To: Robin Nicholas

Statistical Physics 4 Ziyan Li

1. a)

~~type 1 : $\langle V_1^2\rangle^{1/2} =$~~

$$\langle v^2\rangle = \int_0^\infty dv\, v^2 f(v) = \int_0^\infty \frac{4\pi}{(\sqrt{\pi}v_{th})^3} v^4 e^{-v^2/v_{th}^2}\, dv$$

$$= \frac{3}{2} v_{th}^2$$

$$\because v_{th}^2 = \frac{2k_BT}{m}$$

$$\therefore \langle v^2\rangle = \frac{3k_BT}{m}$$

$\therefore$ Type 1 : $V_{1,rms} = \langle V_1^2\rangle^{1/2} = \boxed{\sqrt{\frac{3k_BT}{m_1}}}$

Type 2 : $V_{2,rms} = \langle V_2^2\rangle^{1/2} = \boxed{\sqrt{\frac{3k_BT}{m_2}}}$

b) ~~$dP = \Delta P$~~

$$dP = \Delta P_1\, d\Phi_1(V_z) + \Delta P_2\, d\Phi_2(V_z)$$

$$= 2m_1 V_z^2 n_1 f(V_z)\, dV_z + 2m_2 V_z^2 n_2 f(V_z)\, dV_z$$

$$\therefore P = \int_0^\infty 2m_1 V_z^2 n_1 f(V_z)\, dV_z + \int_0^\infty 2m_2 V_z^2 n_2 f(V_z)\, dV_z$$

$$= \int_{-\infty}^{\infty} m_1 n_1 V_z^2 f(V_z)\, dV_z + \int_{-\infty}^{\infty} m_2 n_2 V_z^2 f(V_z)\, dV_z$$

$f(V_z) = f(-V_z)$

$$= m_1 n_1 \langle V_z^2\rangle_{(m_1)} + m_2 n_2 \langle V_z^2\rangle_{m_2}$$

$$= \frac{1}{3} m_1 n_1 \langle v_1^2 \rangle + \frac{1}{3} m_2 n_2 \langle v_2 \rangle^2$$

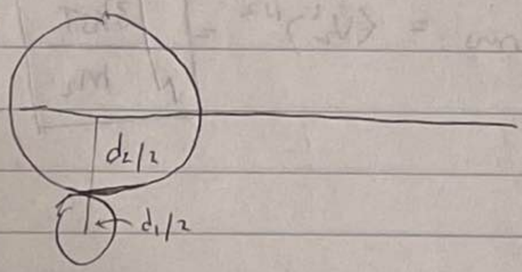
isotropic distribution

$$= \frac{1}{3} m_1 n_1 \frac{3k_BT}{m_1} + \frac{1}{3} m_2 n_2 \frac{3k_BT}{m_2}$$

$$= n_1 k_B T + n_2 k_B T = P_1 + P_2$$

$$\therefore \quad \underline{P = P_1 + P_2} \quad \text{QED.}$$

c)



The ~~crosse~~ cross section

$$\sigma = \pi (r_1 + r_2)^2 = \pi \left(\frac{d_1 + d_2}{2} \right)^2 = \boxed{\frac{\pi}{4} (d_1 + d_2)^2}$$

d) Relative speed $\langle v_r \rangle = \langle |\vec{v_1} - \vec{v_2}| \rangle$

$$\therefore \langle v_r \rangle = \iint d^3\vec{v_1}\, d^3\vec{v_2}\, |\vec{v_1} - \vec{v_2}|\, f(\vec{v_1}, \vec{v_2}) \quad \Big\} \begin{array}{l} \vec{v_1}, \vec{v_2} \\ \text{independent} \end{array}$$

$$= \iint d^3\vec{v_1}\, d^3\vec{v_2}\, |\vec{v_1} - \vec{v_2}|\, f(\vec{v_1})\, f(\vec{v_2})$$

Assume maxwellian distribution for both $\vec{v_1}, \vec{v_2}$

$$= \iint d^3\vec{v_1}\, d^3\vec{v_2}\, |\vec{v_1} - \vec{v_2}| \left(\frac{1}{\sqrt{\pi} v_{th,1}}\right)^3 \left(\frac{1}{\sqrt{\pi} v_{th,2}}\right)^3 \exp\left(-\frac{v_1^2}{v_{th,1}^2} - \frac{v_2^2}{v_{th,2}^2}\right)$$

$$\left(v_{th,1}^2 = \frac{2k_BT}{m_1}, \quad v_{th,2}^2 = \frac{2k_BT}{m_2} \right)$$

~~then~~ Consider : $\vec{V} = \cancel{\vec{v_1} + \vec{v_2}}$

$$\vec{V} = \frac{m_1 \vec{v_1} + m_2 \vec{v_2}}{m_1 + m_2}, \quad \vec{v_r} = \vec{v_1} - \vec{v_2}$$

and change of variable $(\vec{v_1}, \vec{v_2}) \to (\vec{V}, \vec{v_r})$

Calculating Jacobian :

$$dV_x\, dv_{r,x} = \left| \frac{\partial(V_x, v_{r,x})}{\partial(v_{1x}, v_{2x})} \right| dv_{1x}\, dv_{2x}$$

$$\left| \frac{\partial(V_x, v_{r,x})}{\partial(v_{1x}, v_{2x})} \right| = \begin{vmatrix} \frac{\partial V_x}{\partial v_{1x}} & \frac{\partial V_x}{\partial v_{2x}} \\ \frac{\partial v_{rx}}{\partial v_{1x}} & \frac{\partial v_{rx}}{\partial v_{2x}} \end{vmatrix}$$

$$= \left| \left(\frac{m_1}{m_1 + m_2}\right)(-1) - \left(\frac{m_2}{m_1 + m_2}\right)(1) \right|$$

$$= \left| -\frac{m_1 + m_2}{m_1 + m_2} \right| = 1$$

$$\therefore\ d^3\vec{V}\, d^3\vec{v_r} = (1)^3\, d^3\vec{v_1}\, d^3\vec{v_2}$$

$$\therefore\ d^3\vec{V}\, d^3\vec{v_r} = d^3\vec{v_1}\, d^3\vec{v_2}$$

By König's Theorem in classical mechanics

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}MV^2 + \frac{1}{2}\mu v_r^2$$

where $M = m_1 + m_2$ and $\mu = \frac{m_1m_2}{m_1+m_2}$

$$v_{th,1}\cdot v_{th,2} = \frac{2k_BT}{m_1}\cdot\frac{2k_BT}{m_2} = \underbrace{\frac{2k_BT}{M}}_{V_{th}}\ \underbrace{\frac{2k_BT}{\mu}}_{v_{r,th}}$$

$$= V_{th}\cdot v_{r,th}$$

$$\frac{v_1^2}{v_{th,1}^2} + \frac{v_2^2}{v_{th,2}^2} = \frac{1}{2k_BT}(m_1v_1^2 + m_2v_2^2) = \frac{1}{2k_BT}(MV^2 + \mu v_r^2)$$

$$= \frac{V^2}{V_{th}^2} + \frac{v_r^2}{v_{r,th}^2}$$

$$\therefore\ \langle v_r\rangle = \iint d^3\vec{v_1}\, d^3\vec{v_2}\left(\frac{1}{\sqrt{\pi}\,v_{th,1}}\right)^3\left(\frac{1}{\sqrt{\pi}\,v_{th,2}}\right)^3 \exp\left(-\frac{v_1^2}{v_{th,1}^2} - \frac{v_2^2}{v_{th,2}^2}\right)|\vec{v_1}-\vec{v_2}|$$

$$= \iint d^3\vec{V}\, d^3\vec{v_r}\ v_r\left(\frac{1}{\sqrt{\pi}\,V_{th}}\right)^3\left(\frac{1}{\sqrt{\pi}\,v_{r,th}}\right)^3 \exp\left(-\frac{V^2}{V_{th}^2}\right)\exp\left(-\frac{v_r^2}{v_{r,th}^2}\right)$$

$$= \underbrace{\int d^3\vec{V}\left(\frac{1}{\sqrt{\pi}\,V_{th}}\right)^3 \exp\left(-\frac{V^2}{V_{th}^2}\right)}_{1}\int d^3\vec{v_r}\ v_r\,\frac{1}{(\sqrt{\pi}\,v_{r,th})^3}\exp\left(-\frac{v_r^2}{v_{r,th}^2}\right)$$

$$\therefore \langle V_r \rangle = \int d^3\vec{V_r}\, v_r \frac{1}{(\sqrt{\pi}\, V_{r,th})^3} \exp\left(-\frac{V_r^2}{V_{r,th}^2}\right)$$

$$! \quad V_{r,th}^2 = \frac{2k_BT}{\mathcal{N}} = \frac{2k_BT}{m_1}\frac{m_1}{\mathcal{N}} = \frac{m_1}{\mathcal{N}} V_{th,1}^2$$

$$\therefore \langle V_r \rangle = \int d^3\vec{V_r}\, V_r \frac{1}{(\sqrt{\pi}\, V_{th,1})^3} \left(\sqrt{\frac{\mathcal{N}}{m_1}}\right)^3 \exp\left(-\frac{(V_r \sqrt{\mathcal{N}/m_1})^2}{V_{th,1}^2}\right)$$

$$\text{let} \quad \vec{V_r}' = \sqrt{\frac{\mathcal{N}}{m_1}}\, \vec{v_r} \quad \Rightarrow \quad \vec{V_r} = \sqrt{\frac{m_1}{\mathcal{N}}}\, \vec{V_r}' \Rightarrow v_r = \sqrt{\frac{m_1}{\mathcal{N}}}\, v_r'$$

$$\Rightarrow d^3\vec{V_r} = \left(\sqrt{\frac{m_1}{\mathcal{N}}}\right)^3 d\vec{v_r}'$$

$$\therefore \langle V_r \rangle = \int \cancel{\left(\sqrt{\frac{m_1}{\mathcal{N}}}\right)^3} d\vec{V_r}'\, V_r' \left(\sqrt{\frac{m_1}{\mathcal{N}}}\right) \frac{1}{(\sqrt{\pi}\, V_{th,1})^3} \cancel{\left(\sqrt{\frac{\mathcal{N}}{m_1}}\right)^3} \exp\left(-\frac{V_r'^2}{V_{th,1}^2}\right)$$

$$= \sqrt{\frac{m_1}{\mathcal{N}}} \underbrace{\int d^3\vec{V_r}'\, V_r' \frac{1}{(\sqrt{\pi}\, V_{th,1})^3} \exp\left(-\frac{V_r'^2}{V_{th,1}^2}\right)}_{\langle V_1 \rangle}$$

$$= \sqrt{\frac{m_1}{\mathcal{N}}} \langle V_1 \rangle$$

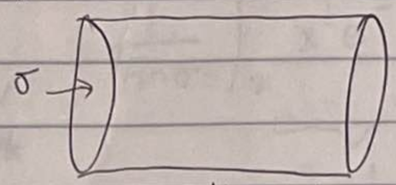
a. The collision rate of 1 with 2

is then

$$\therefore \cancel{\nu_c = \frac{\pi}{4}(d_1^2 + d_2}$$

$$\boxed{\begin{aligned} &\nu_c = \sigma n_2 \langle V_r \rangle \\ &\nu_c = \frac{\pi}{4}(d_1 + d_2)^2 n_2 \sqrt{1 + \frac{m_1}{m_2}} \langle V_1 \rangle \end{aligned}}$$

2 a)



$\langle V_r \rangle dt = \sqrt{2} v dt$

$\langle V_r \rangle$ = average relative velocity

$= \sqrt{2} \langle V \rangle = \sqrt{2} V$

Volume of collision cylinder $= \sqrt{2} \sigma v \, dt$

When dt is small, $n(\sqrt{2}\sigma v \, dt) \ll 1$

$\therefore$ $\sqrt{2} n \sigma v \, dt$, originally representing the number of particle in the collision cylinder, now represents the probability that a collision occurs during time dt. (If there will be 0.5 collisions happen in ~~the~~ time dt we can say that the probability of collision is 50%)

$\therefore$ Probability that collision does not occur is

$$P(dt) = 1 - \sqrt{2} n \sigma v \, dt$$

Probability theory $\Rightarrow$ $P(t+dt) = P(t) P(dt) = P(t)(1 - \sqrt{2} n \sigma v \, dt)$

~~Calculus~~ $\cancel{\Rightarrow \quad P(t+dt) = P(t) + \frac{dP}{dt} dt}$

$$\therefore \quad P(t+dt) - P(t) = -\sqrt{2} n \sigma v P(t) \, dt$$

$$\therefore \quad \frac{P(t+dt) - P(t)}{dt} = -\sqrt{2} n \sigma v P(t)$$

$$\therefore \quad \frac{dP}{dt} = -\sqrt{2} n \sigma v P \qquad \therefore \quad \frac{dP}{P} = -\sqrt{2} n \sigma v \, dt$$

Knowing that $P(t=0) = 1$ (~~lost~~ no collision at $t=0$)

we then have

$$\boxed{P(t) = e^{-\sqrt{2} n \sigma v t}}$$

b) ~~During~~ Between collisions particle travels ~~a~~ through straight lines so $x = vt$

$$\therefore \ \underline{P(x) = e^{-\sqrt{2} n \sigma x}}$$

The probability of surviving without collision up to time t but then colliding in the next dt is

$$\tilde{P}(t) = e^{-\sqrt{2} n \sigma v t} \sqrt{2} n \sigma v \, dt.$$

$\therefore$ Average collision time

$$\tau = \int_0^\infty t \tilde{P}(t) dt = \int_0^\infty t e^{-\sqrt{2} n \sigma v t} \sqrt{2} n \sigma v \, dt.$$

$\delta(x - vt)$

$$= \frac{1}{\sqrt{2} n \sigma v}$$

The mean free path is $\lambda_{mfp} = v\tau = \frac{1}{\sqrt{2} n \sigma}$

OR we say $\underset{(x = vt)}{\tilde{P}(x)} = \cancel{e^{-\sqrt{2}n\sigma x}} \ e^{-\sqrt{2} n \sigma x} \sqrt{2} n \sigma \, dx$

The mean free path is then

$$\lambda_{mfp} = \int_0^\infty x \tilde{P}(x) dx = \int_0^\infty x e^{-\sqrt{2} n \sigma x} \sqrt{2} n \sigma \, dx$$

$$= \underbrace{\frac{1}{\sqrt{2}n\sigma}}_{\substack{\cancel{x'} \\ x'=\sqrt{2}n\sigma x}} \int_0^\infty \underbrace{x' e^{-x'} dx'}_{1} = \frac{1}{\sqrt{2}n\sigma}$$

$$\therefore \sqrt{2}n\sigma = \frac{1}{\lambda_{mfp}}$$

$$\therefore \boxed{P(x) = e^{-x/\lambda_{mfp}}}, \quad \tilde{p}(x) = \frac{1}{\lambda_{mfp}} e^{-x/\lambda_{mfp}}$$

The ~~next~~ root mean square free path:

$$\lambda_{rmsfp}^2 = \int_0^\infty x^2 P(x)\, dx = \int_0^\infty x^2 e^{-\sqrt{2}n\sigma x} \quad \sqrt{2}n\sigma x$$

$$= \frac{1}{2n^2\sigma^2} \int_0^\infty \underbrace{x'^2 e^{-x'} dx'}_{=2! = 2}$$

$$= \boxed{\frac{1}{n^2\sigma^2}}$$

$$\therefore \boxed{\lambda_{rmsfp}} = \boxed{\frac{1}{n\sigma}} = \sqrt{2}\left(\frac{1}{\sqrt{2}n\sigma}\right) = \boxed{\sqrt{2}\lambda_{mfp}}$$

c) The probability density function

$P(x) = e^{-x/\lambda_{mfp}}$ is monotonically decreasing

in $[0, \infty]$ so the most probable free path

$$\therefore \boxed{\lambda_{mpfp} = 0}$$

3

d) (i)

$$\tilde{p}(x > \lambda_{mfp}) = \int_{\lambda_{mfp}}^{\infty} \tilde{p}(x)\,dx = \int_{\lambda_{mfp}}^{\infty} \frac{1}{\lambda_{mfp}} e^{-x/\lambda_{mfp}}\,dx$$

$$= \cancel{\frac{1}{\lambda_{mfp}} e^{-x/\lambda_{mfp}} \Big|_{\lambda_{mfp}}^{\infty}}$$

$$= \frac{-\cancel{\lambda_{mfp}}}{\cancel{\lambda_{mfp}}} e^{-x/\lambda_{mfp}} \Big|_{\lambda_{mfp}}^{\infty}$$

$$= e^{-1} = \boxed{36.8\%}$$

Similarly :

$$\tilde{P}(X > 2\lambda_{mfp}) = e^{-2} = \boxed{13.5\%}$$

$$\tilde{p}(X > 5\lambda_{mfp}) = e^{-5} = \boxed{0.67\%}$$

3. $p = nk_BT$, $\lambda_{mfp} = \frac{1}{\sqrt{2}n\sigma}$

$\therefore n = \frac{p}{k_BT} = \frac{1}{\sqrt{2}\sigma\lambda_{mfp}}$

$\therefore$ Under constant temperature

$$p \propto \frac{1}{\lambda_{mfp}}$$

To avoid collisions we set the mean free path in the chamber to be 10^{-1} m, the length of the ~~cha~~ chamber.

Take atomic radius to be 10^{-10} m

then $\frac{p}{10^5} = \frac{10^{-10} \times 10^3}{10^{-1}}$ (10^5_{pa} = atm pressure)

$= \boxed{10^{-1}\ pa}$ is the highest allowable pressure.

4. using results derived in question 2.

$P(x) = e^{-x/\lambda}$, In this case $x = 10^{-2}m$.

$P(x) = \frac{1}{2.72}$

$\therefore$ $\lambda = \frac{10^{-2}}{\ln(2.72)} m = \boxed{10^{-2} m}$ is the mean free path

Since the beam has its direction of travel,

~~[illegible]~~ in this case the average relative speed ~~velocity~~ ~~is the~~

(~~[illegible]~~ between silver particle and air particle

is the speed of the beam. There is no $\sqrt{2}$

factor in the expression of λ $\therefore$ $\lambda = \frac{1}{\sigma n}$.

$\therefore n = \frac{1}{\lambda\sigma} = \frac{P}{k_B T}$ $\therefore$ $\sigma = \frac{k_B T}{\lambda P}$

$\therefore$ $d = \left(\frac{k_B T}{\pi \lambda P}\right)^{1/2} = \cancel{1.1\times 10}$

$= \left(\frac{1.38\times 10^{-23} \times 273}{\pi (10^{-2})(1)}\right)^{1/2} = \boxed{3.46\times 10^{-10} m}$

is the effective collision radius

~~$\therefore$ the effective collision radius is~~

5. We first estimate the mean free path and

$\because$ Air is mainly composed of nitrogen

$\therefore$ we say $\lambda(\text{air}) \approx \lambda(N_2)$

For N_2, take molecular diameter $d = 0.37\,nm$

cross-section $\sigma = \pi d^2 = 4.3 \times 10^{-19}\,m^2$

For room temperature and pressure

$P \approx 10^5\,Pa$, $T = 300\,K$

Number density $n = \frac{P}{k_B T} = \frac{10^5}{(1.38\times10^{-23})(300)} = 2.4\times10^{25}\,m^{-3}$

$\therefore$ mean free path

$$\lambda = \frac{1}{\sqrt{2}\, n\sigma} = \frac{1}{(\sqrt{2})(4.3\times10^{-19})(2.4\times10^{25})} = 6.85\times10^{-8}\,m$$

Average speed $\langle v \rangle = \frac{2}{\sqrt{\pi}} v_{th} = \frac{2}{\sqrt{\pi}}\sqrt{\frac{2k_B T}{m}}$

$$= \sqrt{\frac{8k_B T}{m\pi}}$$

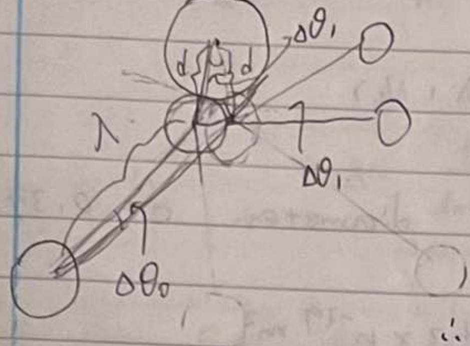
~~Average relative speed $\langle v_r \rangle =$~~

Molecular mass of N_2, $m = \frac{28\times10^{-3}\,kg/mol}{6.02\times10^{23}\,mol^{-1}} = 4.65\times10^{-26}\,kg$

$$\therefore \langle v \rangle = \left(\frac{8(1.38\times10^{-23})(300)}{(4.65\times10^{-26})(\pi)}\right)^{1/2} = 476\,m/s$$

6

The collision time $\tau_c \simeq \frac{\lambda_{mfp}}{\langle v \rangle} = \frac{6.85 \times 10^{-8} m}{476 m/s} = 1.44 \times 10^{-10} s$



$$\Delta x \approx \frac{m_e G}{2L^2} \tau_c^2$$

where $L = 8.8 \times 10^{26} m$ = size of universe

$$\therefore \Delta x = \frac{(9.11 \times 10^{-31})(6.67 \times 10^{-11})}{2(8.8 \times 10^{26})^2} (1.44 \times 10^{-10})^2$$

$$= 8.14 \times 10^{-115} m$$

Initial angular deflection $\Delta\theta_0 \sim \frac{\Delta x}{\lambda}$

Angular deflection after the first collision

$$\Delta\theta_1 \sim \frac{\Delta x}{d} \sim \cancel{\Delta\theta_0} \; \Delta\theta_0 \frac{\lambda}{d}$$

$$\therefore \Delta\theta_n \sim \Delta\theta_0 \left(\frac{\lambda}{d}\right)^n \sim \frac{\Delta x}{\lambda}\left(\frac{\lambda}{d}\right)^n$$

When trajectory has been changed completely, we calculate n such that $\Delta\theta_n \sim 1$

$$\therefore \quad n \sim \frac{\ln(\lambda / \Delta x)}{\ln(\lambda / d)} \sim \boxed{47 \text{ collisions}}$$

$\delta\theta = \frac{d}{\lambda}$

$\delta\phi = \pi/2$

$\frac{\delta\phi}{\delta\theta} = \frac{\pi\lambda}{2d}$

$\frac{\Delta\phi}{\Delta\theta} = \prod_1 \left(\frac{\delta\phi}{\delta\theta}\right)$

$\cancel{\Delta\theta} =$

$\frac{\Delta\phi}{\Delta\theta} = \left(\frac{\pi\lambda}{2d}\right)^N$

$\cancel{\Delta\theta} =$

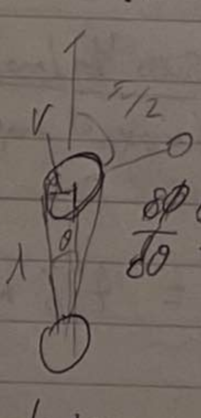
let $\Delta\phi = 1$

$\Delta\theta = \frac{1}{\left(\frac{\pi\lambda}{2d}\right)^N}$

$= (10^{-3})^N$

$\Delta\theta \approx 10^{-200}$

$N = 66$

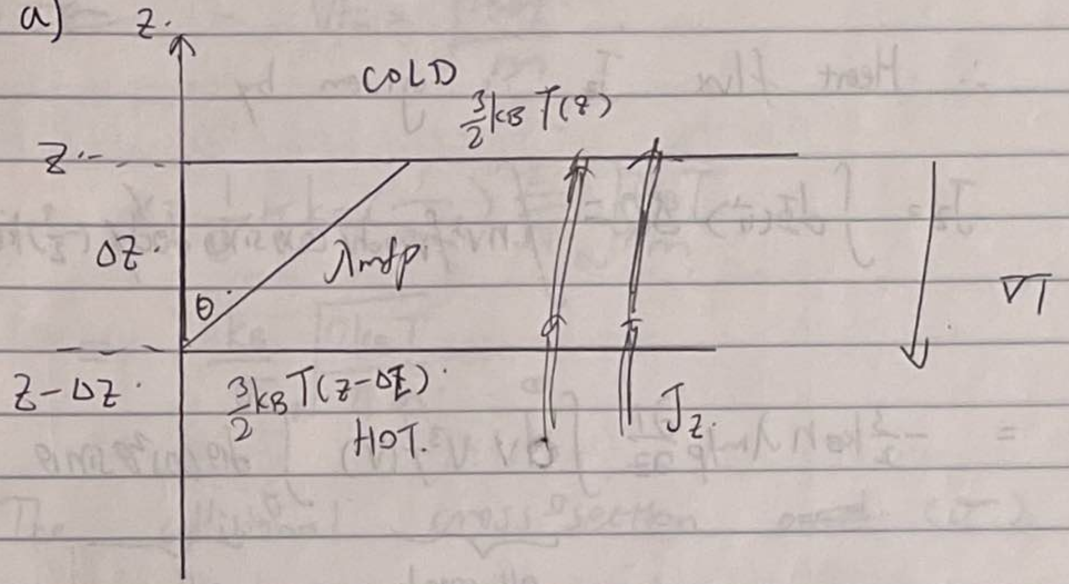


$$\frac{\delta\phi}{\delta\theta} \cancel{\delta\phi} = \frac{\pi/2 - 0}{\lambda/d}$$

(should be r).

d=2r

6 a)



Average Kinetic Energy of an individual particle is $\frac{3}{2}k_B T$

Extra energy brought by each particle travelling across plane z is

$$\Delta Q = \frac{3}{2}k_B T(z-\Delta z) - \frac{3}{2}k_B T(z)$$

$$= \frac{3}{2}k_B\left(T(z) + \frac{\partial T}{\partial z}(-\Delta z)\right) - \frac{3}{2}k_B T(z)$$

$$= -\frac{3}{2}k_B \frac{\partial T}{\partial z}\Delta z = -\frac{3}{2}k_B \frac{\partial T}{\partial z}\lambda_{mfp}\cos\theta$$

Number of particles going through surface z per area per time is

$$d\Phi(\vec{v}) = n v_z f(\vec{v})\, d^3\vec{v} = n v^3 f(v)\, dv \cos\theta \sin\theta\, d\theta\, d\phi$$

$\therefore$ Heat flux J_z is given by

$$J_z = \int d\Phi(\vec{v})\, \Delta Q = \int n v^3 f(v)\, dv \cos\theta \sin\theta\, d\theta\, d\phi \left(-\frac{3}{2}\right) k_B \frac{\partial T}{\partial z} \lambda_{mfp} \cos\theta$$

$$= -\frac{3}{2} k_B n \lambda_{mfp} \frac{\partial T}{\partial z} \underbrace{\int_0^\infty dv\, v^3 f(v)}_{\text{all speeds}} \underbrace{\int_0^\pi d\theta \cos^2\theta \sin\theta}_{\text{all angular direction}} \underbrace{\int_0^{2\pi} d\phi}_{\text{all azimuthal direction}}$$

$$= -\frac{3}{2} k_B n \lambda_{mfp} \frac{\partial T}{\partial z} \left(\frac{\langle v \rangle}{4\pi}\right) \left(\frac{2}{3}\right) (2\pi).$$

$$= -\frac{1}{2} n k_B \lambda_{mfp} \frac{\partial T}{\partial z} \langle v \rangle$$

$\because J_z = -\chi \frac{\partial T}{\partial z} \Rightarrow \chi = \frac{1}{2} n k_B \lambda_{mfp} \langle v \rangle.$

$\therefore \chi = \frac{1}{2} n k_B \left(\frac{1}{n\sigma}\right) \frac{2}{\sqrt{\pi}} (v_{th})$

(Here $\lambda_{mfp} = \frac{1}{n\sigma}$ instead of $\frac{1}{\sqrt{2} n\sigma}$ is because we are considering particles travelling across a stationary surface $z = z$, not a bunch of maxwellian particles collide with each other. So relative velocity needs not to be calculated).

$\therefore$ ~~κ~~ $\because V_{th} = \sqrt{\frac{2k_BT}{m}}$

$$\therefore \kappa = \frac{1}{2}\cancel{n}k_B\left(\frac{1}{\cancel{n}\sigma}\right)\frac{2}{\sqrt{\pi}}\sqrt{\frac{2k_BT}{m}}$$

$$= \frac{k_B}{\sigma}\sqrt{\frac{2k_BT}{\pi m}}$$

The collisional cross-section ~~and~~ (σ) and molecular mass (m) are properties of individual gas molecules, and k_B is a constant. Beside those, The thermal conductivity κ only depend on temperature. $\kappa \propto \sqrt{T}$

b) $\kappa = \frac{1}{2}nk_B\lambda_{mfp}\frac{2}{\sqrt{\pi}}V_{th} = \frac{1}{\sqrt{\pi}}nk_B\lambda_{mfp}V_{th}$.

$$\therefore \lambda_{mfp} = \frac{\sqrt{\pi}\kappa}{nk_BV_{th}}$$

$$V_{th} = \sqrt{\frac{2k_BT}{m}} = \left(\frac{2(1.38\times10^{-23})(273)}{(40\times1.66\times10^{-27}kg)}\right)^{1/2}$$

$$= 337 \text{ m/s}$$

$$n = \frac{P}{k_BT} = \frac{10^5}{(1.13\times10^{-23})(273)} = 2.98\times10^{25}\ m^{-3}$$

$$\lambda_{mfp} = \frac{\sqrt{\pi}\, \kappa}{n k_B V_{th}} = \frac{(\sqrt{\pi})(1.6\times10^{-2})}{(2.98\times10^{25})(1.38\times10^{-23})(337)}$$

$$= 2.05 \times 10^{-7}\ m = \boxed{205\ nm}$$ OR NOT.

S.T.P. $p = 10^5\ P_a$, $T = 300K$.

$$\kappa = \frac{1}{\sqrt{\pi}} n k_B \lambda_{mfp} V_{th} = n k_B \lambda_{mfp} \sqrt{\frac{2 k_B T}{\pi m}}$$

$$= n k_B T \lambda_{mfp} \sqrt{\frac{2k_B}{\pi m T}}$$

$$= P \lambda_{mfp} \sqrt{\frac{2k_B}{\pi m T}}$$

$$\therefore \lambda_{mfp} = \frac{\kappa}{P} \sqrt{\frac{\pi m T}{2k_B}} = \frac{(1.6\times10^{-2})}{(10^5)} \sqrt{\frac{(\pi)(40\times1.66\times10^{-27}\times300)}{2(1.38\times10^{-23})}}$$

$$= \underline{2.4 \times 10^{-7}\ m = 240\ nm}$$

$$\lambda_{mfp} = \frac{1}{n\sigma} = \frac{1}{n\pi d^2}$$

$$\therefore d^2 = \frac{1}{n\pi\lambda_{mfp}} \Rightarrow d = \sqrt{\frac{1}{n\pi\lambda_{mfp}}}$$

$\cancel{n = \frac{P}{k}}$ $$n = \frac{P}{k_B T} = \frac{(10^5)}{(1.38\times10^{-23})(300)} = 2.42\times10^{25}\ m^{-3}$$

$$\therefore d = \sqrt{\frac{1}{(2.42\times10^{25})(\pi)(2.4\times10^{-7})}} = \underline{2.34 \times 10^{-10}\ m}$$

OR if $\lambda = \frac{1}{\sqrt{2} n \sigma}$ then

$$d = \sqrt{\frac{1}{\sqrt{2} n \pi \lambda_{mfp}}} = \underline{1.97 \times 10^{-10}\ m}$$

Density of solid argon is $1.6 \times 10^{-3}\ kg/cm^3$

$\therefore$ In every ~~th~~ $1\ cm^3$ there are

$$\frac{1.6 \times 10^{-3}}{40 \times 1.66 \times 10^{-27}} = 2.41 \times 10^{22} \text{ atoms.}$$

They togather occupy $0.74 \times 1 cm^3 = 0.74\ cm^3$
$= 0.74 \times 10^{-6}\ m^3$

of volume

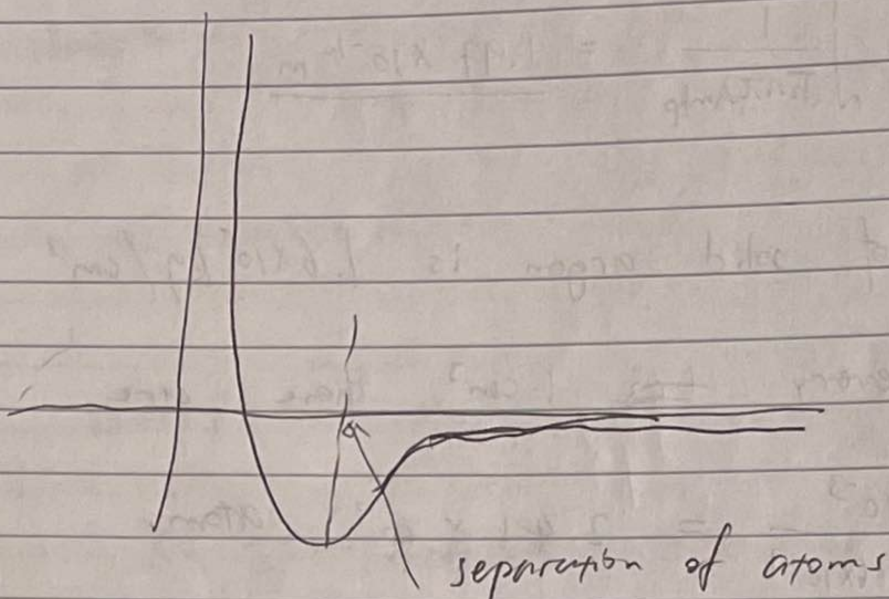
$\therefore$ Volume of one atom

$$= \frac{0.74 \times 10^{-6}}{2.41 \times 10^{22}} = 3.07 \times 10^{-29}\ m^3$$

$$\therefore \frac{4}{3}\pi d^3 = 3.07 \times 10^{-29}$$

$$\therefore d = \left(\frac{3 \times 3.07 \times 10^{-29}}{4\pi} \right)^{\frac{1}{3}} = 1.94 \times 10^{-10}\ m$$

They are consistent.

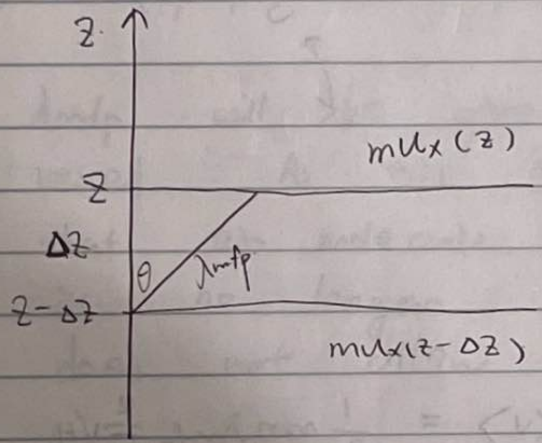

separation of atoms

7. a) Coefficient of viscosity η is given by

$$\Pi_{zx} = -\eta \frac{\partial u_x}{\partial z}$$

where Π_{zx} is the momentum flux of x-component of ~~mean velocity~~ momentum in the z direction.

u_x is the x-component of the mean velocity.



$\Delta z = \lambda_{mfp} \cos\theta$

Extra momentum ~~p~~ brought by each particle

$$\Delta p = mu_x(z-\Delta z) - mu_x(z)$$

$$= \cancel{mu_x(z)} - m\frac{\partial u_x}{\partial z}\Delta z - \cancel{mu_x(z)} = -m\frac{\partial u_x}{\partial z}\Delta z$$

Number of particles travelling across plane z per area per time is

$$d\Phi(\vec{v}) = nv_z f(\vec{v})\, d^3\vec{v} = nv^3 f(v)\, dv \sin\theta \cos\theta \, d\theta \, d\phi$$

∴ By definition the momentum flux Π_{zx}

is given by

$$\Pi_{zx} = \int d\Phi(\vec{v})\, \Delta p = \int n v^3 f(v)\, dv \sin\theta \cos\theta\, d\theta\, d\phi \; -m \frac{\partial u_x}{\partial z} \lambda_{mfp} \cos$$

$$= -mn\lambda_{mfp} \frac{\partial u_x}{\partial z} \underbrace{\int_0^\infty dv\, f(v) v^3}_{\langle v \rangle / 4\pi} \underbrace{\int_0^\pi d\theta \cos^2\theta \sin\theta}_{2/3} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$\cancel{= -2mn\lambda_{mfp} \frac{\partial u_x}{\partial z}}$$

$$= -\frac{1}{3} \underbrace{mn}_{\rho} \lambda_{mfp} \frac{\partial u_x}{\partial z} \langle v \rangle = -\frac{1}{3} \rho \lambda_{mfp} \langle v \rangle \frac{\partial u_x}{\partial z}$$

k

$$\therefore \eta = \frac{1}{3} \rho \lambda_{mfp} \langle v \rangle$$

b) $$\eta = \frac{1}{3} mn \lambda_{mfp} \langle v \rangle = \frac{1}{3} mn \lambda_{mfp} \frac{2}{\sqrt{\pi}} v_{th}.$$

$$= \frac{2}{3\sqrt{\pi}} mn \lambda_{mfp} \sqrt{\frac{2k_BT}{m}} = \frac{2}{3\sqrt{3}} m \cancel{n} \frac{1}{\cancel{n}\sigma} \sqrt{\frac{2k_BT}{m}}$$

$$= \frac{2}{3\sigma} \sqrt{\frac{2mk_BT}{\pi}}$$ which is independent of the number density n.

$\therefore$ Viscosity of air is independent of n

$\therefore$ As air is pumped out, ~~vi~~ vissosity will not change.

$\therefore$ The transfer of momentum between air molecules and the pendulum is the same no matter what n is.

$\therefore$ The rate of damping of swings of the pendulum will not change.

However, since $\lambda_{mfp} \propto \frac{1}{n}$, as pressure gets too low, i.e. as n gets too low, λ_{mfp} will get too ~~high~~ large. This means that

λ_{mfp} will be greater than the size of the vessel. As this happens, the typical distance that air molecule travels since their last collision is no longer λ_{mfp} because the vessel does not allow ~~tha~~ them to travel that far.

$$\therefore \quad \eta = \frac{2}{3\sqrt{\pi}} mn\lambda_{mfp}\sqrt{\frac{2k_BT}{m}}$$

$$= \frac{2m}{3\sqrt{\pi}k_BT}\underbrace{(nk_BT)}_{P}\lambda_{mfp}\sqrt{\frac{2k_BT}{m}} = \frac{2}{3}\sqrt{\frac{2m}{\pi k_BT}}\,P\lambda_{mfp}$$

$\therefore \eta \propto P\lambda_{mfp}$ under constant temperature.

~~$P_1 = 1\,atm = 10^5 Pa \quad \lambda_1 =$~~

when $P_1 = 10^5 Pa = 1\,atm$ $\qquad m \approx m(N_2) = 4.65\times10^{-26}\,kg.$

$$\lambda_{mfp} = \left(\frac{3}{2}\sqrt{\frac{\pi k_BT}{2m}}\right)\frac{\eta}{P} = \left(\frac{3}{2}\right)\left(\frac{\pi(1.38\times10^{-23})(293)}{2(4.65\times10^{-26})}\right)^{1/2}\frac{(18.2\times10^{-6})}{10^5}$$

$$= 1\times10^{-7}$$

For η to not change $P\lambda = \text{const}$

maximum λ is about the size of the vessel

$\therefore \lambda_2 \simeq 1m$

$\therefore P_1\lambda_1 = P_2\lambda_2$

~~$\lambda_2 = \frac{P_1\lambda_1}{P_2}$~~

$$P_2 = \frac{P_1\lambda_1}{\lambda_2} = \frac{(10^5 Pa)(1m)}{(1 \times 10^{-7} m)} = \underline{10^{-2} Pa}$$

Fewer particles collide with the pendulum

but as n becomes small λ gets ~~less~~ large

so ~~it bring~~ particles can travel longer across

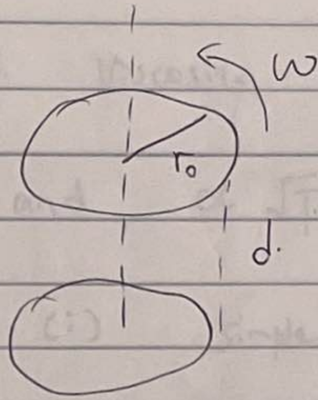
the velocity gradient and thus each colliding

particle brings more extra momentum.

The two effects exactly balances each

other.

8.



$$\frac{F}{A} = \eta \frac{v}{d}$$

$$\frac{dF(r)}{dA} = \eta \frac{\omega r}{d}$$

$$\therefore \quad dF(r) = \eta \frac{\omega r}{d} (2\pi r \, dr)$$

$$\therefore d\tau = r \, dF = \left(\frac{2\pi\eta\omega}{d}\right) r^3 dr$$

$$\therefore \quad \tau = \int_{r=0}^{r=r_0} d\tau = \int_0^{r_0} \left(\frac{2\pi\eta\omega}{d}\right) r^3 dr$$

$$= \frac{\pi\eta\omega r_0^4}{2d} = \frac{(\pi)(2.1\times10^{-5})(10)(0.05^{-4})}{(2)(1\times10^{-3})}$$

$$= \boxed{2\times10^{-6} \ N\cdot m}$$

9. Viscosity η is independent of pressure

and $\propto \sqrt{T}$ from the above measurements.

(i) simple kinetic theory.

$$\eta = \frac{2}{3\sigma}\left(\frac{2mk_BT}{\pi}\right)^{1/2}$$

$\therefore$ This is consistent with measurements.

(ii) For $T = 500k$

$$\cancel{\sigma = \frac{3\eta}{2T}}$$

$$\sigma = \frac{2}{3\eta}\left(\frac{2mk_BT}{\pi}\right)^{\frac{1}{2}}$$

expect factor 2 change in ~~viscosity~~ viscosity but slightly different so expect the size of atom changes

$$\therefore \sigma = \frac{2}{3(3.5\times10^{-5})}\left(\frac{2(40\times1.66\times10^{-27})(1.38\times10^{-23})(500)}{\pi}\right)^{\frac{1}{2}}$$

$$= 3.25\times10^{-19}\ m^2$$

$\sigma = \pi d^2 \Rightarrow d = 3.2\times10^{-10}\ m = 0.32\ nm$

$\simeq 0.34\ nm$

$\therefore$ Consistent with the diameter.

For $T = 2000K$.

$\sigma = \cancel{1.42 \times 10^{-19}}$ 2.84×10^{-19}

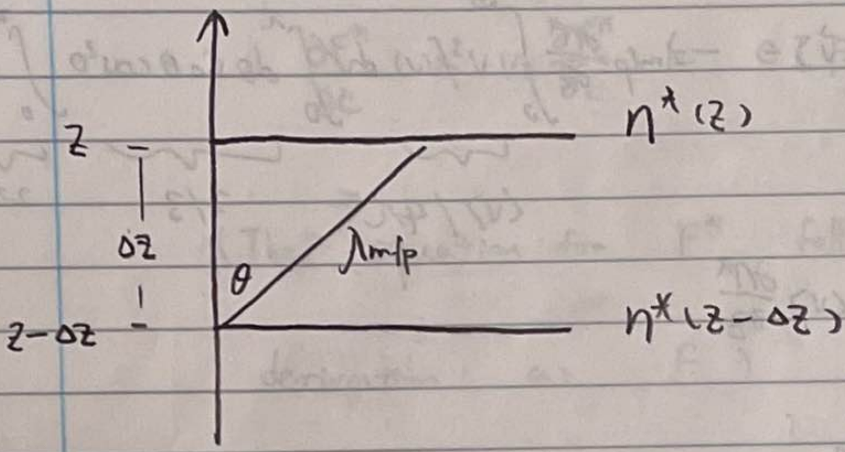
$\pi d^2 = \sigma$

$\Rightarrow d = 3.0 \times 10^{-10}\ m$

$\therefore$ $d = 0.30\ nm$ (Consistent as well).

effective diameter goes down when temperature gets high

10 a)



particles travel a distance λ_{mfp} on average

before reaching the plane z

$$\Delta z = \lambda_{mfp} \cos\theta$$

Extra particle density brought by these particles

is $$\Delta n^* = n^*(z-\Delta z) - n^*(z)$$

$$= \cancel{n^*(z)} - \frac{\partial n^*}{\partial z}\Delta z - \cancel{n^*(z)}$$

$$= -\lambda_{mfp}\cos\theta \frac{\partial n^*}{\partial z}$$

The ~~particle~~ flux of extra particle density

is $$d\Phi^*(\vec{v}) = \Delta n^* v^3 f(v)\, dv \sin\theta\cos\theta\, d\theta\, d\phi$$

$$= -\lambda_{mfp}\frac{\partial n^*}{\partial z} v^3 f(v)\, dv \sin\theta\cos^2\theta\, d\theta\, d\phi$$

$\therefore$ Total particle flux

$$\Phi^* = \int d\Phi^*(\vec{v}) = -\lambda_{mfp} \frac{\partial n^*}{\partial z} \underbrace{\int_0^\infty v^3 f(v)\, dv}_{\langle v \rangle / 4\pi} \underbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}_{2/3} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= -\frac{1}{3} \lambda_{mfp} \langle v \rangle \frac{\partial n^*}{\partial z}$$

$$\because \Phi^* = -D \frac{\partial n^*}{\partial z} \implies D = \frac{1}{3} \lambda_{mfp} \langle v \rangle$$

We know $\kappa = \frac{1}{2} n k_B \lambda_{mfp} \langle v \rangle = \frac{1}{3} \left(\frac{3}{2} n k_B\right) \lambda_{mfp} \langle v \rangle$

$= \frac{1}{3} C_V \lambda_{mfp} \langle v \rangle$

$\eta = \frac{1}{3} \rho \lambda_{mfp} \langle v \rangle$

$$\therefore D \sim \frac{\kappa}{C_V} \sim \frac{\eta}{\rho}$$

b) Starting from the Boltzman kinetic equation

$$\frac{\partial F^*}{\partial t} + V_z \frac{\partial F^*}{\partial z} = C[F^*] = \frac{-F^* - F_m^*}{\tau_c} = -\nu_c^* (F^* - F_m^*)$$

$$= -\nu_c^* \delta F^*$$

(The equation for F^* follows exactly the same derivation as F)

Integrate with respect to $d^3\vec{v}$ $\Rightarrow$

$$\int d^3\vec{v} \frac{\partial F^*}{\partial t} + \int d^3\vec{v}\, V_z \frac{\partial F^*}{\partial z} = \int d^3\vec{v}\, C[F^*]$$

$\int d^3\vec{v}\, C[F^*] = 0$ as collision does not ~~cr~~ change the number density of labelled particles

$$\therefore \frac{\partial}{\partial t} \underbrace{\int d^3\vec{v}\, F^*}_{n^*} + \frac{\partial}{\partial z} \underbrace{\int d^3\vec{v}\, V_z F^*}_{n^* u_z^*} = 0$$

$$\therefore \frac{\partial n^*}{\partial t} + \frac{\partial}{\partial z}(n^* u_z^*) = 0$$

$$n^* u_z^* = \int d^3\vec{v}\, v_z F^* = \underbrace{\int d^3\vec{v}\, v_z F_M^*}_{0} + \int d^3\vec{v}\, v_z \delta F^*$$

where $\quad F_M^* = \frac{n^*}{(\sqrt{\pi} v_{th})^3} e^{-v^2/v_{th}^2}$

(mean velocity $= \vec{0}$, $w = v$).

($v_{th}^2 = \frac{2k_B T}{m^*}$)

and using ~~kf~~ krook's operator $C[F] = -\frac{F^* - F_M^*}{\tau_c}$

for the collision ~~oper~~ operator gives

$$\delta F^* = -\tau_c^* v_z \frac{\partial F_M^*}{\partial z} = -\frac{v_z}{\nu_c^*} \frac{\partial F_M^*}{\partial z}$$

$v_z F_M^*$ is odd in v_z $\therefore \int d^3\vec{v}\, v_z F_M^* = 0$

$$\therefore \int d^3\vec{v}\, v_z F^* = \int d^3\vec{v}\, v_z \delta F^* = \cancel{\int d^3\vec{v} \; -\frac{1}{\nu_c^*}\int d^3\vec{v}\, v_z^2 F_M}$$

$$= -\frac{1}{\nu_c^*}\int d^3\vec{v}\, v_z^2 \frac{\partial F_M^*}{\partial z} = -\frac{1}{\nu_c^*}\frac{\partial}{\partial z}\int d^3\vec{v}\, v_z^2 F_M^*$$

$$= -\frac{1}{\nu_c^*}\frac{\partial}{\partial z}\left[\int d^3\vec{v}\, v_z^2 \frac{n^*}{(\sqrt{\pi} v_{th})^3} e^{-(v_x^2+v_y^2+v_z^2)/v_{th}^2}\right].$$

$$= -\frac{1}{\nu_c^*}\frac{\partial}{\partial z}\Big[n^* \underbrace{\int dv_x \frac{1}{\sqrt{\pi} v_{th}} e^{-v_x^2/v_{th}^2}}_{1} \underbrace{\int dv_y \frac{1}{\sqrt{\pi} v_{th}} e^{-v_y^2/v_{th}^2}}_{1}$$

$$\int dv_z\, v_z^2 \frac{1}{\sqrt{\pi} v_{th}} e^{-v_z^2/v_{th}^2}\Big].$$

$$0 = -\frac{1}{\nu_c^*}\frac{\partial n^*}{\partial z}\int dv_z \frac{1}{\sqrt{\pi} v_{th}} v_z^2 e^{-v_z^2/v_{th}^2}$$

$$= -\frac{1}{\nu_c^*}\frac{\partial n^*}{\partial z}\frac{1}{\sqrt{\pi} v_{th}} \cancel{\frac{1}{2}\sqrt{\pi}} \frac{1}{2}\sqrt{\pi}\, v_{th}^3$$

$$= -\frac{1}{2}\frac{1}{\nu_c^*}\frac{\partial n^*}{\partial z} v_{th}^2$$

$$= -\frac{1}{2}\frac{1}{\nu_c^*}\frac{2k_BT}{m^*}\frac{\partial n^*}{\partial z} = -\frac{k_BT}{m^*\nu_c^*}\frac{\partial n^*}{\partial z}$$

$$\therefore \frac{\partial n^*}{\partial t} = \frac{k_BT}{m^*\nu_c^*}\frac{\partial^2 n^*}{\partial z^2} = D\frac{\partial^2 n^*}{\partial z^2}$$

$$\therefore \boxed{D = \frac{k_BT}{m^*\nu_c^*}}$$

c) kinetic equation :

$$\frac{\partial F^*}{\partial t} + v_z\frac{\partial F^*}{\partial z} = -\nu_c^*\delta F^*$$

multiply by $m^* v_z$ and integrate w.r.t $d^3\vec{v}$ gives

$$\int d^3\vec{v}\, m^* v_z \frac{\partial F^*}{\partial t} + \int d^3\vec{v}\, m^* v_z^2 \frac{\partial F^*}{\partial z} = -\nu_c^*\int d^3\vec{v}\, m^* v_z \delta F^*$$

$$\Rightarrow \frac{\partial}{\partial t} m^* n^* u_z^* = -\frac{\partial}{\partial z}\int d^3\vec{v}\, m^* v_z^2 F^* - \cancel{\nu_c^* m^* \int} - m^*\nu_c^* u_z^* n^*$$

The last term is the expression for drag force $f = - m^* \nu_c^* n^* u_z^*$

and proportionality constant is $\boxed{m^* n^* \nu_c^*}$.

Momentum equation:

$$\frac{\partial}{\partial t} m^* n^* u_z^* = - \frac{\partial}{\partial z} \int d^3\vec{v}\, m^* v_z^2 F^* - m^* n^* \nu_c^* u_z^* \qquad (1)$$

Assume the typical scale of variation of $m^* n^* u_z^*$ w.r.t time, $t = \left(\frac{\partial \ln(m^* n^* u_z^*)}{\partial t} \right)^{-1} \gg \tau_c^*$

then we ~~can~~ have $\nu_c^* \gg \frac{\partial}{\partial t}$

$\therefore$ ~~(1) ⇒~~ LHS of (1) becomes negligible

$$\therefore \ (1) \Rightarrow m^* n^* \nu_c^* u_z^* = - \frac{\partial}{\partial z} \int d^3\vec{v}\, m^* v_z^2 F^*$$

$$\simeq \cancel{-\frac{\partial}{\partial z}} - m^* \frac{\partial}{\partial z} \int d^3\vec{v}\, v_z^2 F_M^* \qquad (F^* \simeq F_M^*)$$

$$= - m^* \frac{\partial}{\partial z} \frac{1}{2} v_{th}^2 n^* = - m^* \frac{\partial}{\partial z} \frac{1}{2} \frac{2 k_B T n^*}{m^*}$$

$$= - \frac{\partial n^* k_B T}{\partial z} = - \frac{\partial P^*}{\partial z}$$

$$\therefore \ n^* u_z^* = - \frac{1}{m^* \nu_c^*} \frac{\partial P^*}{\partial z}$$

11 a) In steady state the heat equation

$$\kappa \vec{\nabla}^2 T + H - C = 0$$

where $C = 0$ and H, the heat source,

is given by $H =$ ohmic heating / volume

$$\Rightarrow H = \frac{I^2 R}{V} = \frac{I^2 \rho \frac{L}{S}}{LS} = \frac{I^2 \rho}{S^2} = \frac{I^2 \rho}{\pi^2 a^4}$$

(we know ρ is uniform)

$$\therefore \kappa \vec{\nabla}^2 T + \frac{I^2 \rho}{\pi^2 a^4} = 0$$

$$\therefore \vec{\nabla}^2 T = -\frac{I^2 \rho}{\pi^2 a^4 \kappa} = \text{const} = -c$$

$$\because T = T(r) \quad \therefore \vec{\nabla}^2 T = \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$\therefore \frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -c \Rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right) = -cr$$

$$\therefore \cancel{r\frac{d^2T}{dr^2} + \frac{dT}{dr} + cr = 0}$$

$$\therefore r\frac{dT}{dr} = \int -cr\,dr = -\frac{cr^2}{2} + A.$$

$$\therefore \cancel{\frac{dT}{dr} = -\frac{cr^2}{2} + \frac{A}{r}} \qquad \frac{dT}{dr} = -\frac{cr}{2} + \frac{A}{r}$$

At $r=0$ $\frac{dT}{dr}$ should be finite

$\therefore A=0$ $\quad\therefore$ ~~$\frac{dT}{dr} = -\frac{cr^2}{2}$~~ $\quad \frac{dT}{dr} = -\frac{cr}{2}$

$$\therefore T = -\frac{cr^2}{4} + B = \underbrace{-c'r^2 + B}_{c' = \frac{c}{4}}$$

Boundary Condition

$$r=a, \quad T(a) = T_0$$

$$\Rightarrow \quad T_0 = -c'a^2 + B \quad \therefore B = T_0 + c'a^2$$

$$\therefore T(r) = T_0 + c'a^2 - c'r^2$$

$$\because c' = \frac{\rho I^2}{4\pi^2 a^4 \chi}$$

$$\therefore \boxed{T(r) = T_0 + \frac{\rho I^2}{4\pi^2 a^4 \chi}(a^2 - r^2)}$$

b) The differential equation is the same as a) so

$$T(r) = -c'r^2 + B$$

But the Boundary Conditions are different

The radial heat flux $J_r = \alpha(T(a) - T_{air})$

$$\therefore \quad J_r\Big|_{r=a} = -k\frac{\partial T}{\partial r}\Big|_{r=a} = \alpha(T(a) - T_{air})$$

$$\Rightarrow \quad 2c'a = \frac{\alpha}{k}(-c'a^2 + B - T_{air})$$

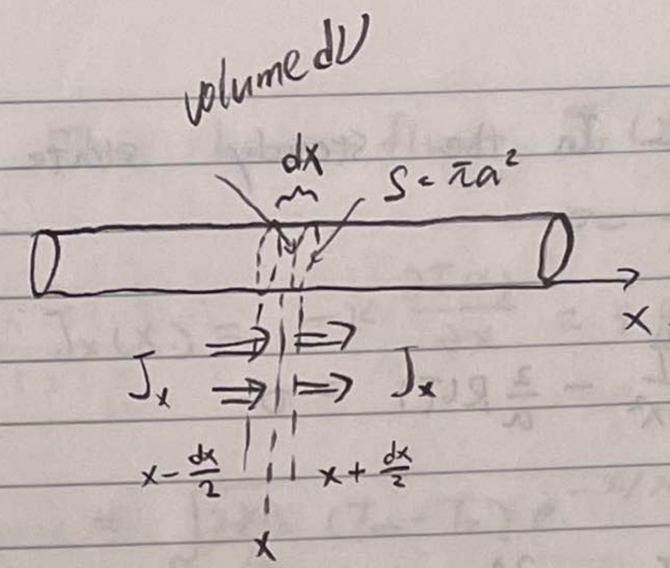
$$\therefore \quad \frac{2c'ka}{\alpha} = -c'a^2 - T_{air} + B$$

$$\therefore \quad B = T_{air} + c'a^2 + \frac{2c'ka}{\alpha}$$

$$\therefore \quad B = T_{air} + \left(\frac{\rho I^2}{4\pi^2 a^4 k}\right)\left(a^2 + \frac{2ka}{\alpha}\right)$$

$$\therefore \quad \boxed{T(r) = T_{air} + \frac{\rho I^2}{4\pi^2 a^4 k}\left(a^2 - r^2 + \frac{2ka}{\alpha}\right)}$$

12)



Total internal energy inside volume dV is $\rho c_m T S\, dx$

Rate of change of internal energy inside dV equals to the net effect of total ~~flux~~ heat flux and the Newton's cooling of the lateral surface

$$\therefore \frac{\partial}{\partial t}(\rho c_m T S\, dx) = S\left(J_x\left(x - \frac{dx}{2}\right) - J_x\left(x + \frac{dx}{2}\right)\right) - 2\pi a\, dx\, R(T)$$

Divide by $S dx$ =>

$$\rho c_m \frac{\partial T}{\partial x} = \underbrace{\frac{J_x(x - \frac{dx}{2}) - J_x(x + \frac{dx}{2})}{dx}}_{= -\frac{\partial J_x}{\partial x}} - \frac{2\pi a}{\pi a^2} R(T)$$

$\because J_x = -k\frac{\partial T}{\partial x}$ $\therefore$ we have

$$\boxed{\rho c_m \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} - \frac{2}{a} R(T)}$$

(a) ~~I~~ In the steady state

$$\frac{\partial T}{\partial t} = 0$$

$$\therefore \quad \kappa \frac{\partial^2 T}{\partial x^2} - \frac{2}{a} R(T) = 0$$

$$\therefore \quad \frac{\partial^2 T}{\partial x^2} - \frac{2A}{\kappa a}(T - T_0) = 0$$

let $\underline{T' = T - T_0}$, $\frac{\partial^2 T'}{\partial x^2} = \frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial^2 T'}{\partial x^2} - \frac{2A}{\kappa a} T' = 0$

let $\underline{\lambda^{-2} = \frac{2A}{\kappa a}}$ then $\frac{\partial^2 T'}{\partial x^2} - \lambda^{-2} T' = 0$

$\cancel{\therefore T' = B_1 e^{-\lambda x} + B}$ $T' = B_1 e^{-x/\lambda} + B_2 e^{x/\lambda}$

As $x \to \infty$ we want finite T thus finite T'

$\Rightarrow \quad B_2 = 0$

$\therefore T' = B_1 e^{-x/\lambda}$

Boundary condition : $x = 0$, $T = T_m$ $\therefore T' = T_m - T_0$

$\therefore \quad T_m - T_0 = B_1 e^{-0} = B_1 \Rightarrow B_1 = T_m - T_0$

$\therefore T'(x) = (T_m - T_0) e^{-x/\lambda}$

$$\therefore \boxed{T(x) = (T_m - T_0) e^{-x/\lambda} + T_0 \qquad \lambda = \sqrt{\frac{\kappa a}{2A}}}$$

(b) The heat flux $J_x(x) = -k\frac{\partial T}{\partial x}$

$$\therefore J_x(x) = -k\frac{\partial T(x)}{\partial x} = +k\sqrt{\frac{2A}{ka}}(T_m - T_0)e^{-x/\lambda}$$

$$= \sqrt{\frac{2kA}{a}}(T_m - T_0)e^{-x/\lambda}$$

Heat flowing through ~~a~~ a cross-section of the cylinder at position x ~~is~~ per unit time, $\frac{dQ}{dt}$, is given by

$$\frac{dQ}{dt} = J_x S = \pi a^2 \sqrt{\frac{2kA}{a}}(T_m - T_0)e^{-x/\lambda}$$

$$= \pi(T_m - T_0)e^{-x/\lambda}\sqrt{2kA}\, a^{3/2} \quad \underline{\propto a^{3/2}}$$

In practice rod isn't infinitely long.

But we can treat it as infinite if

it is long enough compare to the ~~radius~~

~~the~~ characteristic length λ

Let's say ~~\lambda~~ $L = 5\lambda$ is enough

As $e^{-L/\lambda} = e^{-5} = 0.006$ is small

$\therefore L = 5\lambda = 5\sqrt{\frac{\kappa a}{2A}} = 5\left(\frac{(380)(1.5\times10^{-3})}{(2)(250)}\right)^{\frac{1}{2}}$

$= \boxed{0.17\ m}$ is long enough

13 Heat equation $\frac{\partial \varepsilon}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$

$\varepsilon = \frac{3}{2} n k_B T = ncT = CT$ (C = heat capacity)

let $D = \frac{\kappa}{C}$ then

$\cancel{\frac{\partial^2 T}{\partial t}}$ $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$

Looking for wave-like ~~tht~~ solutions $T(x,t) = A\exp(i(kx - \Omega t))$

then $-i\Omega = -Dk^2 \Rightarrow k = \sqrt{\frac{i\Omega}{D}}$

$\Omega > 0 \Rightarrow$ $k = \pm\sqrt{i}\sqrt{\frac{\Omega}{D}} = \pm\frac{1}{\sqrt{2}}(1+i)\sqrt{\frac{\Omega}{D}}$

$= \pm(1+i)\sqrt{\frac{|\Omega|}{2D}}$

$\Omega < 0 \Rightarrow$ $k = \pm\sqrt{-i}\sqrt{\frac{|\Omega|}{D}} = \pm\frac{1}{\sqrt{2}}(1-i)\sqrt{\frac{|\Omega|}{D}}$

$= \pm(1-i)\sqrt{\frac{|\Omega|}{2D}}$

let $\boxed{\delta = \sqrt{\frac{2D}{|\Omega|}} = \cancel{\sqrt{\frac{2\kappa}{|\Omega|C}}}\sqrt{\frac{2\kappa}{|\Omega|C}}}$ then

$k = \pm\frac{(1+i)}{\delta}$ if $\Omega > 0$

$k = \pm\frac{(1-i)}{\delta}$ if $\Omega < 0$

$k = 0$ if $\Omega = 0$

For solution to be finite at ~~x=~~ $x = +\infty$

If $\Omega > 0$:

$$T_\Omega(x,t) = \cancel{A_\Omega \exp(\ldots)} \; A_\Omega \exp(-x/\delta_\Omega) \exp(i(x/\delta_\Omega - \Omega t))$$

If $\Omega < 0$:

$$T_\Omega(x,t) = A_\Omega \exp(-x/\delta_\Omega) \exp(i(-x/\delta_\Omega - \Omega t))$$

General solution :

$$T(x,t) = \sum_\Omega T_\Omega(x,t)$$

Now assume the face with sinusoidal temperature variation is placed at $x=0$ and the variation is given by $T(0,t) = T_0 + T_1 \cos(\omega t)$, then

$$T(0,t) = T_0 + \frac{T_1}{2} e^{i\omega t} + \frac{T_1}{2} e^{-i\omega t}$$

$$= \sum_\Omega T_\Omega(0,t) \; \cancel{= A}$$

$$\Rightarrow \quad A_0 = T_0 \, , \quad A_\omega = \frac{T_1}{2} \, , \quad A_{-\omega} = \frac{T_1}{2}$$

$$\therefore \; T(x,t) = T_0 + \frac{T_1}{2} \exp(-x/\delta_\omega) \exp(i(x/\delta - \omega t))$$

$$+ \frac{T_1}{2} \exp(-x/\delta_{-\omega}) \exp(i(-x/\delta_{-\omega} + \omega t))$$

$$\delta_\omega = \delta_{-\omega} = \sqrt{\frac{2K}{\omega C}} \quad \Rightarrow$$

$$T(x,t) = T_0 + \frac{T_1}{2} (\exp(-x/\delta_\omega)) (\exp(i(x/\delta_\omega - \omega t))$$

$$+ \exp(-i(x/\delta_\omega - \omega t)))$$

$$\therefore \boxed{T(x,t) = T_0 + T_1 \exp(-x/\delta_w)\cos(x/\delta_w - \omega t) \quad \text{where} \quad \delta_w = \sqrt{\frac{2\kappa}{\omega C}}}$$

This is the required damped temperature oscillation and decay length is

$$\delta_w = \sqrt{\frac{2\kappa}{\omega C}}$$

Daily fluctuation : $T_1 = 10°C = \cancel{283\,K}\ 10$, $x = 3m$

$$\omega = \frac{2\pi}{P} = \frac{2\pi}{1\ \text{day}} = \frac{2\pi}{24\times 60\times 60} = 7.3\times 10^{-5}\ s^{-1}$$

$$\delta_w = \sqrt{\frac{2(1.6)}{(7.3\times10^{-5})(2.5\times10^{6})}} = 0.132\ m$$

Amplitude : $A_d = T_1 \exp(-x/\delta_w)$

$$= \cancel{283}^{10} \exp(-3/0.132)$$

$$= \boxed{\cancel{3.8\times10^{-8}\,K}}$$

$10^{-9}\,K$

(P is period)

Annual fluctuation: $T_1 = 20°C = \cancel{293\ K}$ 20, $x = 3m$

$$w = \frac{2\pi}{P} = \frac{2\pi}{1\ year} = \frac{2\pi}{365 \times 24 \times 60 \times 60} = 1.99 \times 10^{-7}\ s^{-1}$$

$$\delta w = \left(\frac{2 \times 1.6}{(1.99 \times 10^{-7})(\cancel{2.6 \times 10^{5}})}\right)^{\frac{1}{2}} = \cancel{7.86\ m}$$

2.5×10^{6} $\qquad$ $2.536\,m$

Amplitude : $A_a = T_1 \exp(-x/\delta w)$

$$= \cancel{293}\ 20 \times e^{-3/\cancel{7.86}\ 2.536}$$

$$= \boxed{\cancel{200\ K}}\quad 6K$$

Daily variation is negligible compare to annual variation

Assume at some point in January ~~wt = 3π~~

$wt = 3\pi$, so $T(0,t) = T_0 + T_1 \cos(wt)$ reaches minimum $T_{min}(0,t) = T - T_1$, then $t = \frac{3\pi}{w}$

If At time $t = t'$, $T(3,t) = T_0 + 200\cos\left(\frac{3}{7.86} - wt\right)$ reaches it minimum, then

$$wt' - \frac{3}{\cancel{7.86}\ 2.536} = 3\pi$$

$$\because\ wt = 3\pi$$

$$\therefore\ w\Delta t = w(t' - t) = \frac{3m}{\cancel{7.86}\,m\ 2.536} \qquad \therefore\ \Delta t = \frac{3}{\cancel{7.86}\,w\ 2.536} = \frac{3P}{\cancel{7.86} \times 2\pi\ 2.536}$$

$$= \cancel{\frac{3}{7}}\ \cancel{2.061} \times 12\,month$$

~~∴ Δt ≈ 0.7 month~~ ~~∴ After about 0.7 month the cellar's temperature will be lowest~~

~~∴ Cellar's Temperature will be lowest in Fe~~

$\therefore \Delta t = \frac{0.188}{\cancel{0.061}} \times 12$ month

$= \cancel{0.7}$ month

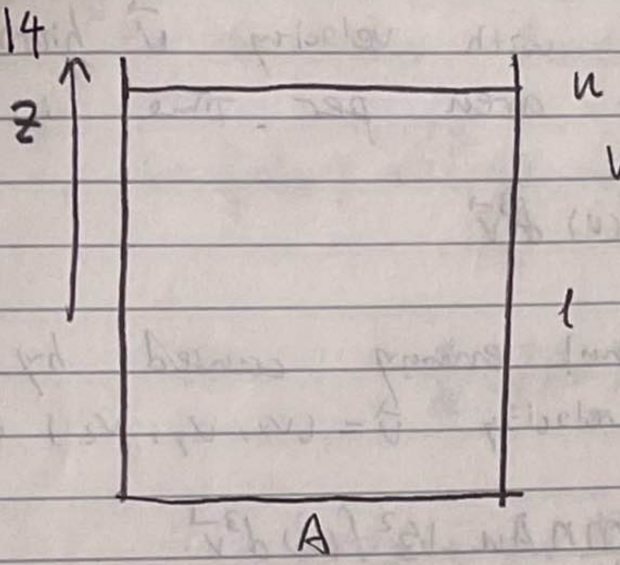
~~$\therefore$ After 2.259 about 0.7 month the collar's temperature will be lowest.~~

$\therefore$ ~~The collar's temperature is lowest in February~~

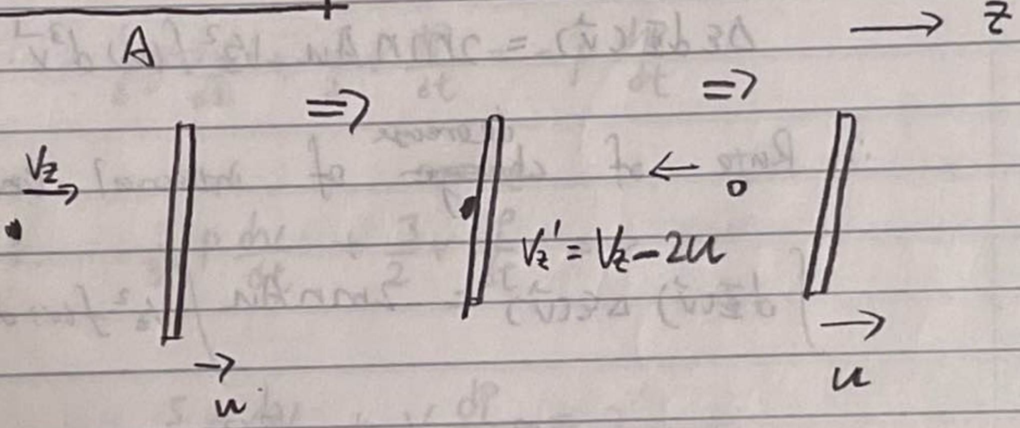
2.259 month

= 70 days

14



$$V = Al \Rightarrow \frac{dV}{dt} = A\frac{dl}{dt} = Au$$



For elastic collision with the piston, only velocity in the z-direction changes, ~~and it~~ Relative velocity in the z-direction reverses direction $\Rightarrow$ $v_z - u = u + v_z' \Rightarrow v_z' = v_z - 2u$

For a particle, ~~change~~ decrease in kinectic ~~a~~ energy after collision with piston is

$$\Delta E = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2) - \frac{1}{2}m(v_x^2 + v_y^2 + (v_z - 2u)^2)$$

$$= \frac{1}{2}m[v_z^2 - (v_z - 2u)^2] = \cancel{\frac{1}{2}mv_z^2} - \cancel{\frac{1}{2}mv_z^2} + 2mv_zu + 2mu^2$$

$2mu^2$ can be neglected since $u \ll v_z$

$$\therefore \Delta E = 2mv_zu$$

Number of particles with velocity $\vec{v}$ hitting the piston per area per time is

$$d\Phi(\vec{v}) = n v_z f(v)\, d^3\vec{v}$$

∴ Decrease of internal energy caused by particles with velocity $\vec{v} = (v_x, v_y, v_z)$ is per time

$$\Delta\varepsilon\, d\Phi(\vec{v}) = 2mnAu\, v_z^2 f(v)\, d^3\vec{v}$$

∴ Rate of ~~change~~ decrease of internal energy is

$$\int d\Phi(\vec{v})\, \Delta\varepsilon(\vec{v}) = 2mnAu \int v_z^2 f(v)\, d^3\vec{v}$$

$$= 2mnAu \underbrace{\int_0^\infty v^4 f(v)\, dv}_{= \frac{\langle v^2 \rangle}{4\pi}} \underbrace{\int_0^{\pi/2} \sin\theta \cos^2\theta\, d\theta}_{\text{only } +z \text{ direction particle can collide with piston} \;=\; \cancel{2}\ 1/3} \underbrace{\int_0^{2\pi} d\phi}_{=2\pi}$$

$$= (2)\left(\frac{1}{3}\right)\left(\frac{2\pi}{4\pi}\right) mn(Au)\, \langle v^2 \rangle$$

$$= \frac{1}{3} mn \frac{dV}{dt} \frac{3}{2} v_{th}^2 = \frac{1}{\cancel{2}} \cancel{m} n \frac{dV}{dt} \frac{\cancel{2} k_B T}{\cancel{m}}$$

$$= n k_B T \frac{dV}{dt} = P \frac{dV}{dt}$$

On the other hand, internal energy density

$$\varepsilon = \frac{3}{2} n k_B T = \frac{3}{2} P \Rightarrow \text{Internal energy } U = \frac{3}{2} PV$$

Hence $\quad \frac{d}{dt}\left(\frac{3}{2}PV\right) = -P\frac{dV}{dt}$

$$\therefore \frac{3}{2}P\frac{dV}{dt} + \frac{3}{2}V\frac{dP}{dt} = -P\frac{dV}{dt}$$

$$\therefore \frac{5}{2}P\frac{dV}{dt} + \frac{3}{2}V\frac{dP}{dt} = 0$$

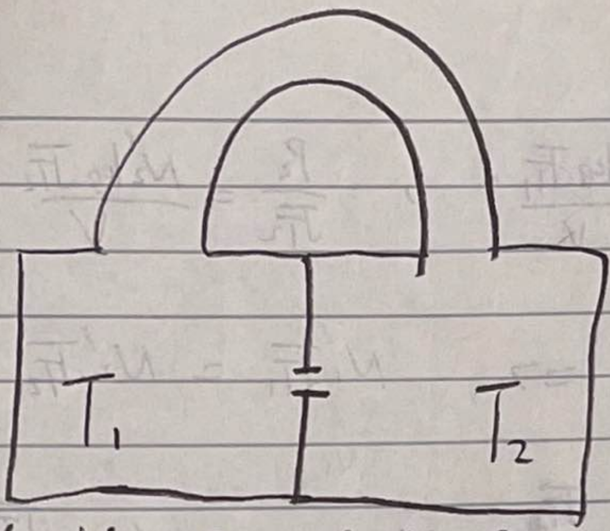
$$\therefore \frac{5}{3}P\frac{dV}{dt} + V\frac{dP}{dt} = 0$$

multiply by $V^{2/3} \Rightarrow \quad \frac{5}{3}PV^{2/3}\frac{dV}{dt} + V^{5/3}\frac{dP}{dt} = 0$

$$\Rightarrow \frac{d}{dt}\left(PV^{5/3}\right) = 0$$

$$\Rightarrow \boxed{PV^{5/3} = \text{constant.}}$$

15



$V, N_1, P_1 \quad V, N_2, P_2$

m = mass of a particle

$T_1 = 225\,K$

$T_2 = 400\,K$

$T_2 > T_1$

$N = N_1 + N_2$

$$P_1 = \frac{N_1 k_B T_1}{V} \qquad P_2 = \frac{N_2 k_B T_2}{V}$$

When connected by tube : (Macroscopic equilibrium)

~~Equilib~~ Equilibrium reached when pressures are balanced

$$\underline{P_1 = P_2} \Rightarrow N_1 T_1 = N_2 T_2$$

$$N = \left(1 + \frac{T_1}{T_2}\right) N_1 \Rightarrow N_1 = \frac{T_2}{T_1 + T_2} N$$

multiply by m $\Rightarrow$ $M_1 = \frac{T_2}{T_1 + T_2} M$

When tube is removed ; (Microscopic equilibrium)

equilibrium reached when the ~~di~~ effusion fluxes are the same for both chambers

The flux $\Phi = \frac{1}{4} n \langle v \rangle = \frac{1}{4} n \sqrt{\frac{8 k_B T}{\pi m}} = \frac{1}{\sqrt{2\pi m k_B}} \frac{P}{\sqrt{T}}$

$$\therefore \Phi_1 = \frac{1}{\sqrt{2\pi m k_B}} \frac{P_1}{\sqrt{T_1}}, \quad \Phi_2 = \frac{1}{\sqrt{2\pi m k_B}} \frac{P_2}{\sqrt{T_2}}$$

$$\Phi_1 = \Phi_2 \Rightarrow \underline{\frac{P_1}{\sqrt{T_1}} = \frac{P_2}{\sqrt{T_2}}}$$

$$\frac{P_1}{\sqrt{T_1}} = \frac{N_1' k_B \sqrt{T_1}}{V} \quad , \quad \frac{P_2}{\sqrt{T_2}} = \frac{N_2' k_B \sqrt{T_2}}{V}$$

$$\frac{P_1}{\sqrt{T_1}} = \frac{P_2}{\sqrt{T_2}} \Rightarrow N_1'\sqrt{T_1} = N_2'\sqrt{T_2}$$

$$\therefore N_1' = \frac{\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}} N \Rightarrow M_1' = \frac{\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}} M$$

$$-|\Delta M| = M_1' - M_1 = \left(\frac{\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}} - \frac{T_2}{T_1+T_2}\right) M$$

$$= \frac{T_2\sqrt{T_2} + T_1\sqrt{T_2} - T_2\sqrt{T_2} - T_2\sqrt{T_1}}{(T_1+T_2)(\sqrt{T_1}+\sqrt{T_2})} M$$

$$= \boxed{\frac{\sqrt{T_1T_2}}{T_1+T_2} \cdot \frac{\sqrt{T_1}-\sqrt{T_2}}{\sqrt{T_1}+\sqrt{T_2}} \cdot M} < 0$$

a) $\because T_2 > T_1$ $\therefore$ mass of chamber 1 decreases ~~increases~~

$\therefore$ Gas flow : $\boxed{1 \to 2}$

b) $\boxed{\Delta M = \frac{\sqrt{T_1T_2}}{T_1+T_2} \frac{\sqrt{T_2}-\sqrt{T_1}}{\sqrt{T_1}+\sqrt{T_2}} M}$ as proven above