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Statistical Mechanics 7 Ziyan Li

1, (a)

From theory of Grand Canonical Ensemble ~~and~~ for indistinguishable particles, mean occupation number

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i-\mu)} \pm 1} \quad (\text{"+" for Fermi Gas, "−" for Bose Gas}).$$

Single particle states are $i = \{\vec{p}, S_z\} = \{\vec{k}, S_z\}$

$$\sum_i = (2S+1)\sum_{\vec{k}} = \frac{(2S+1)V}{(2\pi)^3}\int d^3\vec{k}$$

$$\boxed{= \frac{(2S+1)V}{2\pi^2}\int_0^\infty dk\, k^2 \equiv \int dk\, g(k)}$$

$$\rightarrow \quad g(k) = \frac{(2S+1)V}{2\pi^2}k^2$$

In the ultrarelativistic limit: $\varepsilon(k) \simeq \hbar k c$

$$\rightarrow \quad k = \frac{\varepsilon}{\hbar c} \quad \rightarrow \quad dk = \frac{1}{\hbar c}d\varepsilon$$

$$\rightarrow \quad g(k)\,dk = \frac{(2S+1)V}{2\pi^2}k^2 dk$$

$$= \frac{(2S+1)V}{2\pi^2}\left(\frac{\varepsilon}{\hbar c}\right)^2\frac{1}{\hbar c}d\varepsilon = \underbrace{\frac{(2S+1)V}{2\pi^2(\hbar c)^3}\varepsilon^2}_{g(\varepsilon)}d\varepsilon \equiv g(\varepsilon)\,d\varepsilon$$

$$\rightarrow \quad \sum_i = \int_0^\infty d\varepsilon\, g(\varepsilon)$$

Total number of particles:

$$N = \sum_i \bar{n}_i$$

$$\rightarrow \quad N = \sum_i \bar{n}_i = \int_0^\infty d\varepsilon\, g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

$x = \beta\varepsilon$

$dx = \beta d\varepsilon$

$\varepsilon^2 = \frac{x^2}{\beta^2}$

$$= \int_0^\infty d\varepsilon \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \varepsilon^2 \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

$$N = \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \frac{1}{\beta^3} \int_0^\infty \frac{x^2 dx}{e^{x-\beta\mu} \pm 1}$$

$\longrightarrow$ implicit equation for $\mu(n, T)$ is

$$\boxed{n = \frac{(2s+1)}{2\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^{x-\mu/k_B T} \pm 1}}$$

$\left(n = \frac{N}{V} \right.$; $\left. \beta = \frac{1}{k_B T} \right)$

(b)

$$U = \sum_i n_i \varepsilon_i = \int_0^\infty d\varepsilon\, g(\varepsilon)\, n(\varepsilon)\, \varepsilon$$

$$\cancel{= \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty}$$

$$= \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

$$= \boxed{\frac{(2s+1)V}{2\pi^2(\hbar c)^3} \frac{1}{\beta^4} \int_0^\infty \frac{x^3 dx}{e^{x-\beta\mu} \pm 1}}$$

$$\Phi = -k_B T \ln Z = \mp k_B T \sum_i \ln[1 \pm e^{-\beta(\varepsilon_i - \mu)}].$$

$$\cancel{Z = \prod_i \frac{1}{\quad}}$$

$$\ln Z = \pm \sum_i \ln[1 \pm e^{-\beta(\varepsilon_i - \mu)}].$$

$$= \mp k_B T \int_0^\infty d\varepsilon\, g(\varepsilon) \ln[1 \pm e^{-\beta(\varepsilon - \mu)}]$$

$$= \mp k_B T \frac{(2s+1)V}{2\pi^2 (\hbar c)^3} \int_0^\infty d\varepsilon \cdot \varepsilon^2 \cdot \ln[1 \pm e^{-\beta(\varepsilon - \mu)}]$$

$$= \mp \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \frac{1}{\beta^4} \underbrace{\int_0^\infty dx \cdot x^2 \cdot \ln[1 \pm e^{-x+\beta\mu}]}$$

integration by parts $= \frac{1}{3}\int_0^\infty d(x^3) \ln[1 \pm e^{-x+\beta\mu}]$

$$= \frac{1}{3}\underbrace{\left[x^3 \ln[1 \pm e^{-x+\beta\mu}]\right]_0^\infty}_{0-0=0} - \frac{1}{3}\int_0^\infty dx\, x^3 \cdot \frac{\mp e^{-x+\beta\mu}}{1 \pm e^{-x+\beta\mu}}$$

$$= \pm \frac{1}{3}\int_0^\infty dx \cdot x^3 \frac{1}{e^{x-\beta\mu} \pm 1}$$

$$= \boxed{-\frac{1}{3} \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \frac{1}{\beta^4} \int_0^\infty dx \frac{x^3}{e^{x-\beta\mu} \pm 1}} = -\frac{1}{3} U$$

$\because \Phi = -PV \quad \therefore \quad -PV = -\frac{1}{3} U$

$$\Rightarrow \boxed{PV = \frac{1}{3} U}$$

(c) $$S = \frac{U - \Phi - \mu N}{T} = \frac{\frac{4}{3}U - \mu N}{T}$$

$$\therefore \frac{S}{N} = \frac{4}{3}\frac{U}{NT} - \frac{\mu}{T}$$

Adiabatic process with number of particle held ~~fo~~ fixed

$$\rightarrow \frac{S}{N} = \text{const}$$

$$\rightarrow \frac{4}{3}\frac{U}{NT} - \frac{\mu}{T} = \text{const}$$

$$U = \frac{(2S+1)V}{2\pi^2(\hbar c)^3}(k_BT)^4 \int_0^\infty \frac{x^3 dx}{e^{x-\beta\mu} \pm 1}$$

$$\frac{U}{NT} = \frac{(2S+1)}{2\pi^2(\hbar c)^3} k_B^4 \frac{T^3}{n} \int_0^\infty \frac{x^3 dx}{e^{x-\beta\mu} \pm 1}$$

$$\because n \propto T^3$$

$$\therefore \frac{U}{NT} \propto \int_0^\infty \frac{x^3 dx}{e^{x-\beta\mu} \pm 1}$$

$\therefore \frac{U}{NT}$ is a function of $\beta\mu$ only

$\frac{\mu}{T} \propto \beta\mu \rightarrow \frac{\mu}{T}$ is a function of $\beta\mu$ only

$$\therefore \frac{4}{3}\frac{U}{NT} - \frac{\mu}{T} = \text{const} \rightarrow \beta\mu = \text{const}$$

$$\rightarrow \frac{4}{3}\frac{U}{NT} = \text{const.} \rightarrow VT^3 = \text{const}$$

$\because PV = \frac{1}{3}U$ and $\frac{U}{T}$ is const

$$\therefore \frac{PV}{T} = \text{const.} \quad \therefore \cancel{PT^3 = \text{const} \rightarrow P}$$

$$\therefore VT^3 = \text{const} \rightarrow V(PV)^3 = \text{const} \rightarrow P^3V^4 = \text{const}$$

$$\rightarrow \boxed{PV^{4/3} = \text{const}}$$

(d)

$$n = \frac{(2s+1)}{2\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2\,dx}{e^{x-\beta\mu} \pm 1}$$

$$\rightarrow \quad \frac{n}{2s+1}\left(\frac{\hbar c \pi^{2/3}}{k_B T}\right)^3 = \frac{1}{2}\int_0^\infty \frac{x^2\,dx}{e^{x-\beta\mu}\pm 1} \equiv f(\beta\mu)$$

For hot and dilute, $n \to 0$ and/or $T \to \infty$

$\therefore$ under this limit $\frac{n}{2s+1}\left(\frac{\hbar c\pi^{2/3}}{k_B T}\right)^3 \to 0$

$\therefore \quad f(\beta\mu) \to 0 \quad \Rightarrow \quad e^{-\beta\mu} \to \infty$

$$f(\beta\mu) = \frac{1}{2}\int_0^\infty \frac{x^2\,dx}{e^{x-\beta\mu}\pm 1} \approx \frac{1}{2}e^{\beta\mu}\underbrace{\int_0^\infty x^2 e^{-x}\,dx}_{2!}$$

$$= e^{\beta\mu} \to 0 \quad \Rightarrow \quad e^{\mu/k_B T} \to 0$$

$\therefore$ In the classical limit $e^{\mu/k_B T} \ll 1$

Define $\boxed{\Lambda = \frac{\hbar c \pi^{2/3}}{k_B T}}$, the thermal wavelength for relativistic ~~gas~~ gas. , then

$$f(\beta\mu) = \frac{n\Lambda^3}{2s+1}$$

$\because f(\beta\mu) \to e^{\beta\mu}$ in classical limit

$$\therefore e^{\beta\mu} = \frac{n\Lambda^3}{2s+1}$$

$\because e^{\beta\mu} \ll 1 \quad \therefore \quad \boxed{\frac{n\Lambda^3}{2s+1} \ll 1}$

is the condition for classical limit.

$\because \Lambda$ is a function only of T, this is a condition ~~the~~ involving only n and T.

The gas would ~~see~~ cease to be classical

if $\boxed{n\Lambda^3 \sim 1}$ (Degenerate limit)

total energy $U = \frac{(2s+1)V}{2\pi^2} \frac{(k_B T)^4}{(\hbar c)^3} \int_0^\infty \frac{x^3\,dx}{e^{x-\beta\mu} \pm 1}$

$= N k_B T \left(\frac{2s+1}{n\Lambda^3} \frac{1}{2} \int_0^\infty \frac{x^3\,dx}{e^{x-\beta\mu} \pm 1} \right)$

In ~~the~~ Degenerate limit $\frac{2s+1}{\Lambda} \sim 1$ order of

$\int_0^\infty \frac{x^3\,dx}{e^{x-\beta\mu} \pm 1} \sim \int_0^\infty \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15} \sim 1$

$\therefore \frac{U}{N} \sim k_B T$

For ultra relativistic limit the average energy per particle should be significantly greater than the rest energy of a particle

$\rightarrow k_B T \gg m_e c^2$

$\therefore T \gg \frac{m_e c^2}{k_B}$, say $T = 100 \frac{m_e c^2}{k_B} = 5.94 \times 10^{11}\,K$

$n = \frac{1}{\Lambda^3} = \left(\frac{k_B T}{\hbar c \pi^{2/3}} \right)^3 = \underline{1.8 \times 10^{42}\ m^{-3}}$

(e) At very low T, $\beta \to \infty$, for ~~fermi~~ electrons (Fermions):

$$\bar{n}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \to \begin{cases} 1 & \text{if } \varepsilon < \mu(T=0) = \varepsilon_F \\ 0 & \text{if } \varepsilon > \mu(T=0) = \varepsilon_F \end{cases}$$

$$N = \int_0^\infty d\varepsilon\, g(\varepsilon)\, n(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)$$

$$= \frac{(2s+1)V}{2\pi^2(\hbar c)^3}\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^2 = \frac{(2s+1)V}{2\pi^2(\hbar c)^3}\frac{\varepsilon_F^3}{3}$$

$$\to \boxed{\varepsilon_F = \left(\frac{6\pi^2 n}{2s+1}\right)^{\frac{1}{3}}\hbar c}$$

$$U = \int_0^\infty d\varepsilon\, g(\varepsilon)\, n(\varepsilon)\, \varepsilon = \frac{(2s+1)V}{2\pi^2(\hbar c)^3}\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^3$$

$$\cancel{= \frac{(2s+1)N}{2\pi^2(\hbar c)^3\, n}\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^3}$$

$$\cancel{\to \frac{U}{N} = \frac{(2s+1)}{2\pi^2(\hbar c)^3}\frac{2\pi^2(\hbar c)^3}{(2s+1)}\frac{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^3}{\int_0^{\varepsilon_F} d\varepsilon\, \varepsilon^2} =}$$

$$\to \frac{U}{V} = \frac{2s+1}{2\pi^2(\hbar c)^3}\left(\frac{\varepsilon_F^4}{4}\right)$$

energy density

$$= \frac{3}{4}\underbrace{\left(\frac{2s+1}{2\pi^2(\hbar c)^3}\frac{\varepsilon_F^3}{3}\right)}_{n=\frac{N}{V}}\varepsilon_F = \boxed{\frac{3}{4}n\,\varepsilon_F}$$

The criterion for treating the Fermi gas as a gas at $T \to 0$: The width of step function $\sim k_B T$

$\therefore$ $k_B T \ll \varepsilon_F$ for approximating the step function

$\rightarrow$ condition is $\underline{k_B T \ll \varepsilon_F}$

Heat Capacity. At $k_B T \ll \varepsilon_F$

$$U(T) = U(T=0) + \delta U(T)$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial \delta U}{\partial T}\right)_V$$

for, $\frac{k_B T}{\varepsilon_F} \ll 1$ consider the integral

$$I = \int_0^\infty \frac{d\varepsilon\, f(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}$$, change of variable $x = \beta(\varepsilon - \mu)$

$\rightarrow$ $\varepsilon = \mu + k_B T x$, then $d\varepsilon = k_B T dx$

$f(\varepsilon) = f(\mu + k_B T x)$. $\varepsilon = 0 \Rightarrow x = -\frac{\mu}{k_B T}$

$$I = k_B T \int_{-\frac{\mu}{k_B T}}^\infty \frac{dx\, f(\mu + k_B T x)}{e^x + 1}$$

$$= k_B T \int_0^\infty \frac{dx\, f(\mu + k_B T x)}{e^x + 1} + \underbrace{k_B T \int_{-\frac{\mu}{k_B T}}^0 \frac{dx\, f(\mu + k_B T x)}{e^x + 1}}_{x \to -x \quad dx \to -dx}$$

$$= k_B T \int_0^\infty \frac{dx\, f(\mu + k_B T x)}{e^x + 1} + k_B T \int_0^{\frac{\mu}{k_B T}} \frac{dx\, f(\mu - k_B T x)}{e^{-x} + 1}$$

$$\frac{1}{e^{-x}+1} = 1 - \frac{1}{e^x + 1}$$

$$= k_B T \int_0^{\mu/k_B T} dx\, f(\mu - k_B T x) + k_B T \left[\int_0^\infty \frac{dx\, f(\mu + k_B T x)}{e^x + 1} - \int_0^{\mu/k_B T} \frac{dx\, f(\mu - k_B T x)}{e^x + 1}\right]$$

let $\varepsilon = \mu - k_B T x$.
$d\varepsilon = -k_B T dx$.
$x = 0 \rightarrow \varepsilon = \mu$. $x = \frac{\mu}{k_B T}$, $\varepsilon = 0$

$\mu(T=0) \sim \varepsilon_F > 0$
as $T \to 0$, $\frac{\mu}{k_B T} \to \infty$

$$\simeq \int_0^\mu d\varepsilon\, f(\varepsilon) + k_B T \int_0^\infty \frac{dx}{e^x + 1} \underbrace{[f(\mu + k_B T x) - f(\mu - k_B T x)]}_{\simeq 2 k_B T x f'(\mu) + O[(k_B T x)^3]}$$

$$\approx \int_0^{\mu} d\varepsilon f(\varepsilon) + 2(k_BT)^2 f'(\mu) \underbrace{\int_0^{\infty} \frac{dx \cdot x}{e^x+1}}_{\frac{\pi^2}{12}}$$

$$= \int_0^{\mu} d\varepsilon f(\varepsilon) + \frac{\pi^2}{6} f'(\mu) (k_BT)^2 + \cdots$$

$\longrightarrow$ The sommerfold expansion

$$\because \quad N = \frac{(2s+1)V}{2\pi^2(\hbar c)^3} \frac{\varepsilon_F^3}{3} \quad \therefore \quad \frac{(2s+1)V}{2\pi^2(\hbar c)^3} = \frac{3N}{\varepsilon_F^3}$$

$$N = \int_0^{\infty} d\varepsilon\, g(\varepsilon) n(\varepsilon) = \int_0^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}$$

fermi gas

$$= \underbrace{\frac{(2s+1)V}{2\pi^2(\hbar c)^3}}_{\frac{3N}{\varepsilon_F^3}} \int_0^{\infty} \frac{\varepsilon^2 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= \frac{3N}{\varepsilon_F^3} \left[\int_0^{\mu} \varepsilon^2 d\varepsilon + \frac{\pi^2}{6} \cdot \frac{d}{d\varepsilon}(\varepsilon^2)\Big|_{\mu} \cdot (k_BT)^2 \right]$$

sommerfold expansion

$$= \frac{3N}{\varepsilon_F^2} \left[\frac{\mu^3}{3} + \frac{\pi^2}{6} 2\mu (k_BT)^2 \right]$$

$\because$ $\mu \sim \varepsilon_F$, $k_BT \ll \varepsilon_F$ $\therefore$ we can treat $\mu(k_BT)^2$ as $\varepsilon_F(k_BT)^2$

$$\therefore \quad N = \frac{3N}{\varepsilon_F} \left[\frac{\mu^3}{3} + \frac{\pi^2}{6} \cdot 2\varepsilon_F (k_BT)^2 \right]$$

$$\therefore \quad \mu^3 = \varepsilon_F^3 \left(1 - \pi^2 \left(\frac{k_BT}{\varepsilon_F}\right)^2\right)$$

$$\mu = \varepsilon_F \left(1 - \pi^2 \left(\frac{k_BT}{\varepsilon_F}\right)^2\right)^{\frac{1}{3}} \approx \varepsilon_F \left(1 - \frac{\pi^2}{3} \left(\frac{k_BT}{\varepsilon_F}\right)^2\right)$$

$$U = \int_0^\infty d\varepsilon\, g(\varepsilon)\, n(\varepsilon)\, \varepsilon = \cancel{\frac{3N}{\varepsilon_F}}\ \frac{3N}{\varepsilon_F^3}\int_0^\infty \frac{\varepsilon^3 d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$= \frac{3N}{\varepsilon_F}\left[\frac{\mu^4}{4} + \frac{\pi^2}{6} 3\varepsilon_F^2 (k_B T)^2\right]$$

$$= \frac{3N}{\varepsilon_F^3}\left[\frac{\varepsilon_F^4}{4}\left(1-\frac{\pi^3}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right)^4 + \frac{\pi^2}{6}\cdot 3\, \varepsilon_F^2 (k_B T)^2\right]$$

$$= \frac{3N}{\varepsilon_F^3}\left[\frac{\varepsilon_F^4}{4}\left(1 - 4\cdot\frac{\pi^2}{3}\cdot\left(\frac{k_B T}{\varepsilon_F}\right)^2 + 4\cdot\frac{\pi^2}{6}\cdot 3\left(\frac{k_B T}{\varepsilon_F}\right)^2\right)\right]$$

$$= \frac{3}{4} N\varepsilon_F \left[1 + 4\pi^2 \underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{\frac{1}{6}} \left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

$$= \frac{3}{4} N\varepsilon_F \left[1 + \frac{2\pi^2}{3}\left(\frac{k_B T}{\varepsilon_F}\right)^2\right]$$

$$\Rightarrow \quad C_V = \left(\frac{\partial U}{\partial T}\right)_V$$

$\because$ we consider the situation in which # of particles is fixed $\therefore N \to$ const.

Hold V constant $\Rightarrow n = \frac{N}{V} \to$ const.

$\because \varepsilon_F \propto n^{\frac{1}{3}} \quad \therefore \quad V \to$ const $\Rightarrow \varepsilon_F \to$ const

$$\therefore C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_{N,\varepsilon_F}$$

$$= \frac{\cancel{3}}{\cancel{4}} N\varepsilon_F \cdot \frac{\cancel{2}\pi^2}{\cancel{3}} \cdot \frac{k_B^2}{\varepsilon_F^2} \cdot \cancel{2} T$$

$$= \boxed{N k_B \pi^2 \frac{k_B T}{\varepsilon_F}}$$

Qualitative Calculation:

At $0 < T \ll \frac{\varepsilon_F}{k_B}$, the Fermi distribution has a step like shape with finite width ~ ~~kBT~~ $\Delta\varepsilon \sim k_B T$

→ A small number of Fermions with energies $\sim \varepsilon_F$ can be kicked out of ground state to slightly higer energies

→ This number is $\Delta N_{excited} \sim g(\varepsilon_F)\Delta\varepsilon \sim g(\varepsilon_F) k_B T$

Each of these fermions have ~~ΔE~~ on the order $\Delta\varepsilon \sim k_B T$ more energy than it would have done at $T=0$

→ excess mean energy

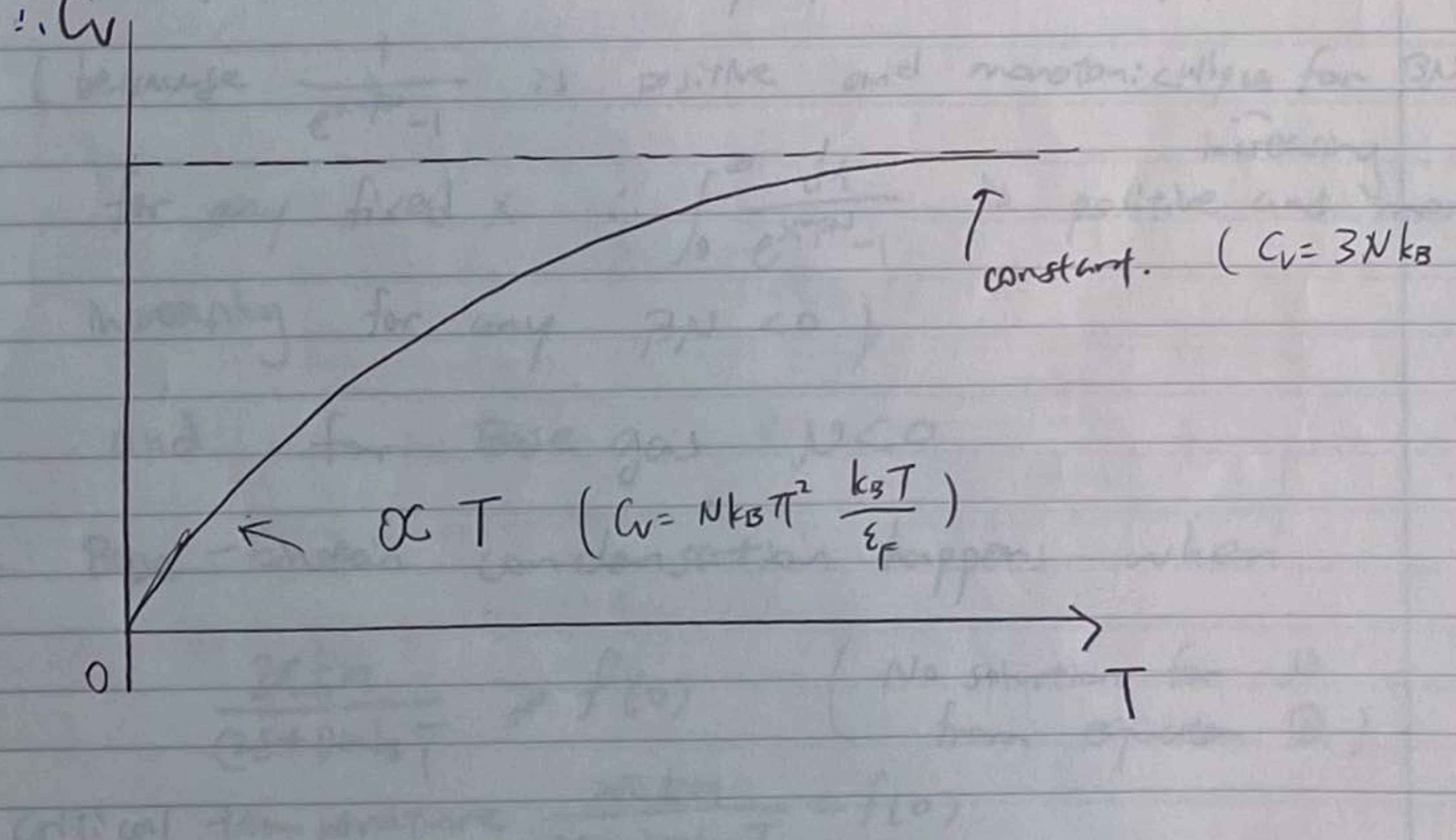
$$\delta U(T) \approx \Delta N_{excited}\, \Delta\varepsilon \sim g(\varepsilon_F)(k_B T)^2 \sim \frac{N}{\varepsilon_F}(k_B T)^2$$

$$\rightarrow\ C_V = \left(\frac{\partial \delta U}{\partial T}\right)_V = \text{Const} \times N k_B \frac{k_B T}{\varepsilon_F}$$

the constant, as determined before, is π^2 for ultrarelativistic ~~quan~~ Fermi gas.

From question 5.7 we know that the classical limit of ultrarelativistic gas' ~~it~~ heat capacity is

$$C_V = 3N k_B \quad \left(T \gg \frac{\varepsilon_F}{k_B}\right)$$



2. (a)

Refer to Q 5.8. Density of states in 2-D is

$$g(k)\,dk = \frac{A k\,dk}{2\pi}(2s+1)$$

$$k = \frac{\sqrt{2m\varepsilon}}{\hbar}, \quad dk = \frac{1}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon$$

$$\therefore g(k)\,dk = \frac{A}{2\pi}\frac{\sqrt{2m\varepsilon}}{\hbar^2}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon\,(2s+1)$$

$$= \frac{\cancel{\pi}}{2\pi}\frac{Am}{\cancel{\pi}\hbar^2}(2s+1)\,d\varepsilon \equiv g(\varepsilon)\,d\varepsilon$$

For Bose gas $\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1}$

$$\therefore N = \sum_i \bar{n}_i = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}$$

$$= \frac{(2s+1)Am}{2\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}$$

$x = \beta\varepsilon$

$d\varepsilon = \frac{1}{\beta}dx$

$$= \frac{(2s+1)Am}{2\pi\hbar^2\beta}\int_0^\infty \frac{dx}{e^{x-\beta\mu}-1}$$

$$\therefore \frac{2\pi\hbar n}{(2s+1)mk_BT} = \int_0^\infty \frac{dx}{e^{x-\beta\mu}-1} \equiv f(\beta\mu) \quad ① \quad \left(n = \frac{N}{A} = \text{density}\right)$$

$f(\beta\mu)$ is bounded by $f(0)$ $\rightarrow$ $f(\beta\mu) \le f(0)$

(because $\frac{1}{e^{x-\beta\mu}-1}$ is positive and monotonically increasing for $\beta\mu<0$ for any fixed x $\therefore \int_0^\infty \frac{dx}{e^{x-\beta\mu}-1}$ is positive and monotonically increasing for any $\beta\mu<0$)

and for Bose gas $\mu<0$

$\therefore$ Bose-Einstein condensation happens when

$$\frac{2\pi\hbar n}{(2s+1)mk_BT} > f(0)$$

(No solution for μ from equation ①)

critical temperature $\frac{2\pi\hbar n}{(2s+1)mk_BT_c} = f(0)$

But in 2-D case

$$f(0) = \int_0^\infty \frac{dx}{e^x - 1} = \int_0^\infty \frac{e^{-x}\,dx}{1 - e^{-x}}$$

$u = e^{-x}$

$du = -e^{-x}dx$

$1 - e^{-x} = 1 - u$

$x = 0 \to u = 1$

$x = \infty \to u = 0$

$$= \cancel{\int_0^\infty} - \int_1^0 \frac{du}{1-u}$$

$$= \int_0^1 \frac{du}{1-u} = -\ln(1-u)\Big|_0^1$$

$$= \underbrace{-\ln(0)}_{-\infty} + \underbrace{\ln(1)}_{0} \Rightarrow +\infty$$ → integral diverges

$\because \frac{1}{T_c} \propto f(0) \quad \therefore T_c = 0\,K$

→ Bose-Einstein condensation does not occur in 2D.

Equation ① always have a solution for μ and the critical temperature is ~~0K~~ 0.

(b)

$$N = \frac{(2S+1)Am}{2\pi\hbar^2\beta}\int_0^\infty \frac{dx}{e^{x-\beta\mu}-1}$$

$$= \frac{(2S+1)Am}{2\pi\hbar^2\beta}\int_0^\infty \frac{e^{\beta\mu - x}}{1-e^{\beta\mu-x}}dx$$

$\cancel{u = e^{\beta\mu - x}}$

$u = e^{\beta\mu - x}$

$du = e^{\beta\mu - x}(-dx)$

$x=0 \quad u = e^{\beta\mu}$

$x=\infty \quad u=0$

$$= \frac{(2S+1)Am}{2\pi\hbar^2\beta}(-)\int_{e^{\beta\mu}}^0 \frac{du}{1-u}$$

$$= \frac{(2S+1)Am}{2\pi\hbar^2\beta}\int_0^{e^{\beta\mu}} \frac{du}{1-u}$$

$$= \frac{(2S+1)Am}{2\pi\hbar^2\beta}\Big[-\ln(1-u)\Big]_0^{e^{\beta\mu}}$$

$$\rightarrow \quad N = -\frac{(2S+1)Am}{2\pi\hbar^2\beta}\ln(1-e^{\beta\mu})$$

$$\rightarrow \quad n = -\frac{(2S+1)\cancel{A}k_B}{2\pi\hbar^2}T\ln(1-e^{\mu/k_BT})$$

$$\rightarrow \quad -\frac{2\pi\hbar^2 n}{(2S+1)\cancel{A}k_BT} = \ln(1-e^{\mu/k_BT})$$

$$1-e^{\mu/k_BT} = \exp\left(-\frac{2\pi\hbar^2 n}{(2S+1)\cancel{A}k_BT}\right)$$

$$e^{\mu/k_BT} = 1-\exp\left(-\frac{2\pi\hbar^2 n}{(2S+1)\cancel{A}k_BT}\right)$$

$$\rightarrow \quad \boxed{\mu = k_BT\ln\left(1-\exp\left(-\frac{2\pi\hbar^2 n}{(2S+1)\cancel{A}k_BT}\right)\right)}$$

let $n_Q \equiv \frac{(2S+1)\cancel{A}k_BT}{2\pi\hbar^2}$, then

$$\underline{\mu = k_BT\ln(1-e^{-n/n_Q})}$$

For small T, $T \to 0$

let $C = \frac{2\pi\hbar^2 n}{(2s+1)k_B}$, $\frac{C}{T} \to \infty$

$\mu = +k_B T \ln(1 - e^{-\frac{c}{T}})$, $e^{-\frac{c}{T}} \to e^{-\infty} \to 0$

very small

$\cancel{= k_B T \ln\left(\frac{1}{1-e^{-\frac{c}{T}}}\right)}$ $e^{-\frac{c}{T}} \ll 1.$

$\cancel{= k_B T \ln\left(\frac{e^{+\frac{c}{T}}}{e^{\frac{c}{T}}-1}\right)}$

For small x, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$

$\approx -x$

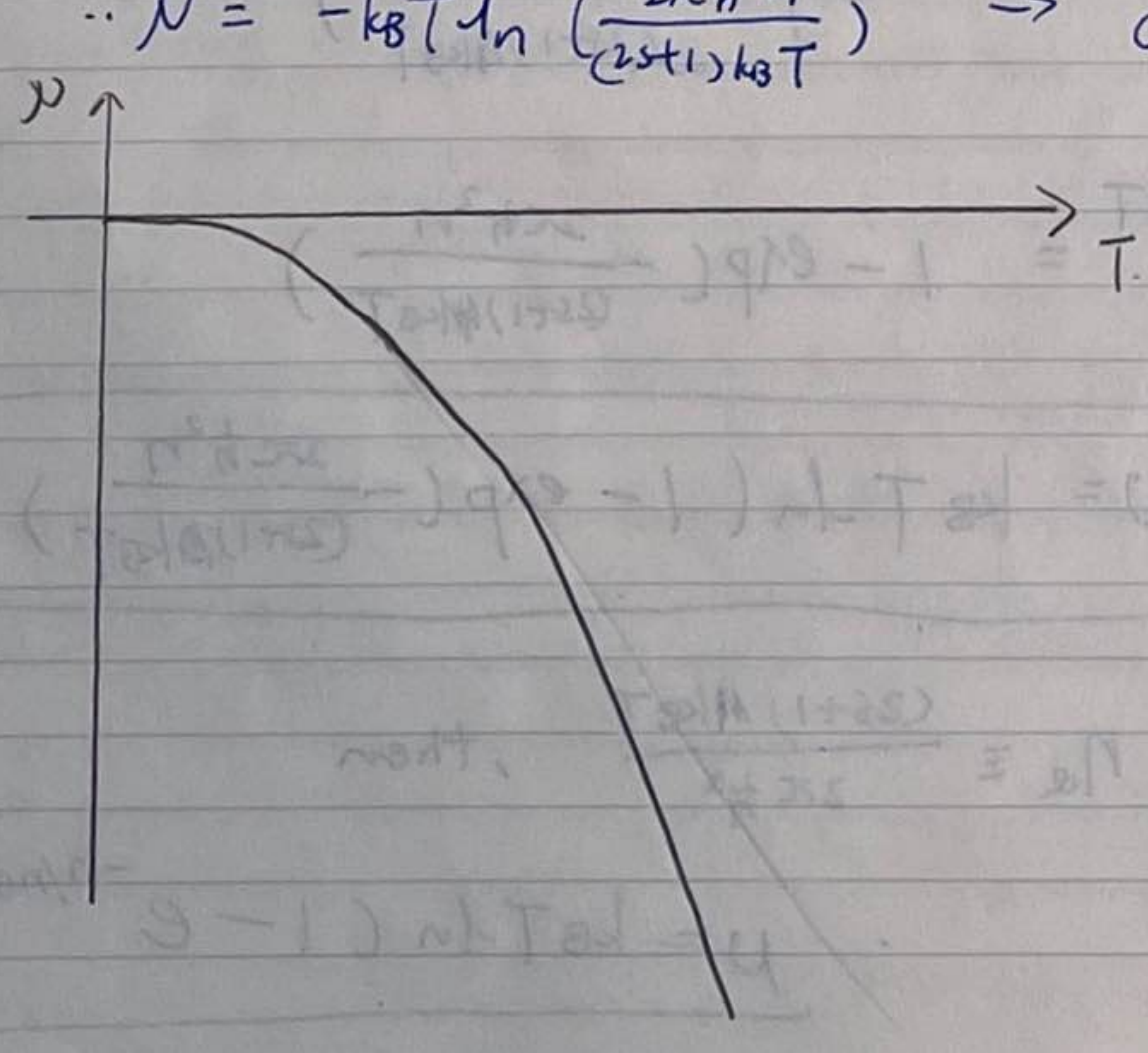
$\therefore \mu = +k_B T \ln(1 - e^{-\frac{c}{T}})$

$\cancel{= -k_B T \ln(e^{-\frac{c}{T}})} = \cancel{-}+ k_B T(-e^{-\frac{c}{T}})$

$\rightarrow \boxed{\mu = \cancel{e} - k_B T \exp\left(-\frac{2\pi\hbar^2 n}{(2s+1)k_B T}\right) \text{ as } T \to 0}$

For large T, $T \to \infty$. $\frac{c}{T} \to 0$ $e^{-\frac{c}{T}} \sim 1 - \frac{c}{T}.$

$\therefore \mu = -k_B T \ln\left(\frac{2\pi\hbar^2 n}{(2s+1)k_B T}\right)$ $\to$ classical limit.



(c)

$$C_{\cancel{A}} = \left(\frac{\partial U}{\partial T}\right)_{\cancel{A}}$$

$$U = \sum_i \bar{n}_i \varepsilon_i = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)\,\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}$$

$$= \cancel{\int_0^\infty} \frac{(2S+1)Am}{2\pi\hbar^2} \int_0^\infty \frac{\varepsilon\, d\varepsilon}{e^{\beta(\varepsilon-\mu)}-1}$$

$$\cancel{= \frac{(2S+1)Am}{2\pi\hbar^2\beta^2} \int_0^\infty x\,dx}$$

$$= \frac{(2S+1)Am}{2\pi\hbar^2\beta^2} \int_0^\infty \frac{x\,dx}{e^{x-\beta\mu}-1}$$

$\because \beta = \frac{1}{k_B T}$, and $\mu = -k_B T \exp\left(-\frac{2\pi\hbar^2 n}{(2S+1)k_B T}\right)$ as $T \to 0$

$\therefore \beta\mu = \cancel{-\exp(-\ldots)} - \exp\left(-\frac{c}{T}\right)$ as $T \to 0$

$= -\exp(-\infty) \Rightarrow -0$

$\therefore e^{-\beta\mu} \to e^0 \to 1$ as $T \to 0$

$$\therefore \int_0^\infty \frac{x\,dx}{e^{x-\beta\mu}-1} \to \int_0^\infty \frac{x\,dx}{e^x-1} = 1.645$$

$$\therefore U \approx \frac{(2S+1)Am}{2\pi\hbar^2\beta^2} \times 1.645 = \frac{1.645(2S+1)Amk_B^2}{2\pi\hbar^2} T^2$$

$$\therefore C_A = \left(\frac{\partial U}{\partial T}\right)_A \approx \boxed{\frac{1.645(2S+1)Amk_B^2}{2\pi\hbar^2} \times 2T} \propto T.$$

as $T \to 0$

$\longrightarrow$ $C_A \propto T$ as $T \to 0$

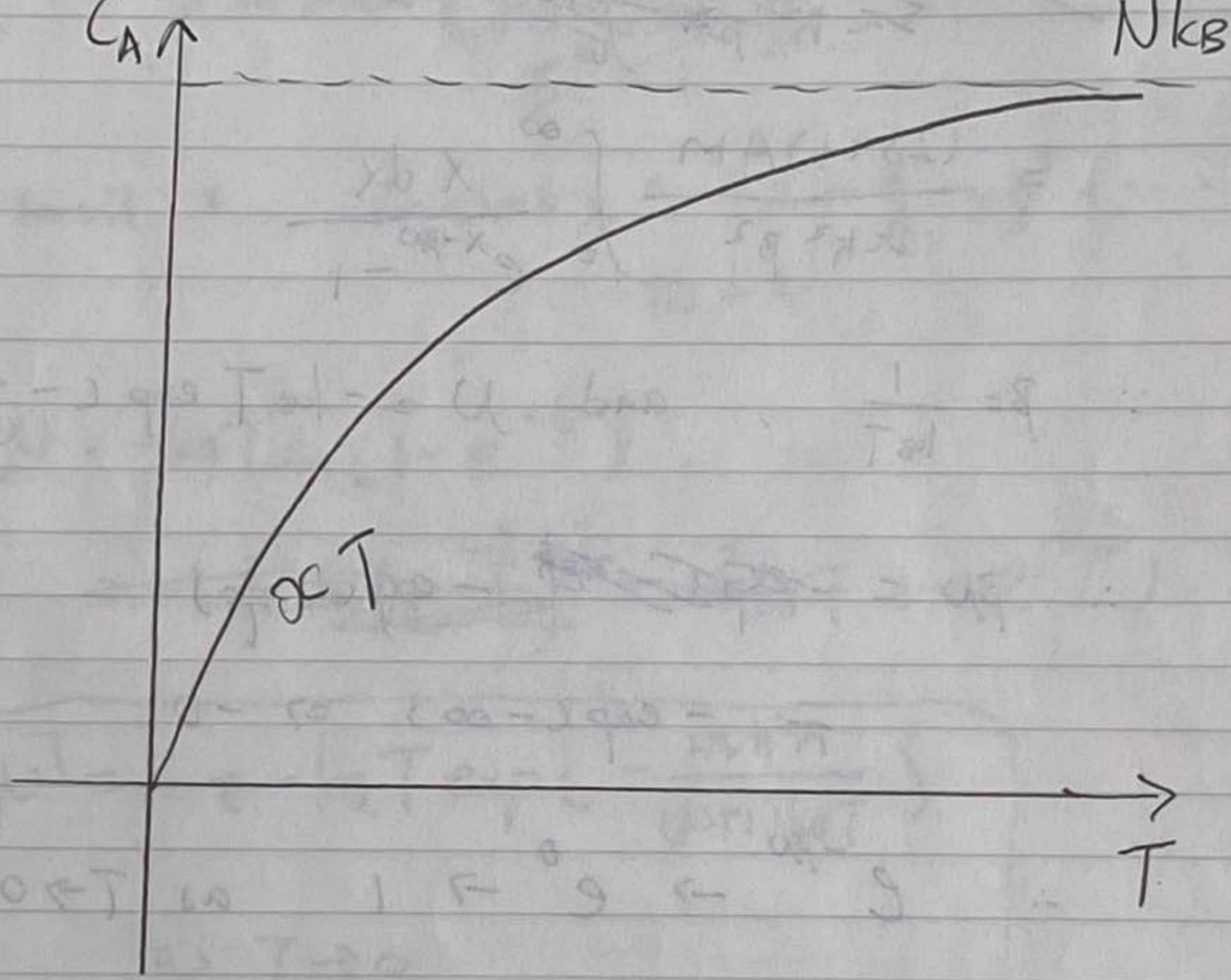
For $T \to \infty$, ~~in~~ the classical limit applies and from equipartition theorem we have

$$U = \frac{2}{2} N k_B T = N k_B T \quad \text{as} \quad T \to \infty$$

$$\therefore \quad C_A = \left(\frac{\partial U}{\partial T}\right)_A = N k_B \quad \underline{\text{independent of } T}$$

as $T \to \infty$

$\therefore$



3.

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{2s+1}\right)^{2/3}, \quad T_F = \frac{\varepsilon_F}{k_B}$$

(a) In Liquid ^3He, the atoms are free to move

$$n = \frac{\rho}{m} = \frac{0.0823\times10^{-3}\,kg/10^{-6}\,m^3}{3.016\times1.661\times10^{-27}}$$

$$= 1.595\times10^{28}\,m^{-3}$$

$$S=\frac{1}{2} \rightarrow 2S+1=2$$

$$\varepsilon_F = \frac{(1.05\times10^{-34})^2}{2\times3.016\times1.661\times10^{-27}}\left(\frac{6\pi^2(1.595\times10^{28})}{2}\right)^{2/3} / (1.6\times10^{-19})$$

$$= \boxed{4.17\times10^{-4}\,eV}$$

$$T_F = \frac{\varepsilon_F}{k_B} = \boxed{4.8\,K}$$

(b)

$$n = n_{electron} = \text{Valence} \times n_{atom}$$

$$= 3\left(\frac{\rho}{m_{Al}}\right) = \frac{3\times2.7\times10^3\,kg/m^3}{26.98\times1.661\times10^{-27}} = 1.81\times10^{29}\,m^{-3}$$

$$S=\frac{1}{2} \rightarrow 2S+1=2$$

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left(\frac{6\pi^2 n}{2}\right)^{2/3} = \boxed{11.6\,eV}$$

$$T_F = \frac{\varepsilon_F}{k_B} = \boxed{1.34\times10^5\,K}$$

(c)

$$2n = \frac{A}{\frac{4}{3}\pi r^3} = \frac{A}{\frac{4}{3}\pi (1.2\times10^{-15})^3 A}$$

density of nucleons is twice the density of neutrons for $^{16}_{8}O$

$$= 1.38\times10^{44}\,m^{-3} \rightarrow n = 6.9\times10^{43}\,m^{-3}$$

$$S = \frac{1}{2} \rightarrow 2S+1 = 2$$

$$\varepsilon_F = \frac{\hbar^2}{2m_p}\left(\frac{6\pi^2 n}{2}\right)^{2/3}$$

$$= \boxed{3.3\times10^7\,eV}$$

$$T_F = \frac{\varepsilon_F}{k_B} = \boxed{3.9\times10^{11}\,K}$$

4. (i) Density of state for 2D.

$$g(k)\,dk = \frac{Ak}{2\pi}\,dk \; a(2s+1)$$

$$k = \frac{\sqrt{2m\varepsilon}}{\hbar} \qquad dk = \frac{1}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon$$

$$\therefore g(k)\,dk = \frac{A}{2\pi}\frac{\sqrt{2m\varepsilon}}{\hbar}\frac{1}{\hbar}\sqrt{\frac{m}{2\varepsilon}}\,d\varepsilon\;(2s+1)$$

$$\rightarrow \quad = \frac{Am(2s+1)}{2\pi\hbar^2}\,d\varepsilon = g(\varepsilon)\,d\varepsilon.$$

$$N = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon) = \frac{Am(2s+1)}{2\pi\hbar^2}\int_0^{\varepsilon_F} d\varepsilon$$

$$= \frac{Am(2s+1)}{2\pi\hbar^2}\varepsilon_F$$

$$\rightarrow \quad \boxed{\varepsilon_F = \frac{2\pi\hbar^2 n}{m(2s+1)}}$$

(ii)

$$\varepsilon_F = \frac{2\pi(1.05\times10^{-34})^2(4\times10^{11}\times10^{4})}{(0.15\times9.11\times10^{-31})(2(\tfrac{1}{2})+1)}$$

$$= 1.01\times10^{-21}\,J = \boxed{6.3\times10^{-3}\,eV}$$

(iii) In 1-D.

$$g(k)\,dk = \frac{L}{\pi}\,dk\,(2s+1)$$

$$= \frac{(2s+1)L}{\pi\hbar}\sqrt{\frac{m}{2}}\frac{d\varepsilon}{\sqrt{\varepsilon}} \equiv g(\varepsilon)\,d\varepsilon$$

$$N = \int_0^{\varepsilon_F} d\varepsilon\,(g(\varepsilon)) = \frac{(2s+1)L}{\pi\hbar}\sqrt{\frac{m}{2}}\cdot 2\sqrt{\varepsilon_F}$$

$$n = \frac{N}{L} \rightarrow \frac{\sqrt{2}\pi\hbar n}{2(2s+1)\sqrt{m}} = \sqrt{\varepsilon_F}$$

$$\rightarrow \boxed{\varepsilon_F = \frac{\pi^2 \hbar^2 n^2}{2(2s+1)^2 m}}$$

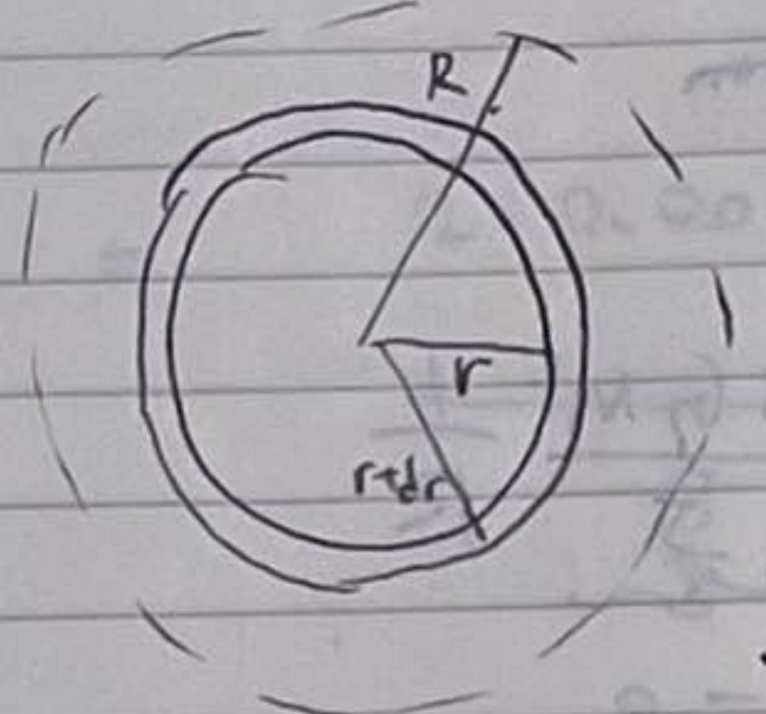
(iv)

$$n = \frac{1}{2.5 \times 10^{-10}\,m} \times 0.5 = 2 \times 10^{9}\ m^{-1}$$

$$\varepsilon_F = \frac{\pi^2 (1.05 \times 10^{-34})^2 (2 \times 10^{9})^2}{8(9.11 \times 10^{-31})} \Big/ (1.6 \times 10^{-19})$$

$$= \boxed{0.37\ eV}$$

5. (i)



$$\rho = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3M}{4\pi R^3}$$

$$dU_g = - \frac{G\left(\frac{4}{3}\pi r^3 \rho\right)\left(\rho 4\pi r^2 dr\right)}{r}$$

$$\therefore U_g = \int_0^R dU_g$$

$$= -G\left(\tfrac{4}{3}\pi\rho\right)(\rho\cdot 4\pi)\int_0^R r^4 dr$$

$$= -G\,\frac{4}{3}\pi\cdot 4\pi\cdot\frac{3^2}{(4\pi)^2}\frac{M^2}{R^6}\cdot\left(\frac{R^5}{5}\right)$$

$$= \boxed{-\frac{3GM^2}{5R}}$$

(ii) mass is mainly contributed by protons and neutrons $\therefore$ # of protons + neutrons $\approx \dfrac{M}{m_p}$

$\because$ Equal # of protons, neutron, and electrons

$\therefore$ # of electrons $N_e = \dfrac{M}{2m_p}$

volume $V = \frac{4}{3}\pi R^3$

$\therefore$ density of electrons $n = \dfrac{N_e}{V} = \dfrac{3M}{8\pi m_p R^3}$

(kinetic energy is mainly contributed by electrons because electrons have lighter mass $\rightarrow$ higher fermi energy)

$$U_{electron} = \frac{3}{5}N_e\varepsilon_F = \frac{3}{5}\left(\frac{M}{2m_p}\right)\left(\frac{\hbar^2}{2m_e}\right)\left(\frac{6\pi^2}{2\times\frac{1}{2}+1}\left(\frac{3M}{8\pi m_p R^3}\right)\right)^{2/3}$$

electron is spin-$\frac{1}{2}$

$$= \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{6\pi^2}{2}\times\frac{3}{8\pi}\right)^{2/3}\times\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2}$$

$$= 0.348\,\frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3} R^2} = \boxed{0.0088\,\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}} \quad (h = 2\pi\hbar)$$

(iii) . (iv)

$$U_{total} = U_{grav} + U_{electron}$$

$$= \underbrace{0.0088 \frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3}}}_{\equiv A} \frac{1}{R^2} - \underbrace{\frac{3GM^2}{5}}_{\equiv B} \frac{1}{R}$$

$$= \frac{A}{R^2} - \frac{B}{R}$$

For the Sun

$M = 2 \times 10^{30}$ kg .

$$A = 5.735 \times 10^{56} \text{ J}\cdot\text{m}^2$$

$$B = 1.601 \times 10^{50} \text{ J}\cdot\text{m}$$

minimise U_{total} , set $\frac{dU_{total}}{dR} = 0$

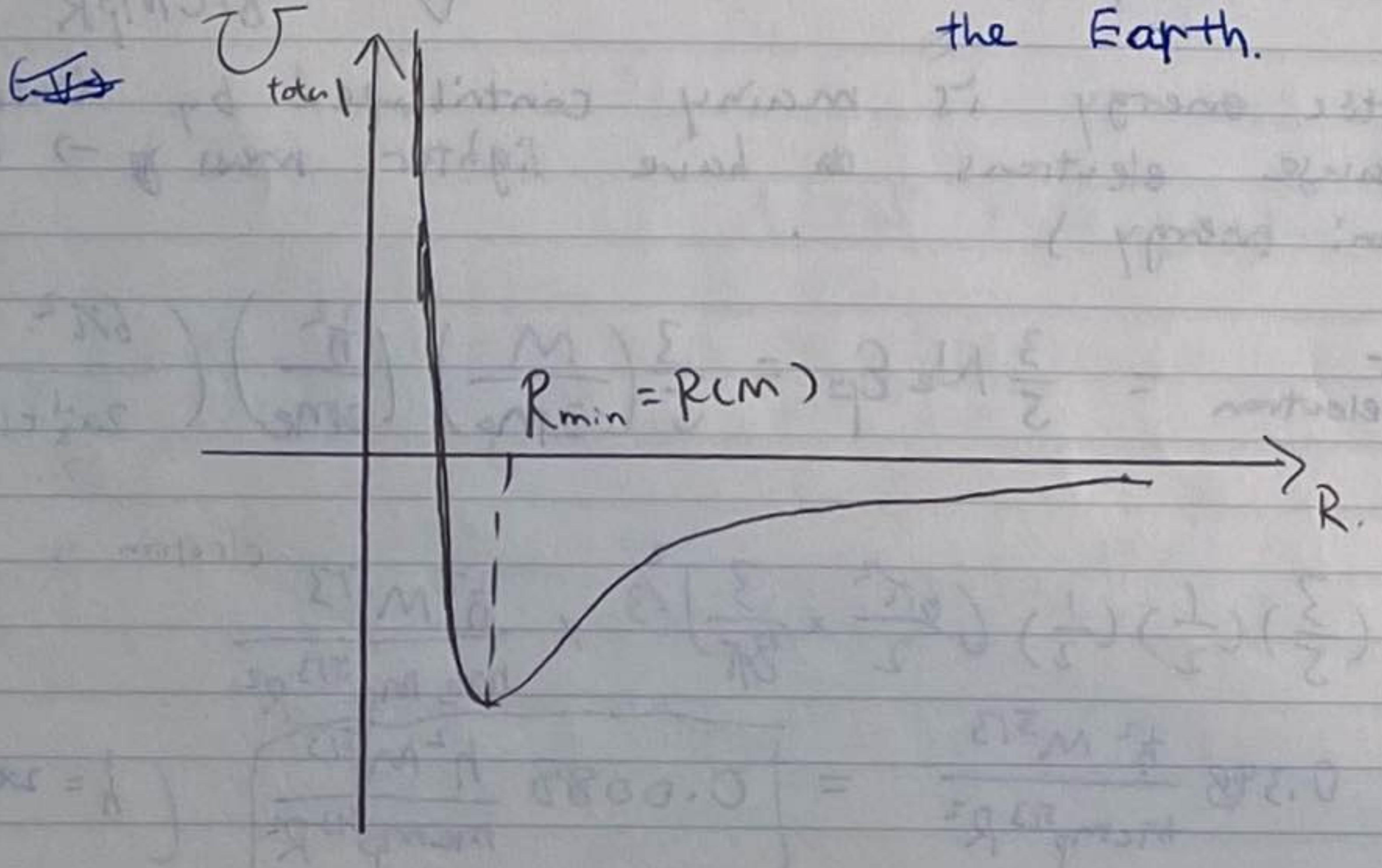
$$\Rightarrow 0 = -2\frac{A}{R_{min}^3} + \frac{B}{R_{min}^2}$$

$$\therefore R_{min} B - 2A = 0$$

$$\rightarrow R_{min} = \boxed{\frac{2A}{B}} \simeq \boxed{7.16 \times 10^6 \text{ m}}$$

$$\sim 10^6 \text{ m}$$

$\sim$ order of radius of the Earth.



$$R(M) = R_{min} = \frac{2A}{B}$$

$$= \frac{2 \times 0.0088 \, \frac{\hbar^2 M^{5/3}}{m_e m_p^{5/3}}}{\frac{3}{5} G M^2}$$

$$= \boxed{0.0293 \frac{\hbar^2}{m_e m_p^{5/3} G} M^{-1/3}}$$

(v) The Fermi energy

$$\varepsilon_F = \frac{\hbar^2}{2m_e} \left(\frac{3\pi^2 \times 3M}{8\pi \, m_p R^3} \right)^{2/3}$$

$M = 2 \times 10^{30}$ kg $\quad m_e = 9.11 \times 10^{-31}$ kg $\quad m_p = 1.67 \times 10^{-27}$ kg

$R = 7.16 \times 10^{6}$ m $\quad , \hbar = 1.05 \times 10^{-34}$ J.s

~~$\varepsilon_F = 2.94 \times 10$~~ $\quad \varepsilon_F = 3.09 \times 10^{-14}$ J

$$= \boxed{1.93 \times 10^5 \text{ eV}}$$

rest energy of electron

$$\varepsilon_r = m_e c^2 = \cancel{8.19}\ 8.20 \times 10^{-14} \text{ J}$$

$$= 5.12 \times 10^5 \text{ eV}$$

$\because$ ε_F is comparable to ε_r

$\therefore$ the relativistic effect is significant

$\therefore$ we were not correct.

6. (i) If electrons are relativistic, then from Q 5.7 we know that the energy levels are $\varepsilon_k = \sqrt{\hbar^2 c^2 k^2 + (mc^2)^2}$

From ~~Q6~~ Q 7.1 we know that

$$g(k)\,dk = \frac{(2s+1)V}{2\pi^2} k^2\,dk$$

$$\varepsilon = \sqrt{\hbar^2 c^2 k^2 + (mc^2)^2} \rightarrow \varepsilon^2 - (mc^2)^2 = \hbar^2 c^2 k^2$$

$$\therefore\ k = \frac{1}{\hbar c}\sqrt{\varepsilon^2 - (mc^2)^2} \quad , k^2 = \cancel{\frac{1}{(\hbar c)^2}} \frac{1}{(\hbar c)^2}(\varepsilon^2 - (mc^2)^2)$$

$$dk = \cancel{\frac{1}{2\hbar c}} \frac{1}{\hbar c} \frac{\varepsilon}{\sqrt{\varepsilon^2 - (mc^2)^2}}\,d\varepsilon$$

$$\rightarrow \quad g(k)\,dk = \frac{(2s+1)V}{2\pi^2} k^2\,dk$$

$$= \frac{(2s+1)V}{2\pi^2} \frac{1}{(\hbar c)^2}(\varepsilon^2 - m^2c^4) \frac{1}{\hbar c} \frac{\varepsilon}{\sqrt{\varepsilon^2 - m^2c^4}}\,d\varepsilon$$

$$= \frac{(2s+1)V}{2\pi^2 (\hbar c)^3} \varepsilon\sqrt{\varepsilon^2 - m^2c^4}\,d\varepsilon \equiv g(\varepsilon)\,d\varepsilon$$

$$N = \int_{mc^2}^{\varepsilon_F} g(\varepsilon)\,d\varepsilon$$

$$= \frac{(2s+1)V}{2\pi^2 (\hbar c)^3} \int_{mc^2}^{\varepsilon_F} \varepsilon\sqrt{\varepsilon^2 - m^2c^4}\,d\varepsilon$$

$$= \frac{(2s+1)V}{2\pi^2 (\hbar c)^3} mc^2 \int_{mc^2}^{\varepsilon_F} \varepsilon\sqrt{\left(\frac{\varepsilon}{mc^2}\right)^2 - 1}\,d\varepsilon$$

let $u = \frac{\varepsilon}{mc^2}$, $u_F = \frac{\varepsilon_F}{mc^2}$

$$\int_{mc^2}^{\varepsilon_F} \varepsilon\sqrt{\left(\frac{\varepsilon}{mc^2}\right)^2 - 1}\,d\varepsilon = \int_1^{u_F} mc^2 u \sqrt{u^2 - 1}\,(mc^2)\,du$$

$$= m^2c^4 \int_1^{u_F} u\sqrt{u^2-1}\,du$$

$$\int u\sqrt{u^2-1}\,du$$

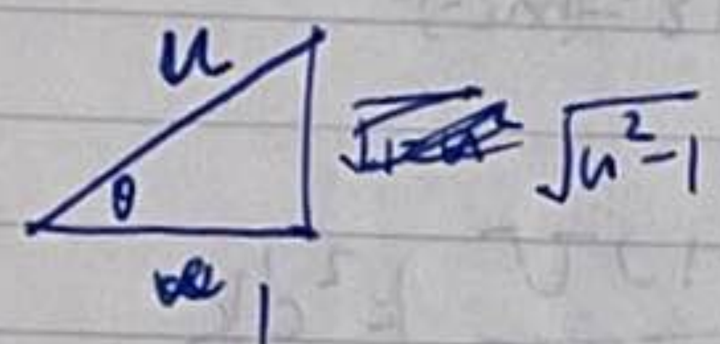
let $u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$
$\sqrt{u^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$= \int \sec\theta^2 \tan^2\theta\, d\theta$$

let $x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$$= \int x^2\,dx = \frac{x^3}{3} + C$$

$$= \frac{(\tan\theta)^3}{3} + C$$



$\sec\theta = u \rightarrow \tan\theta = \sqrt{u^2-1}$

$$\therefore \int u\sqrt{u^2-1}\,du = \frac{(u^2-1)^{3/2}}{3} + C$$

$$\therefore \int_1^{u_f} u\sqrt{u^2-1}\,du = \left.\frac{(u^2-1)^{3/2}}{3}\right|_1^{u_f} = \frac{1}{3}\cancel{\left\{[u_f^2-1]^{3/2} + 1\right\}}$$

$$= \frac{1}{3}[u_f^2-1]^{3/2}$$

$$= \frac{1}{3}\left[\left(\frac{\varepsilon_F}{mc^2}\right)^2 - 1\right]^{3/2}.$$

$$\therefore\ n = \frac{(2s+1)}{2\pi^2(\hbar c)^3}(mc^2)^3\frac{1}{3}\left[\left(\left(\frac{\varepsilon_F}{mc^2}\right)^2 - 1\right)^{3/2}\right]$$

$$= \frac{(2s+1)}{6\pi^2(\hbar c)^3}\left(\varepsilon_F^2 - (mc^2)^2\right)^{3/2}.$$

$$\rightarrow \left(\varepsilon_F^2 - (mc^2)^2\right)^{3/2} = \frac{6\pi^2(\hbar c)^3 n}{2s+1}$$

$$\varepsilon_F = \left(\left(\frac{6\pi^2(\hbar c)^3 n}{2s+1}\right)^{2/3} + (mc^2)^2\right)^{\frac{1}{2}}$$

Considering only the kinetic energies of electron, we scale down the energy levels by mc^2

$$\to \varepsilon_F = \cancel{2S\varepsilon_k}\left(\left(\frac{6\pi^2(\hbar c)^3 n}{2s+1}\right)^{2/3} + (mc^2)^2\right)^{\frac{1}{2}} - mc^2$$

The total energy

$$U = \int_{mc^2}^{\infty} d\varepsilon\, g(\varepsilon)\, n(\varepsilon)\, \varepsilon$$

$$= \frac{(2s+1)V}{2\pi^2(\hbar c)^3}\int_{mc^2}^{\infty} \varepsilon^2\sqrt{\varepsilon^2 - m^2c^4}\, d\varepsilon$$

$$= \frac{(2s+1)V}{2\pi^2(\hbar c)^3}(mc^2)^4\int_1^{u_f} u^2\sqrt{u^2-1}\, du.$$

$$\int u^2\sqrt{u^2-1}\, du = \int \sec^3\theta \tan^2\theta\, d\theta \qquad (u = \sec\theta)$$

$$= \text{let } x = \sec^2\theta \quad dx = 2\sec\theta\tan^2\theta\, d\theta;\ \frac{1}{2}dx = \sec\theta\tan^2\theta\, d\theta$$

$$= \int \underbrace{\sec^2\theta}_{x}\ \underbrace{\sec\theta\tan^2\theta\, d\theta}_{\frac{1}{2}dx} = \frac{1}{2}\int x\, dx = \frac{x^2}{4}$$

$$= \cancel{\frac{\sec^4\theta}{4}}\ \frac{\sec^4\theta}{4} = \frac{u^4}{4}\ \blacksquare$$

$$\therefore \int_1^{u_f} u^2\sqrt{u^2-1}\, du =$$

the analytical result of $\int u^2\sqrt{u^2-1}\,du$ is rather complicated

$\rightarrow$ we expand the integrand

$$\int u^2\sqrt{u^2-1}\,du = \int u^3\sqrt{1-\frac{1}{u^2}}\,du$$

$$= \int u^3\left(1-\frac{1}{2u^2}+\cdots\right)du.$$

$$\approx \int u^3 - \frac{u}{2}\,du$$

$$= \frac{u^4}{4} - \frac{u^2}{4} + \cdots + C.$$

$\therefore$

$$U = \frac{(2s+1)V}{2\pi^2(\hbar c)^3}\frac{(mc^2)^4}{4}\left[\left(\frac{\varepsilon_F}{mc^2}\right)^4 - \left(\frac{\varepsilon_F}{mc^2}\right)^2 + \cdots\right].$$

For electrons $s=\frac{1}{2}$ $\rightarrow$ $2s+1=2$

$$U = \frac{V}{\pi^2\hbar^3c^3}\left[\frac{\varepsilon_F^4}{4} - \frac{m^2c^4}{4}\varepsilon_F^2 + \cdots\right]$$

$$\varepsilon_F = \left((3\pi^2\hbar^3c^3n)^{2/3} + m^2c^4\right)^{\frac{1}{2}}$$

$$n = \frac{3M}{8\pi m_p R^3} \quad \text{(ignoring the constant terms)}$$

$$\varepsilon_F = \left(\left(\frac{9\pi\hbar^3c^3M}{8m_pR^3}\right)^{2/3} + m^2c^4\right)^{\frac{1}{2}}$$

$$= \left(\frac{9\pi\hbar^3c^3M}{8m_pR^3}\right)^{\frac{1}{3}}\left(1 + \frac{m^2c^4}{\left(\frac{9\pi\hbar^3c^3M}{8m_pR^3}\right)^{2/3}}\right)^{\frac{1}{2}}$$

$$\simeq \left(\frac{9\pi\hbar^3c^3M}{8m_pR^3}\right)^{\frac{1}{3}}\left(1 + \frac{1}{2}\frac{m^2c^4}{\left(\frac{9\pi\hbar^3c^3M}{8m_pR^3}\right)^{2/3}}\right)$$

$$= \underbrace{\left(\frac{9\pi \hbar^3 c^3 M}{8m_p}\right)^{\frac{1}{3}}}_{k_1} \frac{1}{R} + \underbrace{\frac{1}{2}\left(\frac{9\pi \hbar^3 c^3 M}{8m_p}\right)^{-\frac{1}{3}} m^2 c^4}_{k_2} R \cancel{R}$$

$$= \frac{k_1}{R} + k_2 R$$

$$\cancel{U \propto V \frac{\varepsilon_F^4}{4} m}$$

$$U_{\text{electron}} = \frac{V}{\pi^2 \hbar^3 c^3}\left[\frac{\varepsilon_F^4}{4} - \frac{m^2c^4}{4}\varepsilon_F^2\right]$$

$$= \frac{\cancel{4} R^3}{3\pi \hbar^3 c^3}\left[\cancel{\left(\frac{k_1}{4k_2}\right.} \left(\frac{k_1}{R} + k_2 R\right)^4 - m^2c^4 \left(\frac{k_1}{R} + k_2 R\right)^2\right]$$

$$= \frac{R^3}{3\pi \hbar^3 c^3}\left[\frac{k_1^4}{R^4} + \frac{4k_1^3 k_2}{R^2} + \cdots - \frac{m^2c^4 k_1^2}{R^2} \cdots\right]$$

$$= \frac{1}{3\pi \hbar^3 c^3}\left[\frac{k_1^4}{R} + (\cancel{4k_1^3k_2}\ \underbrace{4k_1^3k_2 - m^2c^4k_1^2}_{Q}) R\right]$$

$$k_1 k_2 = \frac{1}{2} m^2 c^4$$

$$\therefore \quad 4k_1^3 k_2 = \cancel{2k_1^2 k_1 k_2} \; 2k_1^2(2k_1k_2) = 2m^2c^4 k_1^2$$

$$\therefore \quad (4k_1^3k_2 - m^2c^4k_1^2) = m^2c^4 k_1^2 \geq 0$$

$$\cancel{\therefore\ m>0 \quad c>0 \quad k_1>0,\ k_2>0} \quad \cancel{\therefore m^2c^4k_1^2 > 0}$$

$$\therefore \quad Q > 0$$

→ The coefficient in front of $R > 0$

$$\therefore \quad U_{\text{electron}} = \frac{k_1^4}{3\pi \hbar^3 c^3}\frac{1}{R} + (\text{term } \underline{\text{growing}} \text{ linearly with } R)$$

$+ \text{(higher order of } R\text{)}.$

~~By expanding U and $\mathcal{E}_F$ we assumed that~~

~~$\mathcal{E}_F \gg m_e c^2 \Rightarrow$ ultrarelativistic electrons.~~

For the ~~radius we are interested~~ "small" radius

R^{1} dominate over higher powers of R.

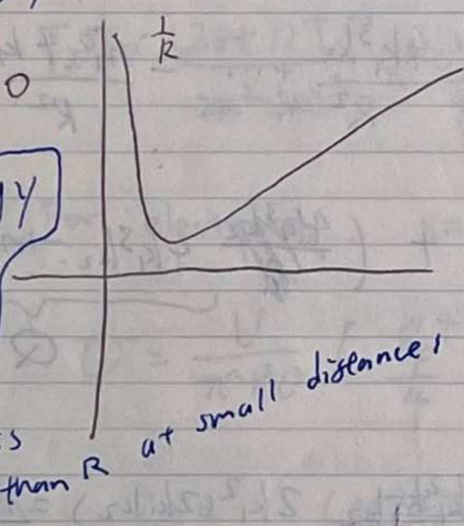
$$U_{total} = U_{electron} + U_{grav}$$

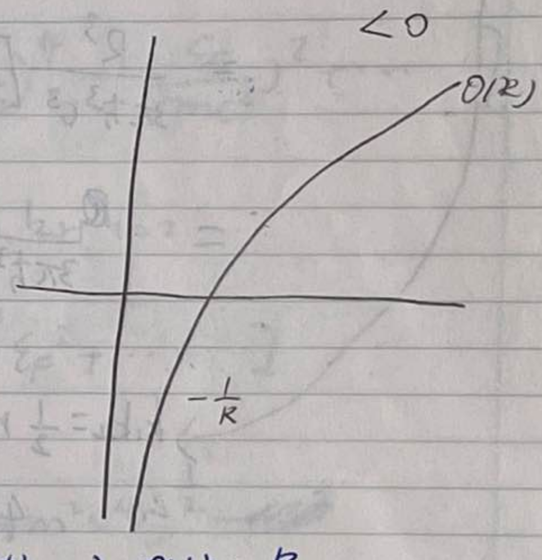
$$= \frac{1}{3\pi\hbar^3 c^3}\left(\frac{9\pi\hbar^3 c^3 M}{8 m_p}\right)^{4/3}\frac{1}{R} - \frac{3}{5}GM^2\frac{1}{R} + O(R)$$

$$= \left[\frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}(\hbar c)\left(\frac{M}{m_p}\right)^{4/3} - \frac{3}{5}GM^2\right]\frac{1}{R} + O(R)$$

total energy scale as $\frac{1}{R}$

because $\frac{1}{R}$ is more dominate than R at small distance!





$\because$ $O(R)$ is increasing with increasing R

(ii) $\therefore$ ~~coeffien~~ coefficient infront of $\frac{1}{R}$ must be +ve

$$\therefore \frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}(\hbar c)\left(\frac{1}{m_p}\right)^{4/3} M^{4/3} - \frac{3}{5}GM^2 \geq 0$$

$\rightarrow$ critical mass M_c when equality holds

$$\frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}(\hbar c)\left(\frac{1}{m_p}\right)^{4/3} M_c^{4/3} = \frac{3}{5}GM^2$$

$$M_c = \left[\frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}\left(\frac{5}{3}\right)\left(\frac{\hbar c}{G}\right)\left(\frac{1}{m_p}\right)^{4/3}\right]^{3/2}$$

$$M_c = \underbrace{\left(\frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}\left(\frac{5}{3}\right)\right)^{3/2}} \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_p^2}$$

this constant is infact not very good because our model is not very good.

$$= 0.929\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_p^2} \approx \boxed{3.419 \times 10^{30}\ kg}$$

$$= 1.7\, M_{sun}$$

assumed to be 2×10^{30} kg

actually is $\underline{\underline{1.44}}$ but close

Above M_c the dwarf star is unstable.

$$\cancel{U^{critical}_{electron} = \frac{1}{3\pi}\left(\frac{9\pi}{8}\right)^{4/3}(\hbar c)\left(\frac{M_c}{m_p}\right)^{4/3}}$$

at ultrarelativistic limit and assume $R = 7.16 \times 10^6$ m still holds.

$\rightarrow U^c_{electron} = \frac{3}{4} N \varepsilon_F$

$$\therefore \frac{U^c_{electron}}{N} = \frac{3}{4}\varepsilon_F$$

$$= \frac{3}{4}\hbar c\,(3\pi^2 n)^{\frac{1}{3}} = \frac{3}{4}\hbar c\left(3\pi^2 \frac{3M_c}{8\pi m_p R^3}\right)^{\frac{1}{3}}$$

$$= 6.38 \times 10^{-14}\ J$$

$$m_e c^2 = 8.20 \times 10^{-14}.$$

$$\rightarrow \frac{U^c_{electron}}{N} \sim m_e c^2.$$

average energy on the order of electron rest energy at Chandrasekhar Mass

7. (i) ~~M_n~~ M_n = mass of neutron $= 1.67\times10^{-27}$ kg

$$U_{grav} = \cancel{3M^2} - \frac{3GM^2}{5R} \quad \text{as before.}$$

$$n = \frac{M}{m_n V} = \frac{3M}{4\pi m_n R^3} \qquad (\text{neutron } s=\tfrac{1}{2})$$

$$U_{neutron} = \frac{3}{5} N_n \varepsilon_F = \frac{3}{5}\left(\frac{M}{m_n}\right)\left(\frac{\hbar^2}{2m_n}\right)\left(3\pi^2\left(\frac{3M}{4\pi m_n R^3}\right)\right)^{2/3}$$

$$= \left(\frac{3}{5}\right)\left(\frac{1}{2}\right)\left(3\pi^2 \times \frac{3}{4\pi}\right)^{2/3} \times \frac{\hbar^2 M^{5/3}}{m_n^{8/3} R^2}$$

$$= 1.105 \times \frac{\hbar^2 M^{5/3}}{m_n^{8/3} R^2}$$

$$\therefore \quad U_{total} = 1.105 \frac{\hbar^2 M^{5/3}}{m_n^{8/3} R^2} - \frac{3GM^2}{5R}$$

$$= \frac{A}{R^2} - \frac{B}{R} \qquad A = 1.105 \frac{\hbar^2 M^{5/3}}{m_n^{8/3}}$$

$$B = \frac{3}{5} GM^2$$

$$\frac{dU_{total}}{dR} = 0 \implies$$

$$R(M) = \frac{2A}{B} = \frac{2\times 1.105 \frac{\hbar^2 M^{5/3}}{m_n^{8/3}}}{\frac{3}{5}GM^2}$$

$$= \boxed{3.683 \frac{\hbar^2}{m_n^{8/3} G} M^{-1/3}}$$

(ii)

$$\boxed{R(M) = 1.23 \times 10^4 \text{ m}}$$

(iii)

For estimation we assume $\frac{3}{4}\varepsilon_F \sim m_n c^2$

then $$\frac{3}{4}\hbar c\left(3\pi^2 \frac{3M_c}{4\pi m_n R^3}\right)^{\frac{1}{3}} = m_n c^2$$

and use $R = 1.23 \times 10^4$ m

then $$M_c = \frac{4 m_n R^3}{9\pi}\left(\frac{4}{3}\frac{c}{\hbar}m_n\right)^3$$

$$= 1.13 \times 10^{32}\ \mathrm{kg}$$

$$= \underline{56.6\,M_{sun}}$$

8. (a) The condition for chemical equalibrium is the balance of chemical potential, but the chemical potential for photons is 0

law of mass action

$\nu_\gamma = \nu_{e^+} = \nu_{e^-} = 1$

$$\therefore \boxed{\mu_{e^+} + \mu_{e^-} = 0}$$

(b) Assume $\mu_{e^+} = \mu_{e^-} = \mu$

then $\boxed{\mu = 0}$

From Q 7.1 we know that when $k_B T \gg m_e c^2$

$$n = \frac{(2s+1)}{2\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^{x-\beta\mu} \pm 1}$$

A electron is a fermion ~~and~~ with spin $s = \frac{1}{2}$, use $\mu = 0$

$$\therefore n^{\pm} = \frac{1}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x + 1}$$

$$= \frac{3}{2\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \underbrace{\zeta(3)}_{1.202}$$

$$= \boxed{0.183\left(\frac{k_B T}{\hbar c}\right)^3}$$

(c) $\because k_B T \gg m_e c^2$

$$\therefore n^{\pm} \gg \left(\frac{m_e c^2}{\hbar c}\right)^3 = \left(\frac{m_e c}{\hbar}\right)^3 ~~= 1.3 \times 10^{38}~~$$

$\sim 10^{37}\ m^{-3}$

upper bound for the density of electrons in ordinary matter is

$$n_0 = a_0^{-3} = \left(\frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}\right)^{-3} \sim 10^{30}\ m^{-3}$$

Bohr radius

$\therefore \underline{n^{\pm} \gg n_0}$ $\rightarrow$ a priori assumption valid

(d) equation for chemical equilibrium

$$\sum_r \nu_s \mu_s = 0$$

$$\rightarrow \quad \mu_{e^+} + \mu_{e^-} = 0$$

classical limit ~~apply~~ applies & since mass is not conserved we must add $m_e c^2$ to the classical chemical potentials to get μ_{e^+} and μ_{e^-} because electrons and positrons have rest energy whereas photons do not. . (~~elect~~ e^- and e^- are spin-$\frac{1}{2}$ particles $\therefore 2s+1 = 2$)

$$\rightarrow \cancel{k_B T \ln\left(\frac{n^+ \lambda_{th}^3}{2}\right) + k_B T}$$

Law of mass action becomes:

$$\left(k_B T \ln\left(\frac{n^+ \lambda_{th}^3}{2}\right) + m_e c^2\right) + \left(k_B T \ln\left(\frac{n^- \lambda_{th}^3}{2}\right) + m_e c^2\right)$$

$$= 0$$

$$\rightarrow \quad k_B T \ln\left(\frac{n^+ n^- \lambda_{th}^6}{4}\right) + 2 m_e c^2 = 0$$

$$\therefore \quad \ln\left(\frac{n^+ n^- \lambda_{th}^6}{4}\right) = -\frac{2 m_e c^2}{k_B T}$$

$$\therefore \quad \frac{n^+ n^- \lambda_{th}^6}{4} = \exp\left(-\frac{2 m_e c^2}{k_B T}\right)$$

$$\lambda_{th} = \hbar \sqrt{\frac{2\pi}{m_e k_B T}} \quad \therefore \quad \lambda_{th}^6 = \hbar^6 \left(\frac{2\pi}{m_e k_B T}\right)^3$$

$$\therefore\ n^+ n^- = 4\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^3 \exp\left(-\frac{2m_e c^2}{k_B T}\right)$$

Density of electrons without pair production is n_0

(electrons that already exist).

$$\therefore\ n^- = n_0 + n^+$$

at $k_B T \ll m_e c^2$, pair production rarely occurs

$$\therefore\ \cancel{n_0 \ll n}\quad n^+ \ll n_0 \sim n^-$$

$$\therefore\ n^+ n^- = n^+(n^+ + n_0) = n^+ n_0 + \underbrace{n^{+2}}_{\text{negligible}} \approx n^+ n_0$$

$$\Rightarrow\ \boxed{n^+ \approx \frac{4}{n_0}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^3 \exp\left(-\frac{2m_e c^2}{k_B T}\right)}$$

9. For fixed volume $dV = 0$

$\therefore dU = TdS - m dB + \mu dN$, $\Phi = U - TS - \mu N$

$d\Phi = TdS - \cancel{SdT} \, m dB + \mu dN - TdS - SdT - \mu dN - N d\mu$

~~$F = U - TS$, $G = U - TS + m dB = \mu N$~~

$d\Phi = -SdT - m dB - N d\mu = -SdT - MV dB - N d\mu$

~~$G = U - TS + mB = \mu N$~~

~~$\Phi = U - TS - \mu N = F - G = -mB = -MVB$~~

$\therefore$ $\boxed{\cancel{M = -\frac{1}{V}\left(\frac{\Phi}{B}\right)}}$ $\boxed{M = -\frac{1}{V}\left(\frac{\partial \Phi}{\partial B}\right)_{V, \mu, T}}$

Half of states have spin $+\frac{1}{2}$ and ~~have~~ half ~~have~~ have $-\frac{1}{2}$

$\rightarrow \Phi(\mu, B) = \frac{1}{2}\Phi_0(\mu + \mu_B B) + \frac{1}{2}\Phi_0(\mu - \mu_B B)$

For ~~For non-degenerate~~ non-relativistic Fermi gas, (electron gas), density of states

$$g(\varepsilon)\, d\varepsilon = \frac{\sqrt{2}\, V m^{3/2}}{\pi^2 \hbar^3} d\varepsilon$$

$$U_0 = \int_0^\infty \frac{d\varepsilon\, g(\varepsilon)\, \varepsilon}{e^{\beta(\varepsilon - \mu)} + 1} = \frac{\sqrt{2}\, V m^{3/2}}{\pi^2 \hbar^3} \int_0^\infty \frac{d\varepsilon\, \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$\Phi_0 = -\frac{2}{3} U_0 = \frac{-2\sqrt{2}}{3\pi^2} \frac{V m^{3/2}}{\hbar^3} \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

$$\therefore \Phi = -\frac{\sqrt{2}}{3\pi^2} \frac{V m^{3/2}}{\hbar^3} \left[\int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon - \mu - \mu_B B)} + 1} + \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon - \mu + \mu_B B)} + 1} \right]$$

For fully-degenerate gas, use Sommerfeld expansion.

$$\int_0^\infty \frac{d\varepsilon f(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu d\varepsilon f(\varepsilon) + \frac{\pi^2}{6} f'(\mu) (k_B T)^2$$

Consider only the first term for now:

$$\therefore \int_0^\infty \frac{d\varepsilon \cdot \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu+\mu_B B)}+1} = \int_0^{\mu-\mu_B B} d\varepsilon\, \varepsilon^{3/2} + \cancel{\frac{\pi^2}{6}\frac{3}{2}(\mu-\mu_B B)^{\frac{1}{2}}(k_B T)^2}$$

$$= \frac{2}{5}(\mu-\mu_B B)^{5/2}$$

similarly $\displaystyle\int_0^\infty \frac{d\varepsilon\, \varepsilon^{3/2}}{e^{\beta(\varepsilon-\mu-\mu_B B)}+1} = \frac{2}{5}(\mu+\mu_B B)^{5/2}.$

$$\mu+\mu_B B \simeq \varepsilon_F^+ \qquad \mu-\mu_B B \simeq \varepsilon_F^-$$

$$\varepsilon_F^+ \simeq \varepsilon_F^- \simeq \varepsilon_F \qquad \varepsilon_F >> \mu_B B$$

$$\therefore \quad \mu_B B << \mu \simeq \varepsilon_F$$

$$\frac{2}{5}(\mu \pm \mu_B B)^{5/2} = \frac{2}{5}\mu^{5/2}\left(1 \pm \frac{\mu_B B}{\mu}\right)^{5/2}$$

$$\simeq \frac{2}{5}\mu^{5/2}\left(1 \pm \frac{5}{2}\frac{\mu_B B}{\mu} + \frac{15}{8}\left(\frac{\mu_B B}{\mu}\right)^2\right)$$

$$= \frac{2}{5}\varepsilon_F^{5/2}\left(1 \pm \frac{5}{2}\frac{\mu_B B}{\mu} + \frac{15}{8}\left(\frac{\mu_B B}{\mu}\right)^2\right)$$

$$\therefore \quad \Phi = -\frac{\sqrt{2}}{3\pi^2}\frac{V m^{3/2}}{\hbar^3}\left[\frac{4}{5}\varepsilon_F^{5/2} + \frac{3}{2}\frac{\mu_B^2 B^2}{\varepsilon_F^2}\varepsilon_F^{5/2}\right]$$

$$M = -\frac{1}{V}\left(\frac{\partial \Phi}{\partial B}\right)_{V,\mu,T} = \frac{\sqrt{2}}{3\pi^2}\frac{m^{3/2}}{\hbar^3}\cdot 3\frac{\mu_B^2}{\varepsilon_F^2}\varepsilon_F^{5/2} B$$

$$= \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\mu_B^2 \varepsilon_F^{1/2} B$$

$$\chi = \left(\frac{\partial M}{\partial B}\right)_{B=0} = \frac{\sqrt{2}}{\pi^2}\frac{m^{3/2}}{\hbar^3}\mu_B^2 \varepsilon_F^{1/2}.$$

$$\varepsilon_F \simeq \frac{\hbar^2}{2m_e}\left(3\pi^2 n\right)^{\frac{2}{3}}$$

$$= \frac{\sqrt{2}}{\pi^2} \frac{m_e^{3/2}}{\hbar^3} \left(\frac{e\hbar}{2m_e c}\right)^2 \left(\frac{\hbar^2}{2m_e}\right)^{\frac{1}{2}} (3\pi^2)^{\frac{1}{3}} n^{\frac{1}{3}}$$

$$= \boxed{\frac{3^{1/3}}{4\pi^{4/3}} \frac{e^2}{m_e c^2} n^{1/3}}$$

(b)

$$\Phi = -\frac{\sqrt{2}}{3\pi^2} \frac{V m_e^{3/2}}{\hbar^3} (k_B T)^{5/2} \cdot \left[\int_0^\infty \frac{x^{3/2}\, dx}{e^{x-\beta(\mu+\mu_B B)}+1} + \int_0^\infty \frac{x^{3/2}\, dx}{e^{x-\beta(\mu-\mu_B B)}+1} \right]$$

classical limit $e^{-\beta(\mu+\mu_B B)} \to \infty$

$$\Phi = -\frac{\sqrt{2}}{3\pi^2} \frac{V m_e^{3/2}}{\hbar^3} (k_B T)^{5/2} \left[e^{\beta(\mu+\mu_B B)} + e^{\beta(\mu-\mu_B B)} \right] \cdot \underbrace{\int_0^\infty x^{3/2} e^{-x} dx}_{\frac{3}{4}\sqrt{\pi}}$$

$$= -\frac{\sqrt{2}}{3\pi^2} \left(\frac{3}{4}\sqrt{\pi}\right) \frac{V m_e^{3/2}}{\hbar^3} k_B^{5/2} T^{5/2} e^{\beta\mu} \left[2\cosh\left(\frac{\mu_B B}{k_B T}\right)\right]$$

For ideal gas in classical limit ~~$e^{\beta\mu}$~~
(electron)

$$\mu = k_B T \ln\left(\frac{n\lambda_{th}^3}{2}\right) \to e^{\beta\mu} = \frac{n\lambda_{th}^3}{2} = \frac{N\lambda_{th}^3}{2V}$$

$$= \frac{N}{2V} \hbar^3 \left(\frac{2\pi}{m_e k_B T}\right)^{3/2}$$

$$\therefore \Phi = -\frac{\sqrt{2}}{3\pi^2} \left(\frac{3}{4}\sqrt{\pi}\right) \cancel{V} \frac{m_e^{3/2}}{\cancel{\hbar^3}} k_B^{5/2} \frac{N}{2\cancel{V}} \frac{2^{3/2}\pi^{3/2}}{m_e^{3/2} k_B^{3/2}} \cancel{\hbar^3} T \cdot 2 \cdot \cosh\left(\frac{\mu_B B}{k_B T}\right)$$

$T \to \infty$, $\mu_B B \ll k_B T$.

$$= -k_B T N \cosh\left(\frac{\mu_B B}{k_B T}\right). \approx -NT k_B \left(1 + \frac{1}{2}\left(\frac{\mu_B B}{k_B T}\right)^2\right)$$

$$M = -\frac{1}{V}\left(\frac{\partial \Phi}{\partial B}\right)_{V,\mu,T} = k_B T \frac{\mu_B^2}{k_B^2 T^2} \frac{1}{2} \cdot 2B \frac{N}{V}$$

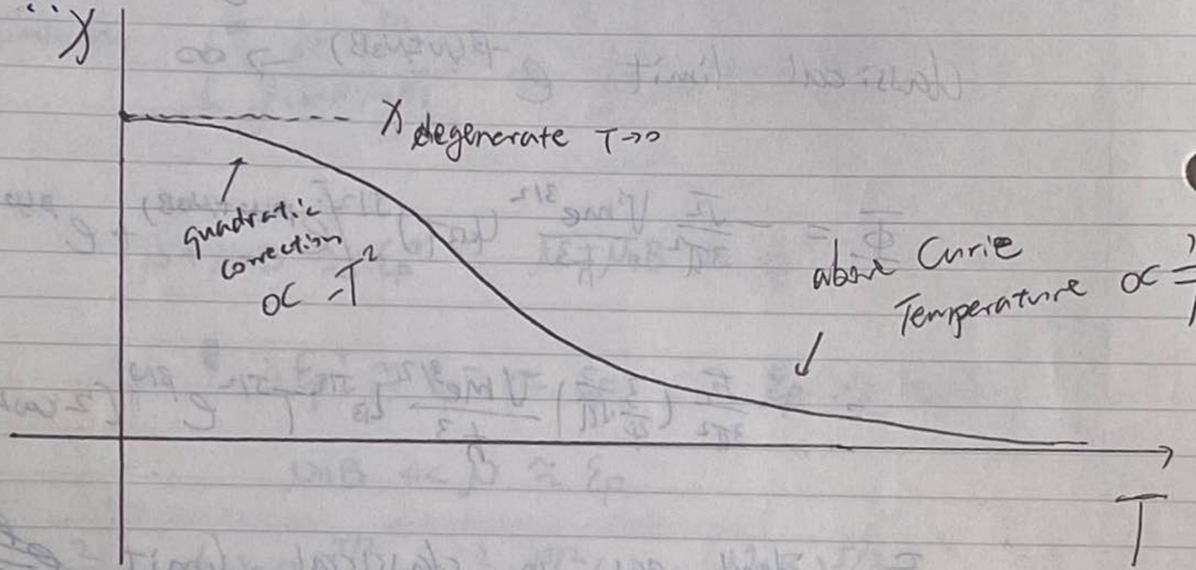
$$= \frac{N\mu_B^2}{V k_B T} B$$

$$\rightarrow \chi = \left(\frac{\partial M}{\partial B}\right)_{B=0} = \frac{N\mu_B^2}{V k_B T} = \frac{n\mu_B^2}{k_B T} = \frac{C}{T}.$$

$\longrightarrow$ Curie's Law recovered.

This ~~is~~ (classical limit) is very ~~(~~small

$\therefore$



(C) To get the finite-temperature correction to χ, we take the second term in the Sommerfeld expansion

$$\delta\Phi = -\frac{\sqrt{2}}{3\pi^2}\frac{V m_e^{3/2}}{\hbar^3}\cdot\frac{\pi^2}{6}k_B^2T^2\left[f'(\mu-\mu_B B)+f'(\mu+\mu_B B)\right]$$

$$(\cancel{f(\mu) = \mu} \quad f(\varepsilon) = \varepsilon^{3/2}, \quad f'(\varepsilon) = \frac{3}{2}\varepsilon^{\frac{1}{2}}$$

$$\delta\Phi = -\frac{\sqrt{2}}{3\pi^2}\frac{V m_e^{3/2}}{\hbar^3}\frac{\pi^2}{6}k_B^2T^2\left[\frac{3}{2}\right]\left[\varepsilon_F^{\frac{1}{2}}\right]\left[\sqrt{1-\frac{\mu_B B}{\varepsilon_F}}+\sqrt{1+\frac{\mu_B B}{\varepsilon_F}}\right]$$

$$= 1 - \cancel{\left(\frac{\mu_B B}{\varepsilon_F}\right)\frac{1}{2}} - \left(\frac{\mu_B B}{\varepsilon_F}\right)^2\frac{1}{8} - \dots \qquad = 2 - \frac{1}{4}\left(\frac{\mu_B B}{\varepsilon_F}\right)^2$$

$$+ 1 + \cancel{\left(\frac{\mu_B B}{\varepsilon_F}\right)\frac{1}{2}} - \left(\frac{\mu_B B}{\varepsilon_F}\right)^2\frac{1}{8}$$

$$= -\frac{\sqrt{2}}{3\pi^2}\frac{\pi^2}{6}\frac{3}{2}\frac{V m_e^{3/2}}{\hbar^3} k_B^2 T^2 \varepsilon_F^{\frac{1}{2}}\left[2 - \frac{1}{4}\frac{\mu_B^2 B^2}{\varepsilon_F^2}\right]$$

$$\delta M = -\frac{1}{V}\left(\frac{\partial \delta \Phi}{\partial B}\right)_{\mu, V, T} = -\frac{\sqrt{2}}{48}\frac{\cancel{V} m_e^{3/2}}{\hbar^3} k_B^2 \varepsilon_F^{-3/2} \mu_B^2 \cdot 2B \cdot T^2$$

$$\therefore \quad \delta\chi = \left(\frac{\partial \delta M}{\partial B}\right)_{B=0} = -\left[\frac{\sqrt{2}}{24}\frac{m_e^{3/2}}{\hbar^3} k_B^2 \varepsilon_F^{-3/2} \mu_B^2\right] T^2$$

the required correction

$\downarrow$

quadratic in T and negative

10. (i) $dU = TdS - pdV + \mu dN = TdS - pdV$ $\quad \underbrace{\mu dN}_{\mu=0}$

define $F = U - TS$ $\quad dF = dU - TdS - SdT = -SdT - pdV$

$\therefore \quad p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{u(T)}{3}$

$\therefore \quad F = -\frac{u}{3}V + C$ $\quad \because F$ is extensive, $\therefore C = 0$
V is extensive

$\therefore F = -\frac{u}{3}V$

$\therefore \quad \cancel{U=T}\ U - TS = uV - TS = -\frac{u}{3}V$

$\therefore \quad S = \frac{4uV}{3T}$

$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT = \frac{4u}{3T}dV + \left(\frac{\partial S}{\partial T}\right)_V dT$

$dU = TdS - pdV = T\left(\frac{4u}{3T}dV + \left(\frac{\partial S}{\partial T}\right)_V dT\right) - pdV$

$= \cancel{\frac{4u}{3T}}\ u dV + T\left(\frac{\partial S}{\partial T}\right)_V dT \quad ①$

$\because U = uV \quad \therefore dU = u dV + V du$

$= u dV + V\frac{du}{dT}dT. \quad ②$

$①, ② \Rightarrow T\left(\frac{\partial S}{\partial T}\right)_V = V\frac{du}{dT}$

$\rightarrow \cancel{T\frac{\partial S}{}}\ T\frac{d}{dT}\left(\frac{4u}{3T}\right) = \frac{du}{dT}.$

$\therefore \quad T \cdot \frac{4}{3} \cdot \frac{T\frac{du}{dT} - u}{T^2} = \frac{du}{dT}.$

$\therefore \frac{4}{3}\left(\frac{du}{dT} - \frac{u}{T}\right) = \frac{du}{dT} \rightarrow \frac{du}{dT} = \frac{4u}{T}.$

$\therefore \frac{du}{u} = \frac{4dT}{T} \rightarrow \ln u = \ln T^4 + C$

$\rightarrow \boxed{u = AT^4}$

(ii) $dU = TdS - pdV$; adiabatic $\Rightarrow$ $dU = -pdV$

$$\rightarrow \quad d(AT^4V) = -\frac{1}{3}AT^4 dV$$

$$\therefore \quad T^4 dV + 4T^3V dT = -\frac{1}{3}T^4 dV$$

$$\therefore \quad 4V dT = \frac{-4}{3} T dV$$

$$\therefore \quad 3V dT = -T dV$$

$$\therefore \quad \frac{3\,dT}{T} = -\frac{dV}{V}\left(\frac{1}{3}\right)$$

$\rightarrow$ ~~ln V = ln~~

$$\ln V^{-\frac{1}{3}} = \ln T + c$$

$$\rightarrow \quad \boxed{T = BV^{-\frac{1}{3}}}$$

A, B, C are constants.

(iii)

$$T_i V_i^{+1/3} = T_f V_f^{+1/3}$$

$$\therefore \quad \frac{T_i}{T_f} = \left(\frac{V_f}{V_i}\right)^{\frac{1}{3}} \approx \left(\frac{R_f}{R_i}\right) \approx$$ the ~~scale~~ cosmic scale factor

$\therefore$ $T_i = T_f \times$ the cosmic scale factor

$$= 2.73\,K \times 1100$$

$$\approx \boxed{3000\,K}$$

11. Photons are bosons with $\mu=0$ $\therefore$ mean occupation # $\bar{n}(\varepsilon,T)$ is $\bar{n}(\varepsilon,T) = \dfrac{1}{\exp(\frac{\varepsilon}{k_BT})-1}$

# of particles between $[\varepsilon, \varepsilon+d\varepsilon]$

is

$$dN(\varepsilon,T) = \bar{n}(\varepsilon,T)\,\underbrace{\rho(\varepsilon)}_{\text{density of state}}\,d\varepsilon$$

total ~~energy~~ # of photons between $[\varepsilon, \varepsilon+d\varepsilon]$

$$\cancel{dU(\varepsilon,T) = \varepsilon\, dN(\varepsilon,T)}$$

$$dN(\varepsilon,T) = \bar{n}(\varepsilon,T)\rho(\varepsilon)\,d\varepsilon$$

(photons cannot have spin 0 $\therefore$ spin degeneracy $=2$)

$$= \frac{1}{\exp(\frac{\varepsilon}{k_BT})-1}\;\underbrace{\frac{V\varepsilon^2}{2\pi^2(\hbar c)^3}}_{\text{calculated in Q7.1}}\,d\varepsilon \times 2$$

$$= 2\times\frac{V}{2\pi^2\hbar^3c^3}\frac{(\varepsilon^2)}{\exp(\frac{\varepsilon}{k_BT})-1}\,d\varepsilon$$

For photons $\varepsilon=\hbar\omega$, $d\varepsilon=\hbar\,d\omega$

$\therefore dN(\omega,T)$ in $[\omega,\omega+d\omega]$ is

$$dN(\omega,T) = 2\times\frac{V}{2\pi^2\hbar^3c^3}\frac{\hbar^2\omega^2}{\exp(\frac{\hbar\omega}{k_BT})-1}\hbar\,d\omega$$

$$= \boxed{2\times\frac{V}{2\pi^2c^3}\frac{\omega^2\,d\omega}{\exp(\frac{\hbar\omega}{k_BT})-1}} \equiv n(\omega)\,d\omega$$

find peak: set $0 = \frac{dn(\omega)}{d\omega}$

$$\rightarrow \quad \frac{d}{d\omega}\left(\frac{\omega^2}{e^{\hbar\omega/k_BT}-1}\right) = 0$$

$$\rightarrow \quad 0 = (e^{\hbar\omega/k_BT}-1)2\omega - \omega^2\left(\frac{\hbar}{k_BT}\right)e^{\hbar\omega/k_BT}$$

$$\rightarrow \quad \frac{2}{\omega} - \frac{\hbar}{k_BT}\frac{1}{1-\exp(-\frac{\hbar\omega}{k_BT})} = 0.$$

solve numerically $\omega = 1.59\frac{k_BT}{\hbar}$

spectral energy density

$$dU = \varepsilon dN = 2\times\frac{V}{2\pi^2\hbar^3c^3}\frac{\varepsilon^3}{\exp(\frac{\varepsilon}{k_BT})-1}d\varepsilon$$

$$dU(\omega,T) = 2\times\frac{V}{2\pi^2c^3}\frac{\hbar\omega^3\,d\omega}{\exp(\frac{\hbar\omega}{k_BT})-1} \qquad (\varepsilon = \hbar\omega)$$

set $0 = \frac{dU}{d\omega}$, then

$$\left(\exp\left(\frac{\hbar\omega}{k_BT}\right)-1\right)3\omega^2 - \omega^3\left(\frac{\hbar}{k_BT}\right)\exp(\hbar\omega/k_BT)$$

$$\rightarrow \quad 0 = \frac{3}{\omega} - \frac{\hbar}{k_BT}\frac{1}{1-\exp(-\frac{\hbar\omega}{k_BT})}$$

solve numerically gives

$$\omega_{max} = 2.82\frac{k_BT}{\hbar}$$

$\cancel{\lambda_{max} = \frac{2\pi c}{\omega_{max}} = \frac{2\pi\hbar c}{k_BT}}$

$$\varepsilon = \frac{hc}{\lambda} \qquad |d\varepsilon| = \left|\frac{-hc}{\lambda^2} d\lambda\right| = \frac{hc}{\lambda^2} d\lambda$$

$$dU(\lambda, T) = 2 \times \frac{V}{2\pi^2 c^3 \hbar^3} \times \frac{\left(\frac{hc}{\lambda}\right)^3}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \frac{hc}{\lambda^2} d\lambda.$$

$$= 8\pi^3 \times 2 \times \frac{V}{\cancel{2\pi^2} c^3 \cancel{\hbar^3}} (hc)^{4} \frac{1}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} d\lambda$$

$$= \frac{8\pi h c V}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{k_B T \lambda}\right) - 1} \equiv \cancel{u_\lambda} \, u_\lambda(\lambda) \, d\lambda$$

set $\frac{dU_\lambda(\lambda)}{d\lambda} = 0$

$$0 = \left(\exp\left(\frac{hc}{k_B T \lambda}\right) - 1\right)(-5\lambda^{-4}) - \lambda^{-5} \exp\left(\frac{hc}{k_B T \lambda}\right) \frac{hc}{k_B T} \left(-\frac{1}{\lambda^2}\right)$$

solve numerically we get

$$\lambda_{max} = \frac{hc}{4.97 k_B T}$$