2662

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

A2: ELECTROMAGNETISM AND OPTICS

TRINITY TERM 2011

Saturday, 25 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

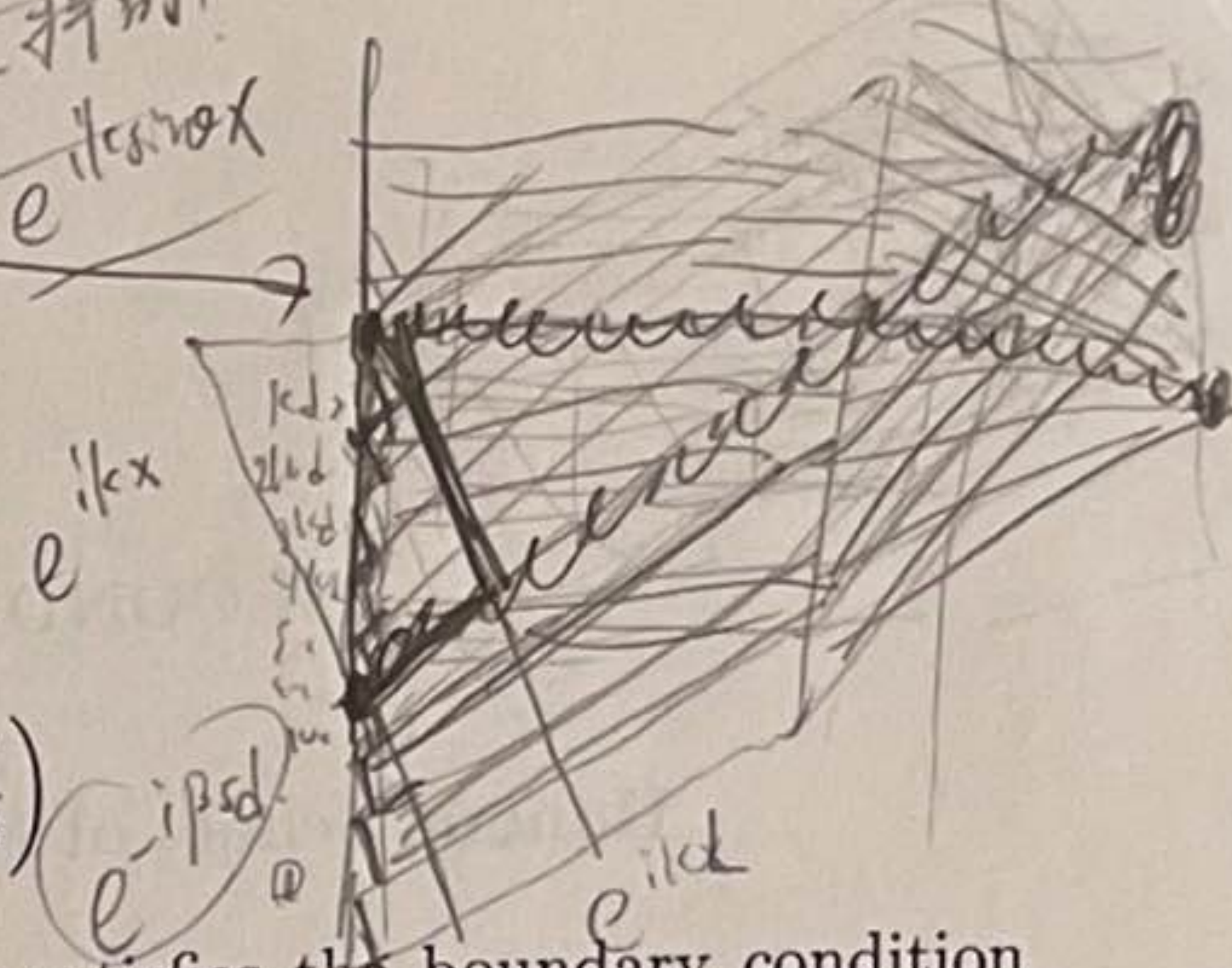
The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

$\frac{58}{100}$

这个点 phase 特别!
对面各光子 $e^{ik\sin\theta x}$

Section A



1. Given an electric field in vacuum:

$$\mathbf{E}(t, \mathbf{r}) = \frac{E_0}{c}\left(0, 0, \frac{y}{t^2}\right)$$

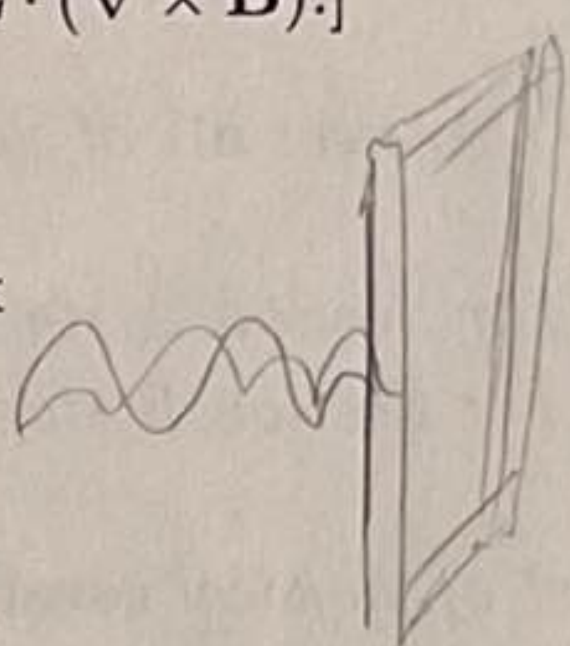
use Maxwell's equations to determine $\mathbf{B}(t, \mathbf{r})$ which satisfies the boundary condition $|\mathbf{B}| \to 0$ when $t \to \infty$. [6]

2. Use Maxwell's equations to show that

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E}$$

where $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$ is the Poynting vector and $U = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2$. Explain the meaning of this equation.
[You may use the identity $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$.] [8]

3. Find the eigenvalues and eigenvectors of the matrix



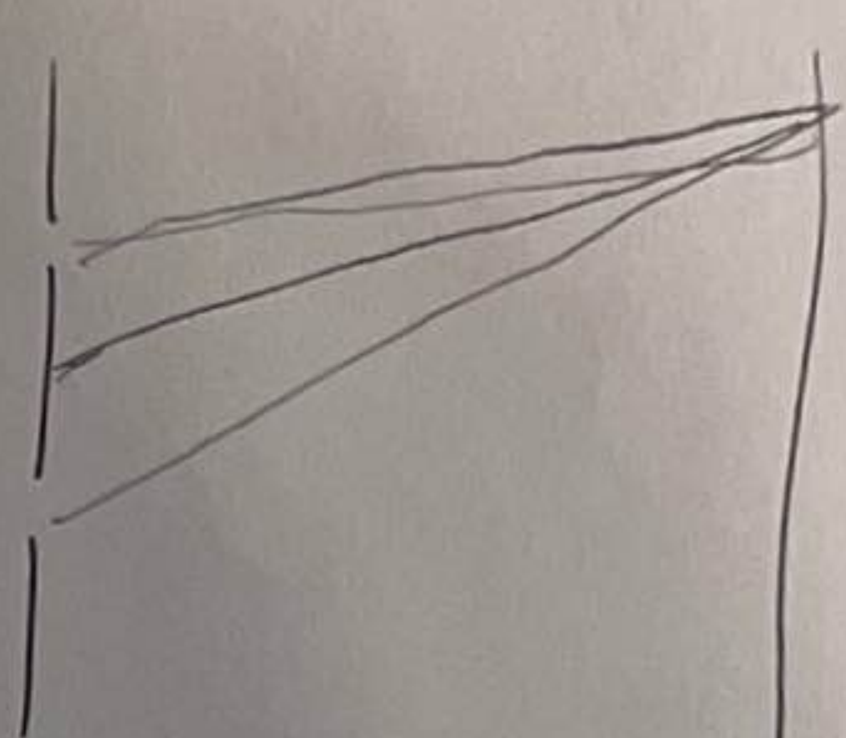
$$U = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find a transformation matrix S such that $S^{-1}US = \Lambda$ where Λ is the diagonal matrix made up of the eigenvalues of U. [7]

4. An electromagnetic wave is incident on a surface which absorbs all the electric field. Use Maxwell's equations to determine the magnetic field on the other side of the surface. [3]

5. Two parallel, long, narrow and identical slits are separated by a distance d and illuminated normally by a plane wave of wavelength λ. Derive the amplitude A and intensity I of the resultant interference pattern in a plane parallel to the slits a long distance L away from them, as a function of the angle θ under which the observation point P appears when viewed from the centre between the slits. State carefully any approximations you make and sketch the geometry of this problem. [5]

Light reaching the observation plane has a phase ϕ_r relative to the phase that the incident plane wave would have had in the observation plane, in the absence of the slits. Show that ϕ_r is zero for small θ. [3]

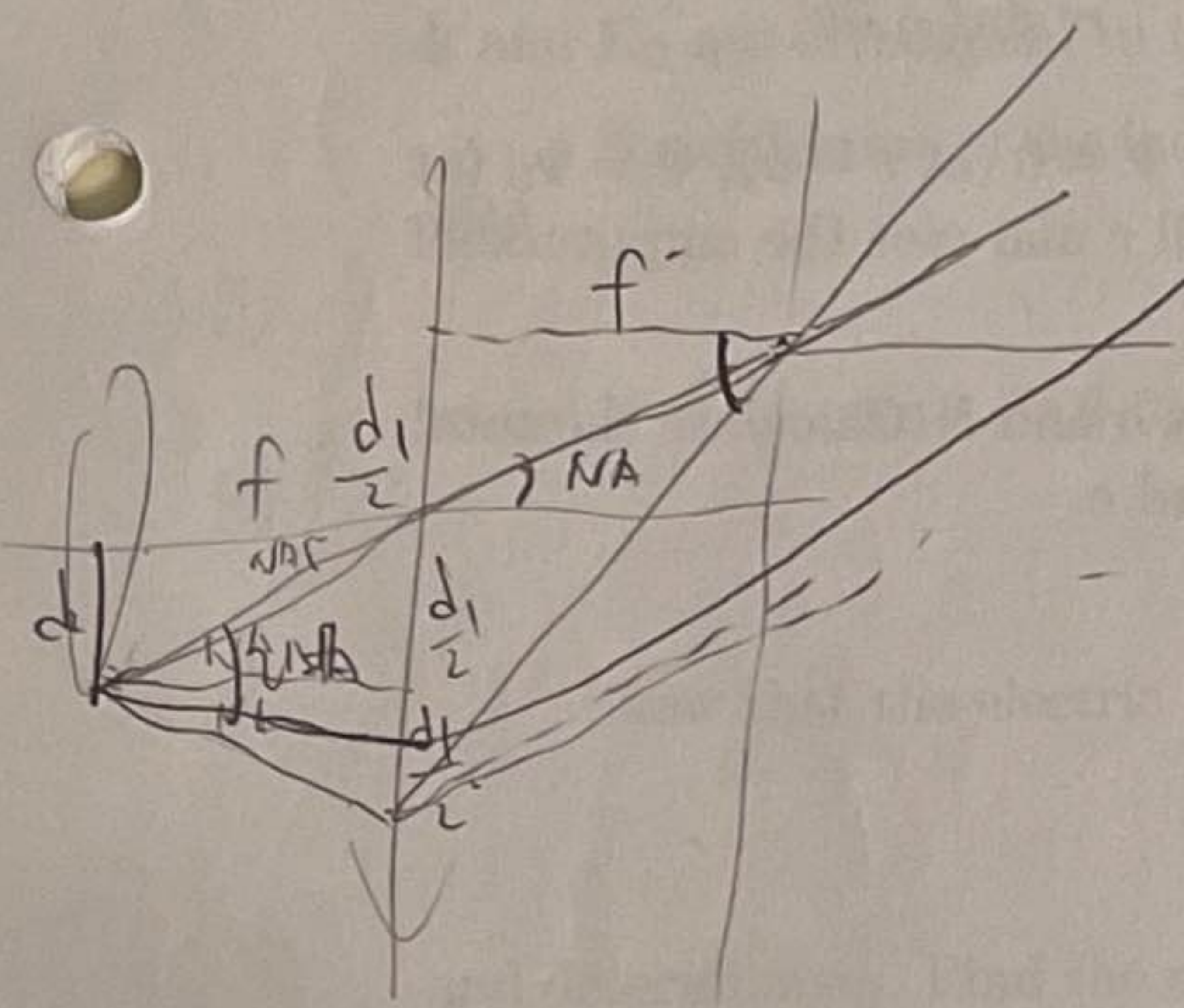


6. A thin lens of focal length f is used to collimate the light emerging from two different optical fibres.

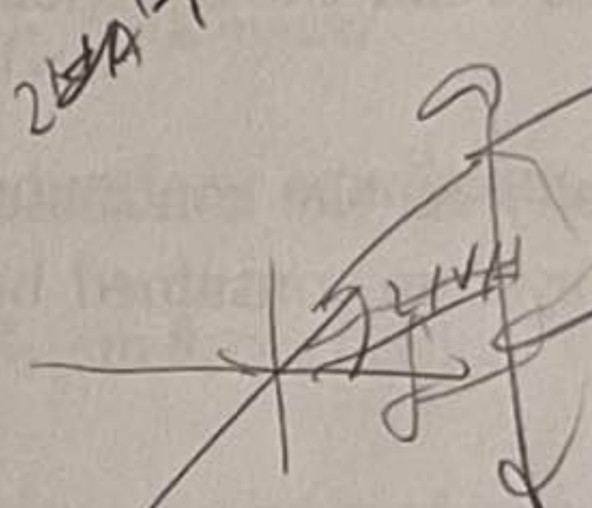
When used with a first fibre of small core diameter and numerical aperture NA it achieves a diffraction limited divergence angle θ_1 for the collimated beam. Find the separation u between the lens and the fibre that provides best collimation. Find θ_1 and the diameter D_1 of the collimated beam immediately behind the lens. [5]

When used with a second fibre of a large core diameter d_2 but the same numerical aperture, the beam diverges much faster than than it did using the first fibre. Calculate this larger divergence angle θ_2 as a function of f and d_2. [3]

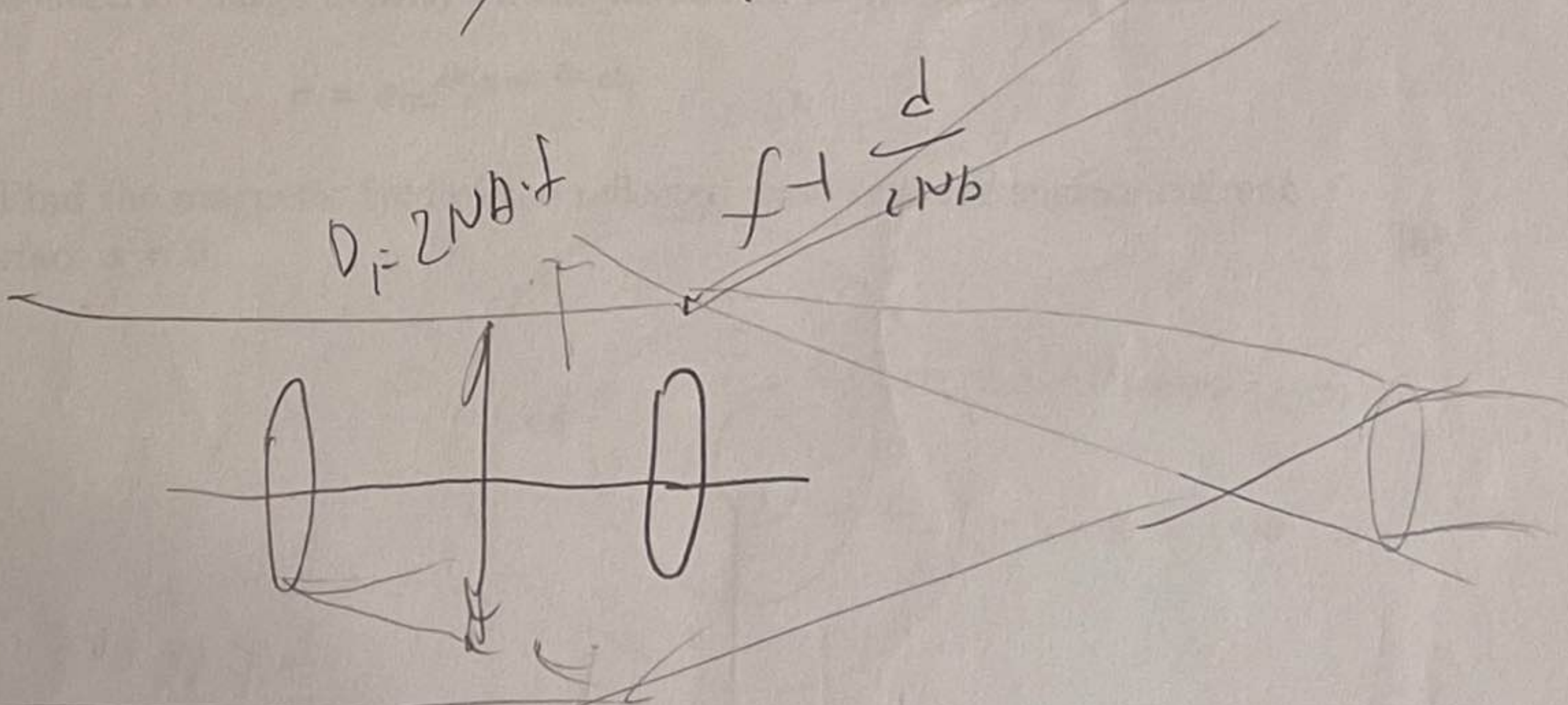
$f \cdot \quad 2NA \cdot f$



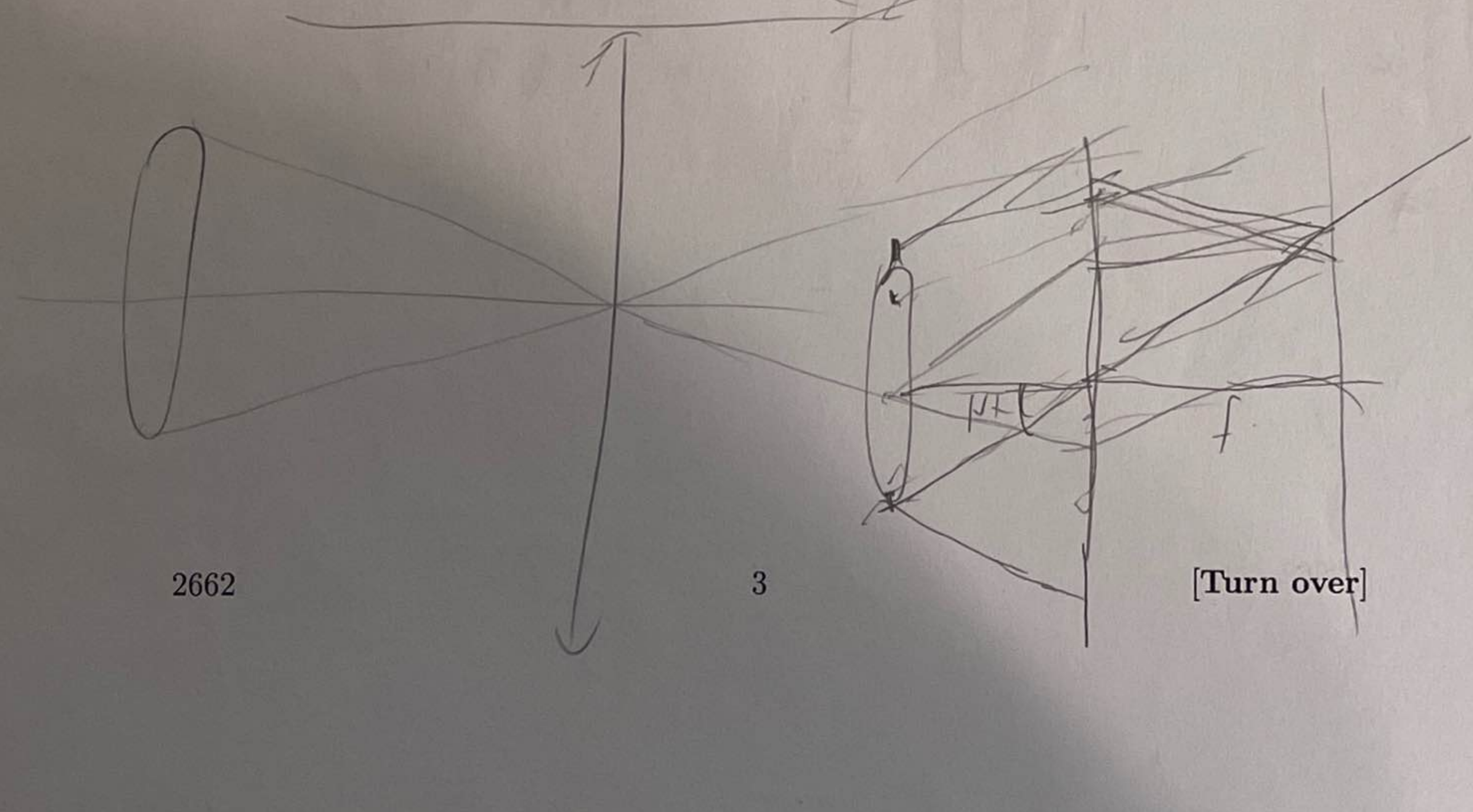
$\theta = \frac{\lambda}{2NA\, D_1}$



$D_1 = 2NA \cdot f \qquad f + \frac{d}{2NA}$



$\tan\theta = \frac{3d_2}{2f}$



Section B

7. Consider three concentric spherical conductors with radii $r = a$, b and c, where $a < b < c$. Under what conditions can the electric field, $\mathbf{E}$, for this configuration be constructed from a potential Φ that satisfies Laplace's equation, $\nabla^2\Phi = 0$? What are the boundary conditions for $\mathbf{E}$? [4]

Assuming spherical symmetry, show that Φ takes the form

$$\Phi = A + \frac{B}{r}\,.$$ [4]

[Hint: the Laplacian in spherical polar coordinates takes the form

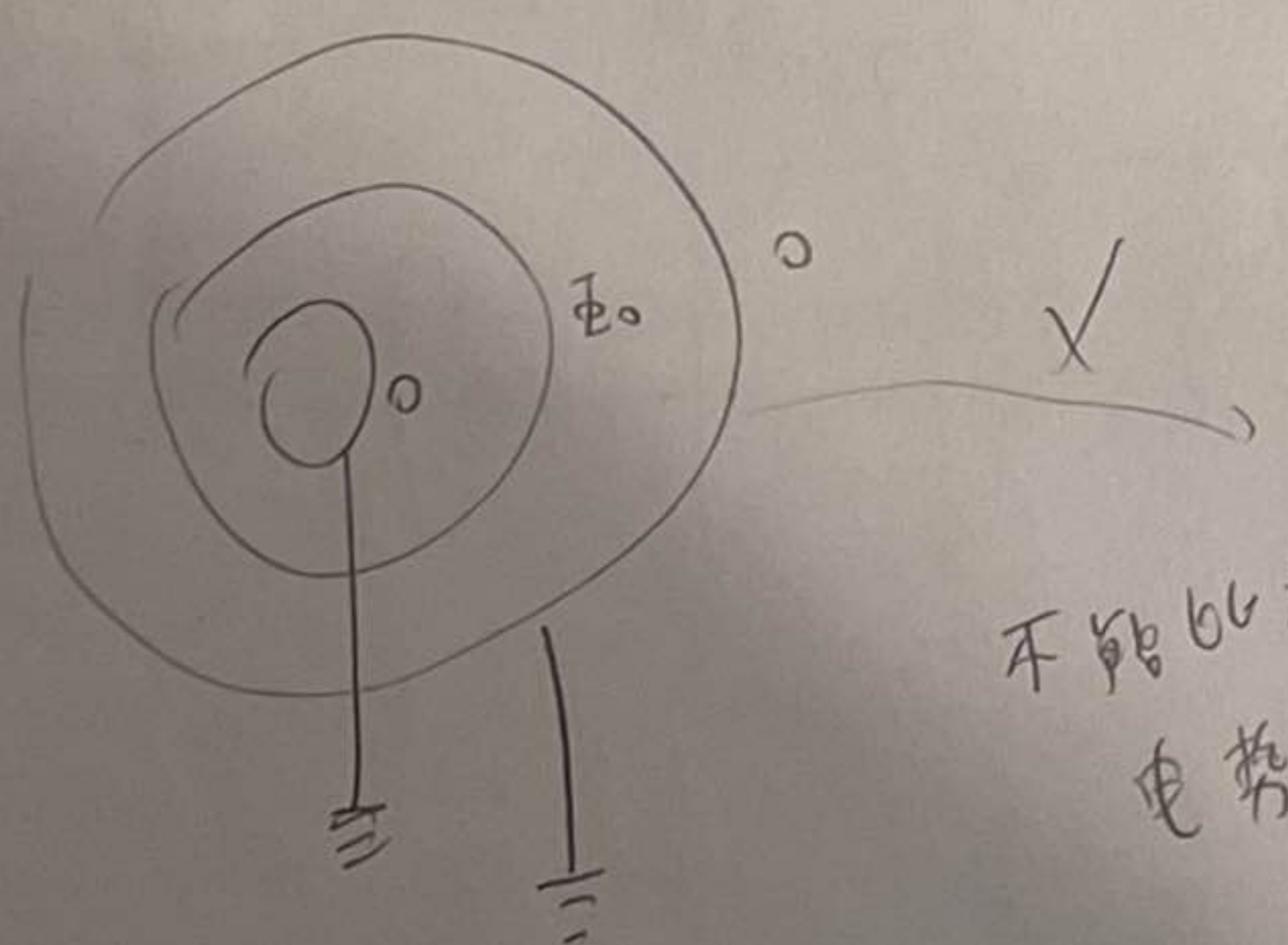
$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial F}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 F}{\partial\theta^2}\quad .]$$

The potentials of the three shells are held at values $\Phi = 0$ (at $r = a$), $\Phi = \Phi_0$ (at $r = b$) and $\Phi = 0$ (at $r = c$). Find the solution to Φ for all r and plot the amplitude of $\mathbf{E}$ as a function of r. [7]

Find the charge density on the conductors of radius a and b. Calculate the total electrostatic energy of the system contained between a and c. [5]

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 0$$

$$r^2\frac{d\Phi}{dr} =$$



不能比无穷远
电势高！

8. A plane electromagnetic wave has the form

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,, \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

for constant vectors $\mathbf{E}_0$, $\mathbf{B}_0$, a positive angular frequency ω and a wave-vector $\mathbf{k}$. Using the vacuum Maxwell equations, prove that

$$\mathbf{k}\cdot\mathbf{E}_0 = 0\,, \quad \mathbf{k}\cdot\mathbf{B}_0 = 0\,, \quad \omega\mathbf{B}_0 = \mathbf{k}\times\mathbf{E}_0$$

and show that $\omega = |\mathbf{k}|c$ where c is the speed of light. What are the boundary conditions on $\mathbf{E}$ and $\mathbf{B}$ on a surface with surface charge density σ and surface current density $\mathbf{s}$? [6]

Consider a plane electromagnetic wave propagating through $x < 0$ onto the surface of a perfect conductor which is placed at $x \geq 0$. Use the forms of the electric and magnetic fields given above to find an expression for $\mathbf{B}$ in terms of $E_0 = |\mathbf{E_0}|$ when both $\mathbf{k}$ and $\mathbf{E_0}$ are orthogonal to the z direction and assume $\mathbf{k} = k(\cos\theta, \sin\theta, 0)$. [4]

The reflection of the incident wave at $x = 0$ produces a reflected wave with electric field

$$\mathbf{E}' = \mathbf{E_0}' e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega t)}$$

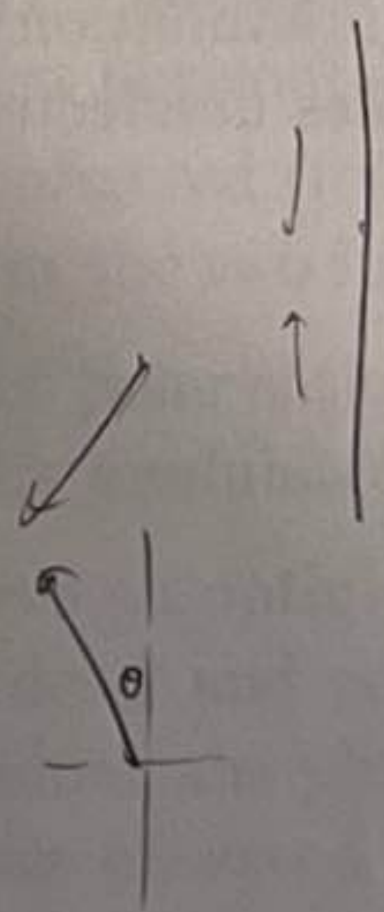
where $\mathbf{k}' = k(-\cos\theta, \sin\theta, 0)$. Use the boundary conditions at $x = 0$ to show that

$$\mathbf{E_0}' = -E_0(\sin\theta, \cos\theta, 0)\,.$$ [5]

Show that the electric charge density on the surface of $x = 0$ takes the form

$$\sigma = \sigma_0 e^{ik(y\sin\theta - ct)}$$

and determine σ_0. Find the magnetic field of the reflected wave and the surface current density $\mathbf{s}$ on the surface $x = 0$. [5]



$\underline{E}_0 = E_0(-\sin\theta, \cos\theta, 0)$

$\underline{E}_0' = E_0(-\sin\theta, -\cos\theta, 0)$

9. A Michelson interferometer with parallel mirrors has a fixed and known optical path length difference (OPD) of L_0. It is illuminated normally to its mirrors with monochromatic plane waves. The frequency of the light varies linearly with time t from $\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)$ to $\left(\langle\nu\rangle + \frac{\Delta\nu}{2}\right)$ as t varies from 0 to T.

Show that the intensity $I(\nu)$ at the output of the interferometer has the form $I(\nu) = I_0(1 + \cos(\Phi_0))$. Find expressions for the interference phase Φ_0 and for the total change of this phase $\Delta\Phi_0 = \Phi_0(T) - \Phi_0(0)$. [5]

A second Michelson interferometer with unknown and fixed OPD $\langle L_1\rangle$ is illuminated simultaneously with the same light. The change of phase of both interferometers is measured with the same relative error $\epsilon = \frac{\sigma_{\Delta\Phi_0}}{\Delta\Phi_0} = \frac{\sigma_{\Delta\Phi_1}}{\Delta\Phi_1}$, where $\sigma_{\Delta\Phi_0}$ and $\sigma_{\Delta\Phi_1}$ are the measurement errors of $\Delta\Phi_0$ and $\Delta\Phi_1$ respectively.

By considering the total phase changes in both interferometers, find an expression for the length ratio $\langle L_1\rangle / L_0$ that is independent of $\langle\nu\rangle$ and $\Delta\nu$. How big is the error $\sigma_{\langle L_1\rangle/L_0}$ of $\langle L_1\rangle / L_0$ which arises from the errors of the total phase changes? [3]

Assume now that, during the time of the illumination, the second interferometer changes its OPD. Its longer arm changes length, linearly with time, by a small amount $0 < \Delta L_1 \ll \langle L_1\rangle$, so that the OPD changes from $\left(\langle L_1\rangle - \frac{\Delta L_1}{2}\right)$ to $\left(\langle L_1\rangle + \frac{\Delta L_1}{2}\right)$. Find an expression for $\Delta\Phi_1$ in this case. Without correcting for such drifts, a measurement of $\langle L_1\rangle$ incurs a drift error σ_{L_1} that is proportional to ΔL_1. Find the constant of this proportionality. [6]

In order to correct the above drift error, a block of material with refractive index $n(\nu)$ can be introduced into the short arm of the second interferometer. Light takes a total geometric path of L_b through this block. Assuming that $n(\nu)$ has the form $n(\nu) = n(0) + \alpha\nu$, find $n(0)$ and α as functions of $\langle\nu\rangle$, L_b, ΔL_1, $\Delta\nu$ and $n_i = n\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)$ (n_i is the refractive index at the start of illumination), such that a measurement of the second interferometer's OPD, $\langle L_1\rangle$ becomes unaffected by long arm drift. [6]

nL_b n'

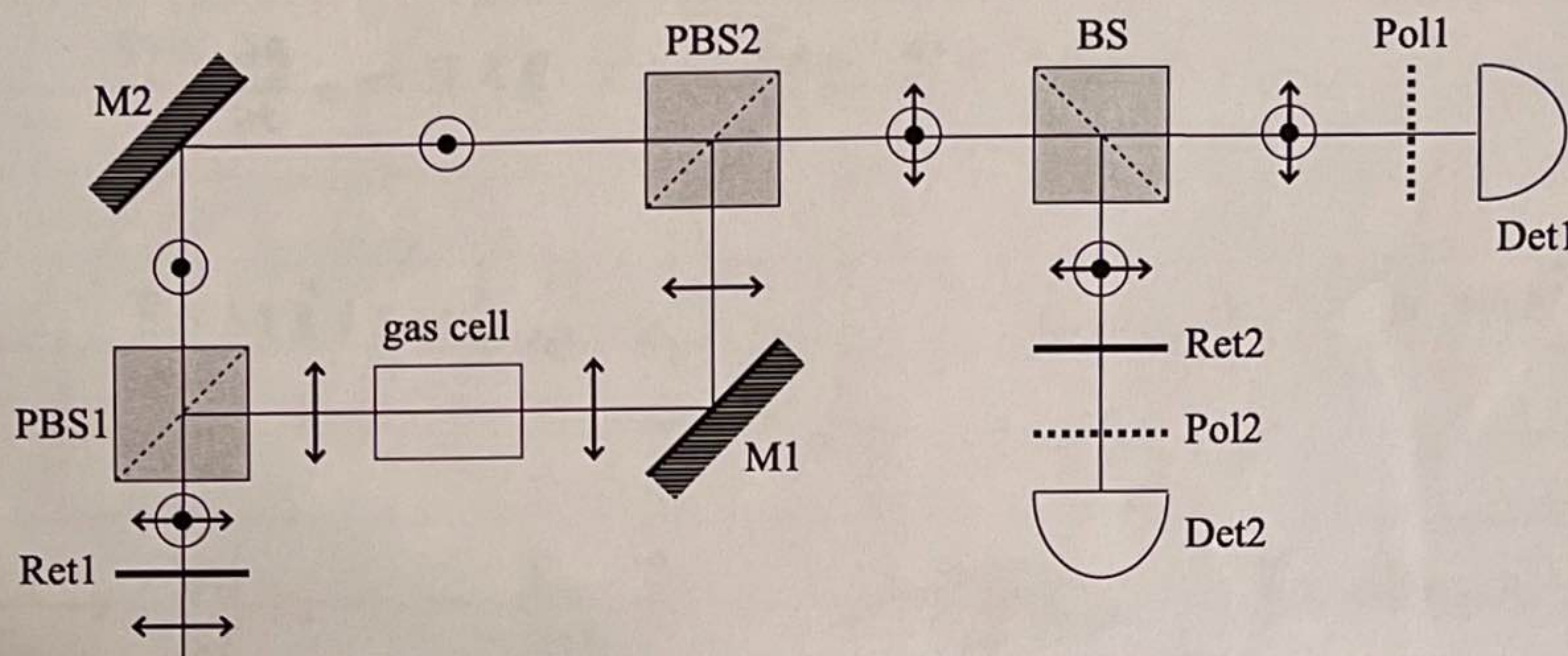
$\nu(n-1)L_b$

$\nu(t)$

$2\pi\nu$

10. How can a uni-axial, birefringent crystal be used to create a retardation plate which retards the phase of one of two perpendicular polarisation directions of a beam of light by a relative phase shift (retardance) $\Delta\phi$? [4]

With the use of a diagram explain how such a crystal can be used as a polarising beam splitter (PBS) which transmits a beam of one linear polarisation direction but reflects a beam of the perpendicular polarisation direction. Indicate the orientation of the optic axis in your diagram. [3]



Ret = Retarder	PBS = Polarising Beam Splitter	Det = Detector
M = Mirror	BS = Beam Splitter	Pol = Polaroid
⊙ = vertically polarised	↔ = horizontally polarised	
	↔⊙ = both polaristion directions present	

The figure above shows a two beam interferometer that can be used to monitor changes of refractive index in a gas cell. It is illuminated with monochromatic light of wavelength λ. The light is linearly, horizontally polarised. The two polarising beam splitters (PBS) pass vertical polarisation and reflect horizontal polarisation.

How should retarder-1 be configured (retardance and orientation of optic axis) to produce beams of equal intensity in the two interferometer arms? [3]

Determine the orientation of polarisation which polaroid-1 should pass so that detector-1 can observe an optimally modulated interference pattern. [2]

Detector-2 should also observe an optimally modulated interference pattern. Determine the configuration of retarder-2 and polariser-2 that will produce an optimally modulated interference signal in which the phase difference of light from the two arms is changed from the phase difference observed on detector-1 by $\delta\phi$. [4]

When $\delta\phi = \pi/2$, the phase difference ϕ_1 of the interference pattern on detector-1 can be determined from just one reading of the two intensities I_1 and I_2. Why is this ability important for measuring changes of refractive index in the gas? Find an expression for ϕ_1 in terms of I_1, I_2 and the average intensities over many interference fringes $\langle I_1 \rangle$ and $\langle I_2 \rangle$. [4]

A2 2011 First Attempt

1. $\underline{E}(t, \underline{r}) = \frac{E_0}{c}(0, 0, \frac{y}{t^2}) = \frac{E_0}{c}\frac{y}{t^2}\hat{\underline{z}}$

$\because \underline{\nabla} \times \underline{E} = -\frac{\partial B}{\partial t}$

$\therefore \frac{\partial B}{\partial t} = -\underline{\nabla} \times \underline{E} = -\frac{E_0}{ct^2} \underline{\nabla} \times (y\hat{\underline{z}})$

$\underline{\nabla} \times (y\hat{\underline{z}}) = \begin{vmatrix} i, & j, & k \\ \partial x & \partial y & \partial z \\ 0 & 0 & y \end{vmatrix} = (1, 0, 0) = \hat{\underline{x}}$

$\rightarrow \frac{\partial B}{\partial t} = -\frac{E_0}{ct^2}\hat{\underline{x}}$ $\quad \cancel{\therefore \frac{\partial |B|}{\partial t} =}$ $\quad \underline{B} = (B_x, 0, 0)$

$\frac{\partial B_x}{\partial t} = -\frac{E_0}{ct^2} \Rightarrow B_x = \frac{E_0}{ct} + K$

As $t \to 0$ $\quad B_x \to 0 \quad \therefore K = 0$

$\therefore B_x = \frac{E_0}{ct}$

$\therefore \underline{B}(t, r) = \cancel{\frac{E_0}{c}(\frac{1}{t},} \frac{E_0}{c}(\frac{1}{t}, 0, 0)$ ✓

2. ~~$\underline{j} \cdot \underline{E}$~~ $\because \underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

$\therefore \underline{j} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B}) - \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$

$\therefore \underline{j} \cdot \underline{E} = \frac{1}{\mu_0}(\underline{\nabla} \times \underline{B}) \cdot \underline{E} - \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \cdot \underline{E}$

$\because U = \frac{\varepsilon_0}{2}\underline{E} \cdot \underline{E} + \frac{1}{2\mu_0}\underline{B} \cdot \underline{B}$

$\therefore \frac{\partial U}{\partial t} = \varepsilon_0 \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} + \frac{1}{\mu_0}\underline{B} \cdot \frac{\partial \underline{B}}{\partial t}$

$$\underline{\nabla}\cdot\underline{S} = \frac{1}{\mu_0}\underline{\nabla}\cdot(\underline{E}\times\underline{B}) = \frac{1}{\mu_0}\underline{B}\cdot(\underbrace{\underline{\nabla}\times\underline{E}}_{=-\frac{\partial B}{\partial t}}) - \frac{1}{\mu_0}\underline{E}\cdot(\nabla\times B)$$

explain.

-3

energy flux

$$\therefore \frac{\partial U}{\partial t} + \underline{\nabla}\cdot\underline{S} = \varepsilon_0 \underline{E}\cdot\frac{\partial \underline{E}}{\partial t} + \cancel{\frac{1}{\mu_0}\underline{B}\cdot\frac{\partial \underline{B}}{\partial t}} - \cancel{\frac{1}{\mu_0}\underline{B}\cdot\frac{\partial \underline{B}}{\partial t}} - \frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B})$$

$$= -\left(\frac{1}{\mu_0}\underline{E}\cdot(\underline{\nabla}\times\underline{B}) - \varepsilon_0\underline{E}\cdot\frac{\partial \underline{E}}{\partial t}\right) = -\underline{j}\cdot\underline{E}$$

change of energy density

ohmic dissipation (work done in moving electrons) (QED)

3. eigenvalues λ, eigenvectors $|v\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\therefore \det\begin{pmatrix} 1-\lambda & 2 & 0 \\ 4 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} = 0$$

$$\therefore (1-\lambda)(3-\lambda)(1-\lambda) - 2(4)(1-\lambda)$$

$$\therefore (\lambda^2-2\lambda+1)(3-\lambda) \overset{-8+8\lambda}{\cancel{-8=0}} \rightarrow -\lambda^3+2\lambda^2-\lambda+3\lambda^2-6\lambda+3-8+8\lambda=0$$

$$\therefore -\lambda^3+5\lambda^2+\lambda-5=0$$

$$\therefore (\lambda-5)(\lambda^2-1)=0 \rightarrow (\lambda-5)(\lambda-1)(\lambda+1)=0$$

$$\therefore \lambda = 5, \pm 1$$

For $\lambda = 5$

$$\begin{pmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightarrow -4a+2b=0$ | $4a-2b=0$ | $-4c=0$

$\rightarrow b=2a$ | $\rightarrow b=2a$ | $\rightarrow c=0$

$\therefore \quad \cancel{|v\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

$$|v\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

✓

For $\lambda = 1$

$$\begin{pmatrix} 0 & 2 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore \quad b = 0 \quad , a = 0 \quad , \quad c$ is arbitrary

$$\therefore \quad |v\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

✓

For $\lambda = -1$

$$\begin{pmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad |v\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

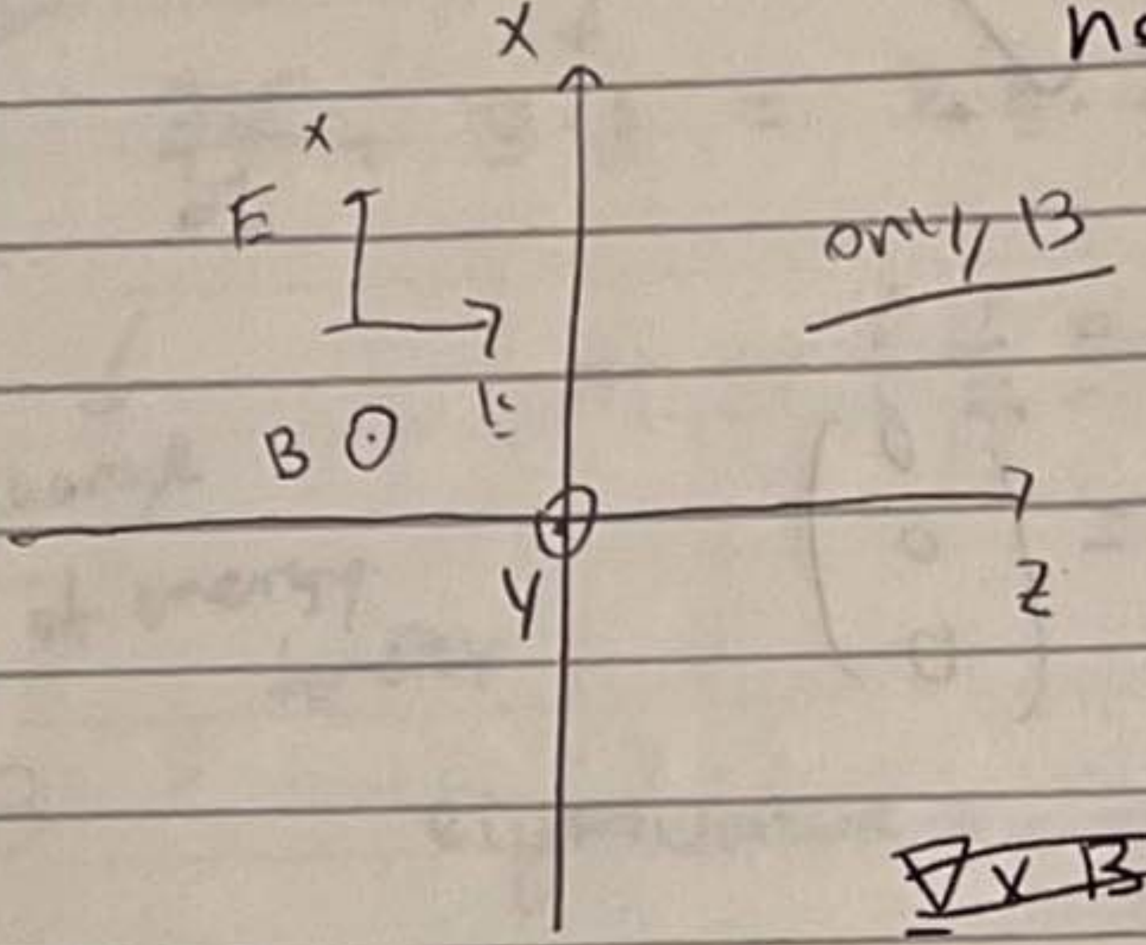
✓

$$\therefore \quad S = \begin{pmatrix} 1/\sqrt{5} & 0 & 1/\sqrt{2} \\ 2/\sqrt{5} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}$$

✓

4. If $\underline{E} = E_0 \cos(kz - \omega t)\,\hat{\underline{x}}$, $\underline{B} = B_0 \cos(kz - \omega t)\,\hat{\underline{y}}$

$\underline{k} = k\hat{\underline{z}}$, ~~then~~ On the other side there is no electric field (all absorbed)



-3 pause time

$\cancel{\nabla \times B}$ $\quad \underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \rightarrow \frac{\partial B}{\partial t} = 0$

$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \rightarrow \underline{\nabla} \times \underline{B} = 0 \rightarrow \underline{B} = \underline{\nabla} \psi$

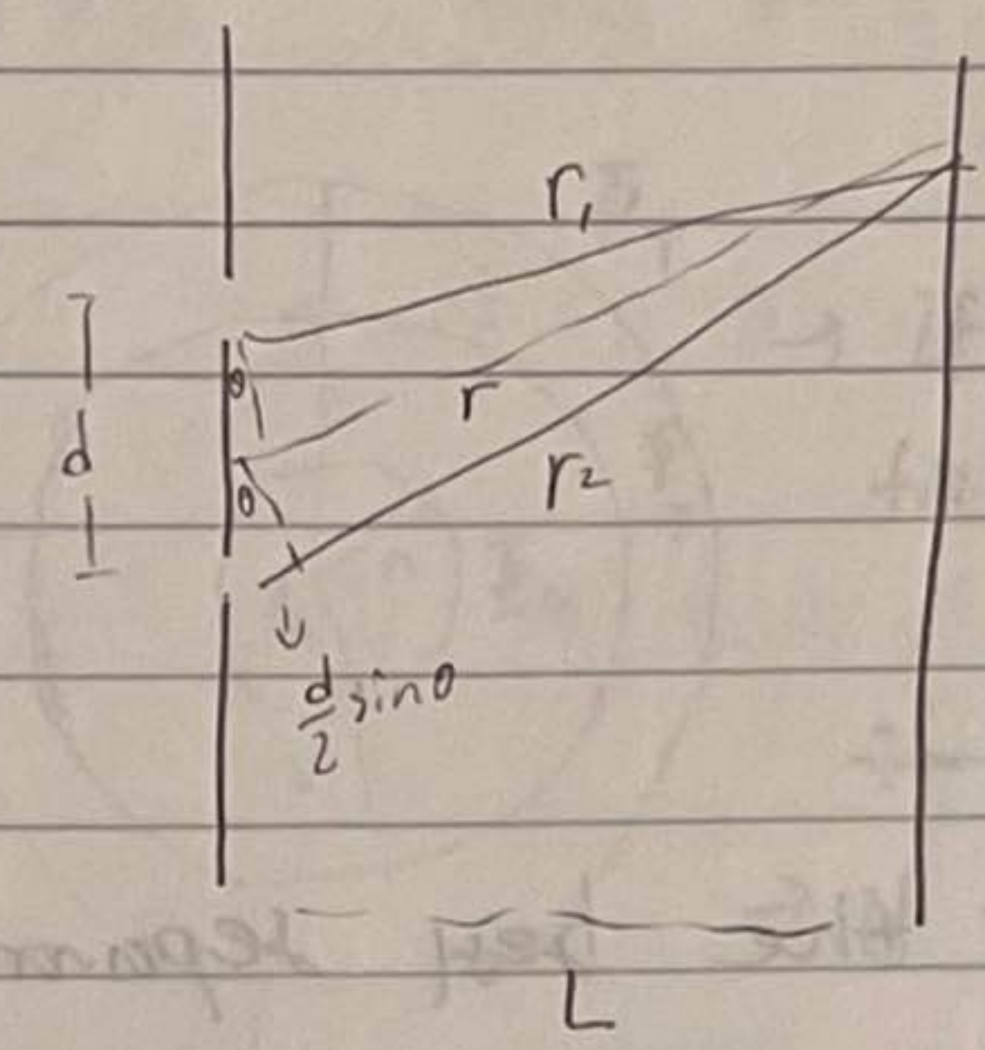
$\underline{\nabla} \cdot \underline{B} = 0 \rightarrow \cancel{\nabla^2 \psi}\ \nabla^2 \psi = 0$

∴ $\underline{B}$ is constant in time. and satisfies the laplace equation $\nabla^2 \psi = 0$ ($\underline{B} = \underline{\nabla}\psi$)

$\underline{B} = 0$ I believe (no infinite energy)

$\frac{10}{20}$

5.



Assume θ is small

$$A = A_0 e^{ikr_1} + A_0 e^{ikr_2}$$

$$= A_0 e^{ik(r - \frac{d}{2}\sin\theta)} + A_0 e^{ik(r + \frac{d}{2}\sin\theta)}$$

$$= A_0 e^{ikr} \left(e^{ik\frac{d}{2}\sin\theta} + e^{-ik\frac{d}{2}\sin\theta} \right)$$

$$\rightarrow \underline{A(\theta) = 2A_0 e^{ikr} \cos\left(\tfrac{1}{2}kd\sin\theta\right)}$$

Intensity $I(\theta) = \cancel{4A_0^2 \cos} \; |A(\theta)|^2$

$$= 4A_0^2 \cos^2\left(\tfrac{1}{2}kd\sin\theta\right)$$

$$= \underline{I_0 \cos^2\left(\tfrac{1}{2}kd\sin\theta\right)}$$

The phase of $A(\theta)$ is e^{ikr}

The phase of plane wave if there is no slit is e^{ikL}

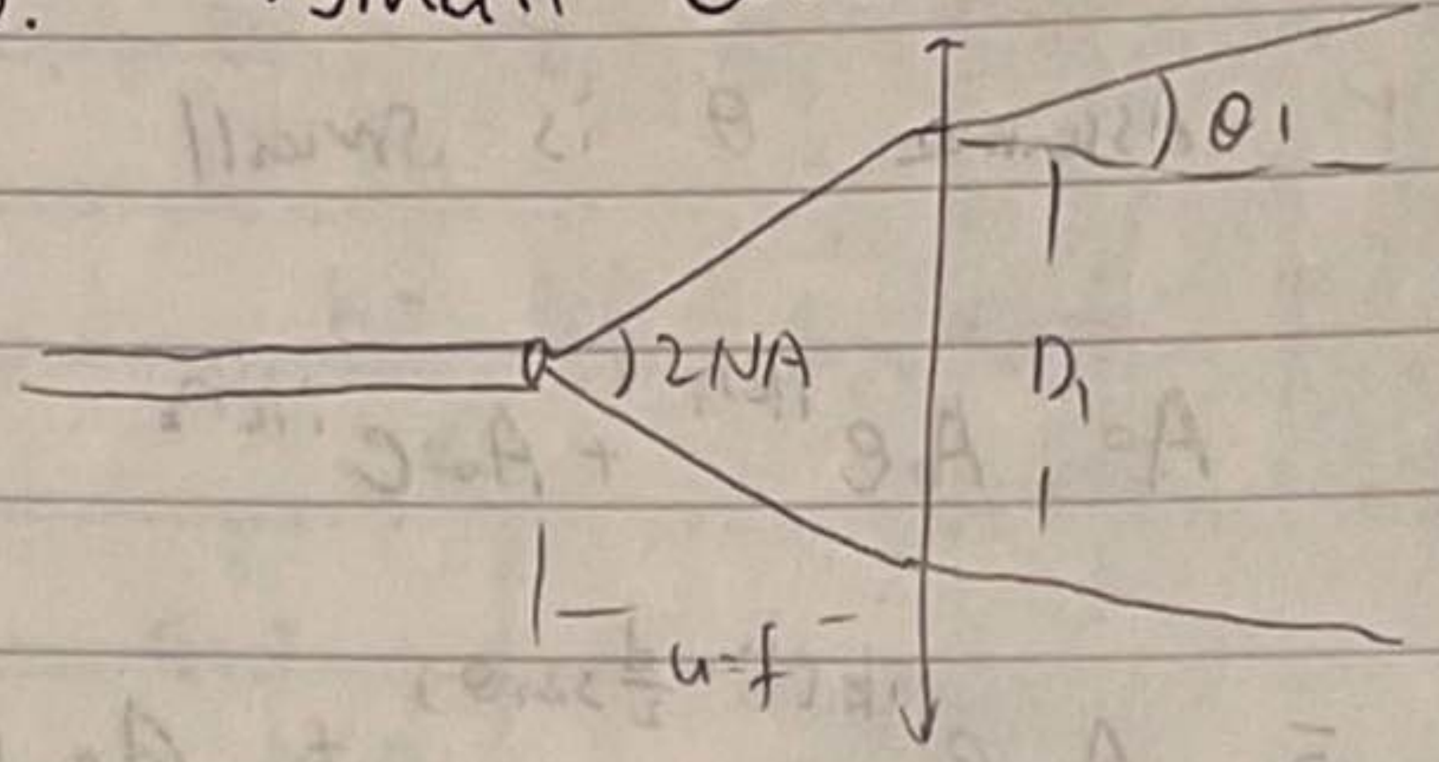
$$\therefore \; \phi_r = k(r - L) = k\left(\frac{L}{\cos\theta} - L\right)$$

$\because \cos\theta \approx 1$ for small θ (first order)

-8

$$\therefore \; \cancel{\phi_r} \; \underline{\phi_r \approx 0} \quad \text{(first order)}$$

6. → small core



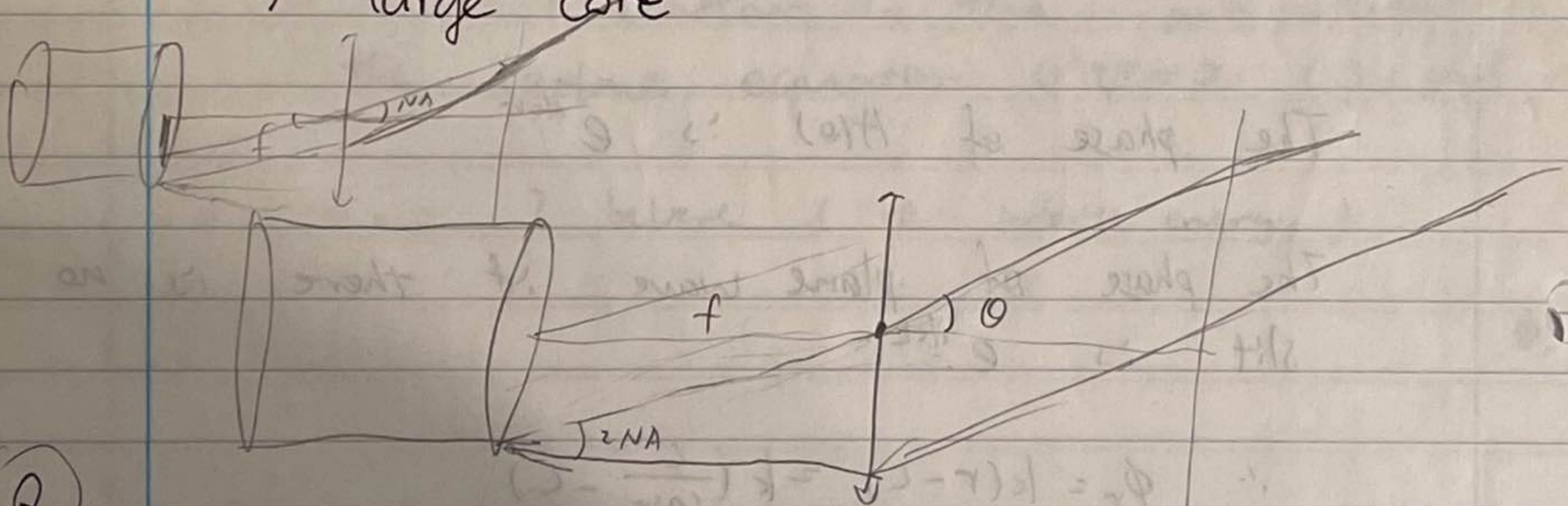
To collimate the beam the best separation

$\underline{u = f}$

Diameter of collimated beam is $\underline{D_1 = 2NA \cdot f}$

the divergence angle $\theta_1 = \frac{\lambda}{D_1} = \frac{\lambda}{2NA \cdot f}$

→ large core



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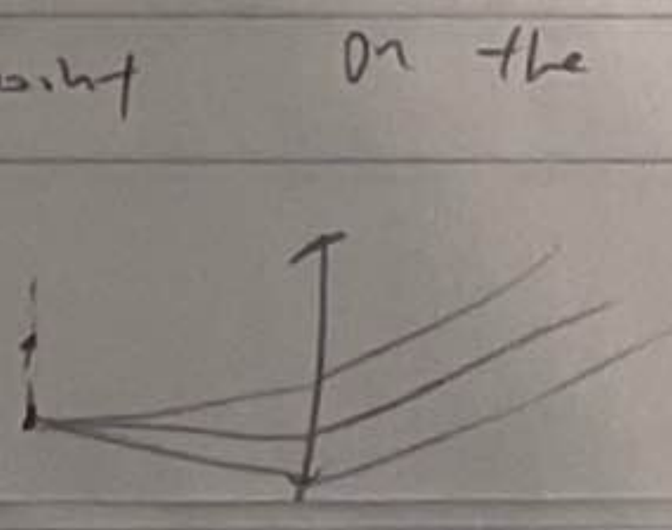
The maximum divergence is caused by light ray coming from the edge of the fibre ~~Assume $\frac{d_2}{f} < NA$ then~~

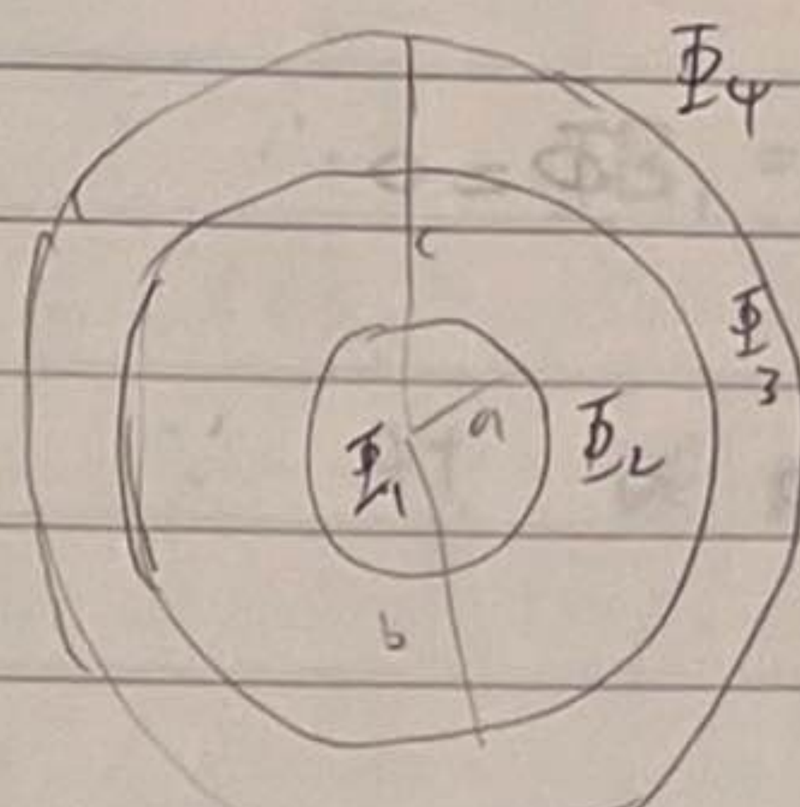
This maximum divergence is $\underline{\theta = \frac{d_2}{2f}}$

light coming from same point on the focal plane will be parallel



10/20

7.



$\cancel{\nabla\times B}$ $\nabla\times\underline{E} = -\frac{\partial B}{\partial t}$

→ If there is no time-varying magnetic field present → $\nabla\times\underline{E} = 0$ → $\underline{E} = -\nabla\Phi$

~~if there is no~~ $\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$

→ If there is no $\cancel{\Phi}$ charge density in space → $\nabla\cdot\underline{E} = 0$

∴ $\cancel{\Phi}$ $\nabla\cdot(-\nabla\Phi) = 0$ → $\underline{\nabla^2\Phi = 0}$

Boundary Conditions :

→ Φ is continuous at $\cancel{\Phi}$ $r = a, b, c$

→ Φ is constant at $r = a, b, c$

~~$\Phi \to 0$ as $r \Rightarrow$~~

→ $\cancel{\Phi}$ Φ doesn't blow up anywhere. (e.g. at $r=0$)

→ Φ vanishes at infinity

$\cancel{\to}$

→ Assuming spherical symmetry → $\Phi = \Phi(r)$

$$\nabla^2\Phi = 0 \to \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 0$$

$$\therefore r^2\frac{d\Phi}{dr} = -B \to B = \text{const}$$

$$\therefore \frac{d\Phi}{dr} = \frac{B}{r^2} \Rightarrow \Phi = B\int -\frac{1}{r^2}dr = \underline{\underline{A + \frac{B}{r}}}$$

(A, B constants)

For $r > c$: $\because \Phi(r \to \infty) \to 0 \quad \therefore A_4 = 0$

$\therefore \Phi_4 = \frac{B_4}{r} \quad \because$ At $r = c \quad \Phi = 0$

$\therefore \cancel{\Phi_4} \frac{B_4}{c} = 0 \to B_4 = 0$

$\therefore \underline{\Phi_4 = 0}$

For $b < r < c$: $\Phi_3 = A_3 + \frac{B_3}{r}$

$\because$ at $r = c \quad \Phi_3 = 0 \quad \therefore A_3 + \frac{B_3}{c} = 0$

$\because$ at $r = b \quad \Phi_3 = \Phi_0 \quad \therefore A_3 + \frac{B_3}{b} = \Phi_0$

$\therefore cA_3 - bA_3 = -b\Phi_0 \quad \therefore A_3 = -\frac{b\Phi_0}{c-b}$

$\therefore B_3 = \frac{cb\Phi_0}{c-b}$

$\therefore \Phi_3 = -\frac{b\Phi_0}{c-b} + \frac{cb\Phi_0}{(c-b)r} = \underline{\frac{b\Phi_0}{c-b}\left(\frac{c}{r} - 1\right)}$

For $a < r < b \quad \Phi_2 = A_2 + \frac{B_2}{r}$

~~at~~ $r = a \quad \Phi_2 = 0 \quad \therefore A_2 + \frac{B_2}{a} = 0$

$r = b \quad \Phi_2 = \Phi_0 \quad \therefore A_2 + \frac{B_2}{b} = \cancel{0} \Phi_0$

$\therefore \underline{\Phi_2 = \frac{b\Phi_0}{b-a}\left(\cancel{-}1 - \frac{a}{r}\right)}$

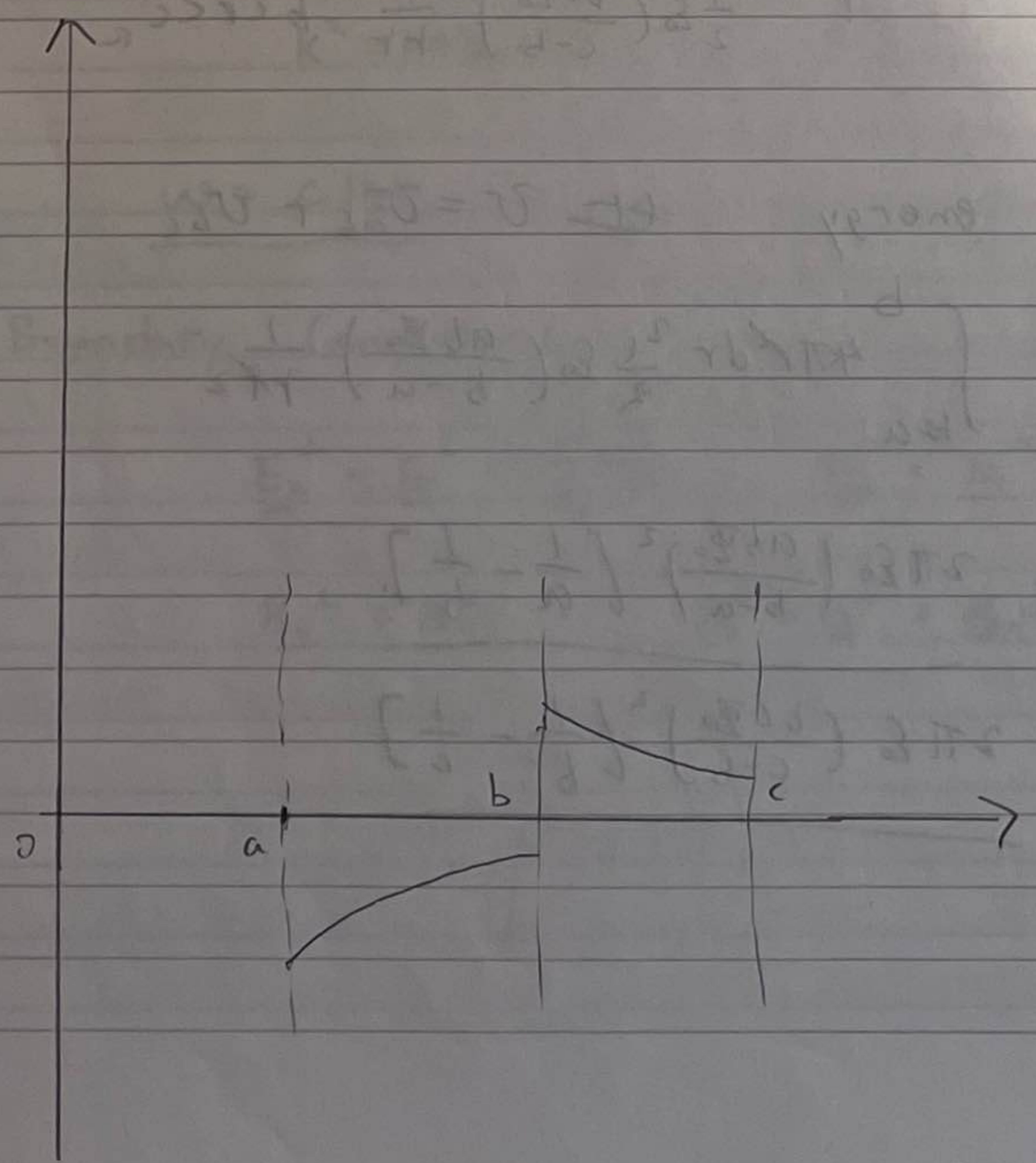
For $\cancel{\alpha}$ $0<r<a$ Φ $\because \frac{1}{r} \to \infty$ as $r \to 0$

$\therefore \quad B_1 = 0 \quad \therefore \Phi_1 = A_1$

$\because$ At $\cancel{\alpha}$ $r=1$ $\quad \Phi_1 = 0 \quad \therefore A_1 = 0 \to \underline{\Phi_1 = 0}$

$\therefore$ the Electric field $\underline{E} = -\nabla\Phi = -\frac{\partial\Phi}{\partial r}\hat{r}$ is

$$\underline{E} = \begin{cases} 0 & 0<r<a \\ -\frac{ab\Phi_0}{b-a}\frac{1}{r^2}\hat{r} & a<r<b \\ \frac{bc\Phi_0}{c-b}\frac{1}{r^2}\hat{r} & \\ 0 & \end{cases}$$



On the surface of conductors, charge density

$$\sigma = \varepsilon_0 E_\perp = \varepsilon_0 E_r = \varepsilon_0 E$$

$$\therefore \sigma(a) = -\frac{ab\varepsilon_0 \Phi_0}{a(b-a)}$$

$$\sigma(b) = \varepsilon_0 \left(\frac{bc\Phi_0}{c-b}\frac{1}{b^2} - \left(-\frac{ab\Phi_0}{b-a}\right)\frac{1}{b^2} \right)$$

$$= \varepsilon_0 \Phi_0 \left(\frac{c}{b(c-b)} + \frac{a}{b(b-a)} \right)$$

Energy density $u = \frac{1}{2}\varepsilon E^2$

$$u = \begin{cases} \frac{1}{2}\varepsilon_0 \left(\frac{ab\Phi_0}{b-a}\right)^2 \frac{1}{r^4} & a<r<b. \\ \frac{1}{2}\varepsilon_0 \left(\frac{bc\Phi_0}{c-b}\right)^2 \frac{1}{r^4} & b<r<c \end{cases}$$

total energy $U = U_{ab} + U_{bc}$

$$U_{ab} = \int_a^b 4\pi r^2 dr \frac{1}{2}\varepsilon_0 \left(\frac{ab\Phi_0}{b-a}\right)^2 \frac{1}{r^4}$$

$$= 2\pi\varepsilon_0 \left(\frac{ab\Phi_0}{b-a}\right)^2 \left[\frac{1}{a} - \frac{1}{b}\right].$$

$$U_{bc} = 2\pi\varepsilon_0 \left(\frac{bc\Phi_0}{c-b}\right)^2 \left[\frac{1}{b} - \frac{1}{c}\right]$$

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$$\underline{E} = \underline{E_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)} \qquad \underline{B} = \underline{B_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

$$\nabla \times \underline{E} = i\underline{k} \times \underline{E_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)} \quad ; \quad -\frac{\partial \underline{B}}{\partial t} = i\omega \underline{B_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

$$\because \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \quad \therefore \quad \underline{\omega \underline{B_0} = \underline{k} \times \underline{E_0}}$$

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} = 0 \rightarrow i\underline{k}\cdot\underline{E_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)} = 0 \rightarrow \underline{\underline{k}\cdot\underline{E_0} = 0}$$

in vacuum

$$\nabla \cdot \underline{B} = 0 \rightarrow i\underline{k}\cdot\underline{B_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)} = 0 \rightarrow \underline{\underline{k}\cdot\underline{B_0} = 0}$$

vacuum wave equations: $\nabla^2 \underline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$.

$$\nabla^2 \underline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

$$\therefore -k^2 \underline{E_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)} = -\mu_0 \varepsilon_0 \omega^2 \underline{E_0} e^{i(\underline{k}\cdot\underline{r} - \omega t)}$$

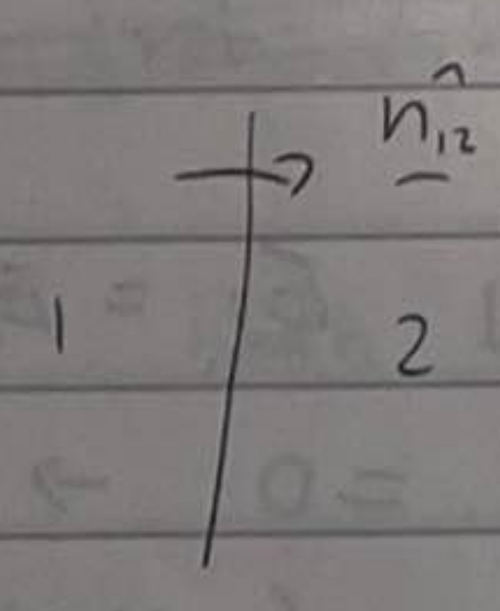
$$\rightarrow k^2 = \mu_0 \varepsilon_0 \omega^2 \quad \therefore \quad k^2 = \frac{\omega^2}{c^2} \qquad \left(c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}\right)$$

$$\therefore \underline{\omega = c|\underline{k}|}$$

Boundary Conditions :

$$\underline{E_2}^{\perp} - \underline{E_1}^{\perp} = \frac{\sigma}{\varepsilon_0} \hat{\underline{n}}_{12} \qquad \underline{E_2}^{\parallel} = \underline{E_1}^{\parallel}$$

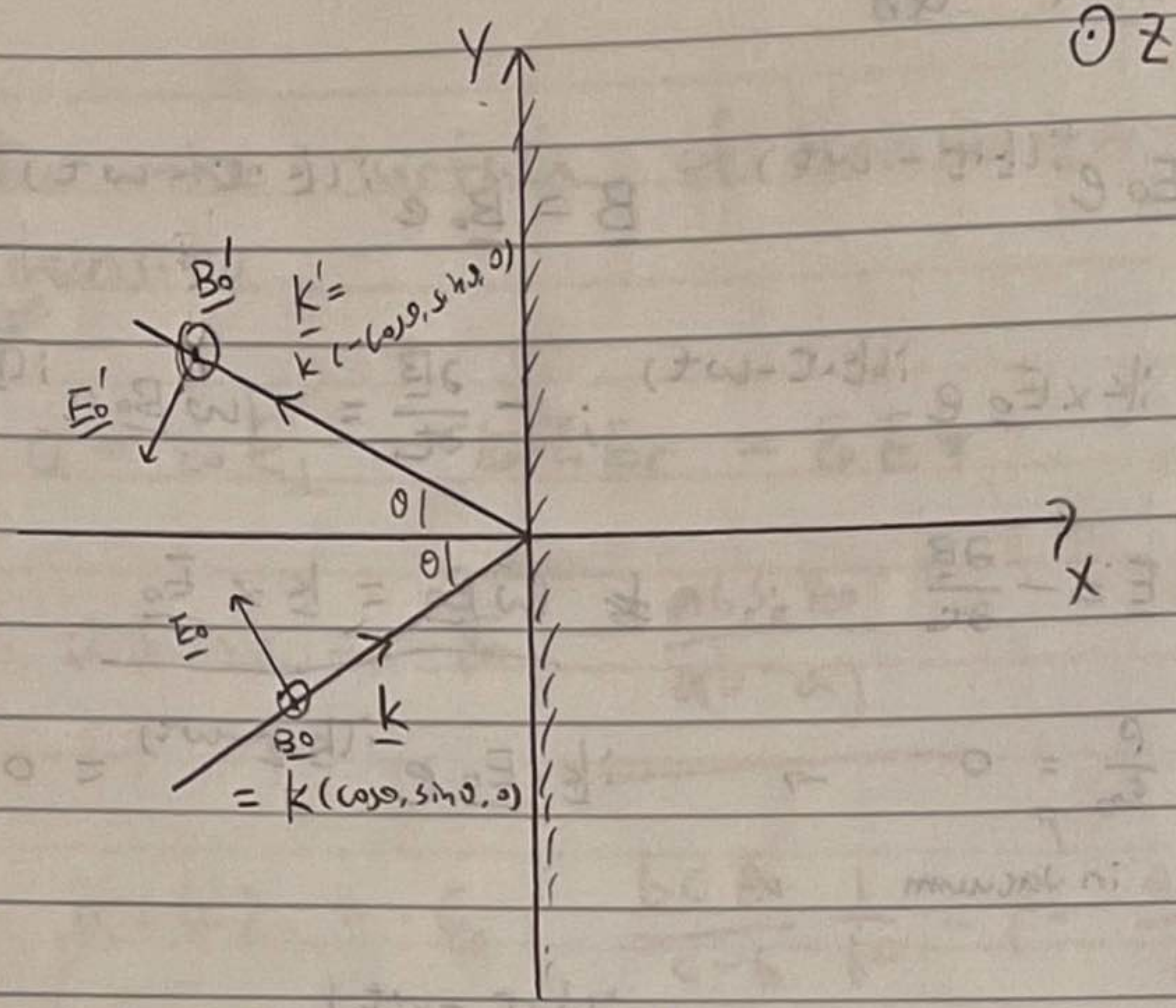
$$\underline{B_2}^{\perp} = \underline{B_1}^{\perp} \qquad \underline{B_2}^{\parallel} - \underline{B_1}^{\parallel} = \mu_0 \underline{S} \times \hat{\underline{n}}_{12}$$



3 waves same plane

$\sigma \to \infty$
$E = 0$, $B = 0$.

E_0, E_0' same polarisation



$\underline{k} \cdot \underline{E}_0 = 0 \quad \underline{k} \cdot \underline{B}_0 = 0$, and $\omega \underline{B_0} = \underline{k} \times \underline{E_0}$

$\rightarrow \omega\, \underline{E_0} \cdot \underline{B_0} = (\underline{k} \times \underline{E_0}) \cdot \underline{E_0} = 0$

$\rightarrow \underline{E_0} \cdot \underline{B_0} = 0$

$\therefore$ $\underline{k}$, $\underline{E_0}$, $\underline{B_0}$ are mutually perpendicular

$\therefore$ $\underline{B_0}$ is along z $\rightarrow \underline{B_0} = |\underline{B_0}| \hat{\underline{z}}$

$\because \underline{k} \cdot \underline{E_0} = 0 \quad \therefore \quad \omega |\underline{B_0}| = |\underline{k}||\underline{E_0}| \sin(90°) = |\underline{k}||\underline{E_0}|$

$\therefore \cancel{|\underline{k}|} c |\underline{B_0}| = \cancel{|\underline{k}|} |\underline{E_0}| \quad \therefore \quad |\underline{B_0}| = \frac{|\underline{E_0}|}{c}$

$\therefore \underline{B_0} = \frac{|\underline{E_0}|}{c} \hat{\underline{z}} = \frac{E_0}{c} \hat{\underline{z}} \quad \therefore \quad \underline{B} = \frac{E_0 \hat{\underline{z}}}{c} \exp(i(\underline{k} \cdot \underline{r} - \omega t))$

$\rightarrow \underline{B} = \frac{E_0 \hat{\underline{z}}}{c} \exp(i(k\cos\theta x + k\sin\theta y - \omega t))$

perfect conductor : $\underline{E_t} = 0$, $\underline{B_t} = 0$ for ~~ ~~ transmitted wave at $x > 0$

(No transmission for perfect conductors)

$\because \underline{E_2^{\parallel}} = \underline{E_1^{\parallel}}$ and $\underline{E_0} = E_0(-\sin\theta, \cos\theta, 0)$

$\therefore \underline{E_0} \cdot \hat{\underline{y}} + \underline{E_0'} \cdot \hat{\underline{y}} = 0 \rightarrow E_0 \cos\theta + \underline{E_0'} \cdot \underline{y} = 0$

$\therefore \ \underline{E_0}' \cdot \hat{\underline{y}} = -E_0 \cos\theta$

Same plane of polarisation $\therefore \ \underline{E_0}' \cdot \hat{\underline{z}} = 0$

$\because \ \underline{E_0}' \cdot \underline{k}' = 0 \ \therefore (\underline{E_0}' \cdot \hat{\underline{x}})(-k\cos\theta) + (\underline{E_0}' \cdot \underline{y})(k\sin\theta) = 0$

$(\underline{k}' = k(-\cos\theta, \sin\theta, 0)$

$$\therefore (\underline{E_0}' \cdot \hat{\underline{x}}) = \frac{E_0 \cos\theta \, k \sin\theta}{-k\cos\theta} = -E_0 \sin\theta$$

$$\therefore \ \underline{E_0}' = -E_0(\sin\theta, \cos\theta, 0)$$

$$\because \ \underline{E_2}'' - \underline{E_1}'' = \frac{\sigma}{\varepsilon_0} \hat{\underline{n}}_{12} = \frac{\sigma}{\varepsilon_0} \hat{\underline{x}} \qquad \underline{E_2}'' = 0 \ \text{(in conductor)}$$

$$\therefore \ \underline{E_1}'' = -\frac{\sigma}{\varepsilon_0} \hat{\underline{x}} \ , \quad E_1'' = \cancel{E_1} \ \underline{E}(0,t) \cdot \hat{\underline{x}} + \underline{E}'(0,t) \cdot \hat{\underline{x}}$$

ie ~~E_1''~~ ~~$\underline{E}(0,t)$~~.

$$E_1'' = -2E_0 \sin\theta \, e^{ik(y\sin\theta - ct)} = \frac{-\sigma}{\varepsilon_0}$$

$$\therefore \ \sigma = 2E_0 \varepsilon_0 \sin\theta \, e^{ik(y\sin\theta - ct)}$$

$$\sigma_0 = 2E_0 \varepsilon_0 \sin\theta$$

Similarly ~~$(\underline{B_0} \cdot \hat{\underline{x}}) = (\underline{B_0}' \cdot \hat{\underline{x}}) = 0$~~

$\underline{B_0}$, $\underline{B_0}'$ both only along $\hat{\underline{z}}$

(same polarisation) ~~and $\underline{B}_1^{\parallel} = \underline{B}_2^{\parallel}$~~

$$\underline{B}_2'' - \underline{B}_1'' = \underline{S} \times \hat{\underline{n}}_{12} = \mu_0 \underline{S} \times \hat{\underline{x}}$$

$\underline{B}_2'' = 0$ (perfect conductor)

$$B_1'' = \frac{2E_0}{c} e^{ik(y\sin\theta - \omega t)}$$

$$\therefore \quad -\frac{2E_0}{c} e^{ik(y\sin\theta - \omega t)} \hat{\underline{z}} = \mu_0 \underline{S} \times \hat{\underline{x}}$$

$$\therefore \quad \underline{S} = \frac{2E_0}{\mu_0 c} e^{ik(y\sin\theta - \omega t)} \hat{\underline{y}}$$

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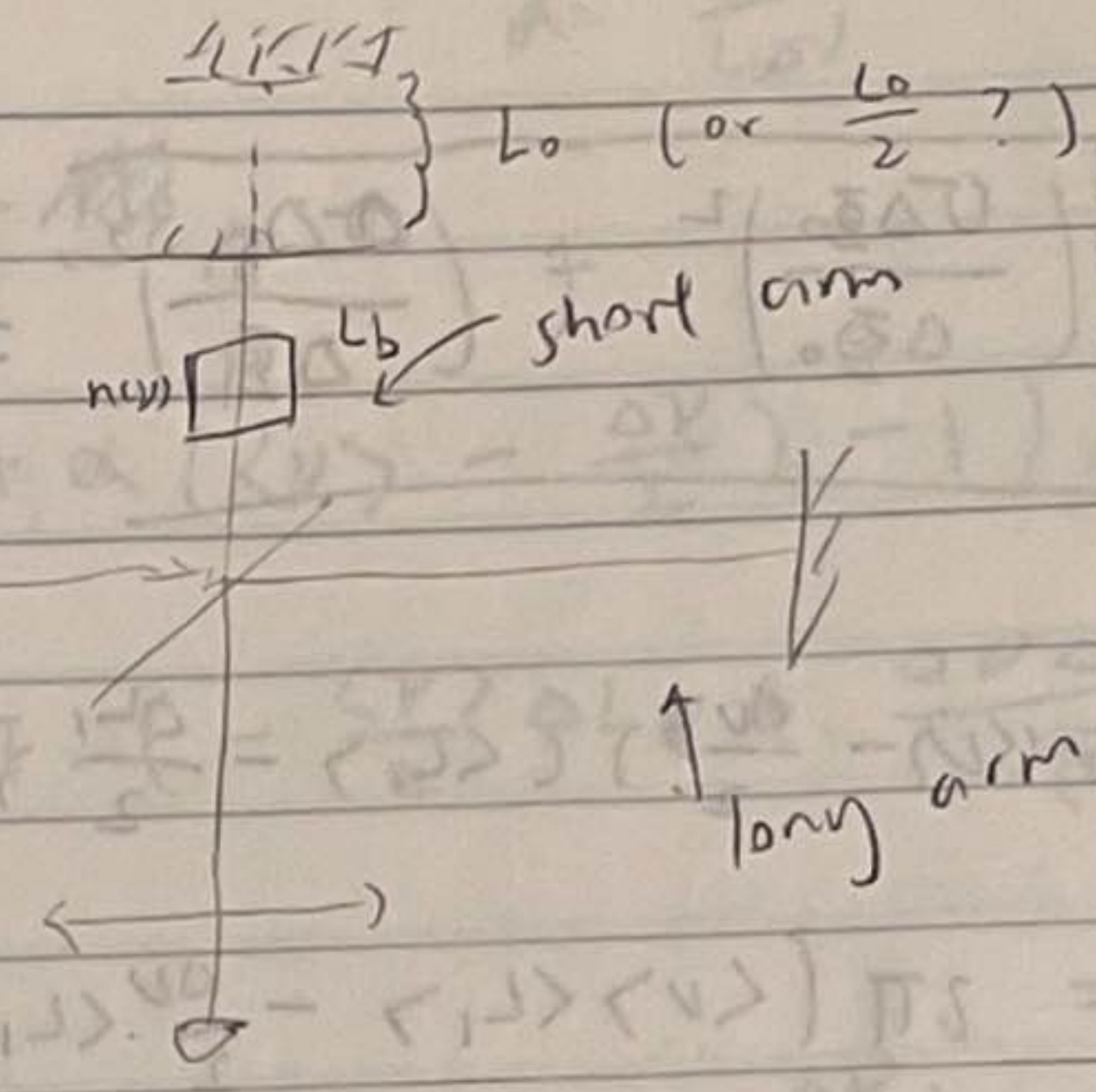
9.



OPD $= L_0$

the frequency $\nu = \nu(t) \rightarrow$ ~~optical path~~

total phase difference $= k \times OPD = 2\pi\nu L_0$

$\therefore$ Amplitude $A = A_0 + A_0 e^{i2\pi\nu L_0}$

$$= A_0(1 + e^{i2\pi\nu L_0}) = \cancel{Ae}$$

intensity $I \propto |A|^2 = |A_0|^2 \,/\, [(1+\cos(2\pi\nu L_0)) + i\sin(2\pi\nu L_0)]$

$$= 2|A_0|^2 + 2|A_0|^2 \cos(2\pi\nu L_0)$$

$\therefore$ $\underline{I = I_0(1+\cos(2\pi\nu L_0))}$ $\qquad \underline{\Phi_0 = 2\pi\nu L_0}$

$$\Delta\Phi_0 = \Phi_0(T) - \Phi_0(0) = 2\pi(\langle\nu\rangle + \tfrac{\Delta\nu}{2})L_0 - 2\pi(\langle\nu\rangle - \tfrac{\Delta\nu}{2})L_0$$

$$= \underline{2\pi\Delta\nu L_0}$$

$\rightarrow$ $\Delta\Phi_0 = 2\pi\Delta\nu L_0 \qquad \Delta\Phi_1 = 2\pi\Delta\nu\langle L_1\rangle$

$\therefore$ $\dfrac{\langle L_1\rangle}{L_0} = \dfrac{\Delta\Phi_1}{\Delta\Phi_0} = \underline{\dfrac{\sigma\Delta\Phi_1}{\sigma\Delta\Phi_0}}$

$$\frac{\sigma_{\langle L_1\rangle/L_0}}{\langle L_1\rangle/L_0} = \sqrt{\left(\frac{\sigma_{\Delta\Phi_0}}{\Delta\Phi_0}\right)^2 + \left(\frac{\sigma_{\Delta\Phi_1}}{\Delta\Phi_1}\right)^2} \Rightarrow \sigma_{\langle L_1\rangle/L_0} = \frac{\langle L_1\rangle}{L_0}\sqrt{2}\,\varepsilon$$

$$\therefore\ \Phi_1(0) = 2\pi\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)\left(\langle L_1\rangle - \frac{\Delta L_1}{2}\right)$$

$$\cancel{\Phi_1(T)} = 2\pi\left(\langle\nu\rangle\langle L_1\rangle - \frac{\Delta\nu}{2}\langle L_1\rangle - \frac{\langle\nu\rangle}{2}\Delta L_1 + \frac{\Delta\nu\Delta L_1}{4}\right)$$

$$\Phi_1(T) = 2\pi\left(\langle\nu\rangle + \frac{\Delta\nu}{2}\right)\left(\langle L_1\rangle + \frac{\Delta L_1}{2}\right)$$

$$= 2\pi\left(\langle\nu\rangle\langle L_1\rangle + \frac{\Delta\nu}{2}\langle L_1\rangle + \frac{\langle\nu\rangle}{2}\Delta L_1 + \frac{\Delta\nu\Delta L_1}{4}\right)$$

$$\therefore\ \Delta\Phi_1 = \Phi_1(T) - \Phi_1(0)$$

$$= 2\pi(\Delta\nu\langle L_1\rangle + \langle\nu\rangle\Delta L_1)$$

Drift error $\sigma_{L_1} = 2\pi(\Delta\nu\langle L_1\rangle + \langle\nu\rangle\Delta L_1) - 2\pi\Delta\nu\langle L_1\rangle$

$$= \underline{2\pi\langle\nu\rangle\Delta L_1}$$

$\rightarrow$ proportionality constant $= 2\pi\langle\nu\rangle$

$$\frac{\sigma_{\langle L_1\rangle}}{L_0} = \cancel{\frac{\langle L_1\rangle}{L_0}} = \frac{2\pi(\langle L_1\rangle\Delta\nu + \langle\nu\rangle\Delta L_1)}{2\pi L_0\Delta\nu} = \frac{\langle L_1\rangle}{L_0} + \frac{\langle\nu\rangle}{\Delta\nu}\frac{\Delta L_1}{L_0}$$

$$\therefore\ \sigma_{L_1} = \frac{\langle\nu\rangle}{\Delta\nu}\Delta L_1$$

Introducing the block, we have

$$\cancel{n\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)L_b\, 2\pi\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)}$$

$$\left(\cancel{n\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right)} - 1\right)L_b\, 2\pi\left(\langle\nu\rangle - \frac{\Delta\nu}{2}\right) =$$

$$\cancel{\rightarrow \pi\Delta\nu\langle L_1\rangle \ast 2\pi\left(-\frac{\langle\nu\rangle}{2}\Delta L_1 + \frac{\Delta\nu\Delta L_1}{4}\right)}$$

$(n(0) + \alpha\langle\nu\rangle - \frac{\Delta\nu}{2}) - 1) L_b = \langle L_1\rangle - \frac{\Delta L_1}{2}$

$(n(0) + \alpha(\langle\nu\rangle + \frac{\Delta\nu}{2}) - 1) L_b = \langle L_1\rangle + \frac{\Delta L_1}{2}$

$\alpha = \frac{\Delta L}{L_b \Delta\nu}$

$\rightarrow$

ignore

$$(n(0) + \alpha(\langle\nu\rangle - \frac{\Delta\nu}{2}) - 1) L_b \, 2\pi (\langle\nu\rangle - \frac{\Delta\nu}{2})$$

$$= 2\pi \left(-\frac{\langle\nu\rangle}{2}\Delta L_1 + \frac{\Delta\nu\Delta L_1}{4}\right) \quad ①$$

ignore

ignore

Also

$$\rightarrow (n(0) + \alpha(\langle\nu\rangle + \frac{\Delta\nu}{2}) - 1) L_b \, 2\pi (\langle\nu\rangle + \frac{\Delta\nu}{2})$$

$$= 2\pi \left(+\frac{\langle\nu\rangle}{2}\Delta L_1 + \frac{\Delta\nu\Delta L_1}{4}\right) \quad ②$$

ignore

I think

② - ①

$\Rightarrow$ ~~$\alpha\Delta\nu L_b 2\pi$~~

$$\alpha\Delta\nu L_b 2\pi\langle\nu\rangle = 2\pi\langle\nu\rangle\Delta L_1$$

$$\therefore \quad \alpha = \frac{\Delta L_1}{\Delta\nu L_b}$$

$$n_0 L_b 2\pi\langle\alpha\rangle + \alpha\langle\nu\rangle^2 L_b 2\pi + \alpha\Delta\nu \pi L_b\langle\alpha\rangle$$

$$= -\pi\langle\alpha\rangle\Delta L_1$$

$$\rightarrow n(0) L_b = -2\alpha\langle\nu\rangle L_b - \alpha\Delta\nu L_b - \Delta L_1$$

$$\therefore \quad n(0) = -\alpha(2\langle\nu\rangle + \Delta\nu) - \frac{\Delta L_1}{L_b}$$

$$L_b(n(\nu) - n_i)\, 2\pi\nu = (L_1(\nu) - L_i)\, 2\pi\nu$$

when L_1 goes from $\langle L_1\rangle - \frac{\Delta L_1}{2}$ to $\langle L_1\rangle + \frac{\Delta L_1}{2}$

ν goes from $\langle\nu\rangle - \frac{\Delta\nu}{2}$ to $\langle\nu\rangle + \frac{\Delta\nu}{2}$

∴ slope $= \frac{\Delta L_1}{\Delta\nu}$, $L_i = \langle L_1\rangle - \frac{\Delta L_1}{2}$

$\nu_i = \langle\nu\rangle - \frac{\Delta\nu}{2}$

$$(L(\nu) - L_i) = \frac{\Delta L_i}{\Delta\nu}(\nu - \nu_i)$$

$$\therefore L(\nu) - \langle L_1\rangle + \frac{\Delta L_1}{\cancel{2}} = \frac{\Delta L_i}{\Delta\nu}\left(\nu - \langle\nu\rangle + \frac{\cancel{\Delta\nu}}{\cancel{2}}\right)$$

$$\therefore L(\nu) = \langle L_i\rangle + \frac{\Delta L_1}{\Delta\nu}(\nu - \langle\nu\rangle)$$

$$\therefore L_b(n(\nu) - n_i) = \cancel{\langle L_i\rangle} + \frac{\Delta L_1}{\Delta\nu}\nu - \frac{\Delta L_1}{\Delta\nu}\langle\nu\rangle - \cancel{\langle L_1\rangle} + \frac{\Delta L_1}{2}$$

$$\therefore n(\nu) = \underbrace{n_i + \frac{\Delta L_1}{2L_b} - \frac{\Delta L_1\langle\nu\rangle}{L_b\Delta\nu}}_{n(0)} + \underbrace{\frac{\Delta L_1}{L_b\Delta\nu}}_{\alpha}\nu$$

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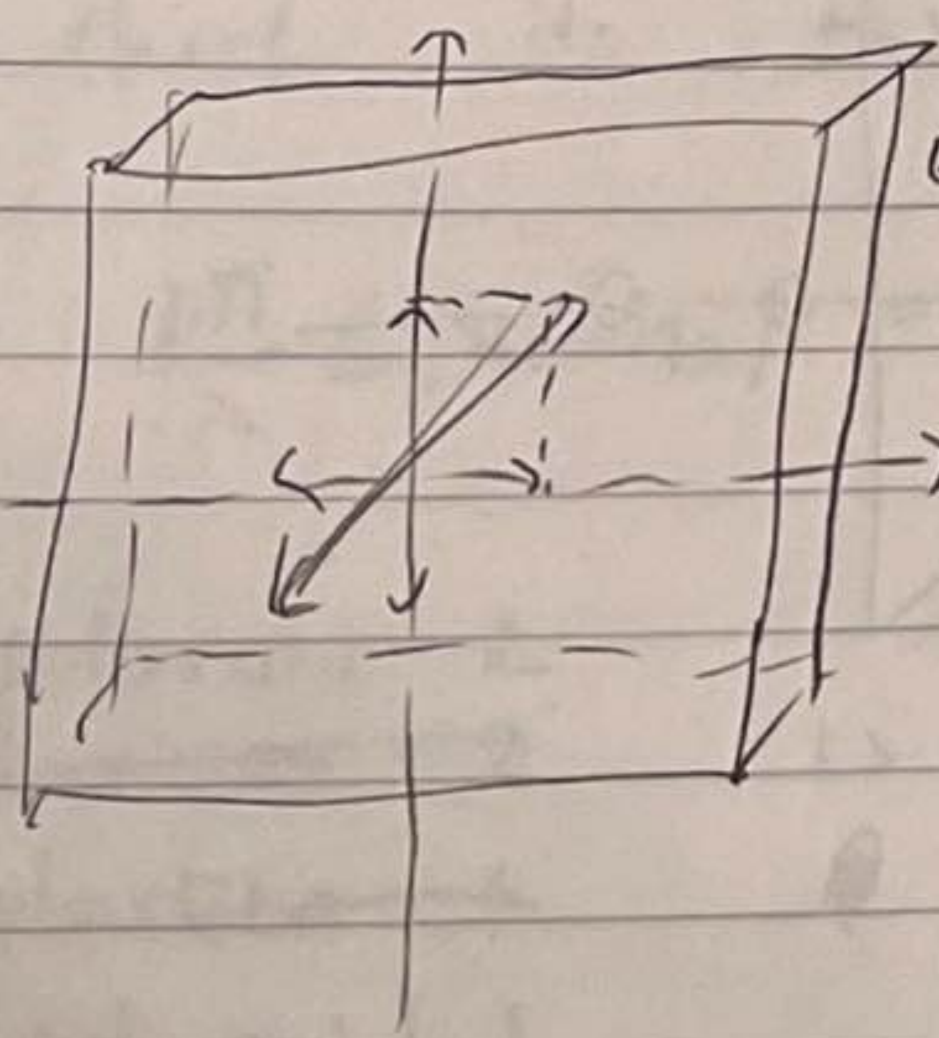
10.



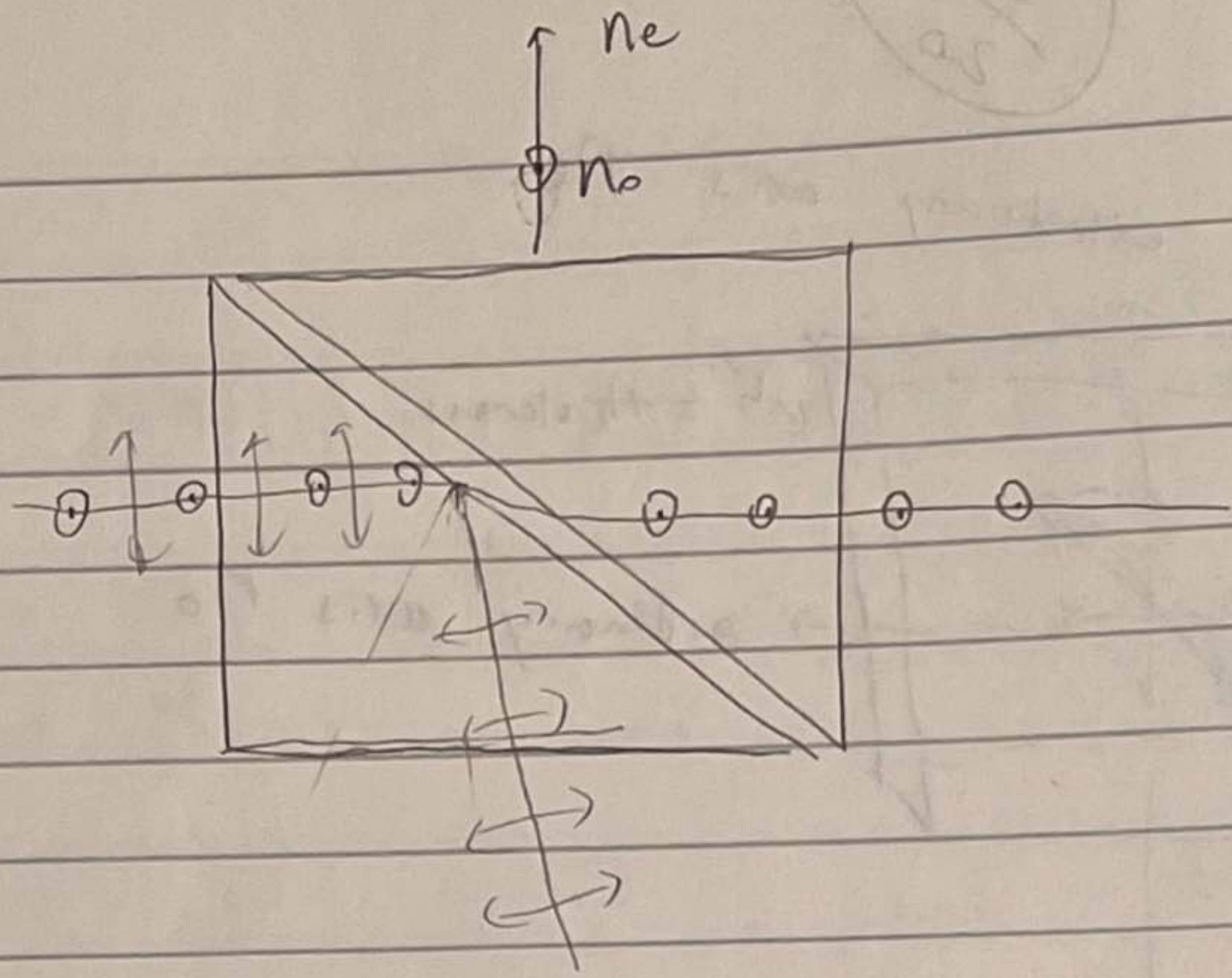
→ Any ~~po~~ incident polarisation can be decomposed into components along ordinary axis and extrodinary axis.

→ Uni-axial, birefringent material has different indices of ~~refre~~ refractions for light polarised in the ordinary (~~e~~) (n_o) and extraordinary (n_e) directions

→ Optical path length traveled are different for the two components, and thus ~~indrot~~ a phase difference is ~~indro~~ introduced.

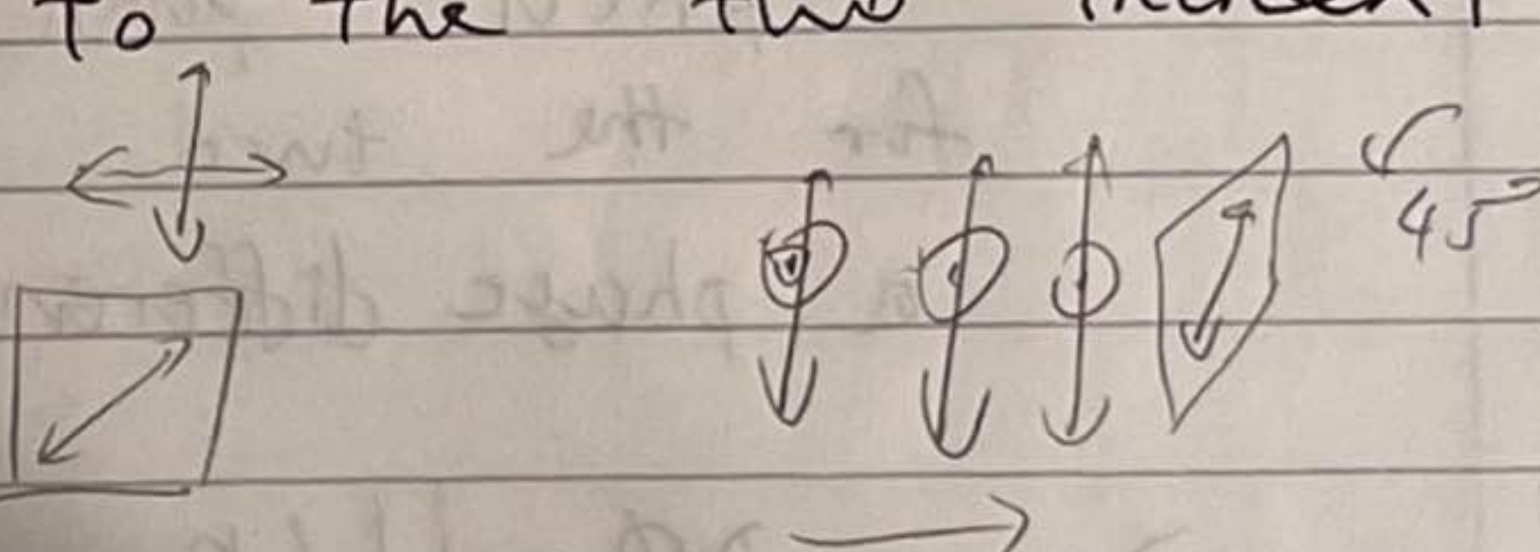
→ $\Delta\phi = |kLn_e - kLn_o| = \frac{2\pi}{\lambda} L\, |n_o - n_e|$

→ A PBS uses the fact that the angle of incidence of light may be ~~the~~ larger than the ~~angle of~~ critical of internal reflection ~~of~~ for n_e but not for n_o



→ ~~Re t~~ Retarder 1 needs to be a half-wave plate, with the fast-axis oriented 22.5° ~~re~~ with respect to the original polarisation direction of the ~~inc~~ incident beam. It rotates the original polarisation by $2 \times 22.5° = 45°$ and thus the beam now has equal amplitude in x, y directions.

→ ~~Polaris~~ Polaroid-1 should be oriented 45° with respect to the two incident polarisations



~~So intensity~~ So the ~~small~~ intensity of the smaller one is the largest

And intensities are equal

→ maximum modulation

→ Retarder-2 needs to be configured so that its thickness L is given by

$$\frac{2\pi}{\lambda} L \,|n_e - n_o| = \delta\phi \quad \therefore \quad L = \frac{\lambda\,\delta\phi}{2\pi\,|n_e - n_o|}$$

polaroid-2

→ ~~polariser-2~~ is the same as ~~polariser-1~~

polaroid-1

Det 1

$$\langle I_1\rangle \cos^2\left(\frac{\phi_1}{2}\right) \Rightarrow I_1 = \langle I_1\rangle\left(1 + \cos(\phi_1)\right)$$

$$\langle I_2\rangle \sin^2\left(\frac{\phi_1}{2}\right) \Rightarrow I_2 = \langle I_2\rangle\left(1 - \cancel{\cos(\phi_1)}\right)$$

$+\sin(\phi_1)$

Det 2

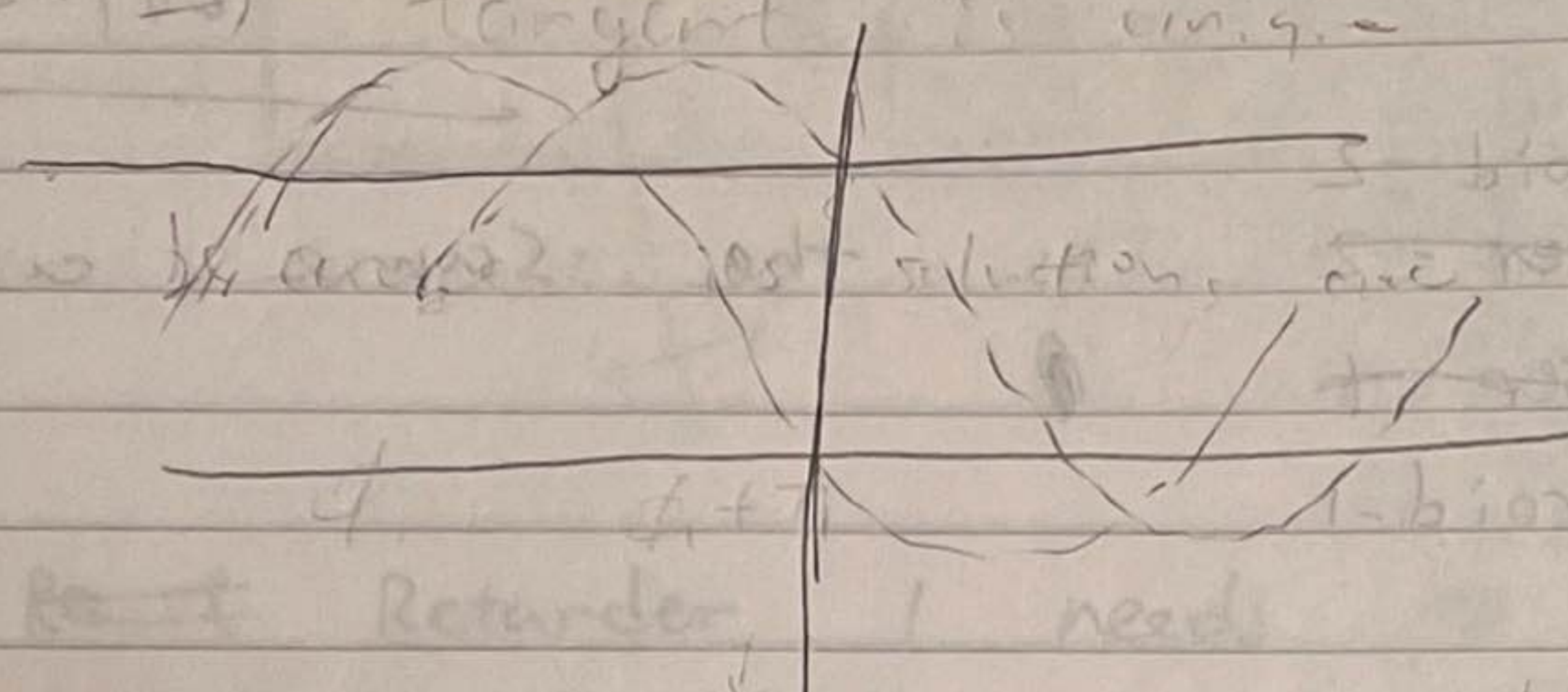
$$\langle I_2\rangle I_1 - \langle I_1\rangle I_2 = \langle I_1\rangle\langle I_2\rangle\cos(\phi_1)$$

$$\cancel{\phi_1 = \cos^{-1}\left(\frac{\langle I_2\rangle I_1 - \langle I_1\rangle I_2}{\langle I_1\rangle\langle I_2\rangle}\right)}$$

~~Knowing cosine and sine simultaneously allows you to determine ϕ_1 exactly~~

~~No multiple roots.~~

$$\cancel{\tan^2\frac{\phi_1}{2} = \frac{I_2/\langle I_2\rangle}{I_1/\langle I_1\rangle}}$$



→ This ability is important because knowing the phase difference ϕ_1 at any ~~insta~~ instant from the instantaneous measurement of I_1 and I_2 ~~is~~ allows us to track the ~~intensities~~ intensity variation as we gradually add gas to the gas cell. If we see any noise that deviates from the cosine and sine ~~pattern then we know~~ variations with time then we know that there are some extrafactors that ~~a~~ are interfering with our measurement.

→ the $\delta/2$ phase shift gives the intensities at Det 1 and Det 2 to be

$$I_1 = \langle I_1\rangle(1+\cos\phi_1) \qquad I_2 = \langle I_2\rangle(1+\sin(\phi_2))$$

→ So knowing the ~~cos~~ cosine and sine of a ~~a~~ phase ~~angle~~ ϕ_1 ~~bet~~ then we can determine ϕ_1 uniquely between 0 and 2π.