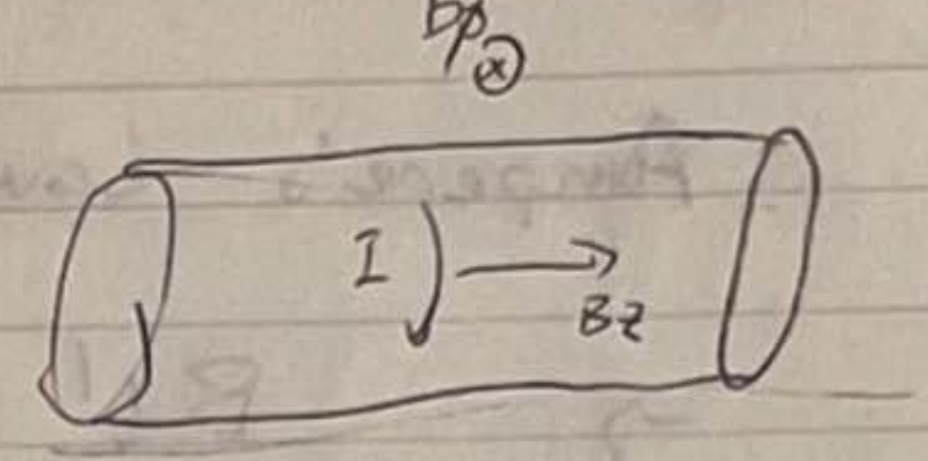
$\therefore \quad \oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{S}$ ~~$= -\frac{d\Phi}{dt}$~~

$\therefore \quad E_\phi (2\pi a) = (\pi a^2)\dot{B} = \pi a^2 \mu_0 n \dot{I}$

$\therefore \quad \underline{E_\phi = n\dot{I}\mu_0 a/2}$

$B_z = \mu_0 n I$





The poynting vector $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ always points <u>into</u> the solenoid

$\because \quad \vec{B} = B_z\hat{z}$, $\vec{E} = E_\phi\hat{\phi}$ and $|\hat{z}\times\hat{\phi}| = 1$

$\therefore \quad S = \frac{1}{\mu_0}E_\phi B_z = \frac{1}{\mu_0} n\dot{I}\mu_0\frac{a}{2}\mu_0 n I$

$$= \frac{1}{2}\mu_0 n^2 a I\dot{I}$$

Poynting vector is energy flux (energy per area per time)

$$\therefore \quad E = \oint dA\int_0^t dt\ \frac{1}{2}\mu_0 n^2 a I\frac{dI}{dt}$$

$$= (2\pi a l)\int_0^t dt\cdot\frac{1}{2}\mu_0 n^2 a I\frac{dI}{dt}$$

~~energ~~ work per length $W = \frac{E}{l}$

$$\therefore \quad W = 2\pi a\cdot\frac{1}{2}\mu_0 n^2 a\underbrace{\int I\,dI}_{\frac{1}{2}I^2}$$

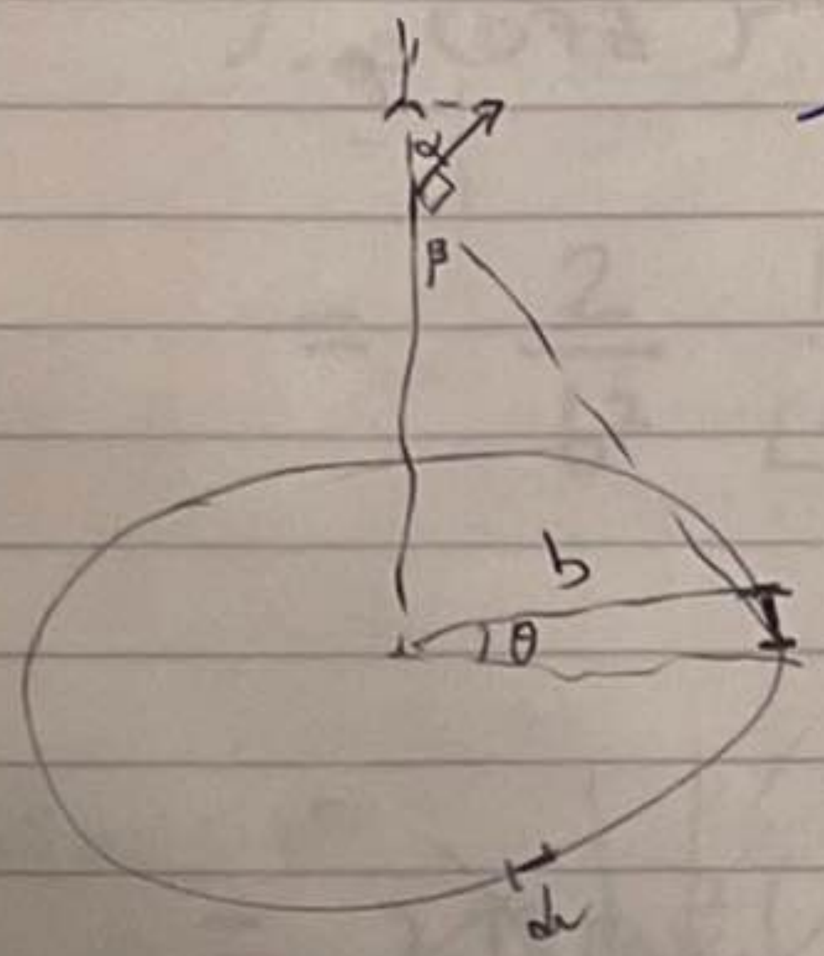
$$= \frac{1}{2}\mu_0\pi a^2 n^2 I^2 \rightarrow \text{consistent.}$$

$$-\frac{dI}{dt} = \alpha \qquad \oint \vec{E}\cdot d\vec{l} = \mathcal{E} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{A}$$

$$\therefore \quad \mathcal{E} = -\frac{d}{dt}\left[(\mu_0 n I)(\pi a^2)\right]$$

$$= \mu_0 n \alpha \pi a^2$$

$$I_1 = \frac{\mathcal{E}}{R} = \frac{\mu_0 n \alpha \pi a^2}{R}$$



Biot-Savart : $d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times \hat{r}}{r^2}$

By symmetry $\vec{B}$ is along $\hat{z}$

$$\therefore \quad B = \cos\alpha \int dB$$

$$\int dB = \cancel{\frac{\mu_0 I}{4\pi}} \frac{\mu_0 I}{2\cancel{4\pi}(z^2+b^2)}(2\pi b)$$

$$\cos\alpha = \sin(\beta) = \frac{b}{\sqrt{b^2+z^2}}$$

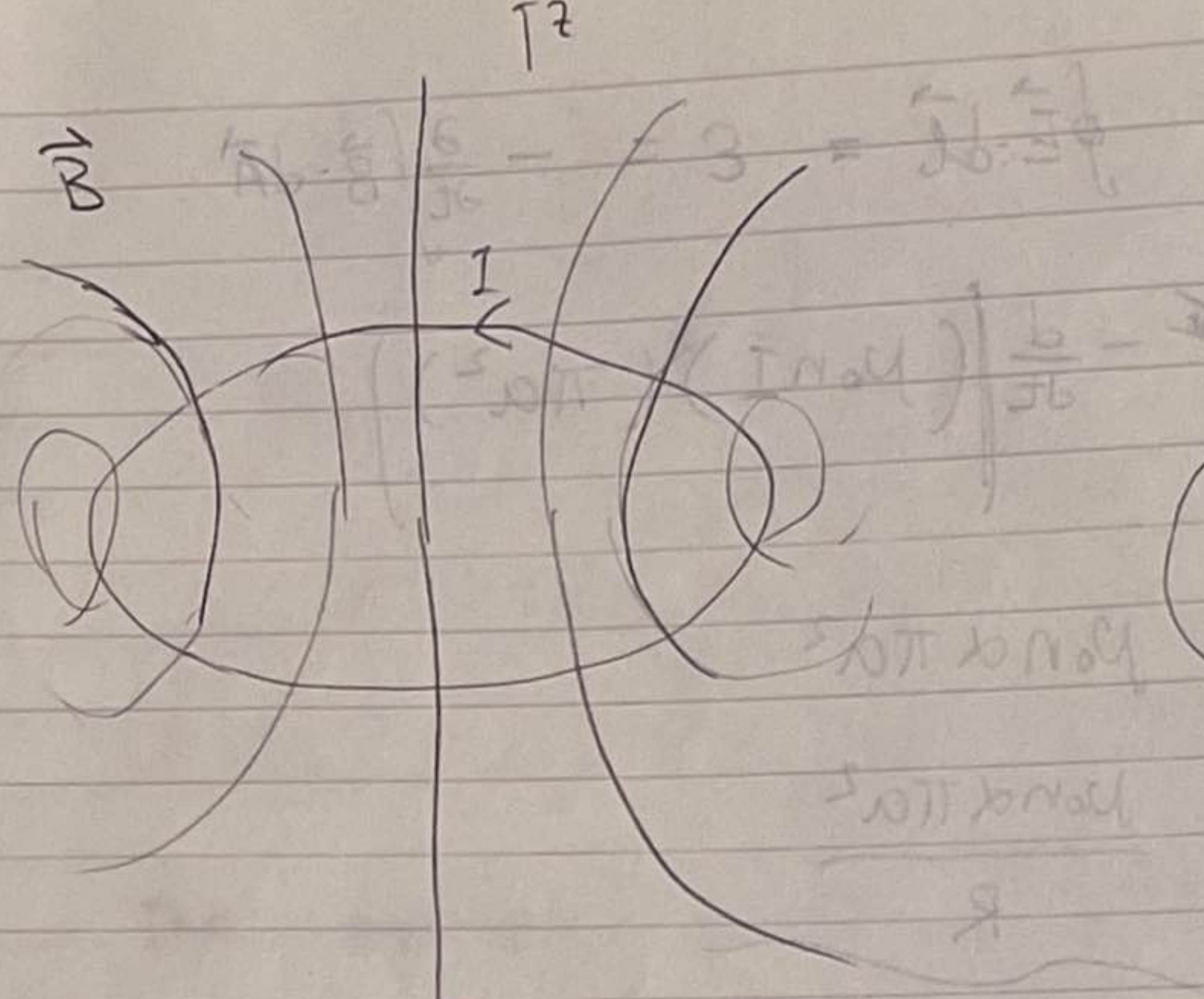
$$\therefore \quad \vec{B} = \frac{\mu_0 I_1 b^2 \hat{z}}{2(b^2+z^2)^{3/2}}$$

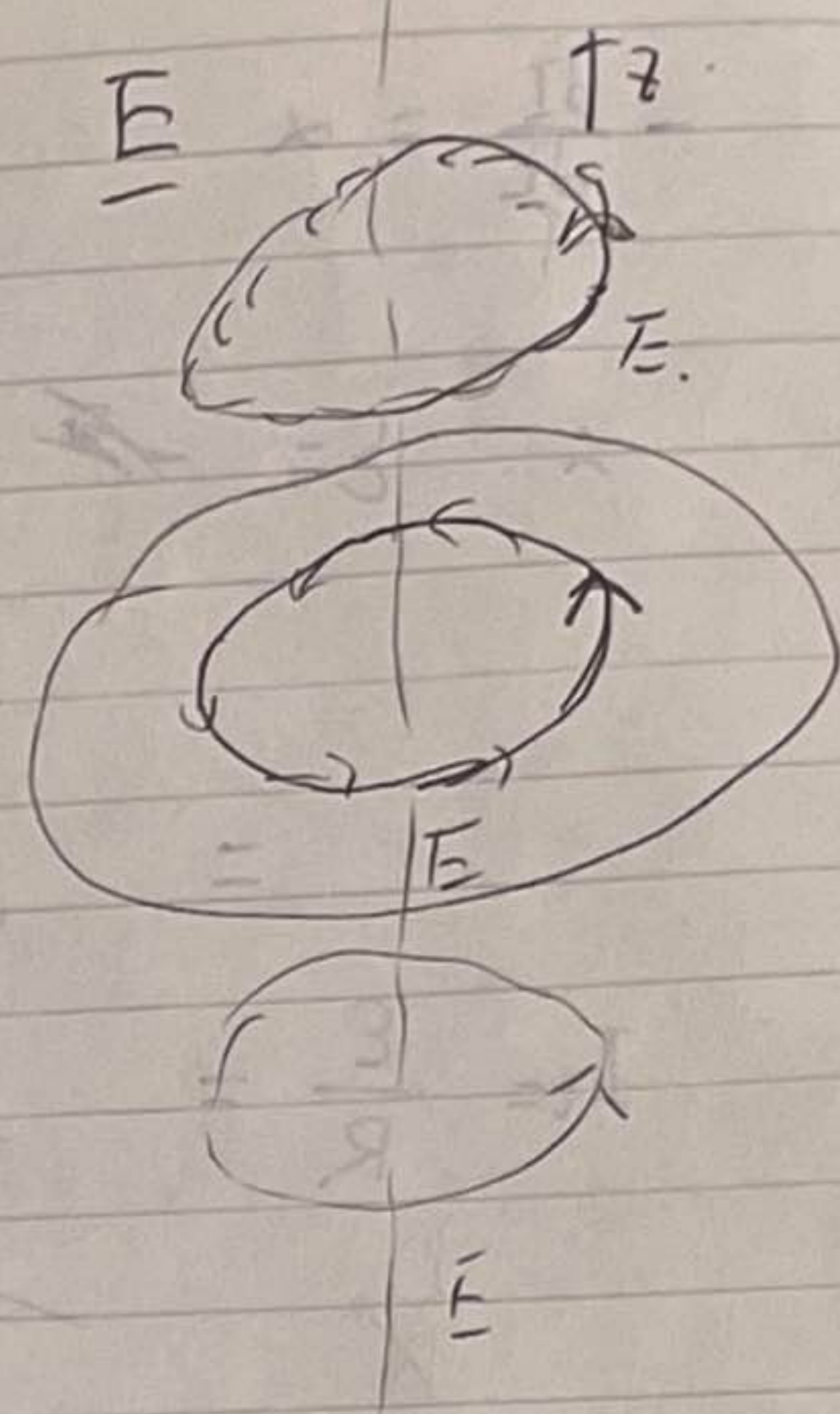
$$E_\phi = n\dot{I}\mu_0 a/2 = n\alpha\mu_0 a/2$$

$$\because \hat{z}\cdot\hat{\phi} = 0$$

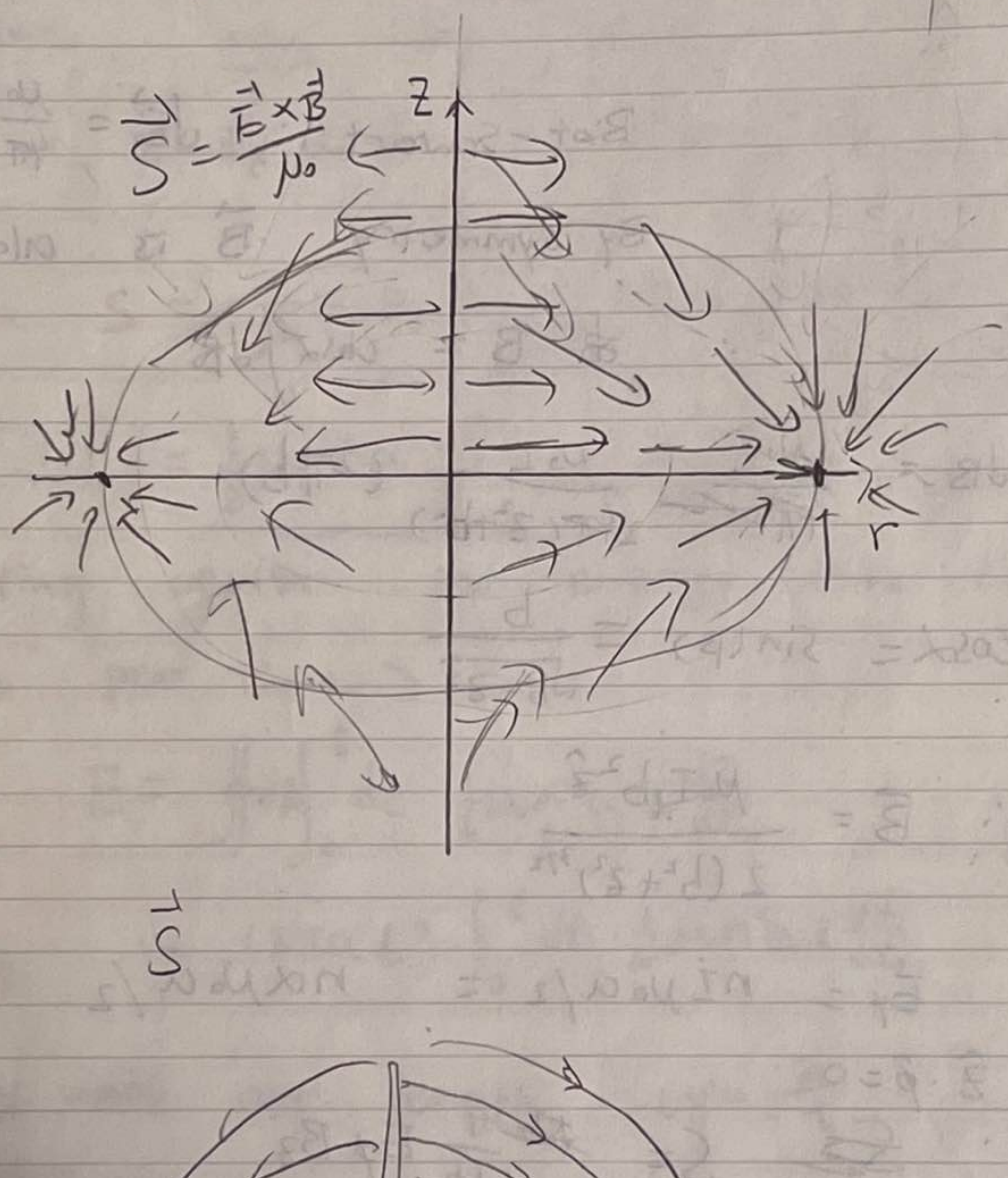
$$\therefore \quad S = \frac{1}{\mu_0} E_\phi B_z$$

$$\rightarrow \quad S = \frac{\mu_0 n a b^2 I_1 \alpha}{4(b^2+z^2)^{3/2}}$$

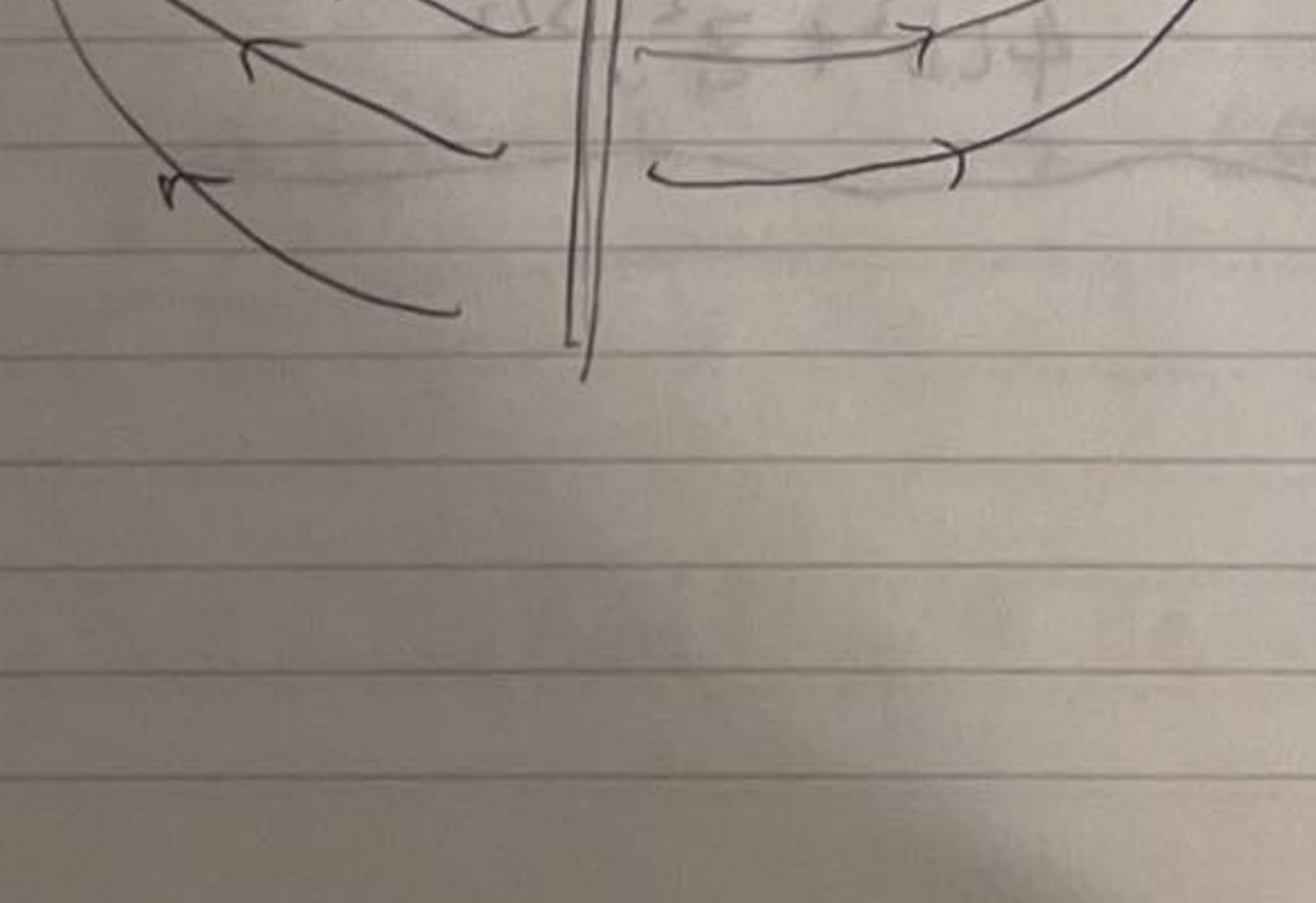




$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$



$\vec{S}$



$$\int \vec{S}\cdot d\vec{A} = \int S\,dA = 2\pi a \int S\,dl = 2\pi a \int S\,dz$$

$$= 2\pi a \int_{-\infty}^{\infty} \left(\frac{\mu_0 n a b^2 I_t \alpha}{4}\right) \frac{dz}{(b^2+z^2)^{3/2}}$$

$$\int_{-\infty}^{\infty} \frac{dz}{(b^2+z^2)^{3/2}} = \left[\frac{z}{(b^2)(b^2+z^2)^{1/2}}\right]_{-L\to-\infty}^{L\to\infty}$$

$$= \frac{2}{b^2} \lim_{L\to\infty} \frac{L}{(b^2+L^2)^{1/2}} = \frac{2}{b^2}$$

$$= \cancel{2}\pi a \left(\frac{\mu_0 n a \cancel{b^2} I_t \alpha}{\cancel{4}}\right) \cdot \frac{\cancel{2}}{\cancel{b^2}}$$

$$= \underline{\pi a^2 \mu_0 n I_t \alpha}$$

$$= (I_t R)\cdot I_t = \underline{I_t^2 R}$$

$I_t R = \mu_0 n \alpha \pi a^2$

$\rightarrow$ QED

8. No free charges ~~currents~~. linear, isotropic, homogeneous

~~$\vec{\nabla}\cdot\vec{E}$~~

~~$\vec{\nabla}\cdot\vec{E}=0$, $\vec{\nabla}\cdot\vec{B}=0$, $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B}=\mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial\vec{E}}{\partial t}$~~

~~$\vec{\nabla}\times$~~

$$\vec{\nabla}\cdot\vec{D}=\rho_{free}=0 \qquad \vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$$

$\vec{B}=\mu_0\mu_r\vec{H}$

$$\hookrightarrow \vec{\nabla}\times\vec{B}=\mu_0\mu_r\vec{J}_f+\mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial\vec{E}}{\partial t}$$

Ohm's Law: $\vec{J}_f=\sigma\vec{E}$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0})-\vec{\nabla}^2\vec{E}=-\vec{\nabla}^2\vec{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=-\mu_0\mu_r\sigma\frac{\partial\vec{E}}{\partial t}-\mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\rightarrow \vec{\nabla}^2\vec{E}-\sigma\mu_0\mu_r\frac{\partial\vec{E}}{\partial t}-\varepsilon_0\varepsilon_r\mu_0\mu_r\frac{\partial^2\vec{E}}{\partial t^2}=0$$

Motion of electron

let $\underline{E}(\underline{x},t)=$ ~~$\underline{E}_0e^{i\omega t}$~~ $\underline{E}_0e^{i(-\underline{k}\cdot\underline{x}+\omega t)}$

$\approx \underline{E}_0e^{i\omega t}$ for small displacement $\underline{x}$

$$m\ddot{\underline{x}}=-\xi\dot{\underline{x}}-e\underline{E}(\underline{x},t)$$

$$\therefore m\ddot{\underline{x}}+\xi\dot{\underline{x}}+e\underline{E}_0e^{+i\omega t}=0$$

try $\underline{x}=\underline{x}_0e^{+i\omega t}$ $(\underline{x}_0 \parallel \underline{E}_0)$ in steady state

$$\therefore (-\omega^2m-i\omega\xi)\underline{x}_0e^{+i\omega t}+e\underline{E}_0e^{+i\omega t}=0$$

$$\therefore (-\omega^2m+i\omega\xi)\underline{x}_0=-e\underline{E}_0$$

$$\therefore \underline{x}=\frac{e}{-m\omega^2+i\omega\xi}\underline{E}_0e^{+i\omega t}$$

$$\dot{\underline{x}}=-\frac{i\omega e}{-m\omega^2+i\omega\xi}E_0e^{+i\omega t}$$

$$=\frac{-i\omega e}{-m\omega^2+i\omega\xi}\underline{E}$$

$\cancel{E = I = J \cdot S \; d}$

$$I = \int \underline{J} \cdot d\underline{S} = \frac{dq}{dt} = \cancel{-env} \; \frac{d}{dt}\int -en \, d\tau$$

$$= \int -en \frac{d\vec{x}}{dt} \cdot d\vec{s}$$

$$\rightarrow \quad \underline{J} = -en \frac{d\vec{x}}{dt} = -en \underline{\dot{x}}$$

$$\rightarrow \underline{J} = -\frac{-in\omega e^2}{-m\omega^2 + i\omega\xi} \underline{E}$$

$$\therefore \sigma = -\frac{-in\omega e^2}{-m\omega^2 + i\omega\xi} \frac{(-m\omega^2 - i\omega\xi)}{(-m\omega^2 - i\omega\xi)}$$

$$= \frac{1}{(m\omega^2)^2 + (\omega\xi)^2} (n\omega^2 e^2 \xi - i n\omega^3 e^2 m)$$

$$= \frac{(ne^2/\xi)}{1 + \left(\frac{m\omega^2}{\omega\xi}\right)^2} \left(1 - i\frac{\omega m}{\xi}\right)$$

let $\sigma_0 = \frac{ne^2}{\xi}$ then $\frac{m\omega^2}{\xi} = \left(\frac{ne^2}{\xi}\right)\left(\frac{m\omega^2}{ne^2}\right) = \frac{\sigma_0 m\omega^2}{ne^2}$

$$\rightarrow \quad \sigma = \frac{\sigma_0}{1 + (\sigma_0 m\omega/ne^2)^2}\left[1 - i\left(\frac{\sigma_0 m\omega}{ne^2}\right)\right].$$

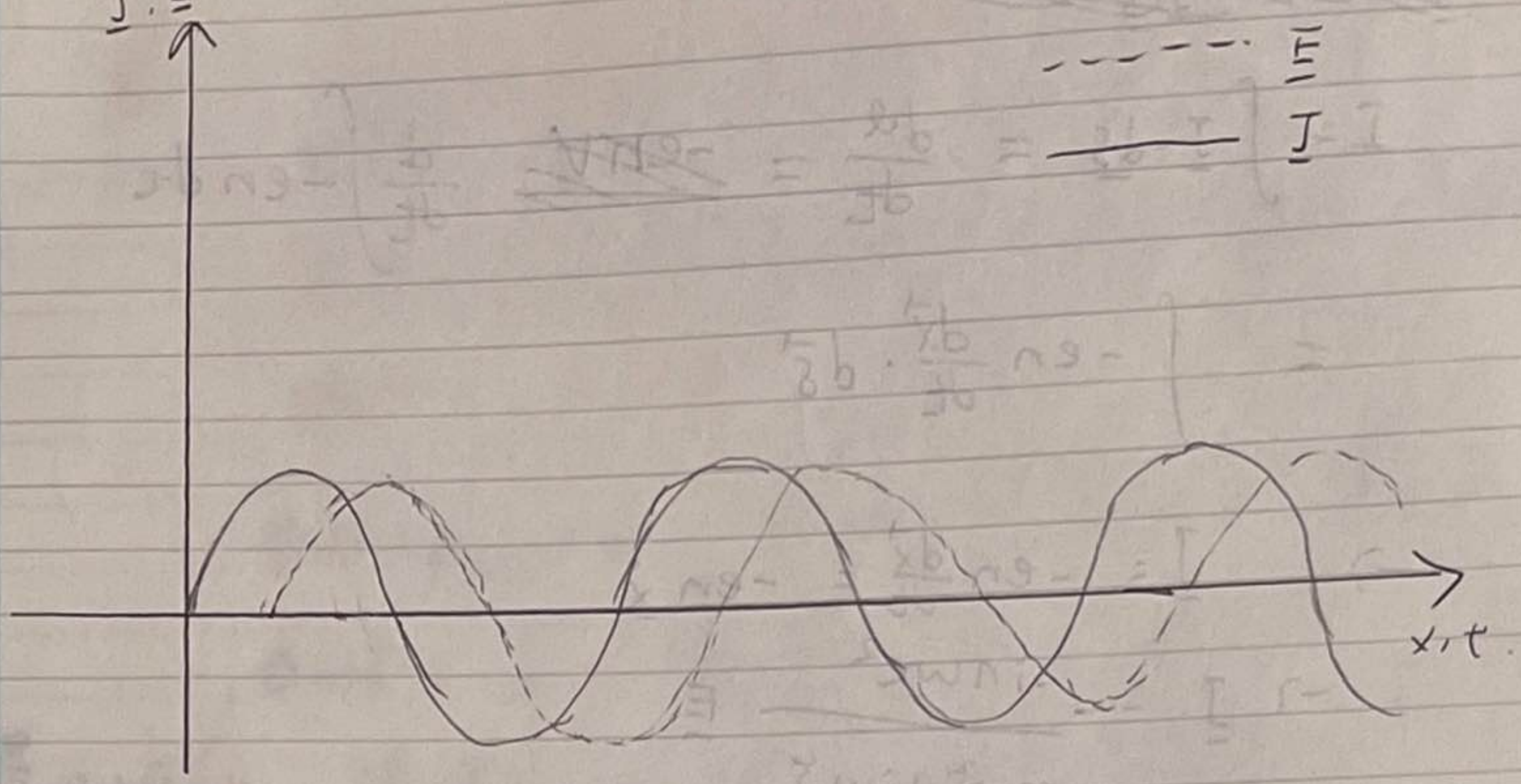
~~low density plasma consider $E(x,t) = E_0 e^{i(kx + \omega t)}$~~

When $\xi = m\omega$, then $\frac{\sigma_0 m\omega^2}{ne^2} = \frac{m\omega^2}{\xi} = \frac{m\omega}{m\omega} = 1$

$\therefore \sigma = \frac{\sigma_0}{2}(1 - i)$ the phase angle θ is

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4} = -45^\circ$$

$\therefore \underline{J} = \sigma \underline{E} \quad \therefore \underline{J}$ lags $\underline{E}$ by $\frac{\pi}{4}$



$$\underline{E}(\underline{x}, t) = \underline{E_0} \exp(i(-\underline{k} \cdot \underline{x} + \omega t))$$

For low density plasma. ω is large
$(\gamma \ll m\omega)$ $\quad$ $\varepsilon_r \approx \mu_r \approx 1$

$$\therefore \frac{\sigma_0 m\omega}{ne^2} = \frac{m\omega}{\gamma} \gg 1$$

$$\therefore \sigma = \frac{\sigma_0}{1 + (\sigma_0 m\omega / ne^2)^2} \left(1 - i\left(\frac{\sigma_0 m\omega}{ne^2}\right)\right)$$

$$\approx \frac{-\sigma_0}{\left(\frac{\sigma_0 m\omega}{ne^2}\right)^2} i\left(\frac{\sigma_0 m\omega}{ne^2}\right)$$

$$= -\frac{\cancel{\sigma_0} i}{\frac{\cancel{\sigma_0} m\omega}{ne^2}} = -\frac{ine^2}{m\omega}$$

$$\therefore \nabla^2 \vec{E} = \sigma \mu_0 \frac{\partial \vec{E}}{\partial t} + \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\therefore -k^2 \cancel{\vec{E}} = -\frac{ine^2}{m\omega}(\mu_0 i\omega) + \varepsilon_0 \mu_0 (-\omega^2)$$

$$\therefore \frac{k^2}{\varepsilon_0 \mu_0} = -\frac{ne^2}{m\varepsilon_0} + \omega^2$$

$$\because c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \qquad \omega_p^2 = \frac{ne^2}{m\varepsilon_0}$$

$$\therefore \underline{c^2 k^2 = \omega^2 - \omega_p^2}$$

If $\omega < \omega_p$, then $k^2 < 0$

$\rightarrow$ k is purely imaginary $\rightarrow$ $k = ik'$ $(k' \in \mathbb{R})$

$$\underline{E}(x,t) = \underline{E_0}\exp(i(-kx+\omega t))$$

$$= \underline{E_0}\exp(k'x)\exp(i\omega t)$$

this wave does not propagate

$\therefore$ The ~~wave~~ wave can propagate ~~can~~ only if

$\omega > \omega_p$

~~ϕ~~ ~~$\partial\partial$~~ $c^2k^2 = \omega^2 - \omega_p^2$ $\rightarrow$ $k = \frac{1}{c}\sqrt{\omega^2-\omega_p^2}$

$\therefore$ $2c^2k\,dk = 2\omega\,d\omega$ $\rightarrow$ group velocity $\frac{d\omega}{dk} = V_g$

$$V_g = \frac{d\omega}{dk} = \frac{c^2k}{\omega} = \frac{c}{\omega}\sqrt{\omega^2-\omega_p^2} = \boxed{c\sqrt{1-\frac{\omega_p^2}{\omega^2}}}$$

Radio wave:

$\omega_1 = 200$ Hz. $\omega_2 = 400$ Hz

$\omega_2 = 2\omega_1$ $L = 530$ light years $= 5.01\times10^{18}$ m

~~t_1~~ $\rightarrow$ ω_p should be small $(\omega_p \ll \omega_1, \omega_2)$

$t_1 = \frac{L}{V_{g_1}}$ $t_2 = \frac{L}{V_{g_2}}$ $\Delta t = 5$ s

$$\therefore \Delta t = |t_1 - t_2| = \frac{L}{c}\left[\left(1-\frac{\omega_p^2}{\omega_1^2}\right)^{-\frac{1}{2}} - \left(1-\frac{\omega_p^2}{\omega_2^2}\right)^{-\frac{1}{2}}\right]$$

$$\approx \frac{L}{c}\left(\cancel{1} + \frac{\omega_p^2}{2\omega_1^2} - \cancel{1} - \frac{\omega_p^2}{2\omega_2^2}\right)$$

$$= \frac{L\omega_p^2}{2c}\left(\frac{1}{\omega_1^2} - \frac{1}{\omega_2^2}\right) = \frac{L\omega_p^2}{2c}\left(\frac{1}{\omega_1^2} - \frac{1}{4\omega_1^2}\right)$$

$$\rightarrow \Delta t = \frac{3L\omega_p^2}{8c\omega_1^2} \rightarrow \omega_p = \left(\frac{8c\omega_1^2\Delta t}{3L}\right)^{1/2}$$

$$\omega_p = \left(\frac{8(3\times10^8)(200\times10^6)^2(5)}{3(5.01\times10^{18})}\right)^{\frac{1}{2}} = (3.19\times10^7)^{1/2}$$

$$= 5.65\times10^3 \text{ Hz}$$

Indeed $\omega_p << \omega_1$,

$$\omega_p^2 = \frac{ne^2}{m\varepsilon_0} \rightarrow n = \frac{m\varepsilon_0\omega_p^2}{e^2}$$

$$n = \frac{(9.11\times10^{-31})(8.854\times10^{-12})(3.19\times10^7)}{(1.6\times10^{-19})^2}$$

$$\approx \boxed{1\times10^4} \ \cancel{m^{-3}}$$

1.6 cm^{-3} ?

$$3\times10^{-10} = \frac{ne^2}{2m\varepsilon_0}\left(\frac{1}{(2\pi\times200\times10^6)^2} - \frac{1}{(2\pi\times400\times10^6)^2}\right)$$

$$\therefore \frac{ne^2}{2m\varepsilon_0} = 6.317\times10^8$$

$$\therefore n = \frac{2(9.11\times10^{-31})(8.854\times10^{-12})(6.317\times10^8)}{(1.6\times10^{-19})^2}$$

$$= 3.98\times10^5 \text{ m}^{-3} \quad ?$$

9.

N slits, incident function $T(x) = \sum_{m=1}^{N} \delta(x-md)$

Fourier transform of $T(x)$ is the Fraunhofer diffraction amplitude distribution.

$$A(u) \propto \int_{-\infty}^{\infty} T(x) e^{iux} dx \qquad (u = k\sin\theta)$$

$$= \int_{-\infty}^{\infty} \sum_{m=1}^{N} \delta(x-md) e^{iux} dx$$

$$= \sum_{m=1}^{N} e^{iumd} = e^{iud} + e^{2iud} + \cdots + e^{Niud}$$

$$= \frac{e^{iud}(1-e^{Nidu})}{1-e^{iud}} = \frac{e^{iud} e^{i\frac{N}{2}ud}}{e^{\frac{iud}{2}}} \left(\frac{e^{\frac{N}{2}iud} - e^{-\frac{N}{2}iud}}{e^{\frac{iud}{2}} - e^{-\frac{iud}{2}}} \right)$$

$$= e^{i\frac{N+1}{2}ud} \cancel{\frac{2i}{2i}} \frac{\sin(Nud/2)}{\sin(ud/2)}$$

$$\propto \frac{\sin(Nud/2)}{\sin(ud/2)}$$

$$I(\theta) \propto |A(u)|^2 = \left(\frac{\sin(Nud/2)}{\sin(ud/2)} \right)^2$$

$$\rightarrow \quad I(\theta) = C \left(\frac{\sin(Nud/2)}{\sin(ud/2)} \right)^2 \qquad (C \text{ is constant})$$

$$\because \quad I(\theta)_{max} = I_0 = I(\theta = 0)$$

$$\therefore \quad I_0 = C \cdot \lim_{u\to 0} \frac{\sin(Nud/2)}{\sin(ud/2)} = C \cdot \left(\frac{Nud/2}{ud/2} \right)^2 = CN^2$$

$$\rightarrow \quad C = \frac{I_0}{N^2}$$

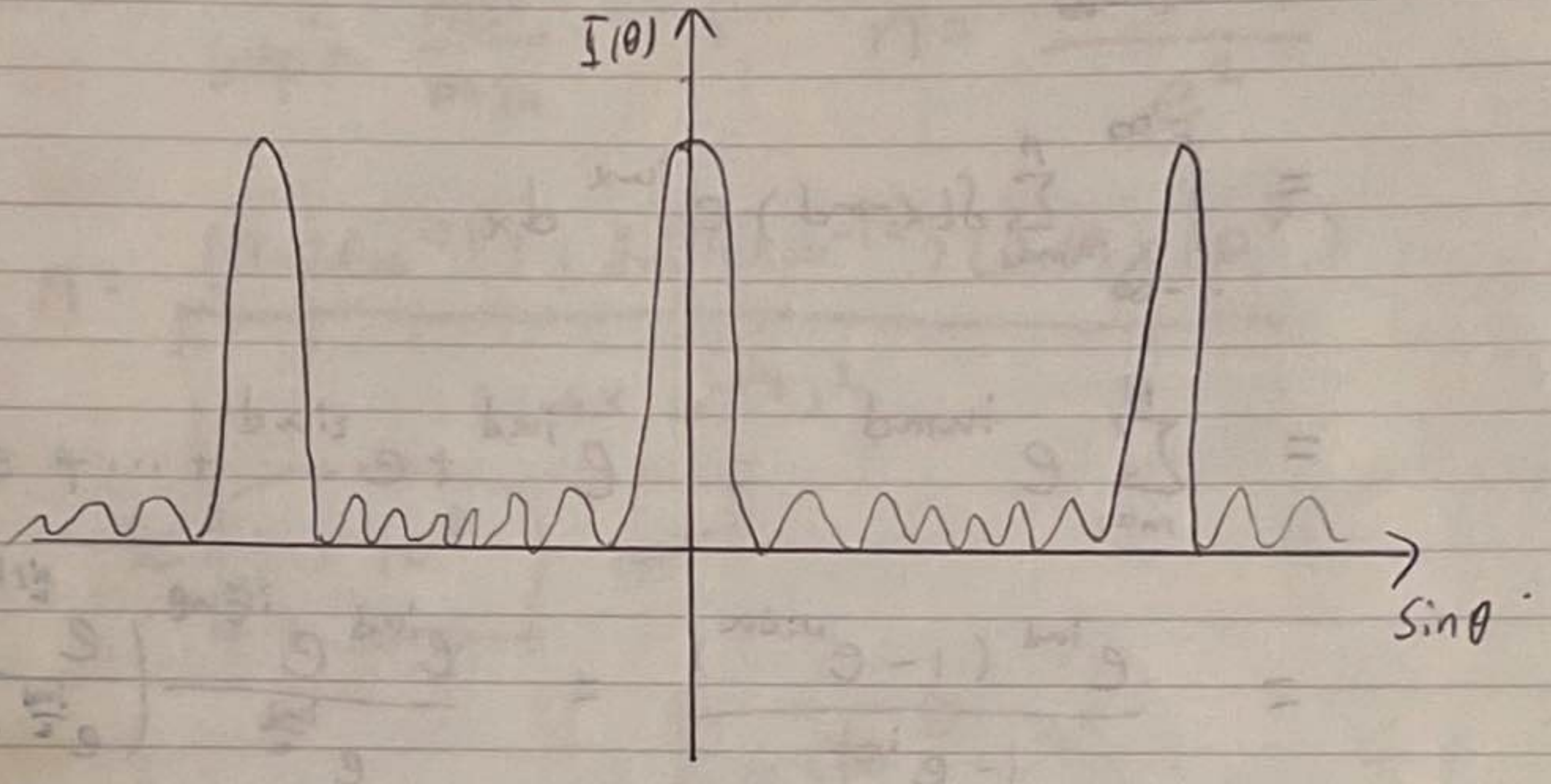
$$\rightarrow \quad I(\theta) = \frac{I_0}{N^2} \frac{\sin^2(Nud/2)}{\sin^2(ud/2)}$$

At maximum intensity we simply add the amplitude from each slit because they are all in phase

$$A_N = N A_1 \quad \because \; I \propto |A|^2$$

$$I_0(N) = \cancel{N^2 |A_1|^2} \; |A_N|^2 = N^2 |A_1|^2$$

$$\therefore \; \underline{I_0 \propto N^2}$$



Convolution "$*$" $\Rightarrow$ if $h = f * g$

then $$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx'$$

~~Convolution theorem~~

Fourier transform of f is $T_F(f)$

$$\rightarrow T_F(w) = T_F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{iwx}\, dx$$

Convolution theorem:

$$T_F(f * g) = T_F(f) \times T_F(g)$$

Proof:

$$T_F(f * g) = \int_{-\infty}^{\infty} (f(x) * g(x))\, e^{iwx}\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cancel{f(x)}\, f(x-x')\, g(x')\, dx'\, e^{iux}\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x')\, g(x')\, dx'\, e^{iu(x-x')}\, e^{iux'}\, dx$$

$$= \cancel{\int_{-\infty}^{\infty} g(x')\, dx' \int_{-\infty}^{\infty} f(x-x')\, e}$$

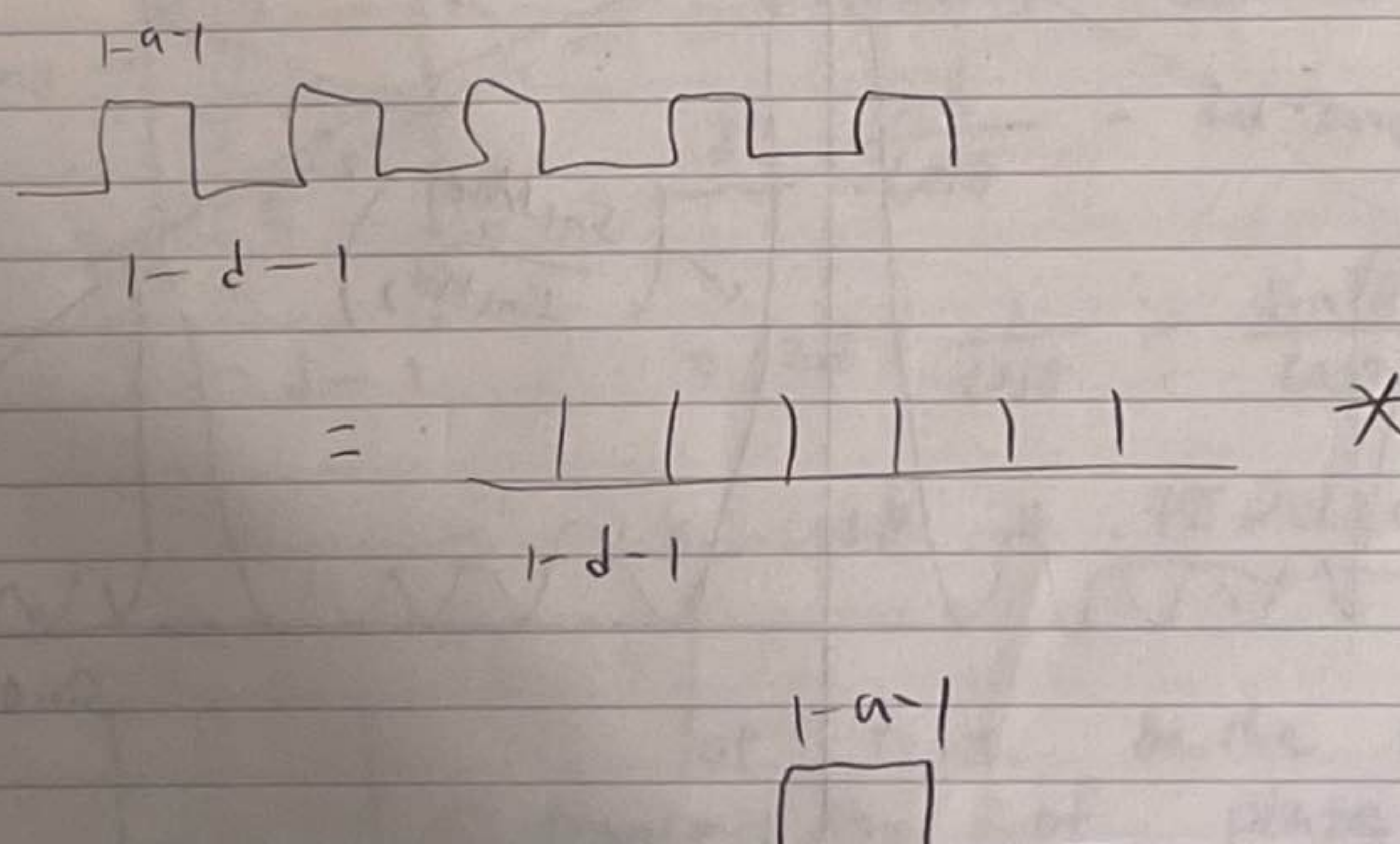
$$= \int_{-\infty}^{\infty} g(x')\, e^{iux'}\, dx' \int_{-\infty}^{\infty} f(x-x')\, e^{iu(x-x')}\, dx$$

$$= \int_{-\infty}^{\infty} g(x')\, e^{iux'}\, dx' \int_{-\infty}^{\infty} f(y)\, e^{iuy}\, dy$$

integrating w.r.t x, x' is a constant
$\therefore$ let $y = x - x'$
then $dy = dx$

$$= \cancel{T_F(f) * T_F(g)} \quad T_F(f) \times T_F(g)$$

QED.



~~TF~~ Fourier Transform of ~~a~~ the single slit is

$$B(u) \propto \int_{-a/2}^{a/2} e^{iux}\, dx = \frac{1}{iu}\left(e^{iua/2} - e^{-iua/2}\right)$$

$$= \frac{2i\sin\left(\frac{ua}{2}\right)}{iua} \cdot a = a\left(\frac{\sin\left(\frac{ua}{2}\right)}{\frac{ua}{2}}\right)$$

The overall amplitude, by convolution theorem, is the product of $A(u)$ and $B(u)$

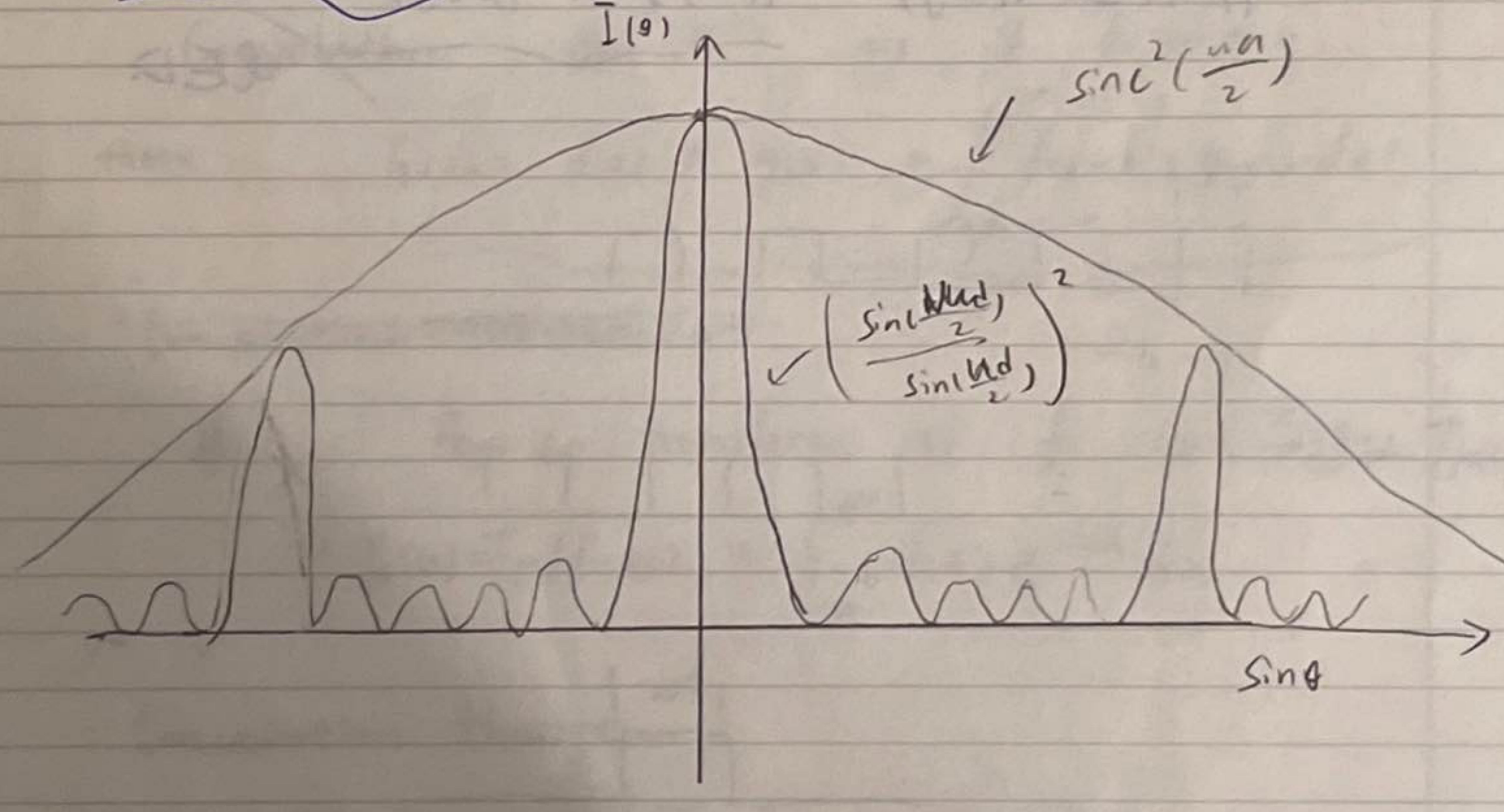
$$\therefore \quad C(u) \propto A(u)\,B(u) = \cancel{\sin} \frac{\sin(\frac{Nud}{2})}{\sin(\frac{ud}{2})} \frac{\sin(\frac{ua}{2})}{\frac{ua}{2}}$$

$\because$ the peak of $B(u)$ is

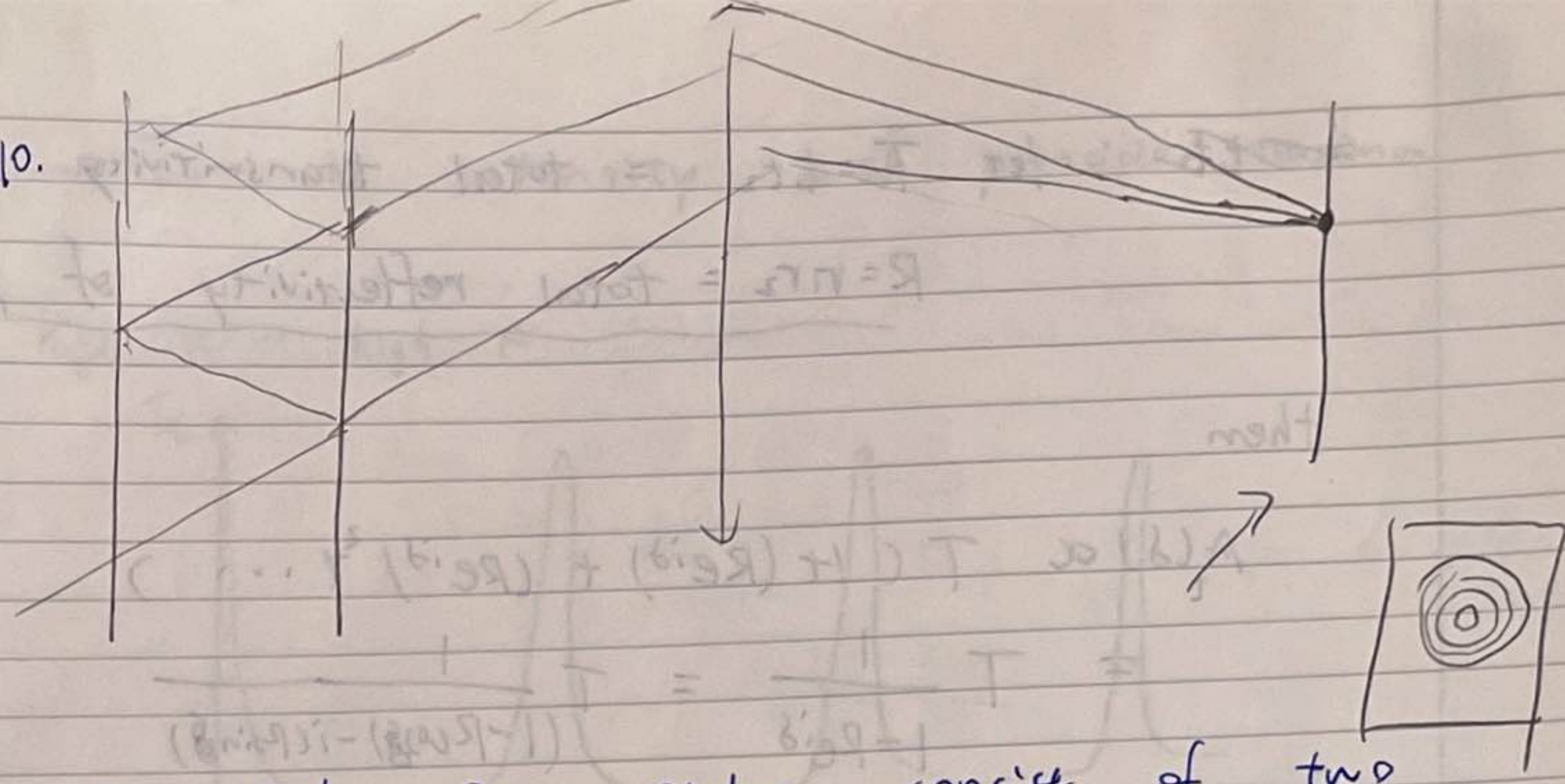
$$\lim_{u \to 0} \frac{\sin(\frac{ua}{2})}{\frac{ua}{2}} = 1 \quad \text{, and} \quad I \propto |A|^2$$

$\therefore$ overall intensity

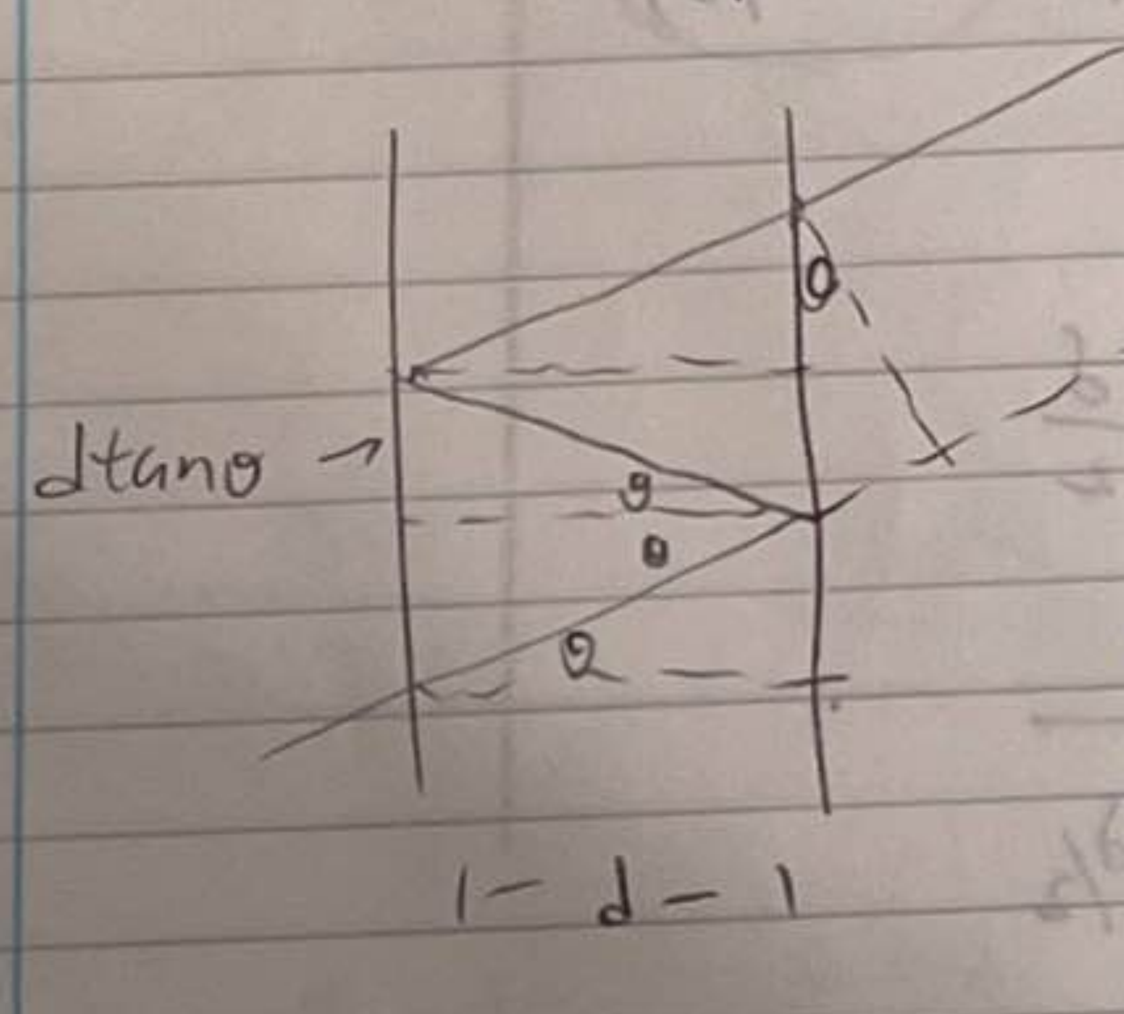
$$I(\theta) = \frac{I_0}{N^2}\left(\frac{\sin(\frac{Nud}{2})}{\sin(\frac{ud}{2})}\right)^2 \left(\frac{\sin(\frac{ua}{2})}{\frac{ua}{2}}\right)^2 \quad (u = k\sin\theta)$$



10.



The Fabry-Perot etalon consists of two parallel plates that can both reflect and transmit light. The ~~the~~ two plates turn the incident beam into multiple beams with certain phase difference and ~~they~~ interfere to give circular fringes.

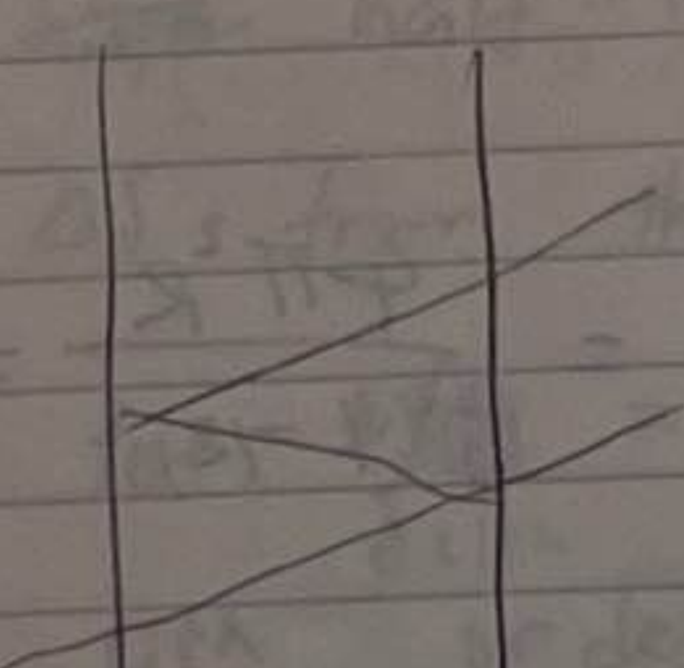


phase difference between two adjacent beams is

$$\delta = \left(\frac{2d}{\cos\theta} - 2d\tan\theta\sin\theta\right)k$$

$$= \left(2d\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right)\right)k$$

$$= 2kd\cos\theta = \underline{4\pi\bar{\nu}d\cos\theta}$$



let r_1, t_1 be the reflectivity and transmitivity of plate 1, and r_2, t_1 be those of 2

then the total amplitude

$$A_t(\delta) \propto t_1t_2 + t_1r_2r_1t_2e^{i\delta} + t_1r_2r_1r_2r_1t_2e^{2i\delta} + \dots$$

$$= t_1t_2\left(1 + (r_1r_2e^{i\delta}) + (r_1r_2e^{i\delta})^2 + \dots\right)$$

~~t_1t_2~~ let $T = t_1 t_2$ = total transmitivity of plates

$R = r_1 r_2$ = total reflectivity of plates

then

$$A_t(\delta) \propto T\left(1 + (Re^{i\delta}) + (Re^{i\delta})^2 + \cdots\right)$$

$$= T\frac{1}{1-Re^{i\delta}} = T\frac{1}{(1-R\cos\delta) - i(R\sin\delta)}$$

$$I_t(\delta) \propto |A_t(\delta)|^2 = |T|^2 \frac{1}{1-2R\cos\delta + R^2(\cos^2\theta + \sin^2\delta)}$$

$$= |T|^2 \frac{1}{(1+R^2) - \underbrace{2R\cos\delta}_{\cos(\delta) = 1-2\sin^2(\delta/2)}} = |T|^2 \frac{1}{1+R^2 - 2R(1-2\sin^2\delta/2)}$$

$$= |T|^2 \frac{1}{(1-2R+R^2) + 4R\sin^2\frac{\delta}{2}}$$

$$= \left|\frac{T}{1-R}\right|^2 \frac{1}{1+\frac{4R}{(1-R)^2}\sin^2\delta/2}$$

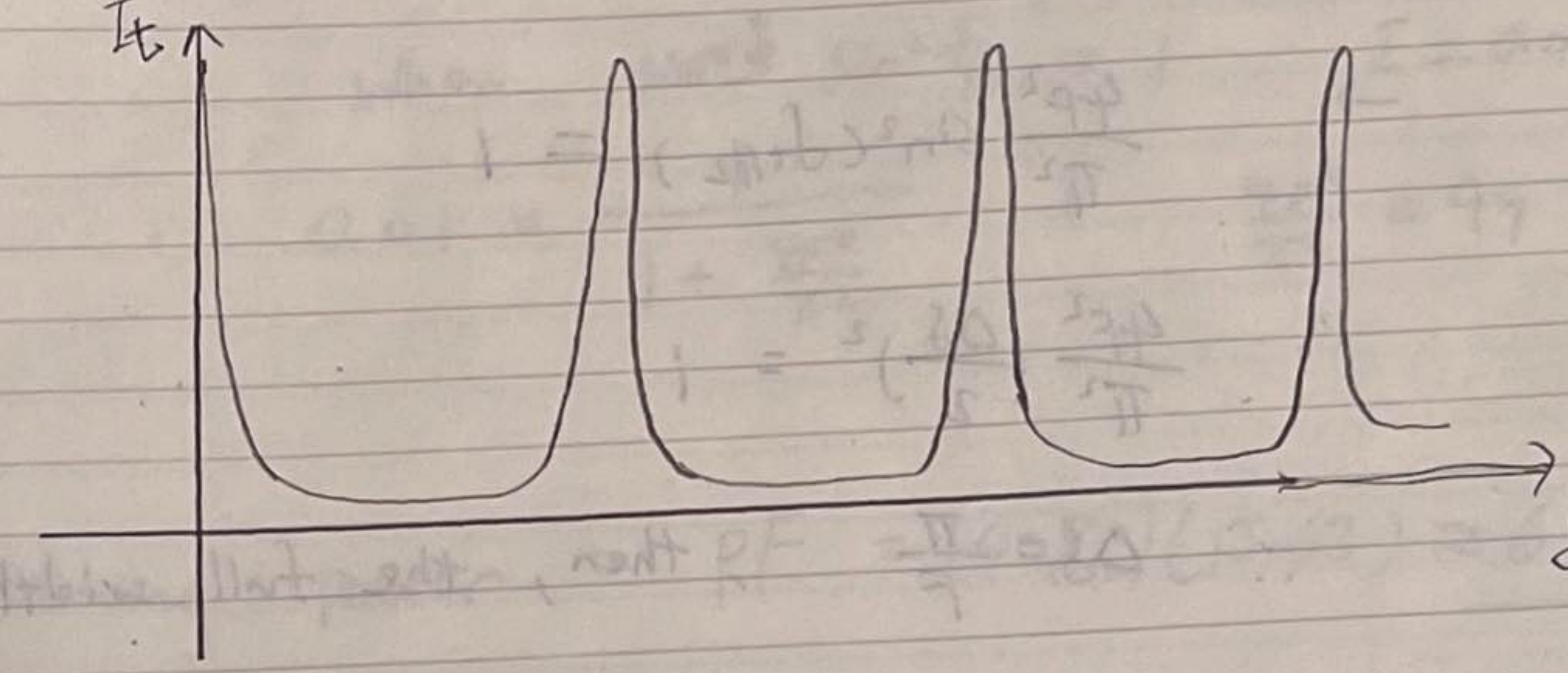
$$\rightarrow I_t(\delta) = \frac{1}{1+\frac{4R}{(1-R)^2}\sin^2\delta/2}$$

let $F = \frac{\pi\sqrt{R}}{1-R}$ then $\frac{4F^2}{\pi^2} = \frac{4\pi^2 R}{\pi^2(1-R)^2} = \frac{4R}{(1-R)^2}$

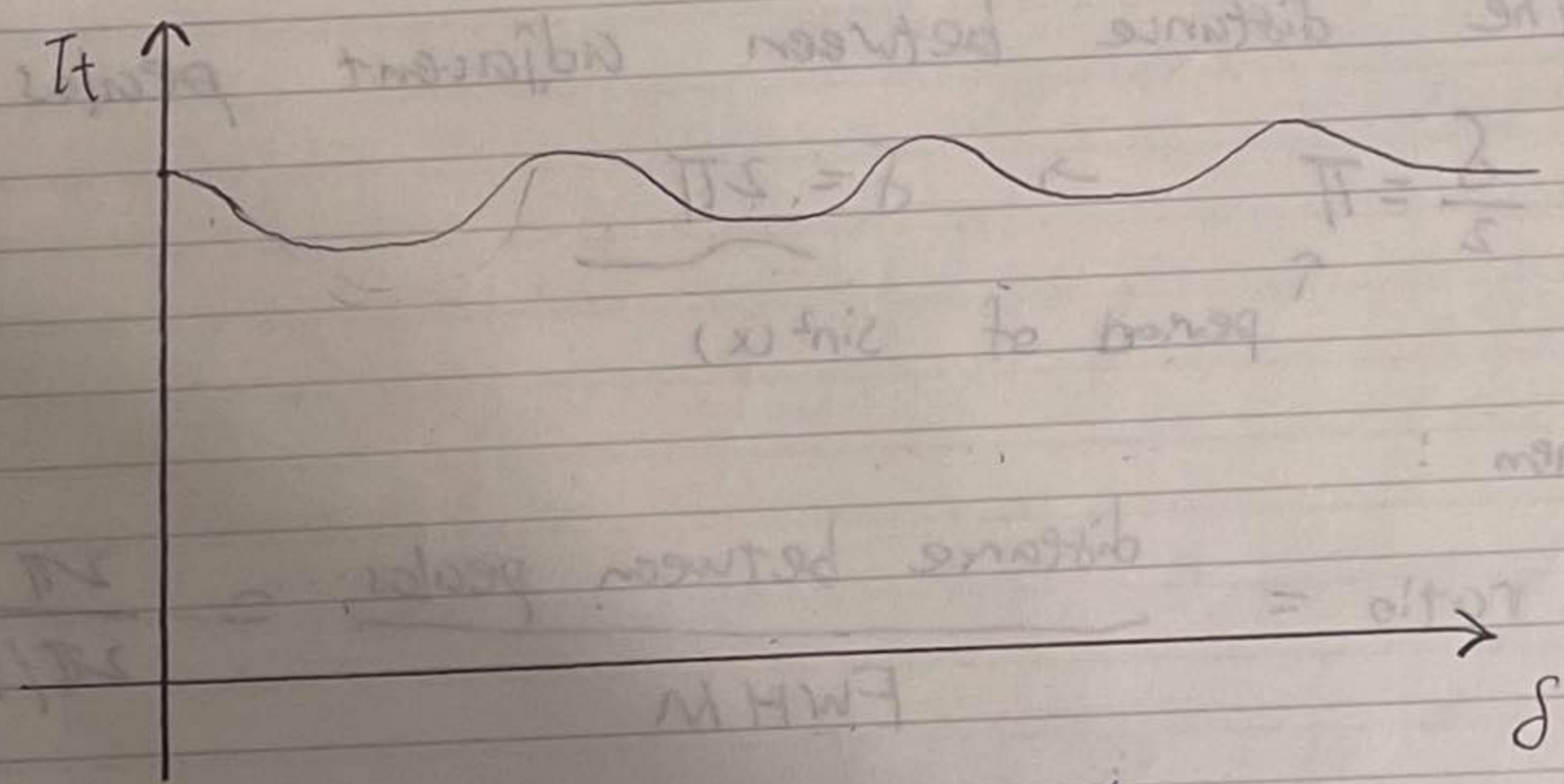
$$\therefore I_t(\delta) = \frac{1}{1+\left(\frac{4F^2}{\pi^2}\right)\sin^2\frac{\delta}{2}}$$

→ I_0 is intensity of principle maxima

high R → high F



low R → low F

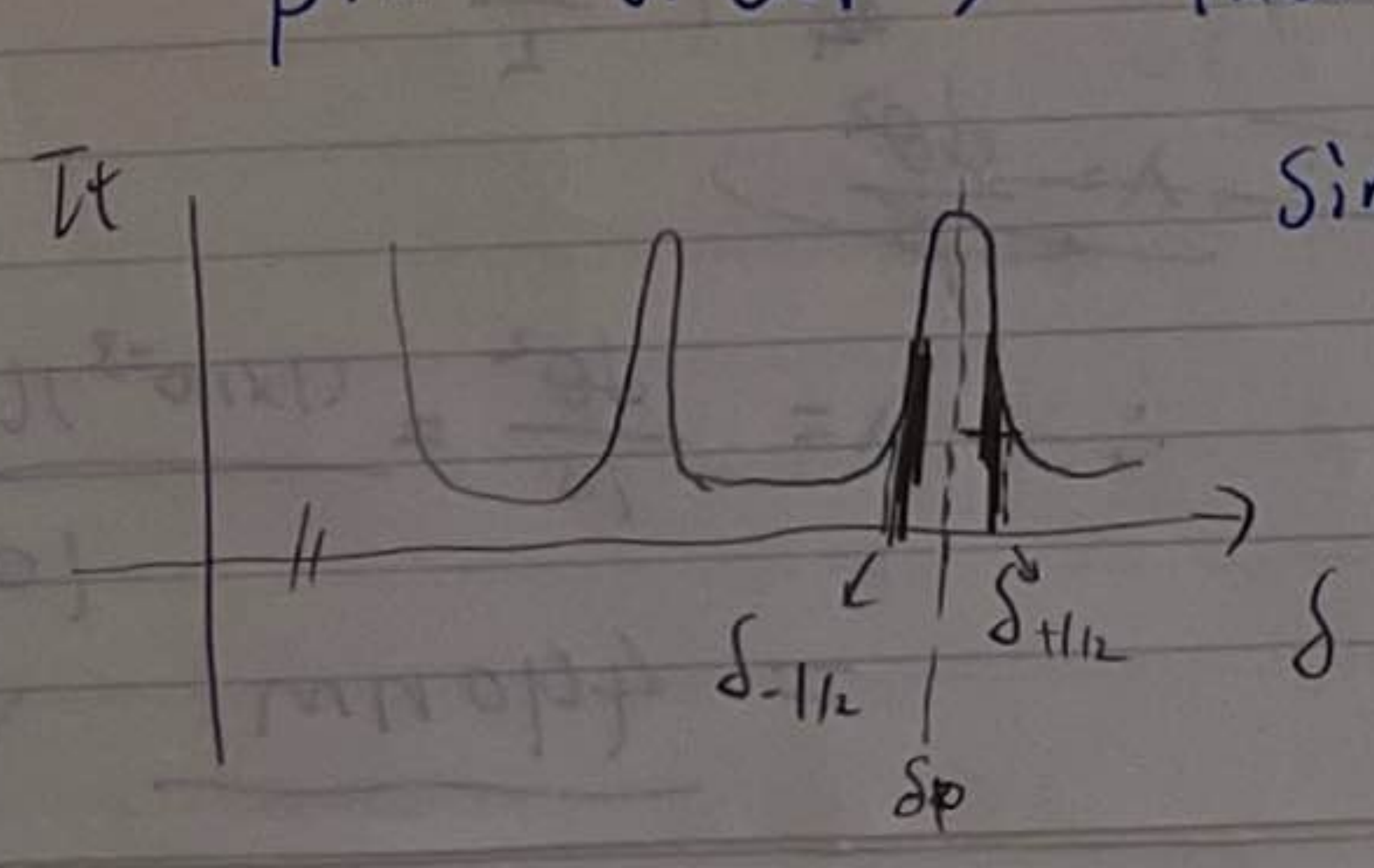


In the limit of high R (high F):

~~high~~ half-width is reached at small deviation $\Delta\delta$ from the peak value of $\delta \rightarrow \delta_p = 2\pi p$

let $\delta_{\pm 1/2} = \pm \delta_p \pm \Delta\delta = 2\pi p \pm \Delta\delta$, for the pth order, then:

$$\sin\left(\frac{2\pi p + \Delta\delta}{2}\right) \approx \frac{\Delta\delta}{2}$$



By definition of half-width :

$$I_t(\delta_{+1/2}) = \frac{I_0}{2} = \frac{I_0}{1+\frac{4F^2}{\pi^2}\sin^2(\delta_{+1/2})}$$

$$\rightarrow \quad \frac{4F^2}{\pi^2}\sin^2(\delta_{+1/2}) = 1$$

$$\therefore \quad \frac{4F^2}{\pi^2}\left(\frac{\Delta\delta}{2}\right)^2 = 1$$

$\therefore \quad \Delta\delta = \frac{\pi}{F}$, then, the full width half maximum

is : $\delta_{FWHM} = \delta_{+1/2} - \delta_{-1/2} = 2\Delta\delta = \underline{\frac{2\pi}{F}}$

The distance between adjacent peaks is

$$\frac{\delta}{2} = \underset{\uparrow}{\pi} \quad \rightarrow \quad \underline{\delta = 2\pi}$$

period of $\sin^2(x)$

then :

$$\text{ratio} = \frac{\text{distance between peaks}}{\text{FWHM}} = \frac{2\pi}{2\pi/F} = \underline{F}$$

$\rightarrow$ Assume the order at $\theta = 0$ is P.

then ~~$4\pi\bar{\nu}d = 2\pi p$~~ $2d = P\lambda$

$\rightarrow$ 10th ~~for~~ order $\theta = 0.07\,\text{rad}$ $\quad 2d\cos\theta = (P-10)\lambda$

$$\therefore \quad 2d(1-\cos\theta) = 10\lambda$$

$\because$ small angle $\cos\theta \simeq$ ~~$\frac{\theta^2}{4}$~~ $1-\frac{\theta^2}{2}$

$\therefore$ ~~$\theta^2 = \frac{P\lambda}{d}$~~ ~~$\therefore \lambda = \frac{d\theta^2}{P}$~~

$\therefore$ ~~θ^2~~ $\theta^2 = \frac{10\lambda}{d}$ $\quad \therefore \quad \lambda = \frac{d\theta^2}{10} = \frac{(1\times10^{-3})(0.07)^2}{10}$

$$= \underline{490\,\text{nm}}$$

maximum order: $p = \frac{2d}{\lambda} \simeq \underline{4081}$

~~Resolving power~~ $\because$ Intensity goes from 100% to 1%

when ~~$\sin^2\delta$~~ $\sin^2 \delta/2 = 1$, $I = 0.01 I_0$

$\therefore \quad 0.01 = \dfrac{1}{1 + \frac{4F^2}{\pi^2}} \rightarrow \frac{4F^2}{\pi^2} = 99 \rightarrow \underline{F \simeq 15.63}$

Resolving power $= pF = (4081)(15.63) \simeq \underline{6.38 \times 10^4}$ ✓

A2 2014 ~~Fot~~ Tutorial Notes

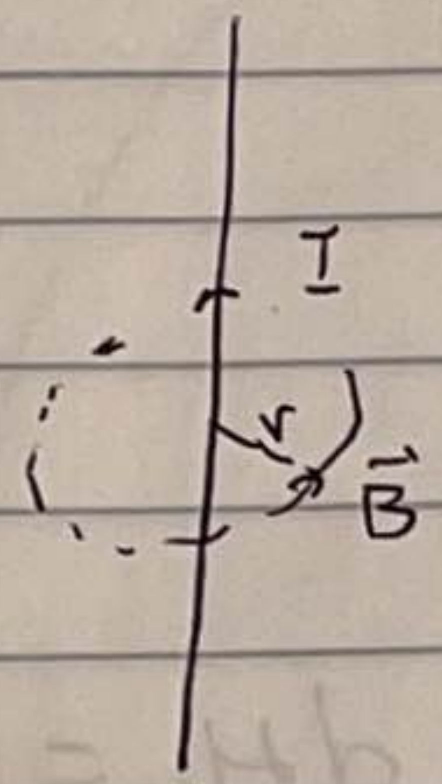
1.

$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$ always true

$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} V$ only if $\frac{\partial \vec{B}}{\partial t} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t}\right) \rightarrow \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\rightarrow \quad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} V \quad \rightarrow \quad \vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$



$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta} = \vec{\nabla} \times \vec{A}$$
$$= \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) \hat{\theta}$$

A has no dependence on z because wire is infinite

$$\rightarrow \quad \frac{\partial A_r}{\partial z} = 0 \quad \therefore \quad -\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

$$\rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi} \ln r + \text{const}$$

$$\vec{A} = \frac{-\mu_0 I}{2\pi} \ln r \, \hat{\underline{z}}$$

In cylindrical $\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{z} \frac{\partial}{\partial z}$

$$\rightarrow \vec{\nabla} \times \vec{A} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} A_r \\ A_\theta \\ A_z \end{pmatrix} \rightarrow$$

rectangle ? $\oint \underline{\nabla \times A} \cdot d\underline{l} = \int \underline{B} \cdot d\underline{s}$

2

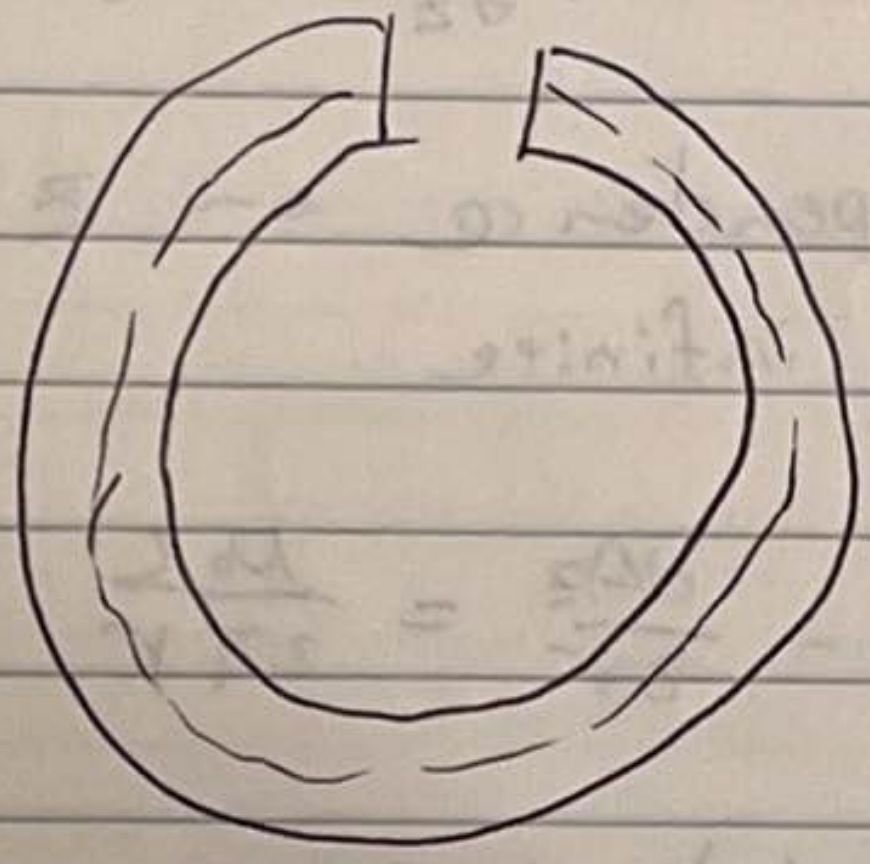
dipole field

$$\vec{E} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$$

$$\vec{T}_1 = -\frac{P_1 P_2}{4\pi\varepsilon_0 d^3}\hat{z}$$

$$\vec{T}_2 = -\frac{2P_1 P_2}{4\pi\varepsilon_0 d^3}\hat{z}$$

3. $\nabla \times \vec{H} = \vec{J}_f$ $\quad \oint \vec{H}\cdot d\vec{l} = I$



$(2\pi R - d)H_r + dH_g =$

$H_r = \frac{B_r}{\mu_0 \mu_r}$ $\quad H_g = \frac{B_g}{\mu_0}$

$B_r = B_g$

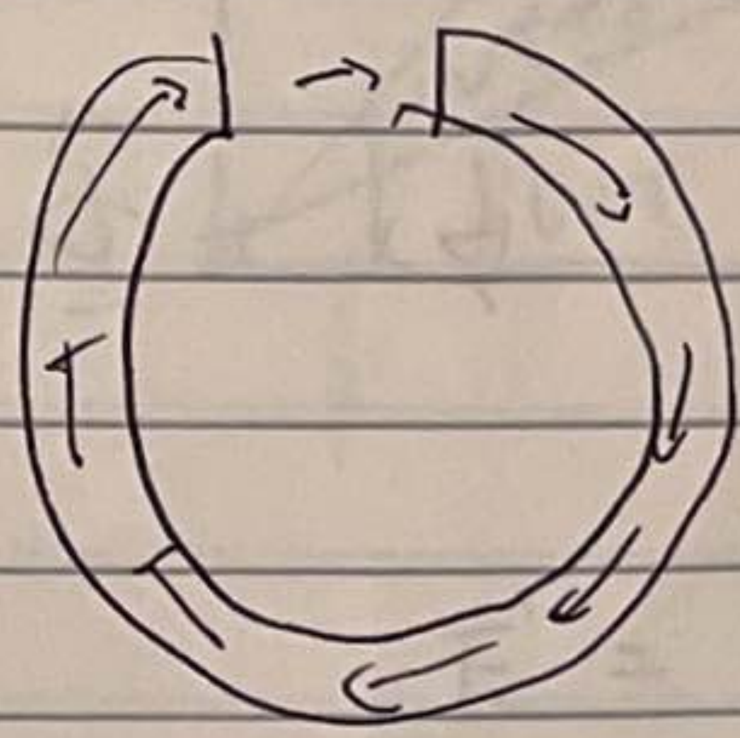
$$B_g = \frac{NI}{(2\pi r - d) + d\mu_r}$$

$$(2\pi R - d)H_r + dH_g = NI$$

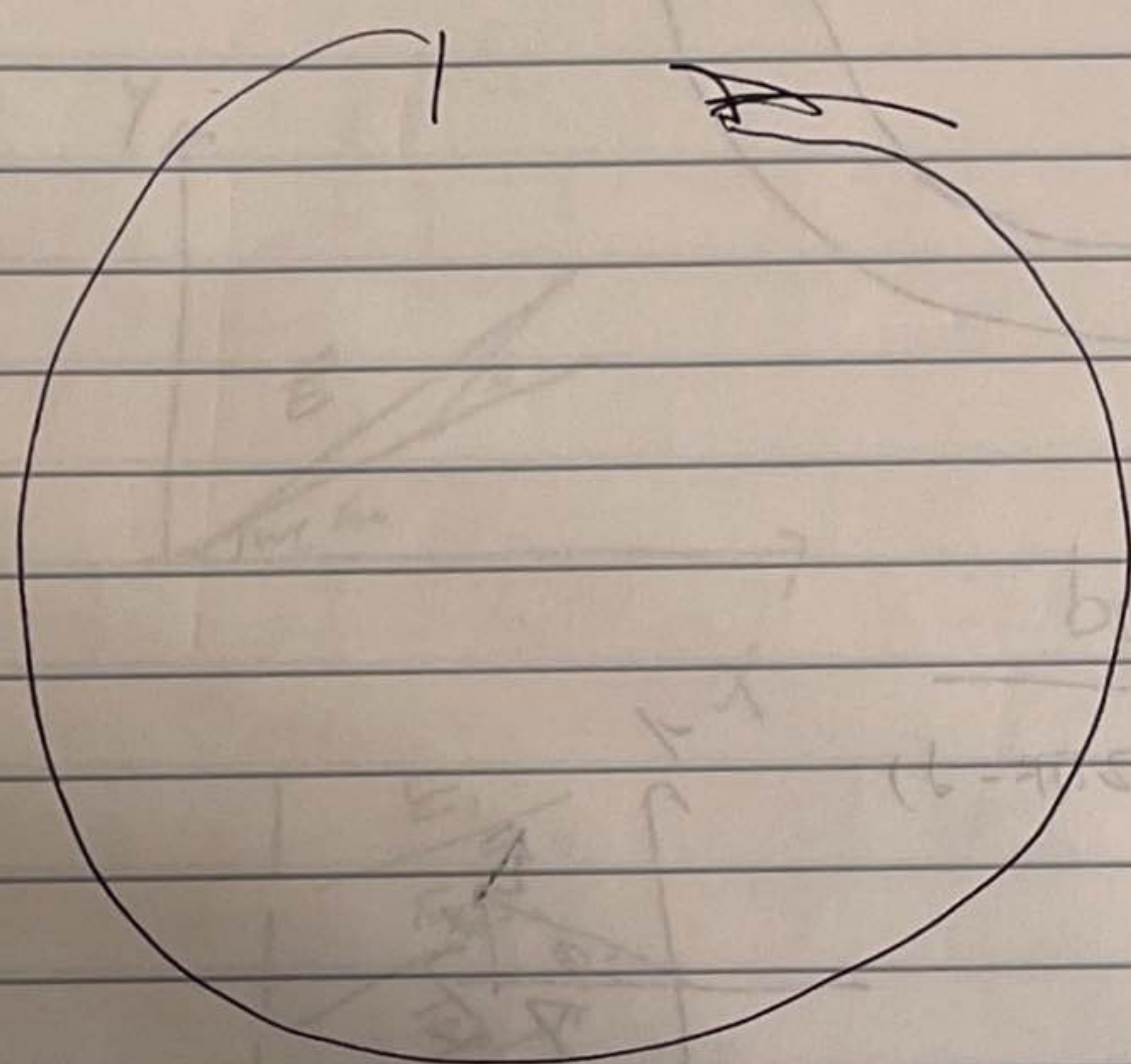
$\hookrightarrow \frac{B_g}{\mu_0}$

$$B_r = \mu_0(H_r + M)$$

$$B_g = \mu_0 M \frac{2\pi R - d}{2\pi R}$$

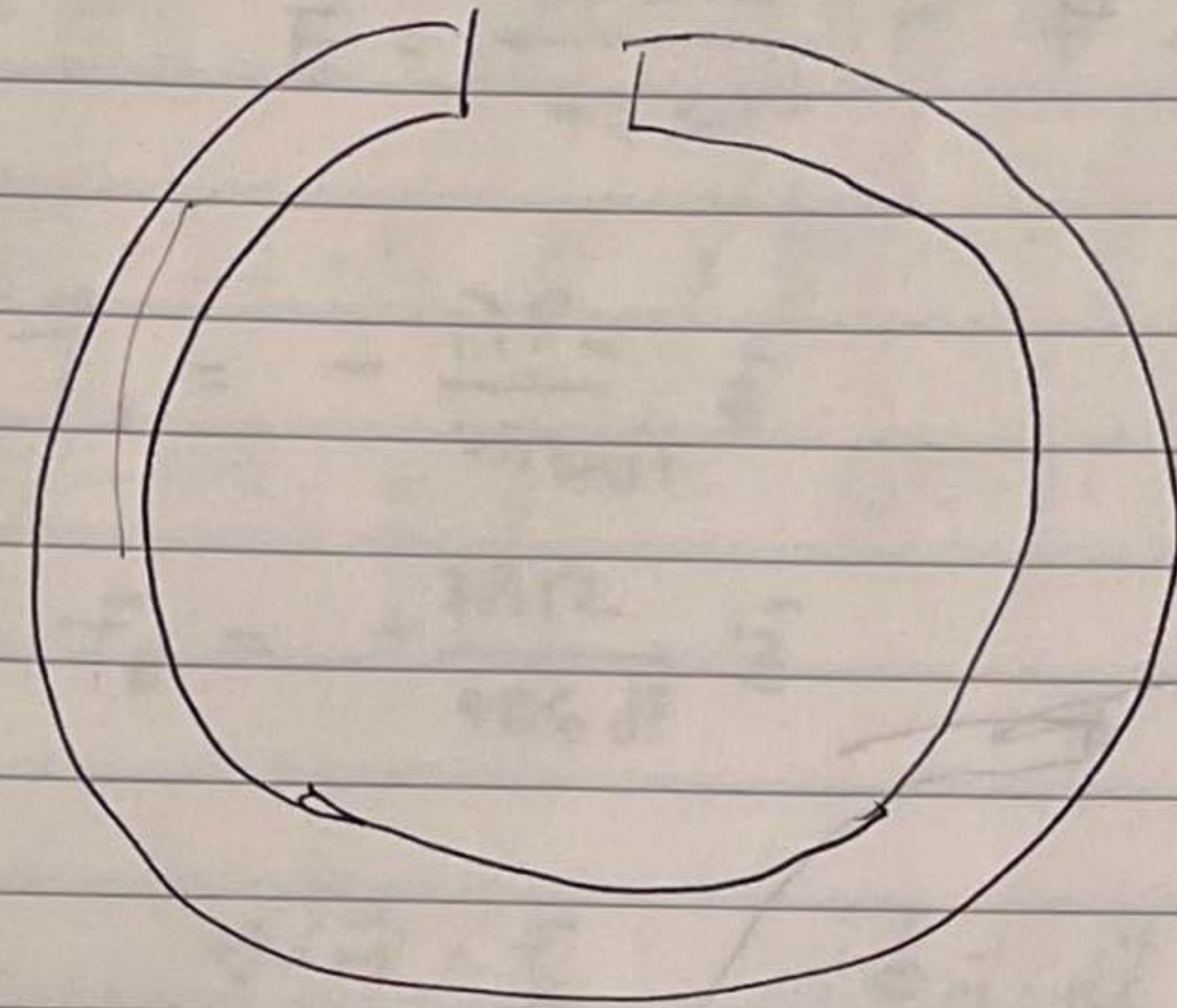


B

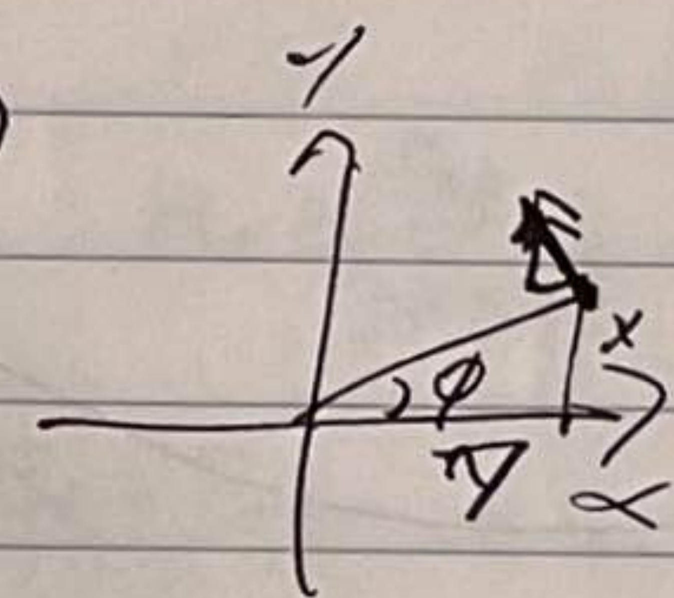


H

$\nabla \times (\hat{\phi}$



$$H_r = \frac{-B_g d}{\mu_0 (2\pi R - d)}$$

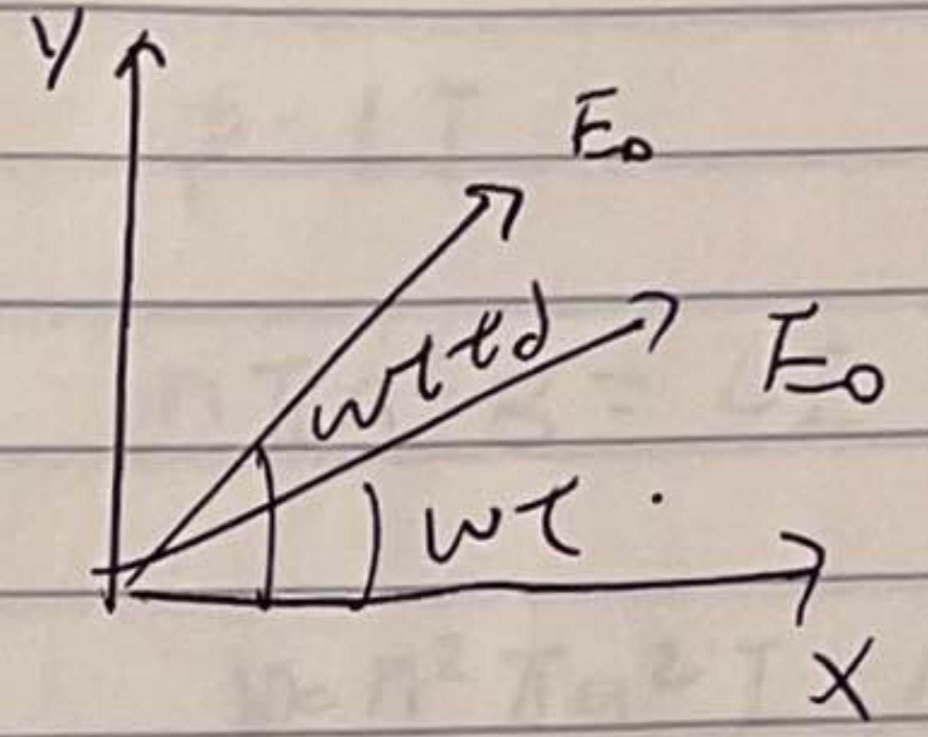


$$\nabla \times \hat{\phi}$$

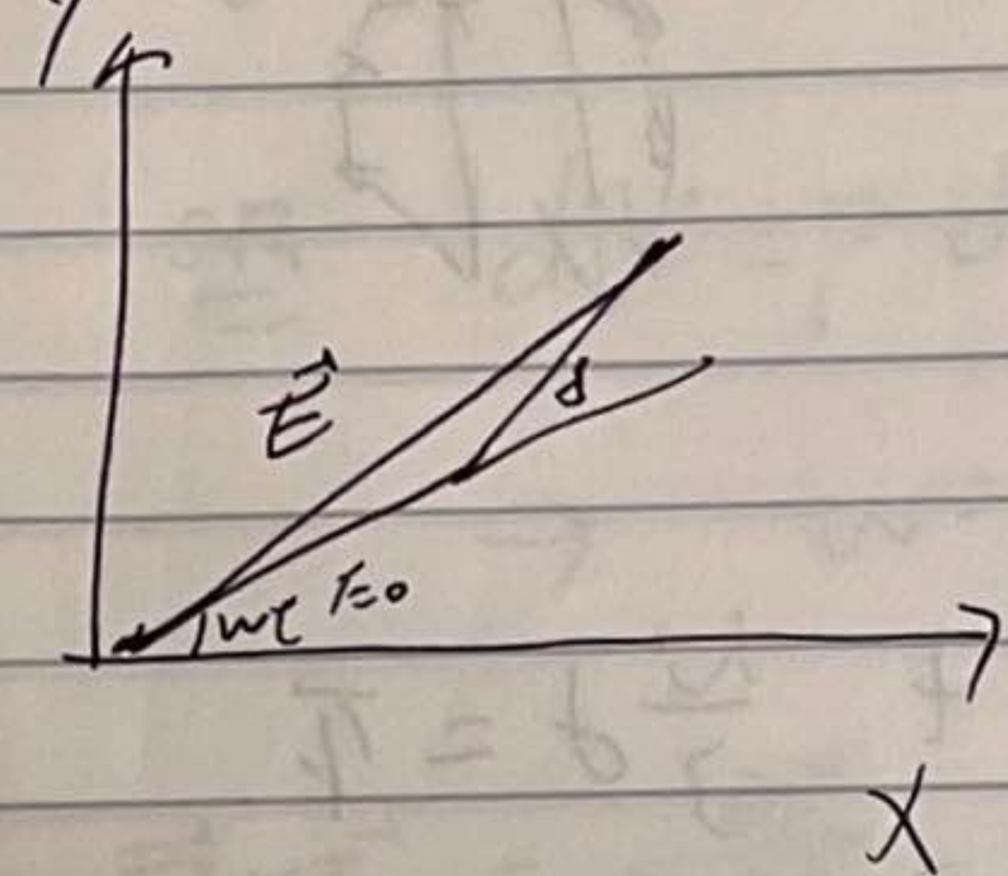
$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \end{vmatrix}$$

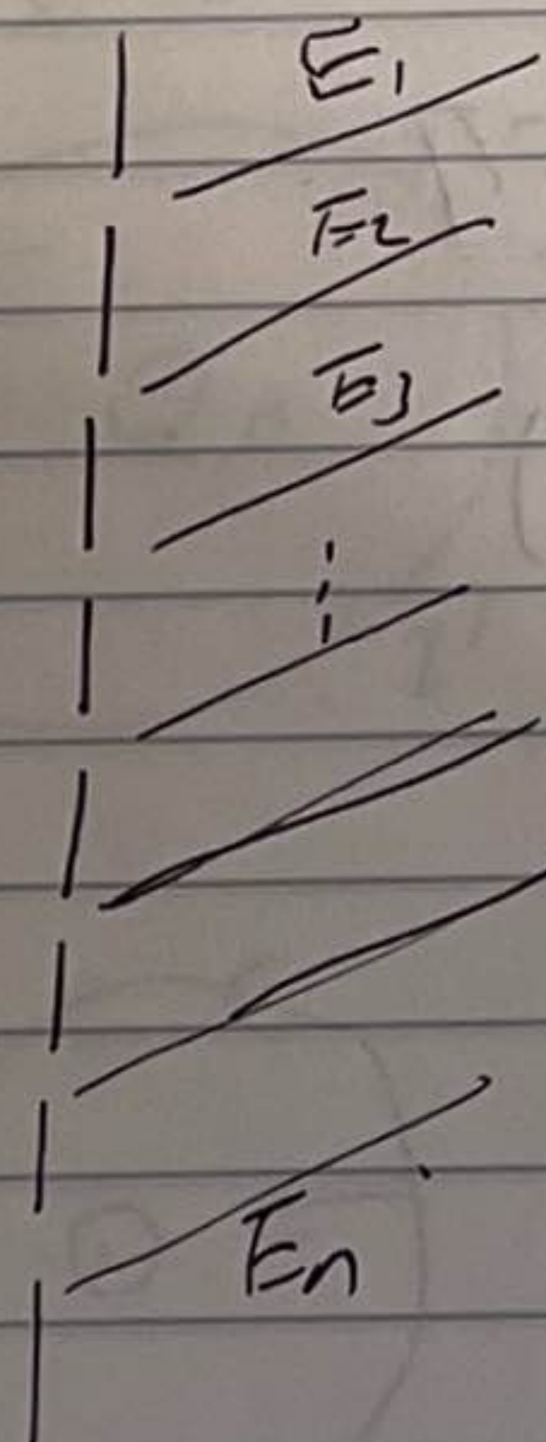
$-\sin\phi \quad \cos\phi$

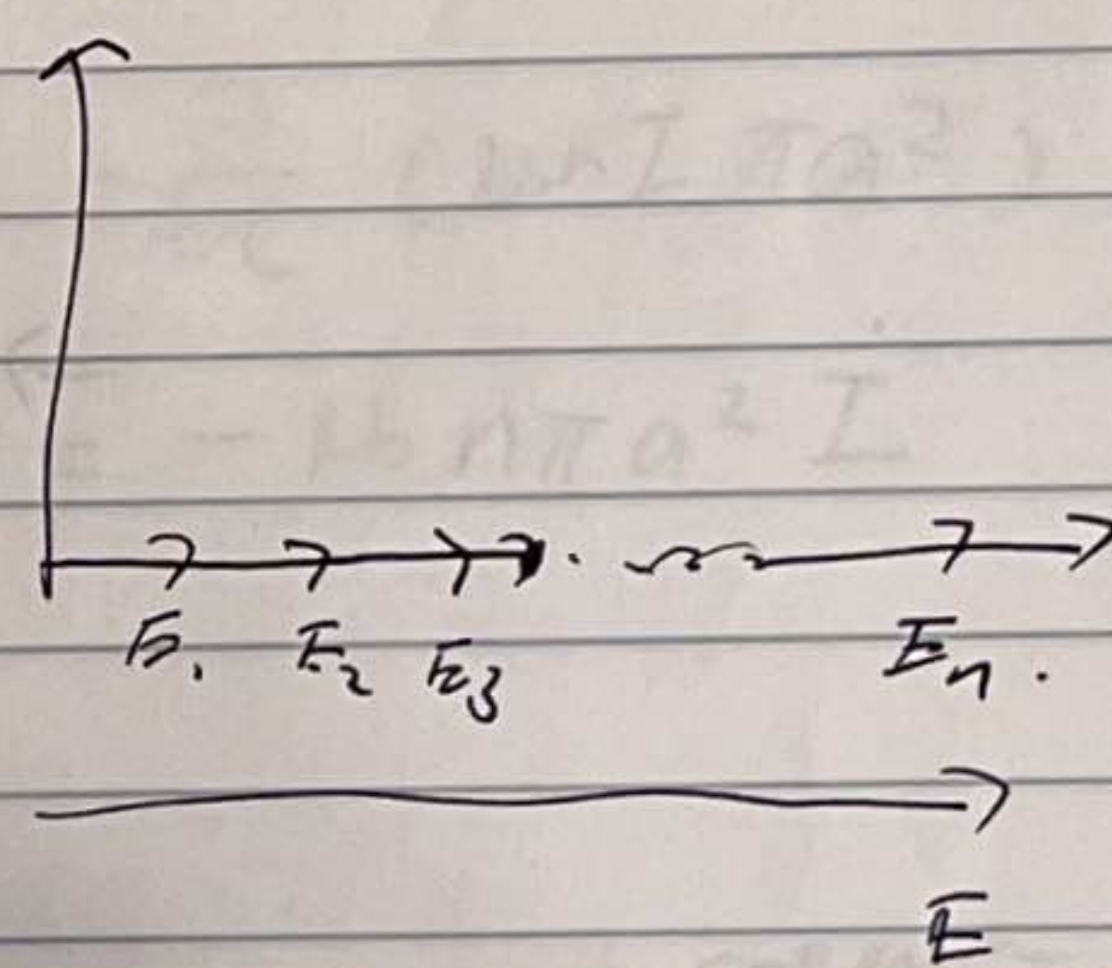
$-\frac{y}{\sqrt{x^2+y^2}} \quad \frac{x}{\sqrt{x^2+y^2}}$

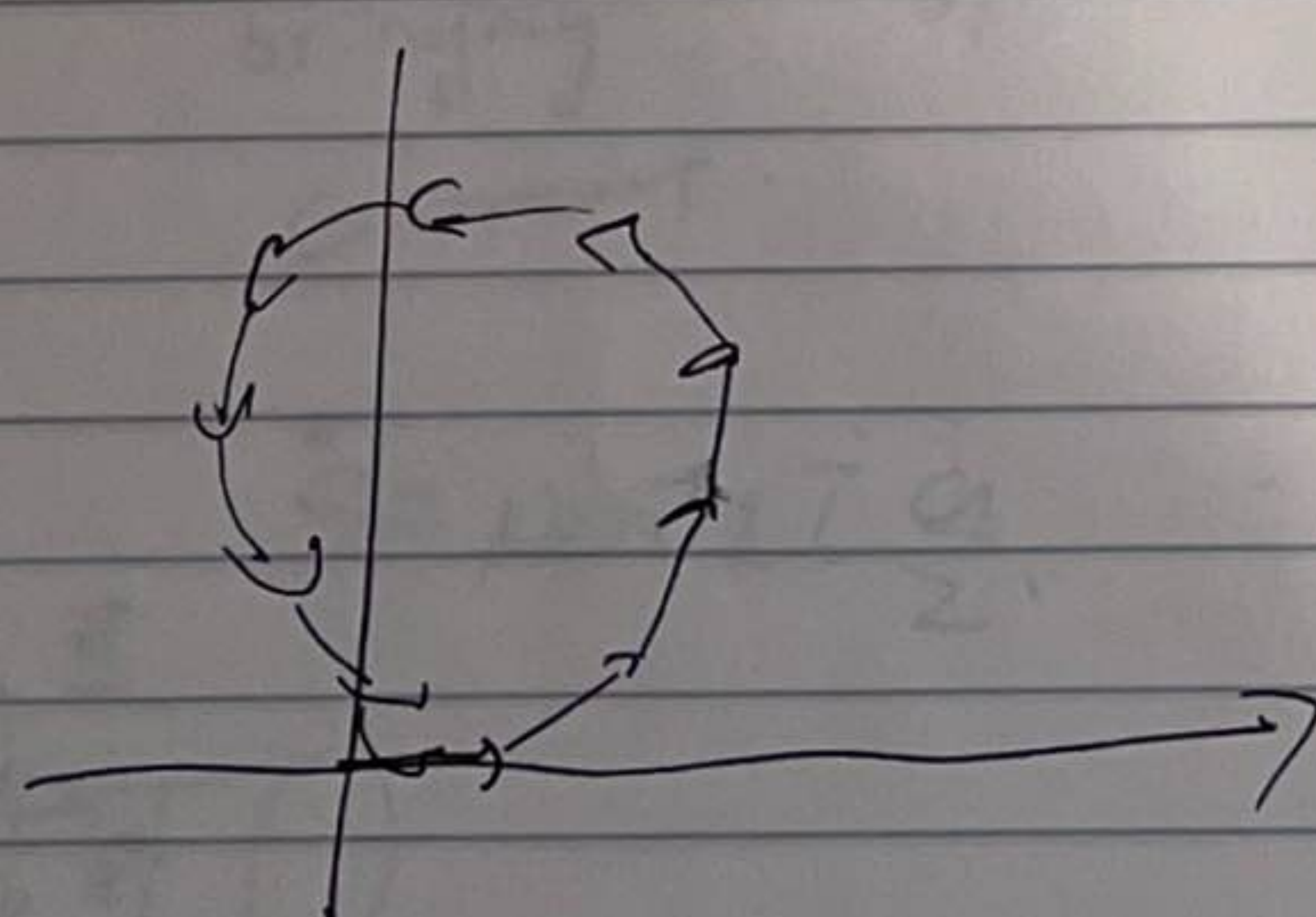


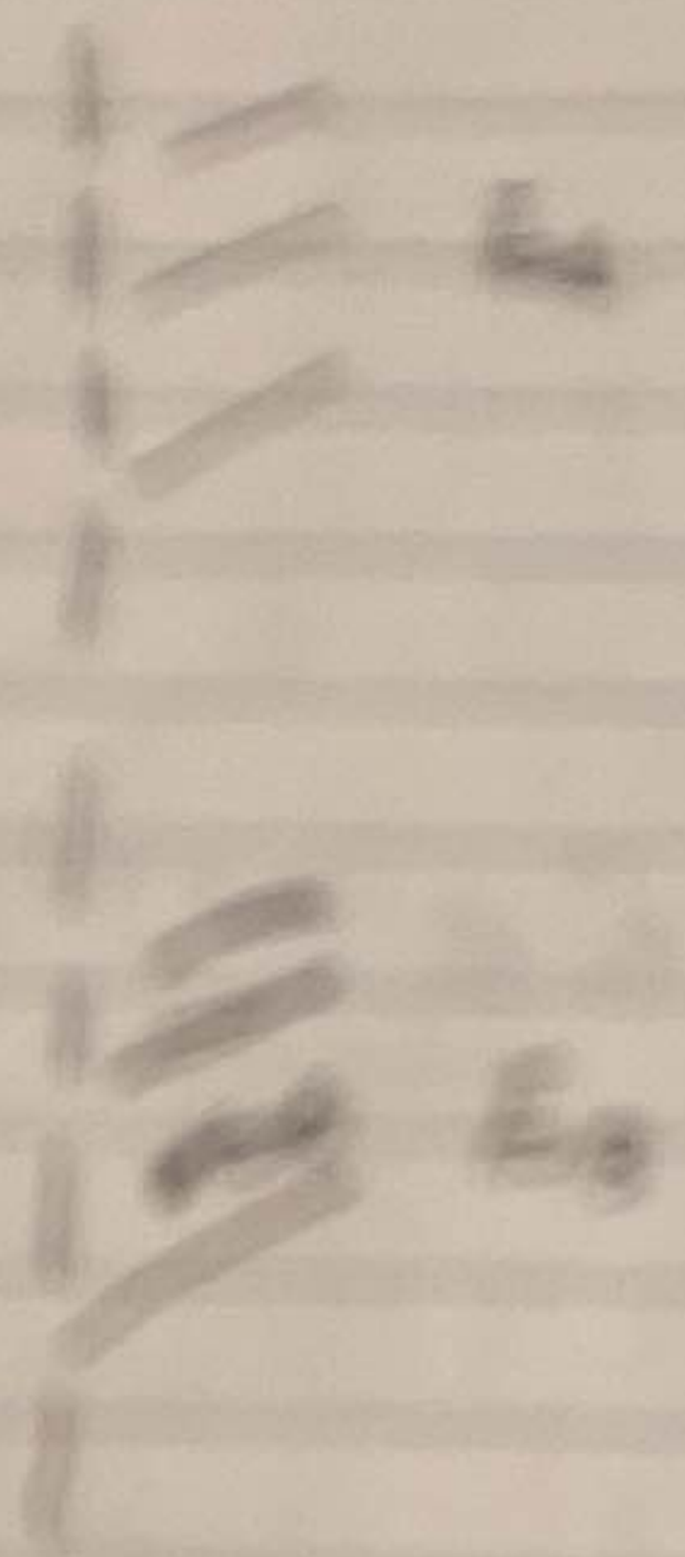
$$\vec{E} = \underbrace{\vec{E}_0 e^{i\omega t}}_{\vec{E}_1} + \underbrace{\vec{E}_0 e^{i(\omega t + \delta)}}_{\vec{E}_2}$$







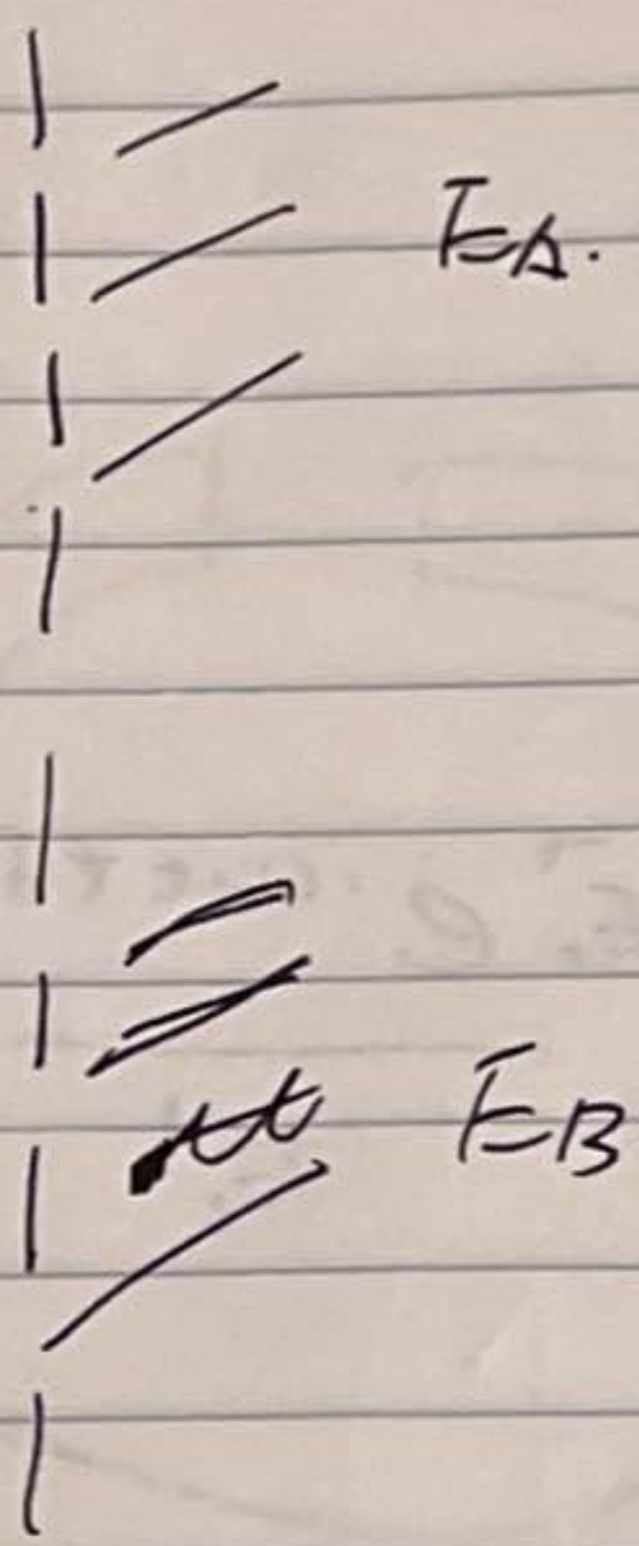


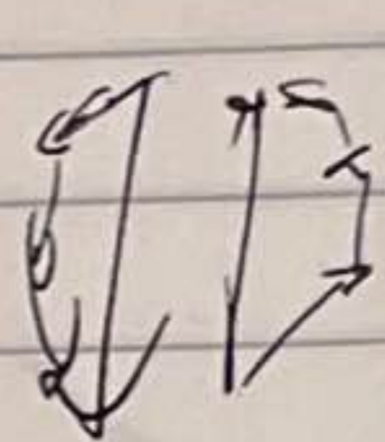


$\frac{w}{2}\delta + \frac{V}{2}\delta = \pi$

$\delta\ \frac{1}{2}(w+V) = \pi$

$\rightarrow \delta = \frac{2}{[illegible]}\left(\frac{[illegible]}{V}\right)$





$$\frac{N}{3}\delta + \frac{N}{3}\delta = \pi$$

$$\frac{2}{3}N\delta = \pi$$

$$\rightarrow \delta = \frac{3}{4}\left(\frac{2\pi}{N}\right)$$

parallel H continuous

at discontinuity. perpendicular

$$\phi = LI$$

$$n\pi a^2 B = LI$$

$$\mu_0 n^2 \pi a^2 I = LI \rightarrow L = \mu_0 n^2 \pi a^2$$

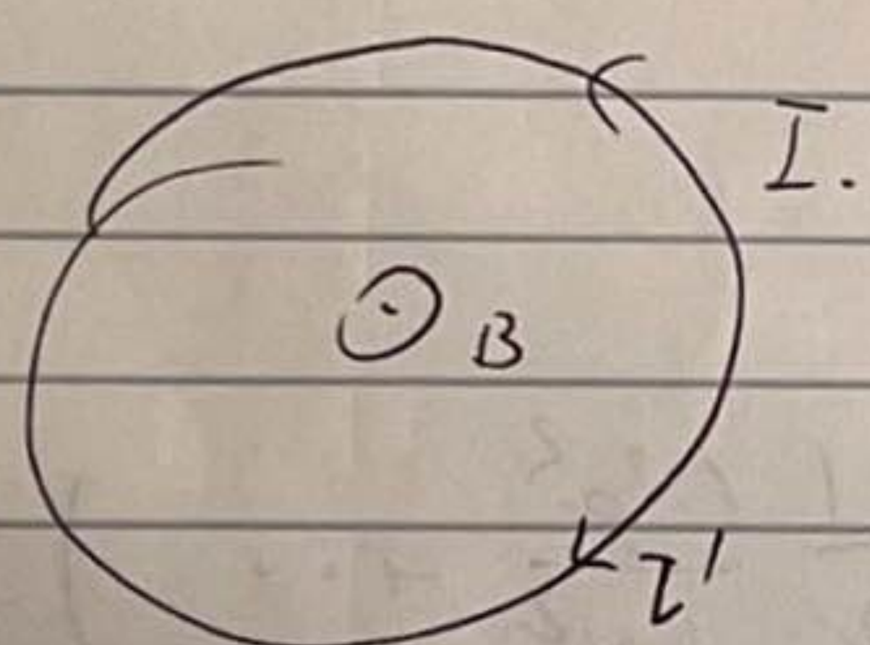
$$\frac{d\phi}{dt} I = P \qquad L\frac{dI}{dt} I = P$$

$$\varepsilon \qquad W = \int_0^? L\frac{dI}{dt} I\,dt.$$

OR $dW = P\,dt = LI\,dI.$

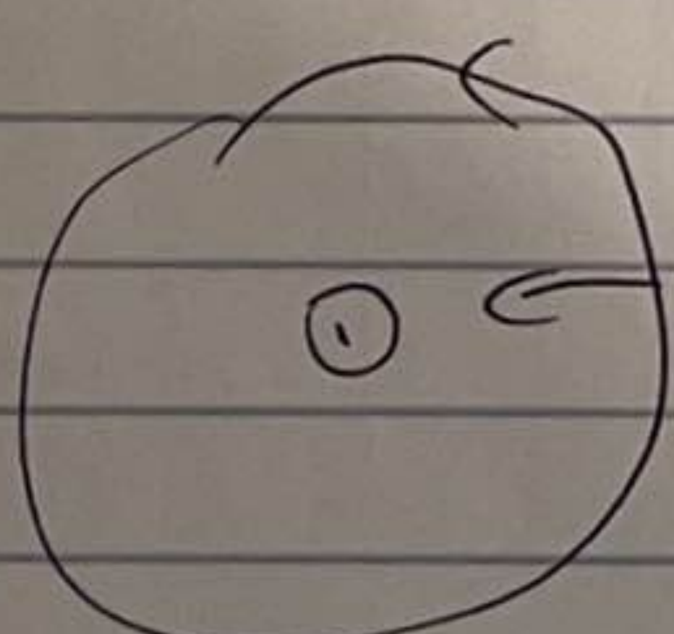
$$\rightarrow W = \int P\,dt = \int LI\,dI$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{S}$$



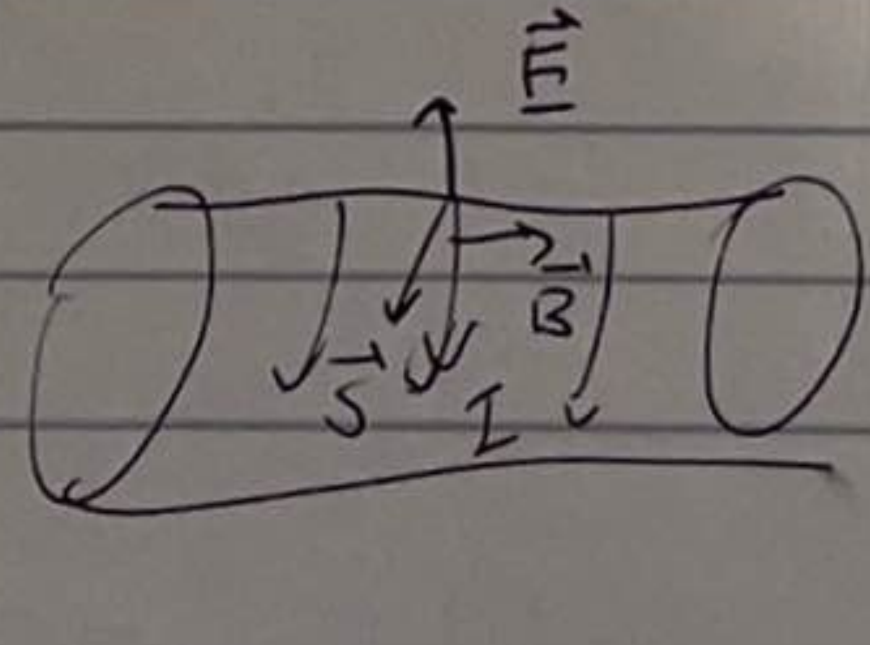
$$= -\frac{\partial}{\partial t}(\mu_0 n I \pi a^2)$$

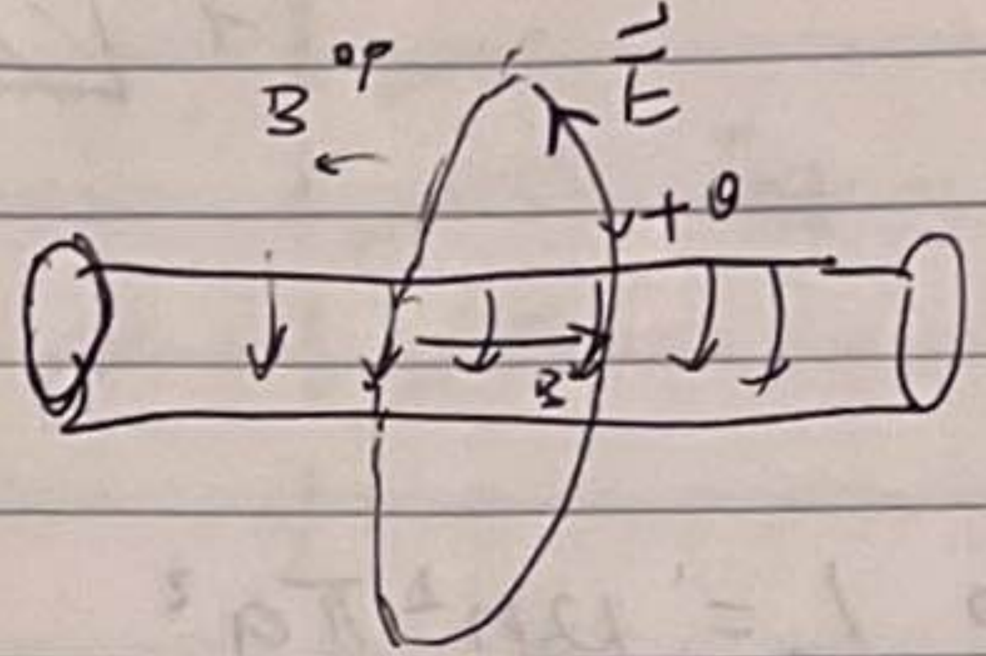
$$E_\theta (2\pi a) = -\mu_0 n \pi a^2 \dot{I}$$



bringing energy to increase current.

$$S = \mu_0 n^2 I \dot{I} \frac{a}{2}.$$





$$2\pi b E = -\frac{\partial}{\partial t}(\pi a^2 B)$$

$$v_g = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

$$t = \frac{l}{v_g} = \frac{l}{c}\left(1+\frac{1}{2}\frac{\omega_p^2}{\omega^2}+\cdots\right)$$