To: Robin Nicholas

Statistical Physics 3 Ziyan Li

1.

∵ $\theta$ is uniformly distributed ∴ $P(\theta) =$ constant.

By definition $\int_0^\pi P(\theta)\,d\theta = 1 \Rightarrow \pi P(\theta) = 1$

$$\Rightarrow \boxed{P(\theta) = \frac{1}{\pi}}$$

(i) $\langle\theta\rangle = \int_0^\pi \theta P(\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \theta\,d\theta = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \boxed{\frac{\pi}{2}}$

(ii) $\langle\theta - \frac{\pi}{2}\rangle = \langle\theta\rangle - \frac{\pi}{2} = \boxed{0}$

(iii) $\langle\theta^2\rangle = \int_0^\pi \theta^2 P(\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \theta^2\,d\theta = \frac{1}{\pi}\left(\frac{\pi^3}{3}\right) = \boxed{\frac{\pi^2}{3}}$

(iv) $\langle\theta^n\rangle = \int_0^\pi \theta^n P(\theta)\,d\theta = \frac{1}{\pi}\int_0^\pi \theta^n\,d\theta = \frac{1}{\pi}\left(\frac{\pi^{n+1}}{n+1}\right) = \boxed{\frac{\pi^n}{n+1}}$

(v) $\langle\cos\theta\rangle = \int_0^\pi P(\theta)\cos\theta\,d\theta = \frac{1}{\pi}\int_0^\pi \cos\theta\,d\theta = \frac{1}{\pi}\left[+\sin(\theta)\right]_0^\pi = \boxed{0}$

(vi) $\langle\sin\theta\rangle = \int_0^\pi P(\theta)\sin\theta\,d\theta = \frac{1}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{1}{\pi}(-\cos\theta)_0^\pi = \boxed{\frac{2}{\pi}}$

(vii) $\langle|\cos\theta|\rangle = \int_0^\pi P(\theta)|\cos\theta|\,d\theta = \frac{1}{\pi}\int_0^{\pi/2} \cos\theta + \frac{1}{\pi}\int_{\pi/2}^{\pi} -\cos\theta\,d\theta.$

$= \frac{1}{\pi} + \frac{1}{\pi} = \boxed{\frac{2}{\pi}}.$

(viii) $\langle\cos^2\theta\rangle = \frac{1}{\pi}\int_0^\pi \cos^2\theta\,d\theta = \boxed{\frac{1}{2}}.$

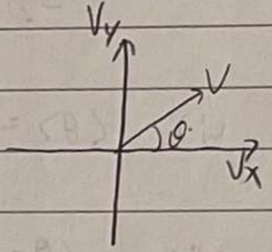
(ix) $\langle\sin^2\theta\rangle = \frac{1}{\pi}\int_0^\pi \sin^2\theta\,d\theta = \boxed{\frac{1}{2}}$

(x) $\langle\cos^2\theta + \sin^2\theta\rangle = \langle 1\rangle = \boxed{1}$

b) Iso-tropic distribution of velocity ($f(\vec{v}) = f(v)$)

doesn't ~~mean~~ necessarily mean the angle between the velocity vector and a fixed axis is distributed uniformly.

For 2-D:



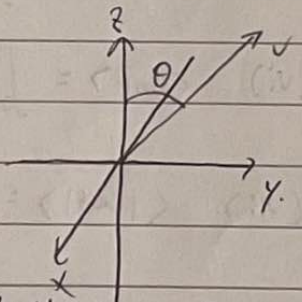
$$f(\vec{v}) = f(v_x, v_y) = f(v)$$

$$f(v)\,dv_x\,dv_y = f(v)\left|\frac{\partial(v_x, v_y)}{\partial(v,\theta)}\right| dv\,d\theta$$

$$= v f(v)\,dv\,d\theta$$

∴ The distribution of $\theta$ is uniform.

For 3-D:



$$f(\vec{v}) = f(v_x, v_y, v_z) = f(v)$$

$$f(v)\,dv_x\,dv_y\,dv_z = f(v)\left|\frac{\partial(v_x, v_y, v_z)}{\partial(v,\theta,\phi)}\right| dv\,d\theta\,d\phi$$

$$= f(v)\,v^2 \sin\theta\,dv\,d\theta\,d\phi = f(v)\,v^2\,dv\,d\Omega$$

∴ Because of the $\sin\theta$ term

The distribution of $\theta$ is not uniform.

The solid angle $\Omega$ is uniformly distributed

2. (a) Isotropic distribution of velocities $f(\vec{v}) = f(v)$

Speed distribution $\tilde{f}(v)$ is given by.

~~$\tilde{f}(v) =$~~ $\tilde{f}(v)\,dv = f(v)\,dv_x\,dv_y\,dv_z$

$=$ ~~$f(v)\,v^2\sin\theta\,dv\,d\phi\,d\theta$~~ $f(v)\,v^2\sin\theta\,dv\,d\theta\,d\phi$

$$= dv\,v^2 f(v) \underbrace{\int_0^{2\pi}\int_0^{\pi} \sin\theta\,d\theta\,d\phi}_{\int d\Omega = 4\pi} = 4\pi v^2 f(v)\,dv$$

$$\therefore \tilde{f}(v) = 4\pi v^2 f(v)$$

(b) $f(\vec{v})$ isotropic $\Rightarrow$ $f$ is even $\Rightarrow f(\vec{v}) = f(-\vec{v})$ ①

~~$f(\vec{v})$ isot~~ $v_x, v_y, v_z$ independent $\Rightarrow f(v) = g(v_x)\,g'(v_y)\,g''(v_z)$

$f(v)$ isotropic $\Rightarrow g = g' = g'' \Rightarrow f(v) = g(v_x)\,g(v_y)\,g(v_z)$ ②

① $\Rightarrow$ ~~$g(v_x) = g(-v_x$~~ $g(v_i) = g(-v_i)$ ($g$ is even)

(i) $$\langle v_i \rangle = \int_{-\infty}^{\infty} v_i\,g(v_i)\,dv_i = \int_{-\infty}^{0} v_i\,g(v_i)\,dv_i + \int_0^{\infty} v_i\,g(v_i)\,dv_i$$

$$= \underbrace{\int_{\infty}^{0}}_{v_i' = -v_i} -v_i' \underbrace{g(-v_i')}_{= g(v_i')} (-dv_i') + \int_0^{\infty} v_i\,g(v_i)\,dv_i$$

$$= -\int_0^{\infty} v_i'\,g(v_i')\,dv_i' + \int_0^{\infty} v_i\,g(v_i)\,dv_i = \boxed{0}$$

~~$\langle |V_i| \rangle = 2\int_0^\infty V_i\, g(V_i)\, dV_i = 2\int_0^\infty V_i\, g(V_i)\, dV_i$~~

~~$\langle v \rangle = \int_0^\infty v \tilde{f}(v)\, dv$~~

~~$= \int_0^\infty$~~

$$\langle |V_i| \rangle = \int_{-\infty}^{\infty} |V_i|\, g(V_i)\, dV_i = 2\int_0^\infty V_i\, g(V_i)\, dV_i$$

$$= 2\int_0^\infty V_i \underbrace{\int_{-\infty}^{\infty} dV_j \int_{-\infty}^{\infty} dV_k\, f(V_i, V_j, V_k)}_{g(V_i)}\, dV_i$$

$$= 2\int_0^\infty V_i\, dV_i \int_{-\infty}^{\infty} dV_j \int_{-\infty}^{\infty} \cancel{f(V)}\, dV_k\, f(V)$$

$$= 2\int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \int_0^\infty V_i \cdot V^2 \sin\theta\, dv$$

$$= 4\pi \int_0^{\pi/2} d\theta \int_0^\infty dv\, (v\cos\theta)\, v^2 \sin\theta\, \cancel{dv}\, f(v)$$

$$= 4\pi \underbrace{\int_0^{\pi/2} d\theta \sin\theta\cos\theta}_{\frac{1}{2}} \int_0^\infty (dv \cdot \underbrace{v^2 f(v)}_{\frac{1}{4\pi}\tilde{f}(v)})\, v$$

$$= \frac{1}{2}\int_0^\infty dv \cdot v \cdot \tilde{f}(v) = \boxed{\frac{1}{2}\langle v \rangle}$$

(iii)

$$\langle v_i^2 \rangle = \int_{-\infty}^{\infty} v_i^2 \, g(v_i) \, dv_i$$

$$= \int_{-\infty}^{\infty} dv_i \, v_i^2 \int_{-\infty}^{\infty} dv_j \underbrace{\int_{-\infty}^{\infty} dv_k \, f(v)}_{g(v_i)}$$

$$= \int_{-\infty}^{\infty} dv_i \int_{-\infty}^{\infty} dv_j \int_{-\infty}^{\infty} dv_k \, f(v) \, v_i^2$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dv \, f(v) \, v^2 \sin\theta \, v_i^2$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{\infty} dv \, f(v) \, v^2 \sin\theta \, v^2 \cos^2\theta$$

$$= (2\pi) \Big( \underbrace{\int_0^{\pi} d\theta \sin\theta \cos^2\theta}_{\frac{2}{3}} \Big) \Big( \int_0^{\infty} dv \underbrace{f(v)}_{\frac{1}{4\pi}\tilde{f}(v)} v^2 \cdot v^2 \Big)$$

$$= \frac{1}{3} \int_0^{\infty} dv \, \tilde{f}(v) \cdot v^2 = \boxed{\frac{1}{3} \langle v^2 \rangle}$$

OR. Isotropic $\Rightarrow$ $\langle v_i^2 \rangle + \langle v_j^2 \rangle + \langle v_k^2 \rangle = 3 \langle v_i^2 \rangle = \langle v^2 \rangle$

$$\therefore \boxed{\langle v_i^2 \rangle = \frac{1}{3} \langle v^2 \rangle}$$

(iv) For 3-D: If $i \neq j$

$$\langle v_i v_j \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i v_j f(v)\, dv_i\, dv_i\, dv_k$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v_i v_j f(v)\, v^2 \sin\theta\, dv\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} (v\sin\theta\sin\phi)(v\sin\theta\cos\phi)\, f(v)\, v^2 \sin\theta\, d\theta\, d\phi$$

$$= \underbrace{\int_0^{2\pi} d\phi \sin\phi\cos\phi}_{0} \times \text{ rest of integral.}$$

$$= 0$$

If $i = j$

$$\langle v_i v_j \rangle = \langle v_i^2 \rangle = \frac{1}{3}\langle v^2 \rangle$$

$$\therefore \boxed{\langle v_i v_j \rangle = \frac{1}{3}\langle v^2 \rangle \delta_{ij}}$$

For 2-D: if $i \neq j$

$$\langle v_i v_j \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i v_j f(v)\, dv_i\, dv_j$$

$$= \int_0^{2\pi}\int_0^{\infty} v\sin\theta\, v\cos\theta\, f(v)\, v\, dv\, d\theta.$$

$$= \underbrace{\int_0^{2\pi} d\theta \sin\theta\cos\theta}_{0} \times \text{ rest of integral} = 0.$$

If $i = j$:

$$\langle v_i^2 \rangle + \langle v_i^2 \rangle = \langle v^2 \rangle = 2\langle v_i^2 \rangle \quad \therefore \langle v_i^2 \rangle = \frac{1}{2}\langle v \rangle$$

$$\therefore \langle v_i v_j \rangle = \frac{1}{2}\langle v \rangle \delta_{ij}$$

Overall $\langle v_i v_j \rangle = \frac{1}{n}\langle v \rangle \delta_{ij}$ (n is dimension of space)

(V) if $i, j, k$ be a permutation of $x, y, z$

~~then~~ $(i \neq j \neq k)$

$$\langle V_i V_j V_k \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i v_j v_k f(v)\, dv_i dv_j dv_k .$$

$$= \cancel{\int_{-\infty}^{\infty} v_i g(v_i} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i v_j v_k g(v_i) g(v_j) g(v_k)\, dv_i dv_j dv_k$$

$$= \underbrace{\int_{-\infty}^{\infty} v_i g(v_i)\, dv_i}_{0} \underbrace{\int_{-\infty}^{\infty} v_j g(v_j)\, dv_j}_{0} \underbrace{\int_{-\infty}^{\infty} v_k g(v_k)\, dv_k}_{0}$$

$$= 0$$

if $i = j \neq k$

$$\langle v_i v_j v_k \rangle = \langle v_i^2 v_k \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i^2 v_k f(v)\, dv_i dv_j dv_k .$$

$$= \cancel{\int_{-\infty}^{\infty}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v^2 \cos^2\theta \, v \sin\theta \sin\phi \; v^2 \sin\theta f(v)\, dv\, d\theta\, d\phi$$

$$= \underbrace{\int_0^{2\pi} d\phi \sin\phi}_{0} \times \text{rest of integral} = 0 .$$

if $i = j = k$

$$\langle v_i v_i v_k \rangle = \langle v_i^3 \rangle = \int_{-\infty}^{\infty} v_i^3 g(v_i)\, dv_i = (v_i \text{ is odd})$$

$$= 0$$

∴ ~~Over~~ Overall : $\langle v_i v_j v_k \rangle = \boxed{0}$

c) $\langle v_i v_j v_k v_l \rangle$ is a symmetric rotationally invariant rank-4 tensor

$$\therefore \langle v_i v_j v_k v_l \rangle = C_1 \delta_{ij}\delta_{kl} + C_2 \delta_{il}\delta_{jk} + C_3 \delta_{ik}\delta_{jl}.$$

For $i=j=k=l$

In -3D:

$$\langle v_i v_i v_k v_l \rangle = \langle v_i^4 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i^4 f(v)\, dv_i dv_j dv_k$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} v^4\cos^4\theta\; v^2\sin\theta\, f(v)\, dv\, d\theta\, d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \cos^4\theta \sin\theta\, d\theta \int_0^{\infty} v^4 \frac{1}{4\pi}\hat{f}(v)\, dv.$$

$$= (2\pi)\left(\frac{1}{4\pi}\right)\left(\frac{2}{5}\right)\langle v^4\rangle$$

$$= \frac{1}{5}\langle v^4\rangle. = (C_1 + C_2 + C_3)\langle v^4\rangle$$

By symmetry $C_1 = C_2 = C_3 = \frac{1}{15}$

$$\therefore \boxed{\langle v_i v_j v_k v_l \rangle = \frac{1}{15}\langle v^4\rangle(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl})}$$

In 2-D: when $i=j=k=l$

$$\langle v_i v_j v_k v_l \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i^4 f(v)\, dv_i dv_j.$$

$$= \int_0^{2\pi}\int_0^{\infty} v^4\cos^4\theta\, f(v)\, v\, dv\, d\theta$$

The ~~vel~~ speed distribution in 2-D is $\tilde{f}(v)$.

then $\hat{f}(v)dv = \int_0^{2\pi}$ ~~$d\theta \int v\,dv\, f(v)$~~ $\left(\int_0^{2\pi} d\theta\right) v\,dv\, f(v)$

$= 2\pi v\, f(v)$

$\therefore$ $\therefore v f(v) = \frac{1}{2\pi}\tilde{f}(v)$

$$\therefore \quad \langle v_i^{4} \rangle = \int_0^{2\pi} d\theta \int_0^{\infty} v^{4} \cos^{4}\theta \ \frac{1}{2\pi} \overset{2}{f}(v)\, dv \cancel{d\theta}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \cos^{4}\theta \quad \int_0^{\infty} v^{4} \overset{2}{f}(v)\, dv.$$

$$= \left(\frac{1}{2\pi}\right) \left(\frac{3}{4}\pi\right) \langle v^{4} \rangle.$$

$$= \frac{3}{8} \langle v^{4} \rangle \quad = (c_1 + c_2 + c_3) \langle v^{4} \rangle.$$

$\therefore$ in-2D $\quad ( c_1 = c_2 = c_3 = \frac{1}{8} ).$

$$\langle v_i v_j v_k v_l \rangle = \boxed{\frac{1}{8} \langle v^{4} \rangle \left( \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right)}$$

3.

$$f(\vec{v})\,d^3\vec{v} \;\alpha\; e^{-v^2/v_{th}^2}\,d^3\vec{V}$$

a) $$\int_0^\infty \tilde{f}(v)\,dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\,dv_x\,dv_y\,dv_z$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} f(v)\,v^2\sin\theta\,dv\,d\theta\,d\phi.$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^{\infty} v^2 f(v)\,dv$$

$$\therefore \quad \tilde{f}(v) \;\alpha\; v^2 f(v)$$

$$\therefore \quad \tilde{f}(v) \;\alpha\; v^2 e^{-v^2/v_{th}^2}$$

let $\tilde{f}(v) = Ae^{-v^2/v_{th}^2}\,v^2$

Normalise : $$\int_0^\infty \tilde{f}(v)\,dv = 1$$

$$\therefore \quad A\int_0^\infty v^2 e^{-\frac{v^2}{v_{th}^2}}\,dv = 1$$

f(v) is even $\Rightarrow$ $$\frac{A}{2}\int_{-\infty}^{\infty} v^2 e^{-v^2/v_{th}^2}\,dv = 1.$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}$$

Sub in $\alpha = \frac{1}{v_{th}^2}$ $\Rightarrow$ $$\int_{-\infty}^{\infty} v^2 e^{-v^2/v_{th}^2}\,dv = \frac{1}{2}\sqrt{\frac{\pi}{\left(\frac{1}{v_{th}^2}\right)^3}}$$

$$= \cancel{\frac{1}{2}\sqrt{\pi v_{th}}} \;\; \frac{1}{2}\sqrt{\pi v_{th}^6}$$

$$\cancel{\therefore \frac{A}{4}\sqrt{\pi V_{th}^{3/2}} = 1}$$

$$\cancel{\therefore \quad A = \frac{4}{\sqrt{\pi V_{th}^{3/2}}}}$$

$$\frac{A}{4}\sqrt{\pi V_{th}^{6}} = 1$$

$$\therefore \quad A = \frac{4}{\sqrt{\pi}\, V_{th}^{3}} = \frac{4\pi}{(\sqrt{\pi})^3 V_{th}^3} = \frac{4\pi}{(\sqrt{\pi} V_{th})^3}$$

$$\therefore \quad \boxed{\tilde{f}(v) = \frac{4\pi v^2}{(\sqrt{\pi} V_{th})^3}\, e^{-v^2/V_{th}^2}.}$$

$$\langle v \rangle = \int_0^\infty v \tilde{f}(v)\, dv$$

$$= A\int_0^\infty v^3 e^{-v^2/V_{th}^2}\, dv$$

$$= A\left(\frac{1}{2\left(\frac{1}{V_{th}^2}\right)^2}\right) = \frac{A}{2} V_{th}^4$$

$$= \frac{1}{2}\frac{4\pi}{(\sqrt{\pi} V_{th})^3} V_{th}^4 = \cancel{4\sqrt{\pi} V_{th}}\ \frac{1}{2}\frac{4}{\sqrt{\pi}} V_{th}. = \boxed{\frac{2}{\sqrt{\pi}} V_{th}}$$

$$\left\langle \frac{1}{v} \right\rangle = A\int_0^\infty \frac{1}{v}\tilde{f}(v)\, dv = A\int_0^\infty v e^{-v^2/V_{th}^2}\, dv$$

$$= A\left(\frac{1}{2\left(\frac{1}{V_{th}^2}\right)}\right) = \frac{A}{2} V_{th}^2 = \frac{1}{2}\frac{4\pi}{(\sqrt{\pi} V_{th})^3} V_{th}^2 = \boxed{\frac{2}{\sqrt{\pi}}\frac{1}{V_{th}}}$$

$$\therefore \langle v \rangle \langle 1/v \rangle = \left(\frac{2}{\sqrt{\pi}}\right)\left(\frac{2}{\sqrt{\pi}}\right)(\cancel{V_{th}})\left(\frac{1}{\cancel{V_{th}}}\right) = \boxed{\frac{4}{\pi}.}$$

b)

$$\langle v^2 \rangle = A\int_0^\infty dv\, v^4 e^{-v^2/v_{th}^2} = A\left(\frac{3}{8}\right)\left(\sqrt{\pi v_{th}^{10}}\right)$$

$$= \cancel{A\frac{3}{8}\sqrt{\pi^{9}}} \quad = A\sqrt{\pi}\left(\frac{3}{8}\right)v_{th}^5$$

$$= \left(\frac{3}{8}\right)\frac{4\pi\sqrt{\pi}}{\pi\sqrt{\pi}\,v_{th}^3}\,v_{th}^5 = \cancel{4v_{th}^2}\ \boxed{\frac{3}{2}v_{th}^2}$$

$$\langle v^3 \rangle = A\int_0^\infty dv\, v^5 e^{-v^2/v_{th}^2} = A\left(v_{th}^6\right).$$

$$= \boxed{\frac{4}{\sqrt{\pi}}v_{th}^3}$$

$$\langle v^4 \rangle = A\int_0^\infty v^6 e^{-v^2/v_{th}^2} = A\frac{6!}{3!\,2^7}\sqrt{\pi v_{th}^{14}} = \frac{15(\cdot)}{16}\sqrt{\pi}\,v_{th}^7$$

$$= \left(\frac{15}{16}\right)(4)\,v_{th}^4 = \boxed{\frac{15}{4}v_{th}^4}$$

$$\langle v^5 \rangle = A\int_0^\infty v^7 e^{-v^2/v_{th}^2} = A\frac{3!}{2}v_{th}^8 = \boxed{\frac{12}{\sqrt{\pi}}v_{th}^5}$$

c) ~~&~~ for odd n. let $n = 2m+1$

$$\langle v^n \rangle = \langle v^{2m+1} \rangle = A\int_0^\infty v^{2m+3} e^{-v^2/v_{th}^2}\, dv.$$

$$= \cancel{A\frac{m!}{2}v_{th}^{m+1}} = A\frac{(m+1)!}{2}v_{th}^{2m+4}$$

$$= \cancel{\frac{2}{\sqrt{\pi}}\left(\frac{n-1}{2}\right)!\,v_{th}^{n}}$$

$$= \boxed{\frac{2}{\sqrt{\pi}}\left(\frac{n+1}{2}\right)!\,v_{th}^{n}}$$

for even $n$, let $n=2m$

~~$\langle v^n \rangle = \langle v^{2m} \rangle = \frac{(2m)!}{m!\, 2^{2m+1}} A\sqrt{\pi}$ [illegible] $(\sqrt{V_{th}}^{4m+2})$.~~

~~$= \frac{n!\sqrt{\pi}}{(\frac{n}{2})!\, 2^{n+1}} A\, V_{th}^{n+1}$~~

$$\langle v^n \rangle = \langle v^{2m} \rangle = \int_{-\infty}^{\infty} v^{2m+2} e^{-v^2/V_{th}^2} dv.$$

$$= \frac{(2m+2)!}{(m+1)!\, 2^{2m+3}} A\sqrt{\pi} \sqrt{V_{th}}^{4m+6}$$

$$= \frac{(n+2)!}{(\frac{n+2}{2})!\, 2^{n+3}} \underset{\overset{\uparrow}{2^2}}{(4)} V_{th}^{2m+3} \frac{1}{V_{th}^3}$$

$$= \boxed{\frac{(n+2)!}{(\frac{n}{2}+1)!\, 2^{n+1}} V_{th}^{n}}$$

$$\langle v^{27} \rangle^{1/27} = \left(\frac{2}{\sqrt{\pi}} \left(\frac{27+1}{2}\right)!\right)^{1/27} V_{th} = 2.55\, V_{th}.$$

$$\langle v^{56} \rangle^{1/56} = \left(\frac{(56+2)!}{(\frac{56}{2}+1)!\, 2^{56+1}}\right)^{\frac{1}{56}} V_{th}.$$

$$= 3.47\, V_{th}.$$

$$\therefore \langle v^{56} \rangle^{1/56} > \langle v^{27} \rangle^{1/27}$$

Do I understand why that is qualitatively?

No I don't!

d) distribution of speeds $\tilde{f}(v)$ in

n-D world. ?

$$f(\vec{v})\, d^n\vec{v} \propto e^{-v^2/v_{th}^2}\, d^n\vec{v}.$$

$$\int_0^{\infty} \tilde{f}(\vec{v})\, dv \propto \int_{-\infty}^{\infty} dv_1 \int_{-\infty}^{\infty} dv_2 \cdots \int_{-\infty}^{\infty} dv_n\, e^{-v^2/v_{th}^2}$$

$$dv_1 dv_2 \cdots dv_n = v^{n-1} \sin^{n-2}(\phi_1) \sin^{n-3}(\phi_2) \cdots \sin(\phi_{n-2})\, dv\, d\phi_1 d\phi_2 \cdots d\phi_{n-1}$$

$$\therefore \int_0^{\infty} \tilde{f}(\vec{v})\, dv \propto \int_0^{\infty} dv\, f(\vec{v})\, v^{n-1} \int_0^{2\pi} d\phi_{n-1} \int_0^{\pi} d\phi_{n-2} \cdots \int_0^{\pi} d\phi_1$$

$$\propto \int_0^{\infty} dv\, f(v)\, v^{n-1}$$

$$\therefore \tilde{f}(v) \propto v^{n-1} f(v)$$

$$\therefore \tilde{f}(v) \propto v^{n-1} e^{-v^2/v_{th}^2}$$ let $\tilde{f}(v) = A v^{n-1} e^{-v^2/v_{th}^2}$

$$A \int_0^{\infty} v^{n-1} e^{-v^2/v_{th}^2}\, dv = 1$$

For even $n$

$$A\left(\frac{(\frac{n}{2}-1)!}{2}\right)\left(V_{th}^{\,n}\right) = 1.$$

$$\therefore \quad A = \frac{2V_{th}^{-n}}{(\frac{n}{2}-1)!}$$

For odd $n$

$$A\left(\frac{(n-1)!}{(\frac{n-1}{2})!\,2^n}\right)\sqrt{\pi}\cdot V_{th}^{\,n} = 1$$

$\therefore$ ~~$A = 2V_{th}^{-n}$~~ $\quad A = \dfrac{(\frac{n-1}{2})!\,2^n}{(n-1)!\,\sqrt{\pi}}\,V_{th}^{-n}.$

$$\therefore \quad \tilde{f}(v) = \begin{cases} \dfrac{2v^{n-1}}{(\frac{n}{2}-1)!\,V_{th}^{\,n}}\,e^{-v^2/v_{th}^2} & \text{for even } n. \\[2ex] \dfrac{(\frac{n-1}{2})!\,2^n}{(n-1)!\,\sqrt{\pi}}\,v^{n-1}e^{-v^2/v_{th}^2} & \text{for odd } n. \end{cases}$$

4. a) The system is only isotropic in the x and y direction $\Rightarrow$ $f(v_x, v_y, v_z) = \boxed{f(v_{xy}, v_z)}$

where $v_{xy}^2 = v_x^2 + v_y^2$

Using cylindrical coordinates in velocity space

$$f(v_{xy}, v_z)\, dv_x\, dv_y\, dv_z = f(v_{xy}, v_z)\, v_{xy}\, dv_{xy}\, dv_z\, d\phi$$

b) $$P_{\parallel} = mn \int d^3\vec{v}\; v_z^2 f(v_{xy}, v_z) = \boxed{mn \langle v_z^2 \rangle}$$

$$P_{\perp} = mn \int d^3\vec{v}\; v_x^2 f(v_{xy}, v_z) = mn \langle v_x^2 \rangle = \boxed{\frac{1}{2} mn \langle v_{xy}^2 \rangle}$$

c) ~~$P = P_{\parallel} \hat{V_z} \cdot \hat{n} + P_{\perp} \hat{V_{xy}} \cdot \hat{n}$~~

~~$= P_{\parallel} \cos\theta + P_{\perp} \sin\theta$~~

$$P = mn \int d^3\vec{v}\, (\vec{v} \cdot \hat{n})^2 f(v_{xy}, v_z).$$

$= mn \langle (\vec{v} \cdot \hat{n})^2 \rangle =$ ~~$mn \langle (\vec{v}_{xy} \cdot \hat{n} + \vec{v}_z \cdot \hat{n})^2 \rangle$~~

~~$= mn \langle (v_{xy} \sin\theta + v_z \cos\theta)^2 \rangle$~~

~~$= mn \langle v_z^2 \rangle \cos^2\theta + mn \langle v_{xy}$~~

$= mn \langle (\hat{v_x} \cos\phi \cdot \hat{n} + \hat{v_y} \sin\phi \cdot \hat{n} + \hat{v_z} \cdot \hat{n})^2 \rangle.$

$$= mn \left\langle \frac{1}{2}(\vec{V_x}\cdot\vec{n})^2 + \frac{1}{2}(\vec{V_y}\cdot\vec{n})^2 + (\vec{V_z}\cdot\vec{n})^2 \right\rangle$$ + cross terms

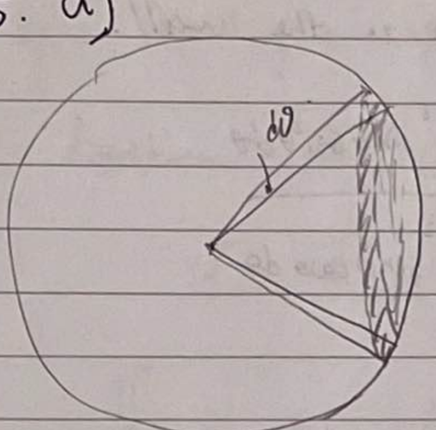
which are zero

$$= mn \left\langle (\vec{V_x}\cdot\hat{n})^2 + (\vec{V_z}\cdot\hat{n})^2 \right\rangle.$$

$$= mn\langle V_z^2\rangle \cos^2\theta + mn\langle V_x^2\rangle \sin^2\theta.$$

$$= \boxed{P_\perp \cos^2\theta + P_{\parallel} \sin^2\theta.}$$

5. a)



# of particle of speed between $v, v+dv$ ~~is~~ per volume is.

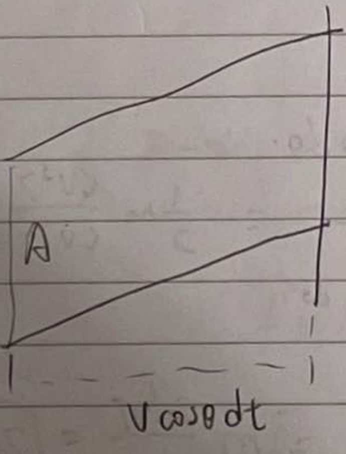
$n\tilde{f}(v)\,dv$

The fraction of particle that has anges between $\theta, \theta+d\theta$ is

$$\frac{d\Omega}{\int d\Omega} = \frac{d\Omega}{4\pi} = \cancel{2\pi\sin\theta}\ \frac{\int_{\phi=0}^{\phi=2\pi} \sin\theta\, d\theta\, d\phi}{4\pi} = \frac{1}{2}\sin\theta\, d\theta$$

∴ # of particles/volume of speed $[v, v+dv]$ and angles $[\theta, \theta+d\theta]$ is

$$\frac{1}{2} n\tilde{f}(v)\, dv \sin\theta\, d\theta$$



In time $dt$, the volume of particles hitting the area A ~~p~~ is

$$A v\cos\theta\, dt$$

∴ # of particle hitting the wall is

$$\frac{1}{2} n v \tilde{f}(v)\, dv \sin\theta\cos\theta\, d\theta\, A\, dt.$$

∴ # of molecules hitting ~~[illegible]~~ per unit Area per time is

$$\boxed{\frac{1}{2} n v \tilde{f}(v)\, dv \sin\theta\cos\theta\, d\theta.}$$

b). For particles ~~flow~~ flying to the wall.

$$\langle \cos\theta \rangle = \frac{\int_0^\infty \cancel{\frac{1}{2}nv}^3 f(v)\,dv \int_0^{\pi/2} \sin\theta\cos^2\theta\,d\theta}{\int_0^\infty \cancel{\frac{1}{2}nv}^2 f(v)\,dv \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta}$$

$$= \frac{1/3}{1/2} = \boxed{\frac{2}{3}}.$$

c) Average energy maxwellian

$$U = \frac{1}{2}m\langle v^2\rangle = \frac{1}{2}m\,\frac{3}{2}v_{th}^2 = \frac{1}{2}m\,\frac{3}{2}\cdot\frac{2k_BT}{m}$$

$$= \boxed{\frac{3}{2}k_BT.}$$

Average energy hitting = ~~# of molecule~~ $\frac{\text{Energy of particles hitting}}{\text{\# of particles hitting}}$

$$= \frac{\text{energy hitting} / A / dt}{\text{\# of particles hitting} / A / dt}$$

$$= \frac{\frac{1}{2}n\int_0^\infty \frac{1}{2}mv^3\tilde{f}(v)\,dv \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}{\frac{1}{2}n\int_0^\infty v\tilde{f}(v)\,dv\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta} = \frac{1}{2}m\frac{\langle v^3\rangle}{\langle v\rangle}$$

$$= \frac{1}{2}m\,\frac{4/\sqrt{\pi}}{2/\sqrt{\pi}}\,\frac{v_{th}^3}{v_{th}} = mv_{th}^2 = m\,\frac{2k_BT}{m} = \boxed{2k_BT}$$

6. According to previous ~~the~~ question the velocity distribution for effusing particles is

$$\tilde{f}(v) \propto v^3 e^{-v^2/v_{th}^2}.$$

$\frac{d\tilde{f}}{dv} = 0$ at $v = v_1$

$$\Rightarrow \quad 3v_1^2 e^{-v_1^2/v_{th}^2} + \left(\frac{1}{v_{th}^2}\right) v_1^3 (-2v_1) e^{-v^2/v_{th}^2}.$$

$$\Rightarrow \quad 3v_1^2 = 2\frac{v_1^4}{v_{th}^2}$$

$$\Rightarrow \quad 3v_{th}^2 = 2v_1^2 \quad \Rightarrow \quad v_1 = \sqrt{\frac{3v_{th}^2}{2}} = \sqrt{\frac{3}{2}}\, v_{th}.$$

$$\Rightarrow \quad v_1 = \sqrt{\frac{3}{2}}\sqrt{\frac{2k_BT}{m}} = \boxed{\sqrt{\frac{3k_BT}{m}}}$$

Particles between the hole and the screen is a maxwellian so.

$$\hat{f}(v) \propto v^2 e^{-v^2/v_{th}^2}.$$

$\frac{d\tilde{f}}{dv} = 0$ at $v = v_2$.

$$\therefore \quad 2v_2 e^{-v_2^4/v_{th}^2} - \frac{2v_2^3}{v_{th}^2} e^{-v_2^2/v_{th}^2} = 0$$

$$\therefore \quad 2v_{th}^2 = 2v_2^2 \quad \Rightarrow \quad v_2 = \sqrt{v_{th}^2}$$

$$\therefore v_2 = v_{th} = \boxed{\sqrt{\frac{2k_BT}{m}}}$$

They are different because collision with the wall favours ~~more~~ faster molecules.

b) $$\langle v \rangle = \frac{\int_0^\infty v^4 e^{-v^2/v_{th}^2} dv}{\int_0^\infty v^3 e^{-v^2/v_{th}^2} dv}$$

$$= \frac{\sqrt{\pi}\left(\frac{3}{8}\right) v_{th}^5}{\frac{1}{2} v_{th}^4} = \boxed{\frac{3}{4}\sqrt{\pi}\, v_{th}} \approx 1.33 v_{th}.$$

$$v_1 = \sqrt{\frac{3}{2}} v_{th} = 1.22 v_{th}$$

$$\therefore \langle v \rangle > v_1$$

The distribution $\hat{f}(v)$ is right skewed so the average is larger than the ~~p~~ value at which peak occurs.

7. Average kinetic energy is

$$u = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \frac{3}{2} v_{th}^2 = \frac{1}{2} \cdot m \cdot \frac{3}{2} \cdot \frac{2k_BT}{m}$$

$$= \boxed{\frac{3}{2} k_B T.}$$

When the time of ~~of~~ opening the hole is short the mean kinetic energy ~~effus~~ of effused particle is (given by question 5)

$$2k_BT.$$

Average kinetic energy ~~is~~ trapped in ~~the~~ the box is $\frac{3}{2}k_BT'$ (maxwellian, since you've waited a little bit).

$$\therefore \frac{3}{2}k_BT' = 2k_BT$$

$$\therefore \boxed{T' = \frac{4}{3}T.}$$ is the box's final temperature.

8. a)

$$d\Phi = \frac{1}{2} n v \hat{f}(v) dv \sin\theta \cos\theta \, d\theta \quad , \theta$$

$$\int_{\text{hole}} d\Phi = \frac{1}{2} n \int v \hat{f}(v) dv \int_0^{a/d} \underset{\theta}{\underbrace{\sin\theta}} \underset{1.}{\underbrace{\cos\theta}} \, d\theta .$$

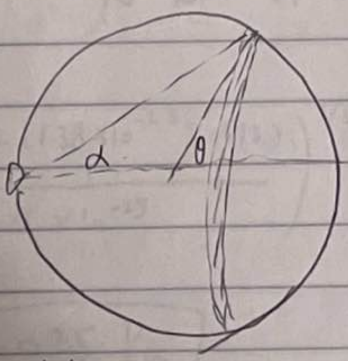
$$= \frac{1}{2} n \langle v \rangle \int_0^{a/d} \theta \, d\theta$$

$$\approx \frac{1}{4} n \langle v \rangle \int_0^{a/d} \theta \, d\theta$$

$$= \boxed{\frac{1}{4} n \langle v \rangle \left(\frac{a^2}{d^2}\right)}$$ per unit area will go through the hole

The rate is $\boxed{\frac{1}{4} n \langle v \rangle A \left(\frac{a^2}{d^2}\right)}$

b)



Integrate $d\phi$ to give $2\pi$ over a ring (for both $d\Phi$ and $d\Omega$)

$d\Phi$ in this ring is

$$d\Phi = v \cos\alpha \, n \hat{f}(v) dv \, \frac{1}{2} \sin\alpha \, d\alpha \quad \propto \cos\alpha \sin\alpha \, d\alpha \quad \propto \sin 2\alpha \, d\alpha$$

$$d\Omega \propto \sin\theta \, d\theta$$

$$\therefore \frac{d\Phi}{d\Omega} \propto \frac{\sin 2\alpha}{\sin\theta} \frac{d\alpha}{d\theta}$$

By Geometry $\theta = 2\alpha$ , $d\alpha = \frac{1}{2} d\theta$

$\therefore \frac{d\Phi}{d\Omega} \propto 1$ independent of $\theta$

↗

flux per solid angle

$\therefore$ The coating is uniform.

9. $$\Phi = \int_{v_z>0} d\Phi = \underbrace{\int_0^{\infty} \frac{1}{2} n v f(v)\, dv}_{\frac{1}{2} n \langle v \rangle} \int_0^{\pi/2} \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}}\, d\theta = \frac{1}{4} n \langle v \rangle$$

$$= \frac{1}{4} n \cdot \frac{2}{\sqrt{\pi}} v_{th} = \frac{n}{2\sqrt{\pi}} \sqrt{\frac{2k_BT}{m}} = \frac{nk_BT}{\sqrt{2\pi m k_B T}} = \frac{P}{\sqrt{2\pi m k_B T}}$$

$p = nk_BT.$

$$\therefore \Phi A = \frac{PA}{\sqrt{2\pi m k_B T}}$$

Rate of change of mass

$$\frac{\Delta m}{\Delta t} = -m \Phi A \leftarrow \text{area.}$$

mass of a molecule

\# of ~~p~~ molecules ~~to~~ effuse per area per time

$$\therefore \quad P = \sqrt{\frac{2\pi k_B T}{m}} \frac{1}{A} \left| \frac{\Delta m}{\Delta t} \right|$$

$$= \left( \frac{2\pi (1.38 \times 10^{-23})(273)}{3.33 \times 10^{-25}} \right)^{1/2} \left( \frac{2.4 \times 10^{-5}}{30 \times 24 \times 60 \times 60} \right) \left( \frac{1}{10^{-7}} \right).$$

$$= \boxed{0.025 \text{ Pa}}$$

10. $m_1 = m(H_2) = 2m$ $\qquad$ $m_2 = m(HD) = 3m$

$$\frac{dn_1}{dt} = -\frac{\Phi A}{V} = \frac{-PA}{V\sqrt{2\pi k_B T m_1}} = \frac{-n k_B T A}{V\sqrt{2\pi k_B T m_1}} = -\frac{1}{\sqrt{2}} c n_1$$

where $c = \frac{k_B T A}{V\sqrt{2\pi k_B T m}}$

Similary

$$\frac{dn_2}{dt} = -\frac{1}{\sqrt{3}} c n_2$$

$\therefore$ $n_1 = n_{10} e^{-\frac{1}{\sqrt{2}}ct}$, $n_2 = n_{20} e^{-\frac{1}{\sqrt{3}}ct}$

$\therefore$ $\frac{n_1}{n_2} = \frac{n_{10}}{n_{20}} e^{-(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}})ct}$ $\quad$ $\because$ $\frac{n_1}{n_2} = \frac{700}{1}$, $\frac{n_{10}}{n_{20}} = \frac{7000}{1}$

$\therefore$ $\frac{1}{10} = (e^{-ct})^{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}}$

$\therefore$ $e^{-ct} = (\frac{1}{10})^{\frac{1}{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}}} = 1.964526 \times 10^{-8}$

$n_1 = n_{10}(e^{-ct})^{\frac{1}{\sqrt{2}}} = 3.5523 \times 10^{-6} n_{10} = (3.55\times10^{-6})(7000) n_{20}$

$n_2 = n_{20}(e^{-ct})^{\frac{1}{\sqrt{3}}} = 3.5523 \times 10^{-5} n_{20}$.

Dalton's Law of Partial Pressure $\Rightarrow$

$$\frac{P}{P_0} = \frac{n_1+n_2}{n_{10}+n_{20}} = \frac{(3.5523\times10^{-6})(7000)+(3.5523\times10^{-5})}{7000+1}$$

$= \cancel{\boxed{3.6\times10^{-6}}}$ $\quad$ $\boxed{3.55\times10^{-6}}$

11. As in Question 10:

$$\frac{dn}{dt} = -\frac{\Phi A}{V} = -\frac{1}{4} n \langle v \rangle \frac{A}{V}$$

$$= -\frac{1}{4} n \frac{2}{\sqrt{\pi}} v_{th} \frac{A}{V} = -\frac{1}{4} n \frac{2}{\sqrt{\pi}} \sqrt{2} \sqrt{\frac{k_B}{m}} T^{1/2} \frac{A}{V}$$

$$= -\frac{\sqrt{2}}{2\sqrt{\pi}} \sqrt{\frac{k_B}{m}} \frac{A}{V} n T^{1/2} = -\left(\frac{2\sqrt{2}}{4} \sqrt{\frac{k_B}{m}} \frac{A}{V}\right) n T^{1/2}$$

As in Question 6:

Energy flux: $\Phi_E = \frac{1}{2} n \int_0^\infty \frac{1}{2} m v^2 \cdot v \hat{f}(v)\, dv \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$

$$= \frac{1}{2} n \cdot \frac{1}{2} m \langle v^3 \rangle \frac{1}{2}$$

$$= \frac{1}{8} n m \langle v^3 \rangle.$$

$$\therefore \cancel{E = \frac{3}{2}} \quad \frac{dE}{dt} = -\Phi_E A$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{3}{2} k_B V n T\right) = -\Phi_E A$$

$$\therefore \quad \frac{3}{2} k_B \frac{d(nT)}{dt} = -\frac{\Phi_E A}{V}$$

$$= -\frac{1}{8} n m \langle v^3 \rangle \frac{A}{V} = -\frac{1}{8} n m \frac{4}{\sqrt{\pi}} v_{th}^3 \frac{A}{V}$$

$$= -\frac{1}{8} n m \frac{4}{\sqrt{\pi}} \left(\frac{2 k_B T}{m}\right)^{3/2} \frac{A}{V}$$

$$= -\frac{1}{8} n \cancel{m} \frac{4}{\sqrt{\pi}} (2\sqrt{2}) \sqrt{\frac{k_B}{\cancel{m}}} \cancel{\frac{k_B}{m}} \frac{A}{V} T^{3/2}$$

$$= -\frac{1}{8} \frac{4}{\sqrt{\pi}} 2\sqrt{2} \sqrt{\frac{k_B}{m}} \frac{A}{V} n T^{3/2} \cdot = -\frac{\sqrt{2}}{\sqrt{\pi}} \sqrt{\frac{k_B}{m}} \frac{A}{V} n T^{3/2}$$

$$\therefore \quad \frac{d}{dt} nT = -\left(\frac{2}{3}\sqrt{2}\sqrt{\frac{k_B}{m}}\frac{A}{V}\right) nT^{3/2}$$

$\therefore$ we have $$\frac{dn}{dt} = -\frac{c}{4} nT^{1/2} \qquad ①$$

$$\frac{d}{dt}(nT) = -\frac{c}{3} nT^{3/2} \qquad ②$$

where $$\boxed{c = 2\sqrt{2}\sqrt{\frac{k_B}{m}}\frac{A}{V}}$$

$$② \Rightarrow \quad n\frac{dT}{dt} + T\frac{dn}{dt} = -\frac{c}{3} nT^{3/2}$$

$$\Rightarrow \quad \cancel{n}\frac{dT}{dt} - \frac{c}{4}\cancel{n}T^{3/2} = -\frac{c}{3}\cancel{n}T^{3/2}.$$

$$\therefore \quad \frac{dT}{dt} = -\frac{c}{12} T^{3/2}$$

$$\therefore \int_{T_0}^{T} \frac{dT}{T^{3/2}} = \int_0^t -\frac{c}{12}\, dt$$

$$-2T^{-1/2}\Big|_{T_0}^{T} = -\frac{c}{12}t\Big|_0^t$$

~~$-2(\sqrt{T} - \sqrt{T_0}) = -\frac{c}{12}t$~~

~~$\therefore \sqrt{T} - \sqrt{T_0} =$~~

$$-2\left(\frac{1}{\sqrt{T}} - \frac{1}{\sqrt{T_0}}\right) = -\frac{c}{12}t.$$

$$\therefore \frac{1}{\sqrt{T}} - \frac{1}{\sqrt{T_0}} = \frac{c}{24} t.$$

$$\therefore \frac{1}{\sqrt{T}} = \frac{c}{24} t + \frac{1}{\sqrt{T_0}}$$

$$\therefore T = \frac{1}{\left(\frac{c}{24} t + \frac{1}{\sqrt{T_0}}\right)^2}$$

$$\therefore T^{1/2} = \frac{1}{\frac{c}{24} t + \frac{1}{\sqrt{T_0}}}$$

$$\frac{dn}{dt} = -\frac{c}{4} n T^{1/2} = -\frac{c}{4} n \frac{1}{\frac{c}{24} t + \frac{1}{\sqrt{T_0}}}$$

$$\therefore \frac{dn}{n} = -\frac{c}{4} \frac{1}{\frac{c}{24} t + \frac{1}{\sqrt{T_0}}} dt.$$

$$\int_{n_0}^{n} \frac{dn}{n} = -\frac{c}{4} \int_0^t \frac{1}{\frac{c}{24} t + \frac{1}{\sqrt{T_0}}} dt$$

$$\therefore \ln \frac{n}{n_0} = -\frac{c}{4} \left[\frac{24}{c}\right] \ln \left(\frac{c}{24} t + \frac{1}{\sqrt{T_0}}\right) \Big|_0^t$$

$$= -6 \ln \left( \frac{\frac{c}{24} t + \frac{1}{\sqrt{T_0}}}{\frac{1}{\sqrt{T_0}}} \right)$$

$$= -6 \ln \left( \frac{c\sqrt{T_0}}{24} t + 1 \right).$$

$$\therefore \; n = n_0 e^{-6 \ln\left(\frac{c}{24}\sqrt{T_0}\, t + 1\right)}.$$

$$\therefore \; \boxed{\begin{aligned} T &= \frac{1}{\left(\frac{c}{24} t + \frac{1}{\sqrt{T_0}}\right)^2} \\ n &= \frac{n_0}{\left(\frac{c}{24}\sqrt{T_0}\, t + 1\right)^6} \end{aligned}}$$

where $\boxed{c = 2\sqrt{2}\sqrt{\frac{k_B}{m}}\frac{A}{V}}$

12 The probability for 1 particle to be in volume V is $p = \frac{V}{\mathcal{V}}$

$\Rightarrow$ The probability for N particular particles to be in volume V is $p^N$

$\Rightarrow$ The probability for $\mathcal{N} - N$ particular particles to be in volume V is $(1-p)^{\mathcal{N}-N}$

$\Rightarrow$ There are $\binom{\mathcal{N}}{N} = \frac{\mathcal{N}!}{(\mathcal{N}-N)!N!}$ different combinations of ~~#~~ particular particles

$$\therefore \boxed{P_N = \frac{\mathcal{N}!}{(\mathcal{N}-N)!N!} \left(\frac{V}{\mathcal{V}}\right)^N \left(1 - \frac{V}{\mathcal{V}}\right)^{\mathcal{N}-N}}$$

a)

$$\lim_{\substack{\mathcal{N}\to\infty \\ \mathcal{V}\to\infty}} \frac{\mathcal{N}}{\mathcal{V}} = n \Rightarrow \lim_{\substack{\mathcal{N}\to\infty \\ \mathcal{V}\to\infty}} \mathcal{N}\frac{V}{\mathcal{V}} = nV$$

$$\Rightarrow \lim_{\mathcal{N}\to\infty} \mathcal{N}p = nV = \langle N \rangle$$

$\therefore p = \frac{\langle N \rangle}{\mathcal{N}}$ in this limit

$$\lim_{\mathcal{N}\to\infty} P_N = \cancel{\lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}!}{N!(\mathcal{N}-N)!}}$$

$$\lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}!}{N!(\mathcal{N}-N)!} \left(\frac{\langle N\rangle}{\mathcal{N}}\right)^N \left(1-\frac{\langle N\rangle}{\mathcal{N}}\right)^{\mathcal{N}-N}$$

$$= \frac{\langle N\rangle^N}{N!} \lim_{\mathcal{N}\to\infty} \underbrace{\frac{\mathcal{N}!}{\cancel{N!}(\mathcal{N}-N)!} \left(\frac{\langle N\rangle}{\mathcal{N}}\right)^N}_{①} \underbrace{\left(1-\frac{\langle N\rangle}{\mathcal{N}}\right)^{\mathcal{N}}}_{②} \underbrace{\left(1-\frac{\langle N\rangle}{\mathcal{N}}\right)^{-N}}_{③}$$

$$\cancel{① \Rightarrow \lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}!}{N!(\mathcal{N}-N)!} \left(\frac{\langle N\rangle}{\mathcal{N}}\right)^N}$$

$$① = \lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}!}{(\mathcal{N}-N)!} \left(\frac{1}{\mathcal{N}}\right)^N$$

$$= \lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}(\mathcal{N}-1)(\mathcal{N}-2)\cdots(1)}{(\mathcal{N}-N)(\mathcal{N}-N-1)\cdots(1)} \left(\frac{1}{\mathcal{N}}\right)^N$$

$$= \lim_{\mathcal{N}\to\infty} \frac{\mathcal{N}(\mathcal{N}-1)\cdots(\mathcal{N}-N+1)}{\mathcal{N}^N}$$

$$= \lim_{\mathcal{N}\to\infty} \left(\frac{\mathcal{N}}{\mathcal{N}}\right)\left(\frac{\mathcal{N}-1}{\mathcal{N}}\right)\left(\frac{\mathcal{N}-2}{\mathcal{N}}\right)\cdots\left(\frac{\mathcal{N}-N+1}{\mathcal{N}}\right)$$

$$= 1.$$

$$② = \lim_{\mathcal{N} \to \infty} \left(1 - \frac{\langle N \rangle}{\mathcal{N}}\right)^{\mathcal{N}} = \lim_{\mathcal{N} \to \infty} \left(1 + \frac{(-\langle N \rangle)}{\mathcal{N}}\right)^{\mathcal{N}}$$

$$= e^{-\langle N \rangle}$$

$$③ = \lim_{\mathcal{N} \to \infty} \left(1 - \frac{\langle N \rangle}{\mathcal{N}}\right)^{-N} = (1)^{-N} = 1.$$

$$\therefore \boxed{\lim_{\mathcal{N} \to \infty} P_N = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} \quad \text{where} \quad \langle N \rangle = nV}$$

b) $\langle N^2 \rangle = \sum_{N=0}^{\infty} N^2 P_N = \sum_{N=0}^{\infty} N^2 \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}$

$$= e^{-\langle N \rangle} \sum_{N=1}^{\infty} N \frac{\langle N \rangle^N}{(N-1)!}$$

$$= e^{-\langle N \rangle} \sum_{N=1}^{\infty} \left[ (N-1) \frac{\langle N \rangle^N}{(N-1)!} + \frac{\langle N \rangle^N}{(N-1)!} \right]$$

$$= e^{-\langle N \rangle} \left[ \sum_{N=2}^{\infty} \frac{\langle N \rangle^N}{(N-2)!} + \sum_{N=1}^{\infty} \frac{\langle N \rangle^N}{(N-1)!} \right].$$

$$= e^{-\langle N \rangle} \left[ \langle N \rangle^2 \sum_{N=2}^{\infty} \frac{\langle N \rangle^{N-2}}{(N-2)!} + \langle N \rangle \sum_{N=1}^{\infty} \frac{\langle N \rangle^{N-1}}{(N-1)!} \right].$$

$$= e^{-\langle N \rangle} \left[ \langle N \rangle^2 e^{+\langle N \rangle} + \langle N \rangle e^{+\langle N \rangle} \right]. = \langle N \rangle^2 + \langle N \rangle$$

$$\therefore \langle N^2 \rangle - \langle N \rangle^2 = \langle N \rangle$$

$$\therefore \frac{\langle (N - \langle N \rangle)^2 \rangle^{1/2}}{\langle N \rangle} = \frac{\langle N^2 - 2N\langle N \rangle + \langle N \rangle^2 \rangle^{1/2}}{\langle N \rangle}$$

$$= \frac{(\langle N^2 \rangle - 2\langle N \rangle^2 + \langle N \rangle^2)^{1/2}}{\langle N \rangle} = \frac{(\langle N^2 \rangle - \langle N \rangle^2)^{1/2}}{\langle N \rangle}$$

$$= \frac{\langle N \rangle^{1/2}}{\langle N \rangle} = \frac{1}{\sqrt{\langle N \rangle}}$$

QED

c)

To find maximum of $P_N$ we require $\frac{dP_N}{dN} = 0$

$$P_N = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} \Rightarrow N!\, P_N = \langle N \rangle^N e^{-\langle N \rangle}$$

$$\Rightarrow \frac{d}{dN}(N!\, P_N) = \frac{d}{dN}(\langle N \rangle^N e^{-\langle N \rangle})$$

$$\Rightarrow P_N \frac{dN!}{dN} + \underbrace{N! \frac{dP_N}{dN}}_{0} = e^{-\langle N \rangle} \frac{d}{dN}(\langle N \rangle^N)$$

$$\Rightarrow P_N \frac{dN!}{dN} = e^{-\langle N \rangle} \frac{d}{dN}(\langle N \rangle^N)$$

Stirling's formula =>

$$N! \approx e^{N\ln N - N}\sqrt{2\pi N} = N^N e^{-N}\sqrt{2\pi N}$$

$$\therefore \frac{dN!}{dN} = \frac{d}{dN}(N^N e^{-N}\sqrt{2\pi N})$$

$$= \frac{d}{dN}(N^N)e^{-N}\sqrt{2\pi N} + N^N\frac{d}{dN}(e^{-N})\sqrt{2\pi N}$$

$$+ N^N e^{-N}\frac{d}{dN}(\sqrt{2\pi N})$$

$$= N^N(1+\ln N)e^{-N}\sqrt{2\pi N} + N^N e^{-N}\sqrt{2\pi N}$$

$$+ N^N e^{-N}\left(\frac{\sqrt{2\pi}}{2}\frac{1}{\sqrt{N}}\right)$$

~~As N → N~~ When N is very large

$$\frac{dN!}{dN} \approx N^N \ln N e^{-N}\sqrt{2\pi N}$$

$$\because p_N\frac{dN!}{dN} = e^{-\langle N\rangle}\frac{d}{dN}(\langle N\rangle^N)$$

$$\therefore \left(\frac{e^{-\langle N\rangle}\langle N\rangle^N}{N!}\right)\frac{dN!}{dN} = e^{-\langle N\rangle}\ln(\langle N\rangle)\langle N\rangle^N$$

$$\therefore \frac{\cancel{e^{-\langle N\rangle}}\langle N\rangle^N}{\cancel{\sqrt{2\pi N}}\cancel{e^{-N}}\cancel{N^N}}\cancel{N^N}\ln N\cancel{e^{-N}}\cancel{\sqrt{2\pi N}} = \cancel{e^{-\langle N\rangle}}\ln(\langle N\rangle)\langle N\rangle^N$$

=> ~~ln(N!)~~ $\ln N\langle N\rangle^N \approx \ln\langle N\rangle\langle N\rangle^N$

=> $\boxed{N \approx \langle N\rangle} = nV$

in the vicinity of $\langle N \rangle$ we have

$$N = \langle N \rangle (1+\delta) \qquad (|\delta| << 1)$$

$$N! = \sqrt{2\pi N} \left(\frac{N}{e}\right)^{N}$$

$$P_N = \frac{\langle N \rangle^{N} e^{-\langle N \rangle}}{N!}$$

$$= \frac{\langle N \rangle^{\langle N \rangle (1+\delta)} e^{-\langle N \rangle}}{\sqrt{2\pi \langle N \rangle (1+\delta)} \left( \frac{\langle N \rangle (1+\delta)}{e} \right)^{\langle N \rangle (1+\delta)}}$$

$$= \frac{\langle N \rangle^{\langle N \rangle (1+\delta)} e^{-\langle N \rangle}}{\sqrt{2\pi} \left[ \langle N \rangle (1+\delta) \right]^{\langle N \rangle (1+\delta) + \frac{1}{2}} e^{-\langle N \rangle (1+\delta)}}$$

$$= \frac{\langle N \rangle^{\langle N \rangle \delta}}{\sqrt{2\pi \langle N \rangle} (1+\delta)^{\langle N \rangle (1+\delta) + \frac{1}{2}}} \equiv \frac{e^{\langle N \rangle \delta}}{\sqrt{2\pi \langle N \rangle}} \cdot \frac{1}{g}$$

where $g = (1+\delta)^{\langle N \rangle (1+\delta) + \frac{1}{2}}$

let $f = \ln g = [\langle N \rangle (1+\delta) + \frac{1}{2}] \ln (1+\delta)$

$$f' = \langle N \rangle \ln (1+\delta) + [\langle N \rangle (1+\delta) + \frac{1}{2}] / (1+\delta)$$

$$f'' = \frac{\langle N \rangle}{1+\delta} + \frac{\langle N \rangle}{1+\delta} - \frac{\langle N \rangle (1+\delta) + 1/2}{(1+\delta)^2}.$$

$$f(0) = 0 \qquad f'(0) = \langle N \rangle + \frac{1}{2} \simeq \langle N \rangle$$

$$f''(0) = \langle N \rangle - \frac{1}{2} \approx \langle N \rangle$$

$$\therefore f \approx f(0) + f'(0)\delta + \frac{f''(0)}{2}\delta^2$$

$$= \langle N \rangle \delta + \frac{\langle N \rangle \delta^2}{2}$$

$$\therefore \quad g = e^f = e^{\langle N \rangle \delta + \frac{\langle N \rangle \delta^2}{2}}$$

$$P_N = \cancel{\frac{e^{\delta \langle N \rangle}}{\sqrt{2\pi \langle N \rangle}}} \frac{e^{\langle N \rangle \delta}}{\sqrt{2\pi \langle N \rangle}} \frac{1}{g}$$

$$= \frac{1}{\sqrt{2\pi \langle N \rangle}} e^{\langle N \rangle \delta - \langle N \rangle \delta - \frac{1}{2}\langle N \rangle \delta^2}$$

$$= \frac{1}{\sqrt{2\pi \langle N \rangle}} e^{-\frac{1}{2}\langle N \rangle \delta^2} = \frac{1}{\sqrt{2\pi \langle N \rangle}} e^{-\frac{(\langle N \rangle \delta)^2}{2\langle N \rangle}}$$

$$= \frac{1}{\sqrt{2\pi \langle N \rangle}} e^{-\frac{(N - \langle N \rangle)^2}{2\langle N \rangle}} \qquad \because \langle N \rangle = nV$$

$$\therefore \boxed{P_N \approx \frac{1}{\sqrt{2\pi nV}} e^{-\frac{(N - nV)^2}{2nV}}}$$

Independent distributions:

$\langle V_x V_y \rangle = \langle V_x \rangle \langle V_y \rangle$

if $x = y$ then $V_x$ and $v_x$ are <u>not</u> independant

$\therefore \langle V_x V_x \rangle \neq \langle v_x \rangle \langle v_x \rangle$

---

c.

i, i, i, i $\qquad \langle v^4 \rangle$

i, i, j, j $\qquad \langle v^2 \rangle \langle v^2 \rangle$

all others are 0.

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.$$

$$+ \underline{\int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2}} =$$

$$\int_x x e^{-\alpha x^2} dx = f(a)$$

$$\langle V^{2m+1}_{\cancel{th}} \rangle \quad V_{th}^{2m} (2m+1)!! / 2^m$$

$$\langle V$$

$$\frac{m!}{2} V_{th}^{2m+2}$$

$$\frac{\sqrt{2}}{2} \frac{(2m-1)!!}{2^m} V_{th}^{2m+1}$$

$$\langle v^{n+1} \rangle^{\frac{1}{n+1}} > \langle n^n \rangle^{\frac{1}{n}}$$

---

4. cylindrical coordinate.

$$P = \frac{1}{3} n m v^2$$

$$= \frac{1}{3} n m (v_x^2 + v_y^2 + v_z^2).$$

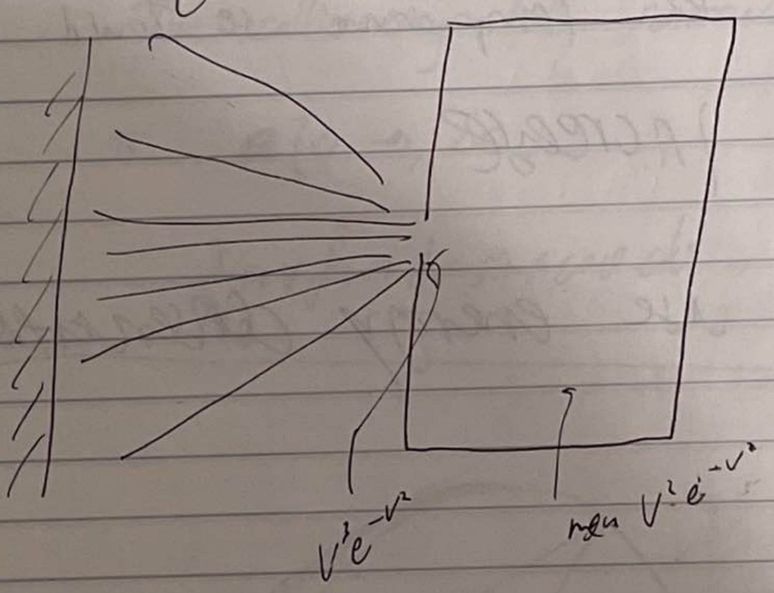
$$= \frac{1}{3} n m (v_x^2 + v_y^2) + \frac{1}{3} n m (v_z^2).$$

$$= \cancel{\frac{1}{3} nm} \quad \frac{2}{3} P_\perp + \frac{1}{3} P_\parallel. \quad \cdots$$

$$\frac{\int \Phi(\theta, v) \cos\theta \cdot dv d\theta}{\int \Phi(\theta \cdot v)\, dv d\theta}$$

---

6



to

~~The median.~~

$\boxed{\text{The mode}}$

every particle comes from hole ends up at the screen. $\therefore f(v_1) = v^3 e^{-v^2}$

Assume no scatter.

Slow atoms are in between.

more faster atoms hits the wall.

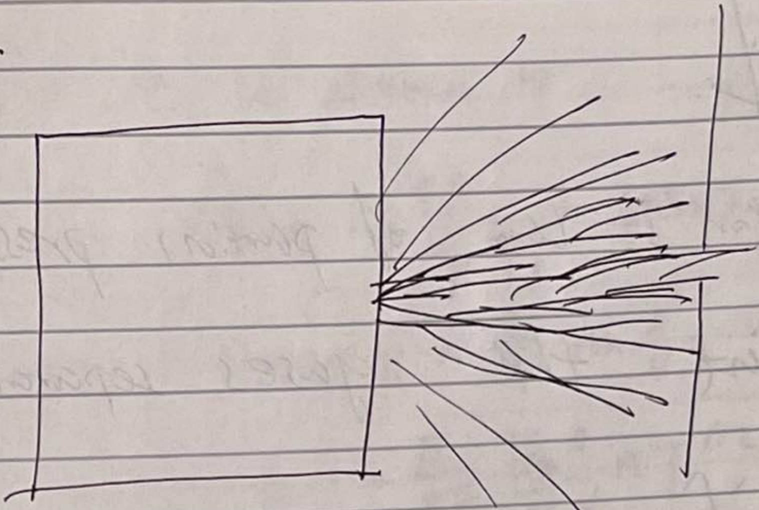
$\therefore$ Distribution at screen favours faster particles than distribution in between.

7. ~~Energy~~ Temperature differences caused by effusion selecting higher energy (faster) particles

∴ Temperature ~~so~~ should increase.

Also use energy conservation.

8.



The collimating hole narrows the distribution

to $[0 \to, \delta\theta]$

$\frac{1}{2} n v \hat{f}(v) dv \sin\theta\cos\theta d\theta$

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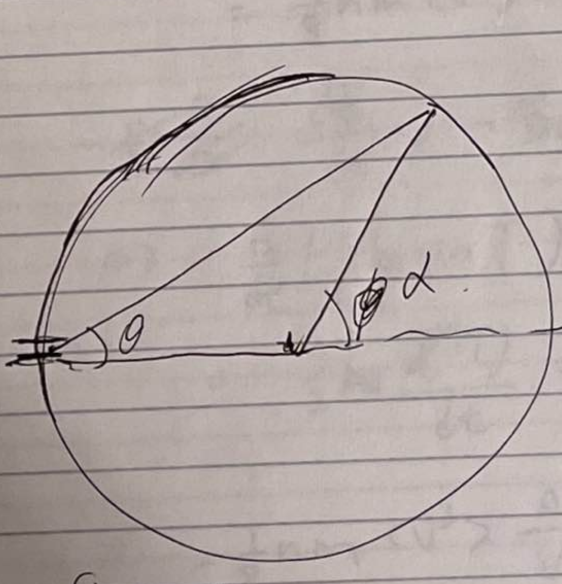

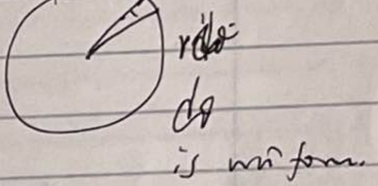

$\sin\theta\cos\theta$

$d\Phi \propto \sin\theta\cos\theta d\theta = \frac{1}{4}\sin\alpha d\alpha \propto d\Omega$

$dA \propto (d\Omega) R^2$

$\therefore d\Phi \propto d\Omega$
$dA \propto d\Omega$

$\therefore \boxed{d\Phi \propto dA}$

[illegible]/10.

Dalton's law of partial pressure

treat the gases separately.

if you have two gases in a box.