

Revision Questions on Statistical Mechanics

R1. (a) The heat capacity of electrons follows from the procedure outlined in section 17.3 on the lecture notes

We have $C_V = \left(\frac{\partial U}{\partial T} \right)_V = \boxed{N k_B \frac{\pi^2}{2} \frac{k_B T}{E_F}}$, where

$\boxed{E_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{2s+1} \right)^{2/3}}$ is the Fermi Energy.

and n is the ^{number} density of ^{electrons in} aluminium metal and $s = \frac{1}{2}$ for electrons.

(b) $\epsilon = \hbar\omega = \hbar kc_s$, phonons have spin 1
~~3~~ (3 possible polarisations +1, 0, -1)

$$\Rightarrow g(k) dk = \frac{(2s+1)V}{2\pi^2} k^2 dk = \frac{3V}{2\pi^2} k^2 dk$$

$$\because \omega = kc_s \rightarrow k = \frac{\omega}{c_s} \rightarrow dk = \frac{1}{c_s} d\omega$$

$$\Rightarrow \cancel{g(\omega) d\omega} = g(k) dk = \frac{3V}{2\pi^2} k^2 dk$$

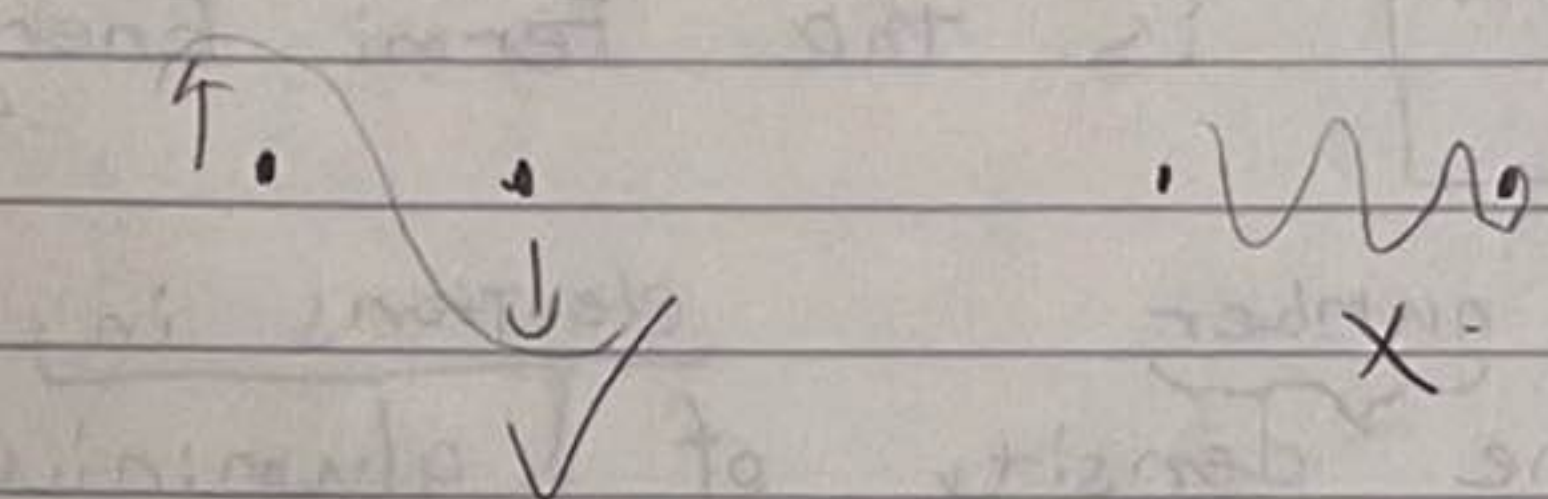
$$= \frac{3V\omega^2}{2\pi^2 c_s^3} d\omega = g(\omega) d\omega$$

High frequency waves have small wavelengths,

$$\lambda = \frac{2\pi c_s}{\omega}, \text{ But there is a minimum wavelength}$$

set by the spacing ~~between~~ between atoms, ~~because~~

It is not possible for sound waves to propagate through a solid with wavelength smaller than the atomic spacing because there's nothing in the middle there to shake.



→ let the maximum allowed phonon frequency be ω_0 , minimum wavelength λ_0 .

Counting the number of single phonon states:

$$\int_0^{\omega_0} d\omega g(\omega) = \frac{3V}{2\pi c_s^3} \int_0^{\omega_0} d\omega \omega^2 = \frac{3V}{2\pi c_s^3} \frac{\omega_0^3}{3}$$

$$= \frac{V \omega_0^3}{2\pi^2 c_s^3}$$

→ Number of degrees of freedom in a lattice of N atoms is $3N$ since each atom can move in 3-dimensions

→ The number of single-phonon states is the number of possible "slots" that each individual phonon in the lattice can occupy, so it is equal to the degree of freedom $3N$

~~So~~ (phonons are indistinguishable so we characterise the state of system by assigning occupation numbers to each single phonon states, which then act like degrees of freedom)

$$\Rightarrow \frac{V \omega_D^3}{2\pi^2 c_s^3} = 3N$$

$$\rightarrow \omega_D = \left(\frac{6\pi^2 N}{V} \right)^{1/3} c_s = c_s (6\pi n)^{1/3}$$

Define the ~~Debye~~ Debye temperature

$$\Theta_D = \frac{\hbar \omega_D}{k_B} = \frac{\hbar c_s}{k_B} (6\pi n)^{1/3}$$

Number of phonons is not conserved so
 Chemical potential $\mu = 0$

\Rightarrow Mean occupation number

$$\bar{n}(\epsilon, T) = \frac{1}{\exp\left(\frac{\epsilon}{k_B T}\right) - 1} \Rightarrow \bar{n}(\omega, T) = \frac{1}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1}$$

$$\rightarrow \cancel{dN(\epsilon, T)} \quad dN(\omega, T) = \bar{n}(\omega, T) g(\omega) d\omega$$

$$\rightarrow \cancel{dU} \quad dU(\omega, T) = \hbar \omega \bar{n}(\omega, T) g(\omega) d\omega$$

$$= \frac{\hbar \omega}{\exp\left(\frac{\hbar \omega}{k_B T}\right) - 1} \times \frac{3V \omega^2}{2\pi^2 c_s^3} d\omega$$

Mean energy \bar{E}

$$U = \frac{3V\hbar}{2\pi^2 c_s^3} \int_0^{\omega_D} d\omega \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}$$

let $x = \frac{\hbar\omega}{k_B T}$, then when $\omega = \omega_D$,

$$x = \frac{\hbar\omega_D}{k_B T} = \frac{\theta_D}{T} \equiv x_D$$

$$d\omega = \frac{k_B T}{\hbar} dx, \quad \omega = \frac{k_B T}{\hbar} x$$

$$\Rightarrow U = \frac{3V\hbar}{2\pi^2 c_s^3} \left(\frac{k_B T}{\hbar}\right)^4 \int_0^{x_D} dx \frac{x^3}{e^x - 1}$$

When temperature is sufficient low such that $\boxed{T \ll \theta_D}$, we have $x_D \rightarrow \infty$
(maximum wavenumber effectively infinite)

$$\text{So } \int_0^{x_D} dx \frac{x^3}{e^x - 1} \approx \int_0^{\infty} dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

$$\rightarrow U \approx \frac{3V k_B^4 \pi^4 T^4}{10\pi^2 c_s^3 \hbar^3}$$

~~$$= \frac{V k_B^4 \pi^3 T^4}{10\pi^2 c_s^3 \hbar^3}$$~~

$$\rightarrow U \approx \frac{V k_B^4 \pi^2 T^4}{10 (\hbar c_s)^3}$$

$$\rightarrow C_V = \left(\frac{\partial U}{\partial T}\right)_V = \boxed{\frac{2\pi^2 V k_B^4 T^3}{5 \hbar^3 c_s^3}} \quad (T \ll \theta_D)$$

$$= \left(\frac{k_B^3}{\hbar^3 c_s^3}\right) \frac{V}{6\pi^2 N} \cdot \frac{2\pi^2 \cdot 6\pi^2 k_B N T^3}{5} = \boxed{N k_B \frac{12\pi^4}{5} \left(\frac{T}{\theta_D}\right)^3}$$

(c) Heat Capacity due to electron gas:

$$C_{v,e} = Nk_B \frac{\pi^2}{2} \left(\frac{T}{\theta_F} \right)$$

\leftarrow Fermi temperature

Heat capacity due to vibration of lattice:

$$C_{v,l} = Nk_B \frac{12\pi^4}{5} \left(\frac{T}{\theta_D} \right)^3$$

\leftarrow Debye temperature

For them to be comparable, set

$$C_{v,e} = C_{v,l} \rightarrow \frac{12\pi^4 T^3}{5\theta_D^3} = \frac{\pi^2 T}{2\theta_F}$$

$$\rightarrow T^2 = \frac{5}{12\pi^2} \frac{\theta_D^3}{\theta_F}$$

$$\theta_F = \frac{\epsilon_F}{k_B}, \quad \text{From Q7.3 we know}$$

$$\text{for aluminium } \theta_F \approx 1.34 \times 10^5 \text{ K}$$

$$\theta_D = \frac{\hbar c_s}{k_B} (6\pi n)^{1/3}, \quad \text{Note that in this case}$$

n is the density of atom rather than

the density of free electron, which we've

used to determine θ_F . So $n = \frac{\rho}{M_{\text{Al}}} = \frac{2.7 \times 10^3 \text{ kg/m}^3}{26.98 \times 1.661 \times 10^{-27}}$

$$\rightarrow n = 6.02 \times 10^{28} \text{ m}^{-3} \quad \text{from data given}$$

in Q7.3

$$\rightarrow \theta_D^3 = \left(\frac{\hbar c s}{k_B} \right)^3 \times 6\pi n$$

$$= \left(\frac{(1.05 \times 10^{-34}) (6000)}{(1.38 \times 10^{-23})} \right)^3 (6\pi \times 6.02 \times 10^{28})$$

$$= ~~1.07~~ 1.08 \times 10^8 \text{ K}^3$$

$$\rightarrow T = \left(\left(\frac{5}{24\pi^2} \right) \left(\frac{1.08 \times 10^8}{1.34 \times 10^5} \right) \right)^{\frac{1}{2}} \approx \boxed{4 \text{ K}}$$

OR use the Debye temperature given. $\theta_D = 394 \text{ K}$

$$\rightarrow T = \left(\left(\frac{5}{24\pi^2} \right) \left(\frac{394^3}{1.34 \times 10^5} \right) \right)^{\frac{1}{2}} \approx \boxed{3 \text{ K}}$$

The result is on the order of a few Kelvins.

R2. From Q 5.6 the energy levels are

$$\epsilon = \frac{\hbar^2 k^2}{2m} \quad \text{where } \underline{k} = \left(\frac{2\pi}{L} n_x, \frac{2\pi}{L} n_y, \frac{2\pi}{L} n_z \right)$$

(n_x, n_y, n_z are integers and assume the box is a cube so $L_x = L_y = L_z = L$)

So the ground state $\epsilon_0 = 0$

the first excited state ϵ_1 has (n_x, n_y, n_z)

$= (\pm 1, 0, 0)$ or $(0, \pm 1, 0)$ or $(0, 0, \pm 1)$, but

$$\epsilon_1 \text{ must be } \epsilon_1 = \left(\frac{2\pi}{L} \right)^2 \frac{\hbar^2}{2m} = \frac{2\pi^2 \hbar^2}{m L^2}$$

$$\rightarrow \epsilon_1 = \frac{2\pi^2 \hbar^2}{m} V^{-2/3} \quad (\text{assume } V = L^3)$$

$$\rightarrow \epsilon_1 = \frac{2\pi^2 \hbar^2 n^{2/3}}{m} N^{-2/3} \rightarrow \epsilon_1 \propto N^{-2/3}$$

the mean occupation number for 1st excited state is

$$\bar{n}_1 = \frac{1}{e^{\beta(\epsilon_1 - \mu)} - 1}$$

At very low temperature, $\mu \rightarrow -\frac{k_B T}{N} \propto \frac{1}{N} = N^{-1}$

$\mu \propto N^{-1}$, $\epsilon_1 \propto N^{-2/3}$ \rightarrow since in the thermodynamic limit N is very large,

$\frac{\mu}{\epsilon_1} \propto \frac{1}{N^{1/3}} \rightarrow 0$ $\therefore \mu$ is negligible compare to ϵ_1

And since $\beta \epsilon_1 = \frac{1}{k_B T} (2\pi \hbar^2 n)^{2/3} N^{-2/3}$

N can be arbitrarily large but T cannot be arbitrarily small $\propto N^{-2/3}$

$N_{BCE} = N(1 - (\frac{T}{T_c})^{3/4})^{-1}$. $\beta \epsilon_1$ is very small compare to 1

If $T=0$ the $N_{BCE} = N$

$\beta N \ll \beta \epsilon_1 \ll 1$

→ No particle in first state.

$\bar{n}_1 = \frac{1}{e^{\beta \epsilon_1 - \beta N} - 1} \approx \frac{1}{e^{\beta \epsilon_1} - 1}$

If T small ($T < T_c$) but $\neq 0$, then

$\frac{N^{-2/3}}{T} \rightarrow 0$ $\approx \frac{1}{(1 + \beta \epsilon_1 - 1)} \approx \frac{1}{\beta \epsilon_1} \propto \frac{1}{\epsilon_1}$

$N \rightarrow \infty$ arbitrarily

$\therefore \epsilon_1 \propto N^{-2/3}$

$\therefore \bar{n}_1 \propto N^{2/3}$

It is large, but not macroscopically occupied

because $\frac{\bar{n}_1}{N} \propto \frac{1}{N^{1/3}} \rightarrow 0$ as $N \rightarrow \infty$

in the thermodynamic limit

First level does not need any special consideration.

R3. (a) $\epsilon(\underline{k}) = \frac{\hbar^2 k^2}{2m} - 2\mu_B S_z B$, $S_z = -1, 0, 1$

By the ~~assumpt~~ principle of maximising Shannon entropy each spin state is equal probable.

With the shift in energy levels the grand potential can be expressed as

$$\Phi(N, B) = \frac{1}{3} \Phi_0(N - 2N_B B) + \frac{1}{3} \Phi_0(N) + \frac{1}{3} \Phi_0(N + 2N_B B)$$

$$= \frac{1}{3} \left(\Phi_0(N) - 2N_B B \frac{\partial \Phi_0}{\partial N} + \frac{4N_B^2 B^2}{2} \frac{\partial^2 \Phi_0}{\partial N^2} \right)$$

$$+ \frac{1}{3} \Phi_0(N) + \frac{1}{3} \left(\Phi_0(N) + 2N_B B \frac{\partial \Phi_0}{\partial N} + \frac{4N_B^2 B^2}{2} \frac{\partial^2 \Phi_0}{\partial N^2} \right)$$

$$= \Phi_0(N) + \frac{4}{3} N_B^2 B^2 \frac{\partial^2 \Phi_0}{\partial N^2}$$

where $\Phi_0(N)$ is the ~~of~~ grand potential without magnetic field ~~at~~ and has chemical potential μ

From Q7.9 we know that the magnetisation

$$M = - \frac{1}{V} \left(\frac{\partial \Phi}{\partial B} \right)_{T, \mu, V} \quad \because \Phi_0(N) \text{ is independent of } B$$

$$\therefore M = - \frac{1}{V} \frac{\partial}{\partial B} M = - \frac{4N_B^2}{3V} \frac{\partial^2 \Phi_0}{\partial N^2} \left(\frac{\partial}{\partial B} B^2 \right)$$

$$\rightarrow M = - \frac{8N_B^2 B}{3V} \frac{\partial^2 \Phi_0}{\partial N^2}$$

We know that ~~$\frac{\partial \Phi_0}{\partial N}$~~

$$N_{\Phi_0} = - \left(\frac{\partial \Phi_0}{\partial \mu} \right)_{T, V}$$

where N_{Φ_0} is the number of particles corresponds to the grand potential without magnetic field $\Phi_0(\mu)$

$$\therefore M = \frac{8N_B^2 B}{3V} \left(\frac{\partial N_{\Phi_0}}{\partial \mu} \right)_{T, V}$$

the magnetic susceptibility $\chi = \lim_{B \rightarrow 0} \frac{\partial M}{\partial B}$

$$\rightarrow \chi = \frac{8N_B^2}{3V} \left(\frac{\partial N_{\Phi_0}}{\partial \mu} \right)_{T, V}$$

In the classical limit:

$$\mu = k_B T \ln \left(\frac{n \lambda_{th}^3}{2s+1} \right)$$

in this case $n = \frac{N_{\Phi_0}}{V}$, $s = 1$, $k_B T = \frac{1}{\beta}$

$$\rightarrow N_{\Phi_0} = \frac{2s+1}{\lambda_{th}^3} V e^{\beta \mu} \quad \rightarrow \frac{2s+1}{\lambda_{th}^3} V e^{\beta \mu}$$

$$\begin{aligned} \rightarrow \left(\frac{\partial N_{\Phi_0}}{\partial \mu} \right)_{T, V} &= \frac{2s+1}{\lambda_{th}^3} V \beta e^{\beta \mu} = \beta N_{\Phi_0} = \frac{N_{\Phi_0}}{k_B T} \\ &= \frac{n V}{k_B T} \end{aligned}$$

$$\rightarrow \chi = \frac{8N_B^2}{3V} \frac{n V}{k_B T} = \frac{8n N_B^2}{3k_B T} \quad \text{classically}$$

From Curie's Law: $\chi = \frac{C}{T}$ where

$$C = \frac{N_B^2}{3k_B} n g^2 J(J+1)$$

~~where~~ In this case $J=1$, $g=2$

$$C = \frac{8N_B^2 n}{3k_B}$$

$$\rightarrow \chi = \frac{8N_B^2 n}{3k_B T}$$

~~Curie's~~ Curie's Law is recovered.

(b) As $T \rightarrow T_c + 0$:

① the equation for N holds since $T > T_c$

② $\because T \sim T_c \therefore \mu = 0$ (for Φ_0)

$$\rightarrow N = \frac{2(2s+1)V}{\sqrt{\pi}} \frac{1}{\lambda_{th}^3} \int_0^\infty \frac{dx \sqrt{x}}{e^x - 1}$$

$$= \frac{3V}{\lambda_{th}^3} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx \sqrt{x}}{e^x - 1} \right)$$

$$\rightarrow N = \frac{2(2s+1)V}{\sqrt{\pi} \lambda_{th}^3} \int_0^\infty \frac{dx \sqrt{x}}{e^{x-\beta\mu} - 1}$$

$$\rightarrow \left(\frac{\partial N}{\partial \mu} \right)_{T,V} = \frac{6V}{\sqrt{\pi} \lambda_{th}^3} \int_0^\infty dx \sqrt{x} \frac{\partial}{\partial \mu} \left(\frac{1}{e^{x-\beta\mu} - 1} \right)$$

$$\rightarrow = \frac{-1}{(e^{x-\beta\mu} - 1)^2} (e^{x-\beta\mu}) (-\beta)$$

$$\therefore N=0 \quad \lambda_{th} = h \sqrt{\frac{2\pi}{m k_B T_c}}$$

$$\therefore \left(\frac{\partial N}{\partial \mu} \right)_{T,V} = \frac{6V}{\sqrt{\pi} \lambda_{th}^3} k_B T \int_0^{\infty} \frac{dx e^{-x} \sqrt{x}}{(e^{-x} - 1)^2}$$

$$\rightarrow \left(\frac{\partial N}{\partial \mu} \right)_{T,V} = \frac{6V}{\sqrt{\pi} \lambda_{th}^3} \left[\frac{\partial}{\partial \mu} \left(\frac{dx \sqrt{x}}{e^{-x} - 1} \right) \right]_{\mu=0} \left(\text{for } N=0, \lambda_{th} = h \sqrt{\frac{2\pi}{m k_B T}} \right)$$

$$= \frac{6V}{\sqrt{\pi} \lambda_{th}^3} \int_0^{\infty} \frac{dx e^{-x} \sqrt{x}}{(e^{-x} - 1)^2}$$

↳ this integral diverges

$$\therefore \left(\frac{\partial N}{\partial \mu} \right)_{T,V} \rightarrow \infty$$

$$\therefore \chi \rightarrow \infty \text{ as } T \rightarrow T_c + 0$$

For temperatures close to $T_c + 0$, we calculate ~~the~~ $\left(\frac{\partial N}{\partial \mu} \right)_{T,V}$ at T such that $T - T_c$ is small and positive. Remember in the calculation of N we always work in the non-magnetised $\Phi = \Phi_0$.

~~When $T = T_c$, let $N = N_0$, $\mu = 0$~~

~~At~~ When $N=0$, let $N = N_0$

we have
$$N_0 = \frac{6V}{\sqrt{\pi} \lambda_{th}^3} \int_0^{\infty} \frac{dx \sqrt{x}}{e^x - 1}$$

$$N = \frac{6TV}{\pi^3 \lambda_{th}^3} \int_0^{\infty} \frac{dx \bar{J}_x}{e^{x-\beta\mu} - 1}$$

$$\rightarrow N = N_0(T) + \frac{6TV}{\pi^3 \lambda_{th}^3} \left[\int_0^{\infty} \frac{dx \bar{J}_x}{e^{x-\beta\mu} - 1} - \int_0^{\infty} \frac{dx \bar{J}_x}{e^x - 1} \right]$$

μ is small near $T = T_0$, so the important part of the integral arises from the region where x is small

\rightarrow when x is large, the integrand

$$\bar{J}_x \left(\frac{1}{e^{x-\beta\mu} - 1} - \frac{1}{e^x - 1} \right) = \frac{-\bar{J}_x [(e^{x-\beta\mu} - 1) - (e^x - 1)]}{(e^{x-\beta\mu} - 1)(e^x - 1)}$$

$$= \frac{\bar{J}_x e^x (1 - e^{-\beta\mu})}{(e^{x-\beta\mu} - 1)(e^x - 1)}$$

$$\approx \frac{\bar{J}_x e^x (\beta\mu)}{(e^x - 1)^2} \rightarrow 0 \text{ as } x \rightarrow \infty \text{ and } \mu \rightarrow 0$$

When x is small, the integrand

$$\bar{J}_x \left(\frac{1}{e^{x-\beta\mu} - 1} - \frac{1}{e^x - 1} \right) = \bar{J}_x \left(\frac{1}{1 + x - \beta\mu - 1} - \frac{1}{1 + x - 1} \right)$$

$$= \bar{J}_x \left(\frac{1}{x - \beta\mu} - \frac{1}{x} \right) = \bar{J}_x \left(\frac{\beta\mu}{(x - \beta\mu)x} \right)$$

$$= \frac{\beta\mu}{\bar{J}_x (x - \beta\mu)x}$$

Hence when x is small the contribution is significant

$$\rightarrow N - N_0 = \frac{6T\beta N}{\sqrt{\pi}\lambda_{th}^3} \int_0^{\infty} \frac{dx}{\sqrt{x}(x-\beta N)}$$

let $y = |\beta N| = -\beta N \quad \because N < 0, \beta > 0$

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x+y)} = 2 \int \frac{du}{u^2+y} = \frac{2}{y} \int \frac{1}{\frac{u^2}{y}+1} du$$

$u = \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}}$

$$= \frac{2}{\sqrt{y}} \int \frac{1}{s^2+1} ds = \frac{2 \tan^{-1}(s)}{\sqrt{y}} + C$$

$$s = \frac{u}{\sqrt{y}}, \quad ds = \frac{1}{\sqrt{y}} du$$

$$= \frac{2 \tan^{-1} \sqrt{\frac{x}{y}}}{\sqrt{y}} + C$$

$$\rightarrow N - N_0 = \frac{6T\beta N}{\sqrt{\pi}\lambda_{th}^3} \left(\frac{2 \tan^{-1} \left(\sqrt{\frac{x}{|\beta N|}} \right)}{\sqrt{|\beta N|}} \right) \Bigg|_{x=0}^{\infty}$$

$$= \frac{6T\beta N}{\sqrt{\pi}\lambda_{th}^3} \left(\frac{2 \times \frac{\pi}{2}}{\sqrt{\beta N}} - 0 \right)$$

$$= \frac{6\sqrt{\pi} T \beta N}{\lambda_{th}^3} \left(\frac{\pi}{\sqrt{|\beta N|}} \right)$$

$$= -\frac{6\sqrt{\pi} T}{\lambda_{th}^3} \sqrt{|\beta N|} \rightarrow \frac{(N - N_0) \lambda_{th}^3}{6\sqrt{\pi} T \sqrt{\beta}} = -\sqrt{|\omega|}$$

$$\rightarrow \left(\frac{\lambda_{th}^3}{6\sqrt{\pi}\beta V} \right)^2 (N - N_0)^2 = -N$$

$$\left(\frac{\partial N}{\partial N} \right)_{T,V} \rightarrow \frac{\lambda_{th}^3}{6\sqrt{\pi}\beta V} 2(N - N_0) \left(\frac{\partial N}{\partial N} \right)_{T,V} = - \frac{\partial N}{\partial N} = -1$$

$$\rightarrow \left(\frac{\partial N}{\partial N} \right)_{T,V} = \frac{6\sqrt{\pi}\beta V}{\lambda_{th}^3 (N_0 - N)}$$

$$\therefore \chi = \frac{8N\beta^2}{3V} \left(\frac{\partial N}{\partial N} \right)_{T,V}$$

$$\therefore \chi = \frac{8N\beta^2 \sqrt{\pi}\beta}{\lambda_{th}^3 (N_0 - N)}$$

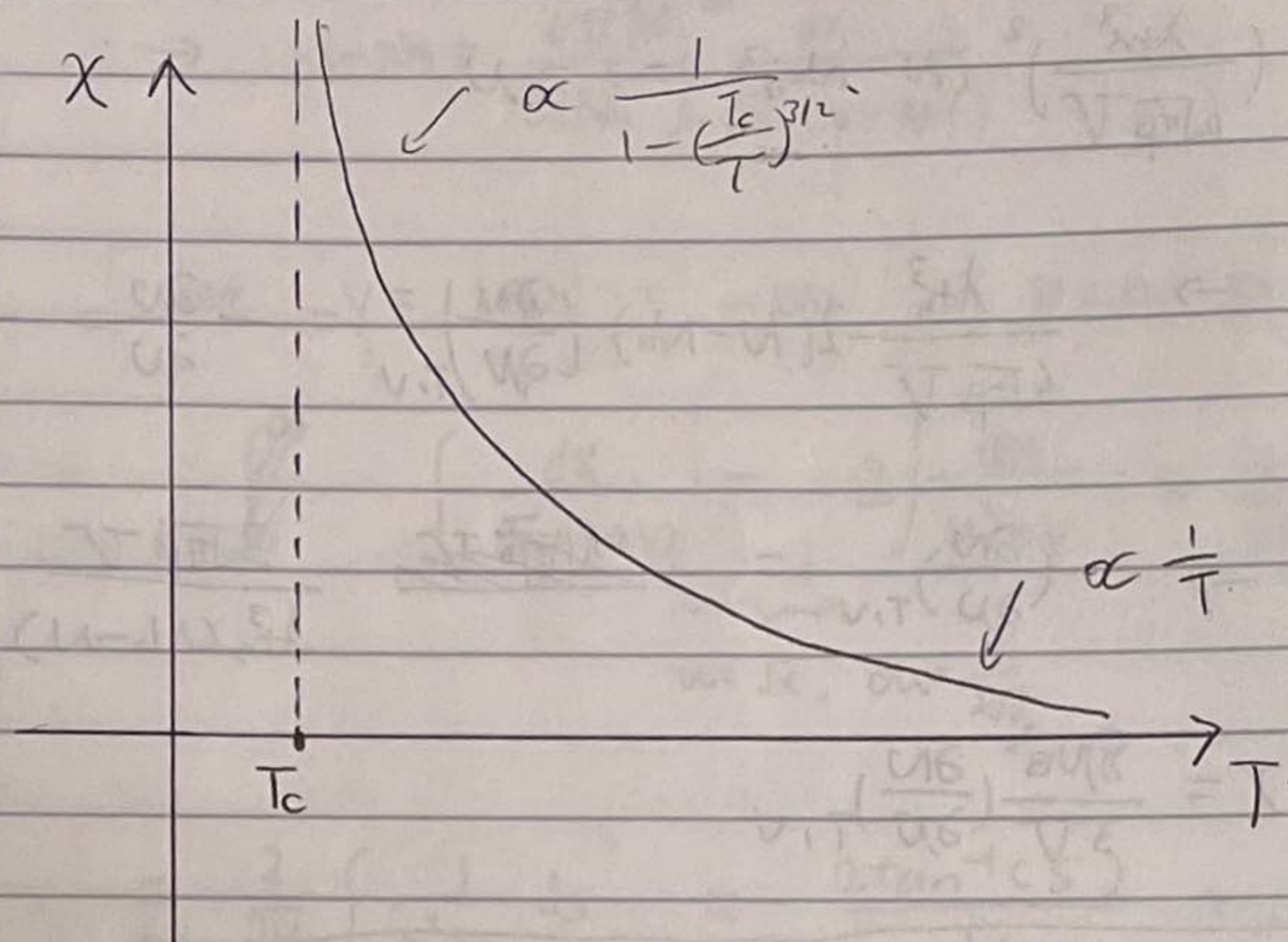
where $\beta = \frac{1}{k_B T}$, $\lambda_{th} = h \sqrt{\frac{2\pi}{m k_B T}}$,

$$N_0(T) = \frac{6V}{\sqrt{\pi}\lambda_{th}^3} \int_0^\infty \frac{dx \sqrt{x}}{e^x - 1} \approx \frac{7.837 V}{\lambda_{th}^3}$$

When $T > T_c$, $N_0 > N$ as T decreases N_0 approaches $N + 0$, χ grows larger and larger towards infinity

When $T \rightarrow T_c + 0$, $N_0 \rightarrow N + 0$

Hence $\chi \rightarrow \infty$



(c) At $T < T_c$, the ~~state~~ macroscopically occupied state will be the state with lowest energy level \rightarrow this is the state with

$$k=0 \quad \text{and} \quad S_z = -1 \quad \rightarrow \quad \epsilon = -2N_3 B$$

(this T_c depends on B but B is small, unlike the T_c used in (b))

Macroscopic occupation requires at $T=0$

$$\bar{N}_{k=0, S_z=-1} = \frac{1}{e^{\beta(-2N_3 B - N)} - 1} \rightarrow N$$

$$\therefore e^{\beta(-2N_3 B - N)} - 1 = \frac{1}{N}$$

$$\beta(-2N_3 B - N) = 1 + \frac{1}{N}$$

$$\rightarrow \mu = -2N_3 B - k_B T \ln \left(1 + \frac{1}{N} \right) \rightarrow -2N_3 B$$

as $N \rightarrow \infty$ in the thermodynamic limit

When let $f(\beta\mu) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{dx J_x}{e^{x-\beta\mu} - 1}$

when $T > T_c(B)$

$$N = \frac{(2s+1)V}{\lambda_{th}^3} f(\beta\mu)$$

when $T = T_c(B)$, $f(\beta\mu) = f(-2N_3 B)$

$T_c(B)$ is solved from the equation

$$N = \frac{(2s+1)V}{(\lambda_{th}(T=T_c(B)))^3} f(-2N_3 B)$$

For small B , $f(-2N_3 B) \approx f(0)$

$\therefore T_c(B) \approx T_c(0) = T_c$ without magnetic field.

when $T < T_c(B)$

$$N_{excited} = \frac{(2s+1)V}{\lambda_{th}^3} f(-2N_3 B)$$

$$\therefore \frac{N_{excited}}{N} = \frac{\lambda_{th}^3(T_c(B))}{\lambda_{th}^3(T)} = \left(\frac{T}{T_c} \right) \left(\frac{T}{T_c(B)} \right)^{3/2}$$

$$\rightarrow \mathcal{N} = -2N_3 B - k_B T \ln \left(1 + \frac{1}{N} \right) \rightarrow -2N_3 B$$

as $N \rightarrow \infty$ in the thermodynamic limit

~~When~~ let $f(\beta N) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{dx J_x}{e^{x - \beta N} - 1}$

when $T > T_c(B)$

$$N = \frac{(2s+1)V}{\lambda_{th}^3} f(\beta N)$$

when $T = T_c(B)$, $f(\beta N) = f(-2N_3 B)$

$T_c(B)$ is solved from the equation

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→ Number of particles condensed in state $|k=0, S_z=-1\rangle$ is

$$\bar{N}_0 = N - N_{\text{excited}} = N \left(1 - \left(\frac{T}{T_c(B)} \right)^{3/2} \right)$$

$$\approx N \left(1 - \left(\frac{T}{T_c} \right)^{3/2} \right)$$

↑ for small B ($B \rightarrow 0$)
and $T_c = \text{non-magnetised } T_c$

When $T < T_c(B)$, μ is constant so $\frac{\partial N}{\partial \mu} = 0$, there is no magnetisation due

to the excited particles.

But the ground state condensate has magnetisation.

Each particle has energy $-2\mu_B B$
so each particle has magnetic moment

$$m = 2\mu_B \quad \left(\begin{array}{l} \text{of magnetic dipole} \\ \text{since energy } \psi \text{ in magnetic field} \\ \text{is } (U = -\underline{m} \cdot \underline{B}) \end{array} \right)$$

$\bar{N}_0 \rightarrow$ particles in ground states so

total magnetic moment

$$m = 2\mu_B \bar{N}_0$$

~~Magnet~~ Magnetisation :

$$M = \frac{m}{V} = \frac{2\mu_B N_0}{V} = \left[\frac{2\mu_B N}{V} \left(1 - \left(\frac{T}{T_c(B)} \right)^{3/2} \right) \right]$$

$$\text{As } B \rightarrow 0 \quad T_c(B) \rightarrow T_c(0) = T_c$$

\therefore Spontaneous magnetisation is

$$M_0(n, T) = \lim_{B \rightarrow 0} M(n, T, B)$$

$$= \frac{2\mu_B n}{V} \left(1 - \left(\frac{T}{T_c} \right)^{3/2} \right)$$

where

$$T_c = \frac{2\pi\hbar^2}{mk_B} \left[\frac{n}{2.612(2.5+1)} \right]^{2/3}$$

Magnetisation is non-zero even if B is vanishingly small. This is because ~~as long~~ no matter how small ~~the~~ B is, in the Bose-Einstein condensation state a macroscopically significant number of particles are aligned with the magnetic field and produce a net dipole moment and thus ~~magne~~ non-zero magnetisation.

the magnetic moment is due to spin, not finite B . And the critical temperature T_c is finite as $B \rightarrow 0$. So M_0 never vanishes.

The result of (b) makes sense because we can say that for $T \leq T_c$, we have $\chi \rightarrow \infty$

→ For $B=0$, $M=0$, For $B=0^+$

$$M = M_0 = 2NB^2 \left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$$

∴ there is a discontinuity between the value of M ~~and~~ at $B=0$ and $B=0^+$

and ∴ $\chi = \left(\frac{\partial M}{\partial B}\right)_{B \rightarrow 0}$

∴ $\chi \rightarrow \infty$ as $T \leq T_c$

(b) and (c) are consistent

R4. When N is fixed (average number of particles \bar{N} is fixed, that is), the equation for N for quantum gases is

$$N = \frac{2(2s+1)V}{\sqrt{\pi} \lambda_{th}^3} \int_0^{\infty} \frac{dx \sqrt{x}}{e^{x-\beta\mu} \pm 1} \quad (1)$$

where $\beta = \frac{1}{k_B T}$, $\lambda_{th} = \frac{h}{\sqrt{2\pi m k_B T}}$, and "+" for

Fermi gases and "-" for Bose gases.

Now N is not fixed, so we allow the creation and annihilation of matter. Conservation of mass no longer holds and is replaced by the conservation of mass-energy. Thus we need to add the rest mass of a particle, mc^2 , to the chemical potential μ , because the physical meaning of chemical potential is the energy cost of adding one particle to the system. Adding the rest mass to the chemical potential (and ^{to} internal energy as well) ensures that creation and annihilation of matter (thus mass) is also considered. Equalisation of μ is still a condition for equilibrium in processes that involve mass change, such as pair production.

Hence, equation (1) has to be modified,
This is because in the derivation, ~~it~~

$$N = \sum_i \bar{n}_i, \quad \bar{n}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} \pm 1}$$

and we've ~~use~~ converted \sum_i to

$$\int_0^{\infty} d\epsilon g(\epsilon)$$

The integration limit is from 0 to ∞ ,
which means that the energy levels ϵ_i
must not contain the rest energy mc^2 .

So in equation (1) μ is ~~the~~ and must be
the chemical potential without mc^2 .

After modification, the implicit equation relating
 ~~μ~~ μ (with mc^2) to N, V, T, S, m is

$$N = \frac{2(2S+1)V}{\sqrt{\pi} \lambda_{th}^3} \int_0^{\infty} \frac{dx \sqrt{x}}{e^{x - \beta(\mu - mc^2)} \pm 1} \quad (2)$$

If we allow the creation and annihilation of particles, then fixed particle number is no longer a conservation law. So the conservation laws putting on the system are just fixed volume and fixed temperature, because now only temperature and volume are controlled parameters in equation (2)

In this circumstance, the system will try to minimise its availability and in this case of fixed V and T the availability is F .

$$\text{So we have } \left(\frac{\partial F}{\partial N} \right)_{V, T} = 0 \Rightarrow \mu = 0$$

→ For an open system $\mu = 0$

$$\rightarrow N = \frac{2(2s+1)V}{\sqrt{\pi} \lambda_{th}^3} \int_0^{\infty} \frac{dx \sqrt{x}}{e^{x+\beta mc^2} \pm 1}$$

(N is no longer fixed, it is determined by T and V)

$$\text{So we have } f(\beta mc^2) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{dx \sqrt{x}}{e^{x+\beta mc^2} \pm 1}$$

$$\text{and } f(\beta mc^2) = \frac{N}{V} \frac{\lambda_{th}^3}{2s+1} = \frac{n \lambda_{th}^3}{2s+1} \left(\frac{2\pi}{mk_B T} \right)^{3/2}$$

Ideal gas at room temperature ($T \sim 300\text{K}$) is hot and dilute, \leftarrow so n is small, T is large.

$$\therefore f(\beta mc^2) \propto \frac{n}{T^{3/2}} \quad \& \quad \therefore f(\beta mc^2) \rightarrow 0$$

$$\rightarrow e^{\beta mc^2} \rightarrow \infty$$

$$\rightarrow f(\beta mc^2) = \frac{2}{\sqrt{2}} \int_0^{\infty} \frac{dx Jx}{e^x e^{\beta mc^2} \pm 1} \approx \frac{2}{\sqrt{\pi}} e^{-\beta mc^2} \int_0^{\infty} Jx e^{-x} dx$$

$$= e^{-\beta mc^2} \rightarrow \frac{1}{\infty} \rightarrow 0$$

$$\rightarrow e^{-\beta mc^2} \approx \frac{n \lambda_{th}^3}{2s+1} \rightarrow n = \frac{2s+1}{\lambda_{th}^3} e^{-\frac{mc^2}{k_B T}}$$

~~$$\rightarrow mc^2 \approx k_B T \ln \left(\frac{n \lambda_{th}^3}{2s+1} \right)$$~~

ignore the spin internal degrees of freedom.

$$n \approx \frac{1}{h^3} \left(\frac{m k_B T}{2\pi} \right)^{3/2} e^{-\frac{mc^2}{k_B T}}$$

use $m \approx m_p = 1.67 \times 10^{-27} \text{ kg}$
 $T \approx 300 \text{ K}$

we get $n \sim 0$

This result does not accurately describe the room I am sitting in. This is because in this room the creation/annihilation of matter is not allowed and number of particles is conserved in equilibrium. Also even if we can create and destroy particles at will, we are still subject to conservation laws such as conservation of charge, which & makes n not necessarily 0.

R5. (a) \rightarrow For fermions, each copy of the system in the ensemble can have only one or none of the N_i particles available for each state i , and particles are non-distinguishable.

\therefore Number of ways to assign the N_i particles to N copies in the ensemble is

$$\Omega_i = \frac{N!}{N_i!(N-N_i)!}$$

$\underbrace{ N_i}_{N_i \text{ particles}}$
 $\underbrace{ N-N_i}_{N \text{ copies}}$

$\Omega_N(N_1, N_2, \dots) = \prod_i \Omega_i$ Ω is the number of ways such that the system can be constructed

$$S_B = \ln \Omega_N, \quad S_G = \frac{S_B}{N} = \frac{\ln \Omega_N}{N}$$

$$\rightarrow S_G = \frac{1}{N} \ln \Omega_N = \frac{1}{N} \ln \prod_i \Omega_i = \frac{1}{N} \sum_i \ln \Omega_i$$

$$\ln \Omega_i = \ln N! - \ln N_i! - \ln (N-N_i)!$$

$$= N \ln N - N - N_i \ln N_i + N_i - (N-N_i) \ln (N-N_i) + (N-N_i)$$

Stirling's
formula \rightarrow

$$= N \ln N - N_i \ln N_i - (N-N_i) \ln (N-N_i)$$

$$+ \cancel{(N_i + (N-N_i) - N)}$$

$$= N \left[\ln N - \frac{N_i}{N} \left(\ln \frac{N_i}{N} + \ln N \right) - \frac{(N-N_i)}{N} \left(\ln \frac{N-N_i}{N} + \ln N \right) \right]$$

$$= N \left[\ln N - \left(\frac{N_i}{N} + \frac{N-N_i}{N} \right) \ln N - \bar{n}_i \ln \bar{n}_i - (1-\bar{n}_i) \ln (1-\bar{n}_i) \right]$$

$$\Rightarrow S_G = -N \left[\bar{n}_i \ln \bar{n}_i + (1-\bar{n}_i) \ln (1-\bar{n}_i) \right]$$

$$\rightarrow S_G = \frac{1}{N} \sum_i \ln \Omega_i = - \sum_i [\bar{n}_i \ln \bar{n}_i + (1 - \bar{n}_i) \ln (1 - \bar{n}_i)]$$

for fermions

For Bosons, the N_i particles in each state i can be distributed completely arbitrarily between the N copies, and the particles are indistinguishable

$S_G = \frac{1}{N} \sum_i \ln \Omega_i$ still holds, we now we have a different Ω_i :

To calculate this, consider " \square " as the N copies of the system and " 0 " as the ~~per~~ N_i particles

We can arrange ^{them} to form an array like this

$\square 0000 \square 0 \square 00 \square \square 0000 \square 000 \dots$

always stays where it is

can be permuted

the number of " 0 " to the right of the j th " \square " represent the assigned number of particles to the j th copy of the system.

the total number of such a configuration is

$$\Omega_i = \frac{(N + N_i - 1)!}{(N - 1)! N_i!}$$

$$\rightarrow \ln \Omega_i = \ln(N + N_i - 1)! - \ln(N - 1)! - \ln N_i!$$

$$= \left[\cancel{-(N + N_i - 1)} + N - 1 + N_i \right] + (N + N_i - 1) \ln(N + N_i - 1) - (N - 1) \ln(N - 1) - N_i \ln N_i$$

$$= N \left[\left(1 + \frac{N_i}{N} - \frac{1}{N}\right) \left(\ln \left(\frac{N + N_i - 1}{N} \right) + \ln N \right) - \left(1 - \frac{1}{N}\right) \left(\ln \left(1 - \frac{1}{N}\right) + \ln N \right) - \frac{N_i}{N} \left(\ln \frac{N_i}{N} + \ln N \right) \right]$$

$$\xrightarrow{\text{as } N \rightarrow \infty} = N \left[(1 + \bar{n}_i) \ln(1 + \bar{n}_i) - \bar{n}_i \ln \bar{n}_i + \ln N (1 + \bar{n}_i - 1 - \bar{n}_i) \right]$$

$$\xrightarrow{\frac{1}{N} \rightarrow 0} = -N \left[\bar{n}_i \ln \bar{n}_i - (1 + \bar{n}_i) \ln(1 + \bar{n}_i) \right]$$

$$\rightarrow S_G = \frac{1}{N} \sum_i \ln \Omega_i = - \sum_i \left[\bar{n}_i \ln \bar{n}_i - (1 + \bar{n}_i) \ln(1 + \bar{n}_i) \right]$$

for bosons

(b) fixed average energy : $\sum_i \bar{n}_i \epsilon_i = U = \text{const}$ (1)

fixed number of particles : $\sum_i \bar{n}_i = \bar{N} = \text{const}$ (2)

(ϵ_i is the energy of the i th ^{state} ~~energy level~~)

For equilibrium $\rightarrow S_G \rightarrow \text{max}$.

subject to (1), (2)

So $S'_G = S_G - \beta U - \beta N$ (1) + βN (2) $\rightarrow \text{max}$

or

unconditionally. β and $-\beta N$ are the two Lagrange multipliers.

For fermions :

$$S'_G = - \sum_i [\bar{n}_i \ln \bar{n}_i + (1 - \bar{n}_i) \ln (1 - \bar{n}_i)]$$

$$- \beta (\sum_i \bar{n}_i \epsilon_i - U) + \beta N (\sum_i \bar{n}_i - \bar{N})$$

$$dS'_G = 0 = - \sum_i [1 + \ln \bar{n}_i + (-1) + (-1) \ln (1 - \bar{n}_i)] d\bar{n}_i$$

$$- \beta \sum_i \epsilon_i d\bar{n}_i - (\sum_i \bar{n}_i \epsilon_i - U) d\beta$$

$$+ \beta N \sum_i d\bar{n}_i + (\sum_i \bar{n}_i - \bar{N}) d(\beta N)$$

$$\equiv \sum_i$$

term in front of $d\bar{n}_i$ should be 0

$$\rightarrow \ln \bar{n}_i - \ln (1 - \bar{n}_i) + \beta (\epsilon_i - \mu) = 0$$

$$\rightarrow \frac{\bar{n}_i}{1 - \bar{n}_i} = e^{\beta(\epsilon_i - \mu)}$$

$$\rightarrow \bar{n}_i = e^{\beta(\epsilon_i - \mu)} - \bar{n}_i e^{\beta(\epsilon_i - \mu)}$$

$$\rightarrow \bar{n}_i = \frac{e^{-\beta(\epsilon_i - \mu)}}{1 + e^{-\beta(\epsilon_i - \mu)}}$$

$$\rightarrow \bar{n}_i = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \quad \text{for fermions}$$

For Bosons:

$$S_G' = -\sum_i [\bar{n}_i \ln \bar{n}_i - (1 + \bar{n}_i) \ln (1 + \bar{n}_i)] - \beta (\sum_i \bar{n}_i \epsilon_i - U) + \beta N (\sum_i \bar{n}_i - \bar{N})$$

$$0 = dS_G' = -\sum_i [1 + \ln \bar{n}_i - 1 - \ln (1 + \bar{n}_i)] d\bar{n}_i - \beta \sum_i \epsilon_i d\bar{n}_i - \sum_i (\bar{n}_i \epsilon_i - U) d\beta + \beta N \sum_i d\bar{n}_i + (\sum_i \bar{n}_i - \bar{N}) d(\beta N)$$

term in front of $d\bar{n}_i$ should be 0

$$\rightarrow \ln \bar{n}_i - \ln (1 + \bar{n}_i) + \beta (\epsilon_i - \mu) = 0$$

$$\rightarrow \frac{\bar{n}_i}{1 + \bar{n}_i} = e^{-\beta (\epsilon_i - \mu)}$$

$$\rightarrow \bar{n}_i = e^{-\beta (\epsilon_i - \mu)} + \bar{n}_i e^{-\beta (\epsilon_i - \mu)}$$

$$\rightarrow \bar{n}_i = \frac{e^{-\beta (\epsilon_i - \mu)}}{1 - e^{-\beta (\epsilon_i - \mu)}}$$

$$\rightarrow \bar{n}_i = \frac{1}{e^{\beta (\epsilon_i - \mu)} - 1} \quad \text{for bosons}$$

(c) For a classical gas, we make the assumption that $N \gg N_i$ because the gas is very dilute

then effectively each particle being distributed ~~effectively~~ sees ~~the~~ N copies of systems available to it, we have ~~N^{N_i}~~ N^{N_i} possibilities

But particles are identical so we need to divide ~~the~~ the number of possibilities by $N_i!$, the number of permutations of N_i ~~copies of the~~ particles. In doing so we assumed that since $N \gg N_i$, no more than ~~1~~ one particle will fall into the same copy of the system.

$$\rightarrow \Omega_i = \frac{N^{N_i}}{N_i!}$$

$$\rightarrow \ln \Omega_i = N_i \ln N - \ln N_i!$$

$$= N_i \ln N - N_i \ln N_i + N_i$$

$$= N \left(\frac{N_i}{N} \ln N - \frac{N_i}{N} \ln N_i + \frac{N_i}{N} \right)$$

$$= -N (\bar{n}_i \ln \bar{n}_i - \bar{n}_i)$$

$$\rightarrow S_G = \frac{1}{N} \sum_i \ln \Omega_i = \underbrace{-\sum_i [\bar{n}_i \ln \bar{n}_i - \bar{n}_i]}$$

$$0 = dS_G' = - \sum_i \left[\lambda + \ln \bar{n}_i + \bar{\lambda} \right] d\bar{n}_i - \beta \sum_i \epsilon_i d\bar{n}_i + \beta \mu \sum_i d\bar{n}_i - d\beta \left(\sum_i \bar{n}_i \epsilon_i - U \right) + d(\beta \mu) \left(\sum_i \bar{n}_i - N \right)$$

term in front of $d\bar{n}_i$ should be 0

$$\rightarrow \ln \bar{n}_i + \beta(\epsilon_i - \mu) = 0$$

$$\rightarrow \underline{\bar{n}_i = e^{-\beta(\epsilon_i - \mu)}}$$

for classical gas, this is the

maxwell - boltzmann distribution.