

A10429W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

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A1: THERMAL PHYSICS

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TRINITY TERM 2015

Thursday, 18 June, 9.30 am – 12.30 pm

*Answer all of Section A and three questions from Section B.*

*For Section A start the answer to each question on a fresh page.  
For Section B start the answer to each question in a fresh book.*

*A list of physical constants and conversion factors accompanies this paper.*

*The numbers in the margin indicate the weight that the Examiners expect to  
assign to each part of the question.*

**Do NOT turn over until told that you may do so.**

$$S = \frac{U}{T} + k_B \ln Z$$

$$F = -k_B T \ln Z$$

$$U - TS = -k_B T \ln Z$$

### Section A

1. State when a thermodynamic processes is *reversible*. Justify which of the following processes for an ideal gas are reversible and which are irreversible:

- (i) expansion in a thermally insulated container by slowly moving a piston;
- (ii) expansion in a container in thermal contact with a large heat reservoir, by slowly moving a piston;
- (iii) a thermally insulated container is separated into two identical partitions, one filled with gas and the other evacuated. The partition is removed allowing the gas to fill the whole container.

[5]

2 levels?

2. A physical system has two accessible energy levels, a singlet at energy 0 and a triplet at energy  $\Delta > 0$ . Using physical arguments or otherwise find the values of the internal energy  $U$ , the specific heat  $C$  and the entropy  $S$  in the limit of very low and very high temperature  $T$ . Draw carefully labelled plots of the temperature dependence of  $U$ ,  $C$  and  $S$ .

[7]

-2

3. Find the mean kinetic energy per particle in the gas effusing out through a small hole of a vessel containing an ideal classical gas in thermal equilibrium at temperature  $T$ .

[5]

4. Show that in an adiabatic expansion of an ideal monoatomic gas  $pV^{5/3}$  is constant, where  $p$  is the pressure and  $V$  the volume of the gas.

$$dU = Tds - PdV \quad dU = -PdV \quad [4]$$

5. The entropy  $S$  of a certain physical system varies with its energy  $U$  at fixed volume  $V$  as  $S = AU^{3/4}$ , where  $A = 1.26 \times 10^{-4} \text{ J}^{1/4} \text{ K}^{-1}$ . When the system's energy is  $U = 4 \times 10^{-8} \text{ J}$ , what is the system's temperature?

$$dU = Tds - PdV \quad [4]$$

$$T = \left(\frac{\partial U}{\partial S}\right)_V$$

6. A non-relativistic ideal gas of  $N$  fermions of mass  $m$  and spin  $1/2$  is confined to move in a two-dimensional plane of area  $A$ . Find the expression for the Fermi energy  $\epsilon_F$ . At  $T = 0$  find the average energy per particle  $\langle \epsilon \rangle$  in terms of  $\epsilon_F$ . At high temperatures  $T \gg \epsilon_F/k_B$  find how  $\langle \epsilon \rangle$  varies with  $T$ . Draw a carefully labelled sketch of the temperature dependence of  $\langle \epsilon \rangle$  from  $T = 0$  to  $T \gg \epsilon_F/k_B$ .

[8]

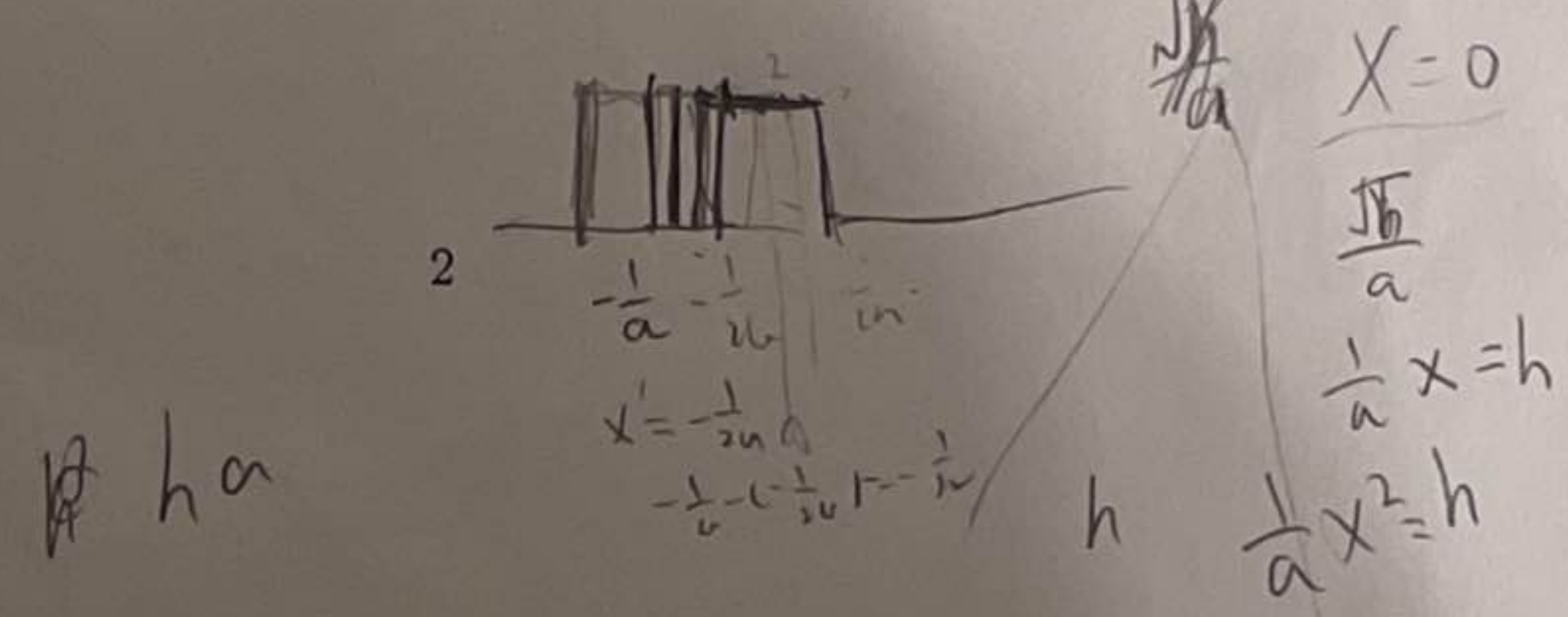
7. Find the Fourier transform,  $g(\omega)$ , of the triangular function

$$f(t) = \begin{cases} h(1 - a|t|), & |t| \leq 1/a \\ 0, & |t| > 1/a \end{cases}$$

$$\int_{-\infty}^{\infty} f(x-x')g(x')dx'$$

where  $h$  and  $a$  are positive constants. Plot  $g(\omega)$  in the range  $[-6\pi a, 6\pi a]$ .

[7]



## Section B

8. Draw the phase diagram in the  $(P, T)$  plane for a typical pure substance showing where the gas, liquid and solid phases occur. Indicate the *critical* and *triple* points and state their properties. [4]

Derive the Clausius–Clapeyron relation for the equilibrium phase boundary between two phases,

$$\frac{dP}{dT} = \frac{L}{T(V_2 - V_1)},$$

where  $L$  is the latent heat of the phase transformation and  $V_2 - V_1$  is the volume change, for one mole of substance. [6]

In the vicinity of the triple point the vapour pressure of liquid and solid ammonia  $\text{NH}_3$  varies with temperature as:

$$\text{liquid: } \ln P = 24.38 - \frac{3063}{T},$$

$$\text{solid: } \ln P = 27.92 - \frac{3754}{T},$$

where the pressure  $P$  is in units of Pa and the temperature  $T$  is in units of K.

(i) Determine the pressure and temperature of the triple point.  
 (ii) Determine the latent heat of vaporization  $L_{\text{vap}}$ , sublimation  $L_{\text{sub}}$  and melting  $L_{\text{melt}}$  at the triple point, explaining any approximations that you make.

(iii) The outer atmosphere of the planet Jupiter contains a thick layer of frozen ammonia crystals at an average temperature of 165 K and pressure of  $5 \times 10^4$  Pa. Estimate by how much the temperature would need to increase for the ammonia crystals to change into the vapour phase under the same pressure. [10]

check



above?  
below?

9. Give one definition of the *chemical potential* for a gas of  $N$  identical particles. Show that for an ideal gas of indistinguishable particles the partition function is

$$Z_N = \frac{Z_1^N}{N!},$$

and hence show that the chemical potential is

$$\mu = -k_B T \ln \left( \frac{Z_1}{N} \right),$$

where  $Z_1$  is the single particle partition function and  $T$  the temperature. [6]

An initially evacuated thin-walled container of volume  $V$  and wall area  $A$  is filled with a total of  $N$  atoms of argon of mass  $m$  each. The walls are held at a constant temperature  $T$  by contact with an outside reservoir. A fraction of the argon atoms become adsorbed onto the walls and the rest remain in the gas. An adsorbed atom is free to move parallel to the smooth surface of the internal walls of the container, but it requires an energy cost  $\epsilon_0 > 0$  to overcome the attraction of the surface and escape into the gas. Write down the expression for the energy of an adsorbed atom moving with velocity  $\mathbf{v}$  parallel to the surface. Hence, or otherwise, show that the partition function for an adsorbed atom is

$$Z_{1,a} = a A T e^{b/T},$$

whereas for an atom in the gas it is

$$Z_{1,g} = c V T^{3/2},$$

where  $a$ ,  $b$  and  $c$  are temperature-independent constants to be determined. [8]

Determine what fraction of the total number of atoms  $N_a/N$  is adsorbed onto the surface at thermal equilibrium, plot this as a function of temperature and comment on the physical origin of this dependence. [6]

[You may use Stirling's formula  $\ln N! \simeq N \ln N - N$  for large  $N$ .]

Handwritten derivation for the partition function of an adsorbed atom:

$$\sum_{\alpha} e^{-\beta E_{\alpha}} = \int d\epsilon g(\epsilon) e^{-\beta \epsilon}$$

$$g(\epsilon) = \frac{A k}{2\pi} \int x^2 e^{-x^2} dx \quad d\epsilon = \frac{\hbar^2 k}{m} dk$$

$$g(\epsilon) \frac{\hbar^2 k}{m} = \frac{A k}{2\pi}$$

$$g(\epsilon) = \frac{mA}{2\pi \hbar^2}$$

$$dU = C_L dT + f dL$$

$$dU = \bar{Q} f dL$$

$$\therefore C_L dT = 2f dL$$

$$U + fL$$

$$U - TS =$$

10. Write the first law of thermodynamics in differential form for a stretched piece of rubber of length  $L$  and tension  $f$ . [2]

The tension in a stretched rubber is found to vary with temperature  $T$  as

$$f = aT^2(L - L_0),$$

$$\left(\frac{\partial U}{\partial L}\right)_T =$$

where  $L_0$  is the unstretched length and  $a > 0$  is a constant.

- (i) If the rubber is stretched isothermally will the entropy increase or decrease?  
 (ii) If the rubber is stretched adiabatically will it cool or warm? Comment on the physical origin of this effect. [6]

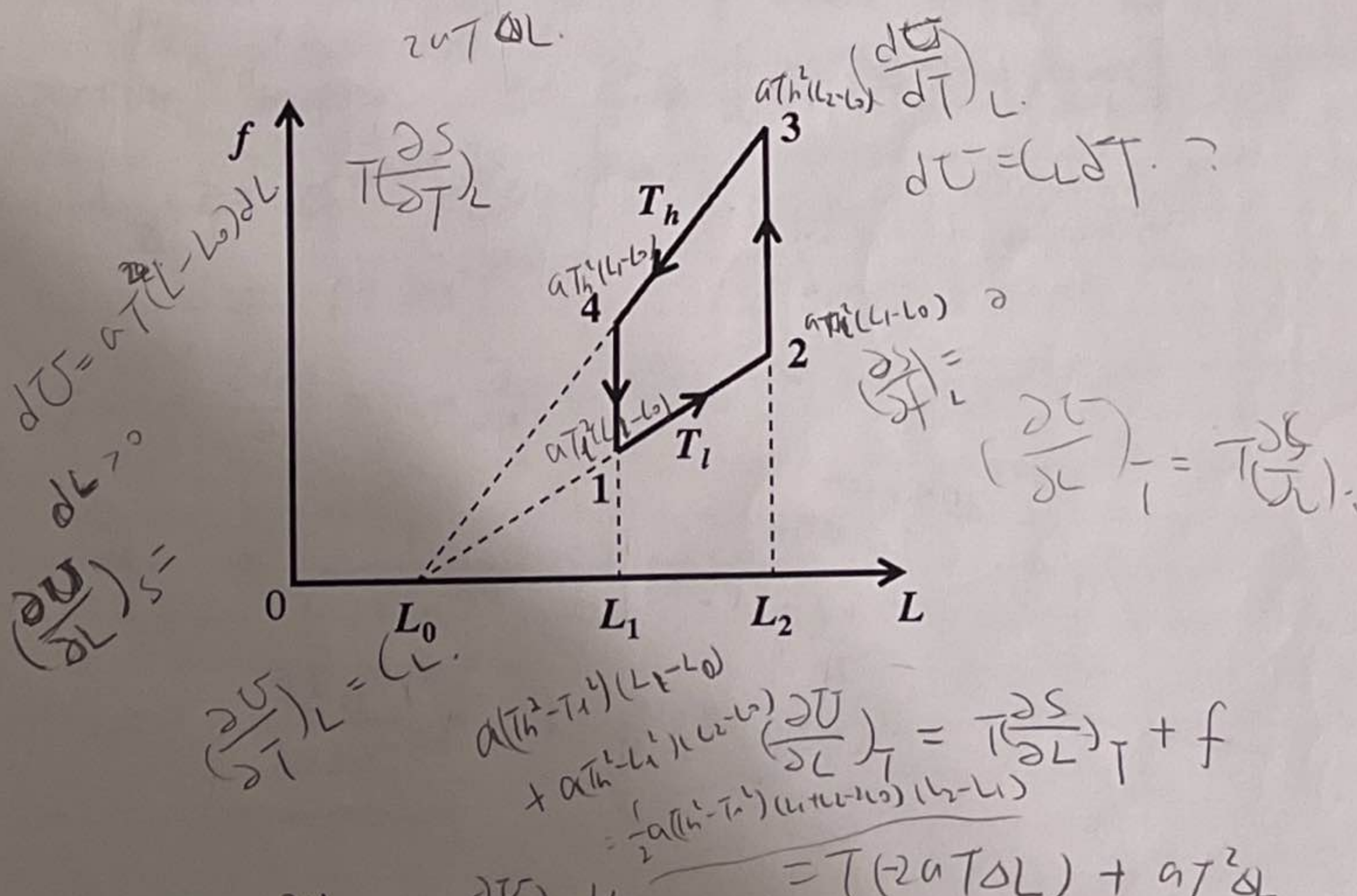
The specific heat at constant length when  $L = L_0$  is found to be  $C_L(L_0, T) = bT$ , where  $b$  is a constant. Show that the entropy at a general length  $L$  and temperature  $T$  is of the form

$$S(L, T) = [b - a(L - L_0)^2]T + S_0,$$

$$C_L = T \left(\frac{\partial S}{\partial T}\right)_L = bT$$

where  $S_0$  is a constant. Find the specific heat at constant length  $C_L$  for a general length  $L$  and temperature  $T$ . [4]

The diagram below shows the cycle of an idealised heat engine operated with the above rubber between a hot reservoir at temperature  $T_h$  and a cold reservoir at temperature  $T_l$ . Explain why the cycle is run counter-clockwise to operate as an engine. Over which parts of the cycle is heat absorbed and over which is it released? Find an expression for the efficiency of the engine in terms of the parameters of the problem. [8]



$$dU = \left(\frac{\partial U}{\partial T}\right)_L dT + \left(\frac{\partial U}{\partial L}\right)_T dL$$

$$= T(2aT \Delta L) + aT^2 \Delta L$$

A10429W1

$$= C_L dT - f dL$$

$$= -aT^2 \Delta L \quad [\text{Turn over}]$$

$$= (b - a(L - L_0)^2)T dT - f dL$$

$$= -f$$

11. Using the fact that photons obey Bose-Einstein statistics show that for thermal radiation inside a black-body cavity at temperature  $T$  the number of photons with angular frequency between  $\omega$  and  $\omega + d\omega$  is

$$n(\omega) d\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/k_B T} - 1},$$

where  $V$  is the cavity volume and the other symbols have their usual meaning.

[5]

Show that the energy density of the thermal radiation is

$$u = AT^4,$$

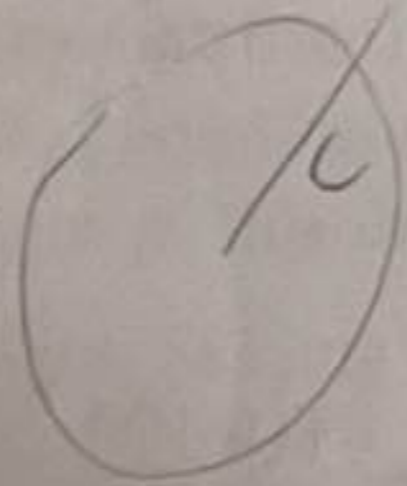
and that the power of thermal radiation incident per unit area of the cavity walls (equal to the power radiated by the walls) is  $\sigma T^4$ . Obtain expressions for  $A$  and  $\sigma$ .

[7]

A satellite powered by a Carnot engine uses heat from a nuclear reactor at a fixed temperature  $T_0$ . Heat is released into outer space via thermal radiation emitted by a set of fins at temperature  $T$  located on the outside of the satellite. The fins can be considered as ideal black bodies. Find the optimum temperature of the fins for which the Carnot engine will have the maximum power output and determine the engine efficiency in this case.

[8]

[ You may use  $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} .$  ]



$$\int \left( \frac{uV}{N} \right) \left( \frac{N}{V} \right) c \omega d\omega A dt \sin^2 \theta d\theta$$

$$\frac{1}{2} \int_0^{\pi/2} d\theta \sin(2\theta)$$

$$-\frac{1}{4} \cos(2\theta) \Big|_0^{\pi/2}$$

$$= +\frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$$

1. A reversible process is a process that we can run it in reverse. ~~We can~~ In a reversible process the system ~~is~~ remains in equilibrium throughout the entire process.

(i) Is reversible  $\rightarrow$  because process is quasi-static ✓

(ii) Is reversible  $\rightarrow$  process is quasi-static ✓

quasi-static & no-friction  $\Rightarrow$  reversible

(iii) Is ~~reversible~~ irreversible  $\rightarrow$  Joule expansion is

sudden, not quasi-static. Process involves a non-zero change, not in equilibrium all the time. ✓

2. 1 level energy 0, 3 levels energy  $\Delta$

partition function  $Z = \sum_{\alpha} e^{-\beta E_{\alpha}}$

$$Z = e^{-\beta(0)} + 3e^{-\beta(\Delta)}$$

$$= 1 + 3e^{-\beta\Delta}$$

$$U = - \frac{\partial \ln Z}{\partial \beta} = - \frac{\partial}{\partial \beta} (1 + 3e^{-\beta\Delta})$$

$$= + \frac{1}{1 + 3e^{-\beta\Delta}} (3(\Delta)e^{-\beta\Delta})$$

$$= \frac{3\Delta e^{-\beta\Delta}}{1 + 3e^{-\beta\Delta}} \quad (\beta = \frac{1}{k_B T})$$

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V \approx = \frac{\partial U}{\partial \beta} \frac{\partial \beta}{\partial T}$$

$$\frac{\partial \beta}{\partial T} = - \frac{1}{k_B T^2}$$

$$\frac{\partial U}{\partial \beta} = \frac{\partial}{\partial \beta} \left( \frac{\Delta 3e^{-\beta \Delta}}{1+3e^{-\beta \Delta}} \right)$$

$$= \frac{\partial}{\partial \beta} \left( 1 - \frac{1}{1+3e^{-\beta \Delta}} \right) \Delta$$

$$= - \frac{\Delta^2}{(1+3e^{-\beta \Delta})^2} (-3e^{-\beta \Delta})$$

$$= \frac{3\Delta^2 e^{-\beta \Delta}}{(1+3e^{-\beta \Delta})^2}$$

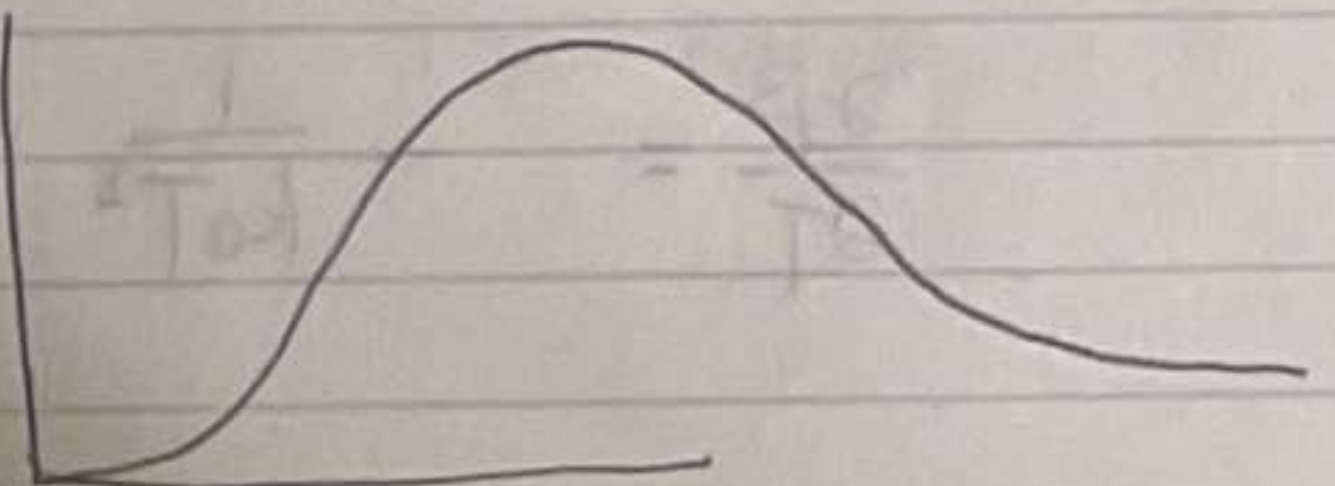
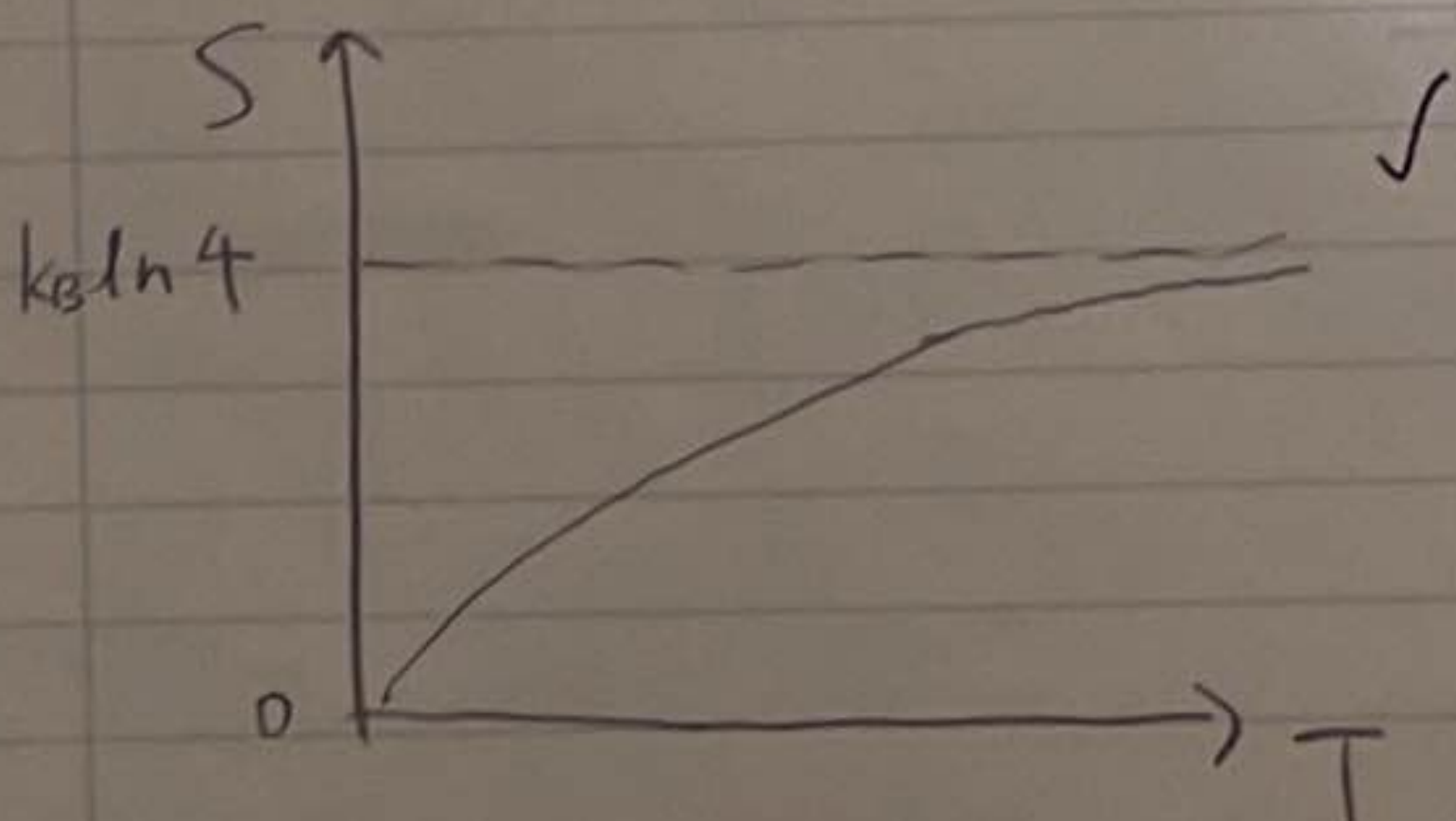
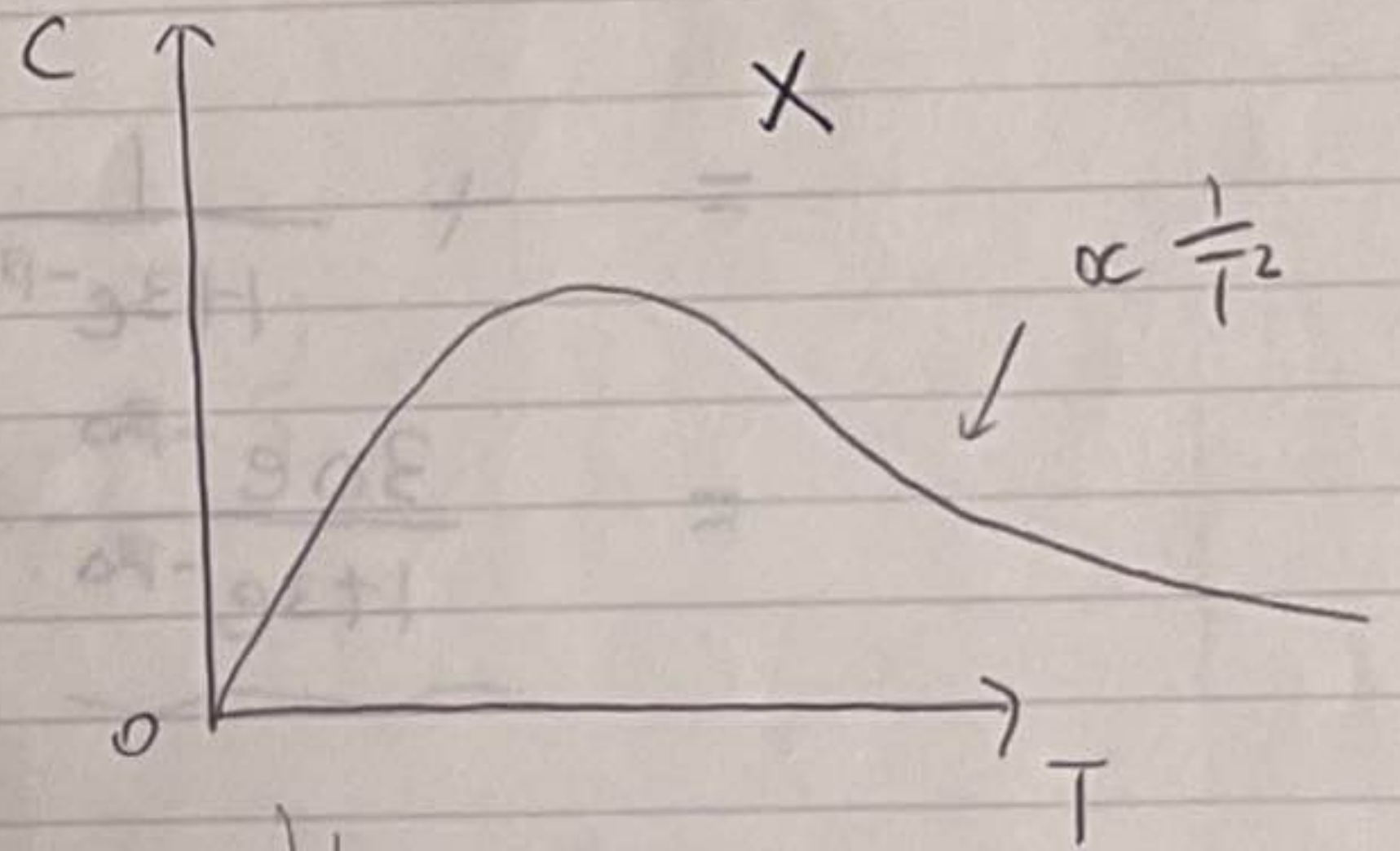
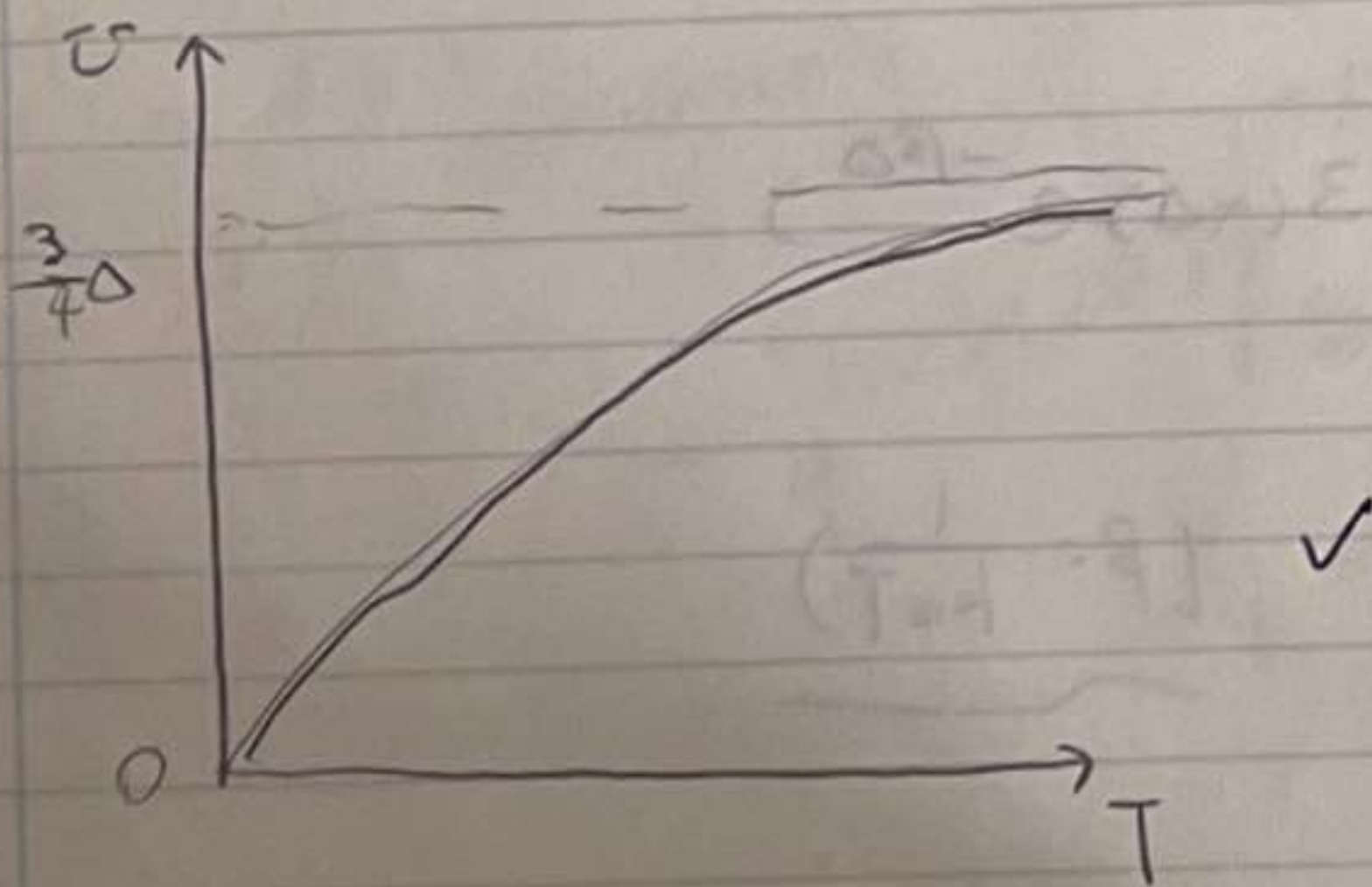
$$\therefore C = \frac{3\Delta^2}{k_B T^2} \frac{e^{-\frac{\Delta}{k_B T}}}{(1+3e^{-\frac{\Delta}{k_B T}})^2}$$

$$S = \frac{U}{T} + k_B \ln Z$$

$$= \frac{3\Delta}{T} \left( \frac{e^{-\frac{\Delta}{k_B T}}}{1+3e^{-\frac{\Delta}{k_B T}}} \right) + k_B \ln (1+3e^{-\frac{\Delta}{k_B T}})$$

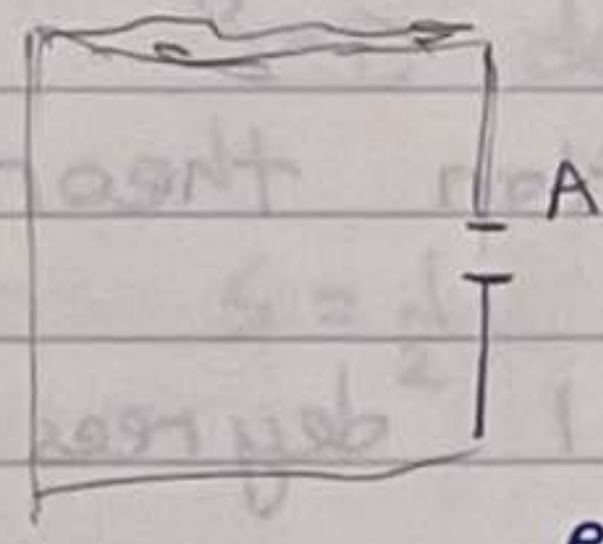
$$\beta \rightarrow 0 \quad T \rightarrow \infty \quad \beta \rightarrow 0 \rightarrow S \rightarrow \underline{k_B \ln 4}$$

$$T \rightarrow 0 \quad \beta \rightarrow \infty \rightarrow S \rightarrow \underline{0}$$





3.



Flux of particles hitting the hole / Area · time

$$d\Phi(\vec{v}) = n v_z f(\vec{v}) d^3\vec{v} \quad (f(\vec{v}) = f(v) \text{ is isotropic velocity distribution})$$

$$= n v \cos\theta f(v) v^2 \sin\theta dv d\theta d\phi$$

energy flux of  $\vec{v}$

$$dE(\vec{v}) = \left( \frac{1}{2} m v^2 (v_z f(\vec{v}) d^3\vec{v}) \right) n$$

$$= \frac{1}{2} m v^3 \cos\theta f(v) d^3\vec{v} = \frac{1}{2} m v^3 \cos\theta f(v) v^2 \sin\theta dv d\theta d\phi n$$

~~Average energy~~

Total Energy flux :

$$E = \int dE(\vec{v}) = \frac{1}{2} m n \int_0^\infty dv \cdot v^5 f(v) \int_0^{\pi/2} d\theta \sin\theta \cos\theta \int_0^{2\pi} d\phi$$

↑ hitting the surface

Total # particles flux :

$$\Phi = \int d\Phi(\vec{v}) = n \int_0^\infty dv v^3 f(v) \int_0^{\pi/2} d\theta \sin\theta \cos\theta \int_0^{2\pi} d\phi$$

mean energy

$$\langle E \rangle = \frac{E}{\Phi} = \frac{1}{2} m \frac{\int_0^\infty dv v^5 f(v)}{\int_0^\infty dv v^3 f(v)}$$

$$= \frac{1}{2} m \frac{\frac{1}{4\pi} \int \tilde{f}(v) v^3 dv}{\frac{1}{4\pi} \int \tilde{f}(v) v dv} = \frac{1}{2} m \frac{\langle v^3 \rangle}{\langle v \rangle}$$

here we use Maxwellian distribution for classical ideal gas

$$= \frac{1}{2} m \frac{\frac{4}{\sqrt{\pi}} v_{th}^3}{\frac{3}{\sqrt{\pi}} v_{th}^2} = m v_{th}^2 = m \frac{2k_B T}{m} = \underline{2k_B T}$$

✓

4. Internal energy of ideal monatomic gas is

$$U = \frac{3}{2} Nk_B T \quad \text{by equipartition theorem}$$

(each particle has 3 translational degrees of freedom, there are  $N$  particles)

Ideal gas equation of state:  $PV = Nk_B T$

$$\therefore U = \frac{3}{2} PV$$

$dU = Tds - PdV$  in adiabatic process  $ds = 0$

$$\rightarrow dU = -PdV$$

$$\therefore d\left(\frac{3}{2}PV\right) = -PdV \rightarrow \frac{3}{2}PdV + \frac{3}{2}VdP = -PdV$$

$$\therefore \frac{5}{3}PdV + VdP = 0$$

$$\therefore \int_{V_1}^{V_2} \frac{5}{3} \frac{dV}{V} = - \int_{P_1}^{P_2} \frac{dP}{P} \rightarrow \frac{5}{3} \ln \frac{V_2}{V_1} = - \ln \frac{P_2}{P_1}$$

$$\rightarrow \left(\frac{V_2}{V_1}\right)^{5/3} = \frac{P_1}{P_2} \rightarrow P_1 V_1^{5/3} = P_2 V_2^{5/3}$$

$$\rightarrow \underline{PV^{5/3} = \text{const}} \quad \phi \rightarrow \text{QED} \quad \checkmark$$

5.  $\therefore dU = Tds - PdV \quad \therefore T = \left(\frac{\partial U}{\partial S}\right)_V$

$$S = AU^{3/4} \quad \text{at constant } V$$

$$\therefore \left(\frac{\partial S}{\partial U}\right)_V = \frac{3}{4} AU^{-1/4} = \frac{1}{T}$$

$$\therefore T = \frac{4U^{1/4}}{3A} = \frac{(4)(4 \times 10^{-8})^{1/4}}{3(1.26 \times 10^{-4})}$$

$$= \underline{149.7 \text{ K}} \quad \checkmark$$

6.

2-D density of states

$$g(k) dk = \frac{A(2s+1)}{2\pi} k dk$$

$$s = \frac{1}{2} \rightarrow 2s+1 = 2 \quad (\text{spin} = \frac{1}{2})$$

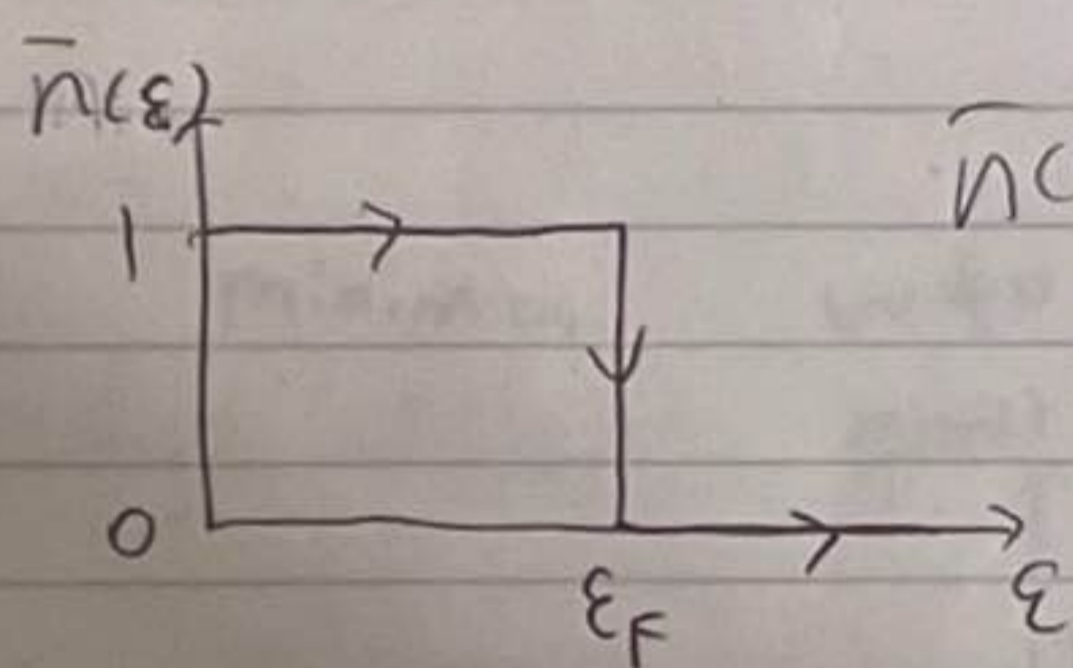
$$g(k) dk = \frac{A}{\pi} k dk = g(\epsilon) d\epsilon \quad \because \epsilon = \frac{\hbar^2 k^2}{2m}$$

non-relativistic

$$\therefore d\epsilon = \frac{\hbar^2 k}{m} dk \rightarrow k dk = \frac{m d\epsilon}{\hbar^2}$$

$$\rightarrow g(k) dk = \frac{A}{\pi} \left(\frac{m}{\hbar^2}\right) d\epsilon = g(\epsilon) d\epsilon \rightarrow g(\epsilon) = \frac{mA}{\pi \hbar^2}$$

When  $T \ll \frac{\epsilon_F}{k_B}$ , the mean occupation number has shape



$$\bar{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1}$$

if  $\epsilon = \mu$ ,  
 $\bar{n}(\epsilon) = \frac{1}{2}$

$$\therefore N = \sum_i \bar{n}_i = \int d\epsilon g(\epsilon) \bar{n}(\epsilon)$$

$$= \int_0^{\epsilon_F} d\epsilon g(\epsilon) = \frac{mA}{\pi \hbar^2} \int_0^{\epsilon_F} d\epsilon = \frac{mA}{\pi \hbar^2} \epsilon_F$$

$$\rightarrow \epsilon_F = \frac{\pi \hbar^2 N}{mA}$$

energy  $[\epsilon, \epsilon + d\epsilon]$ how many states have energy  $\epsilon$ ?

Total energy

$$U = \int d\epsilon g(\epsilon) \bar{n}(\epsilon) \epsilon$$

the energy  $\epsilon$ how many particles in each state with energy  $\epsilon$ ?

$$= \frac{mA}{\pi \hbar^2} \int_0^{\epsilon_F} \epsilon d\epsilon$$

use  $F$  $\bar{n}_i$ at  $T=0$ 

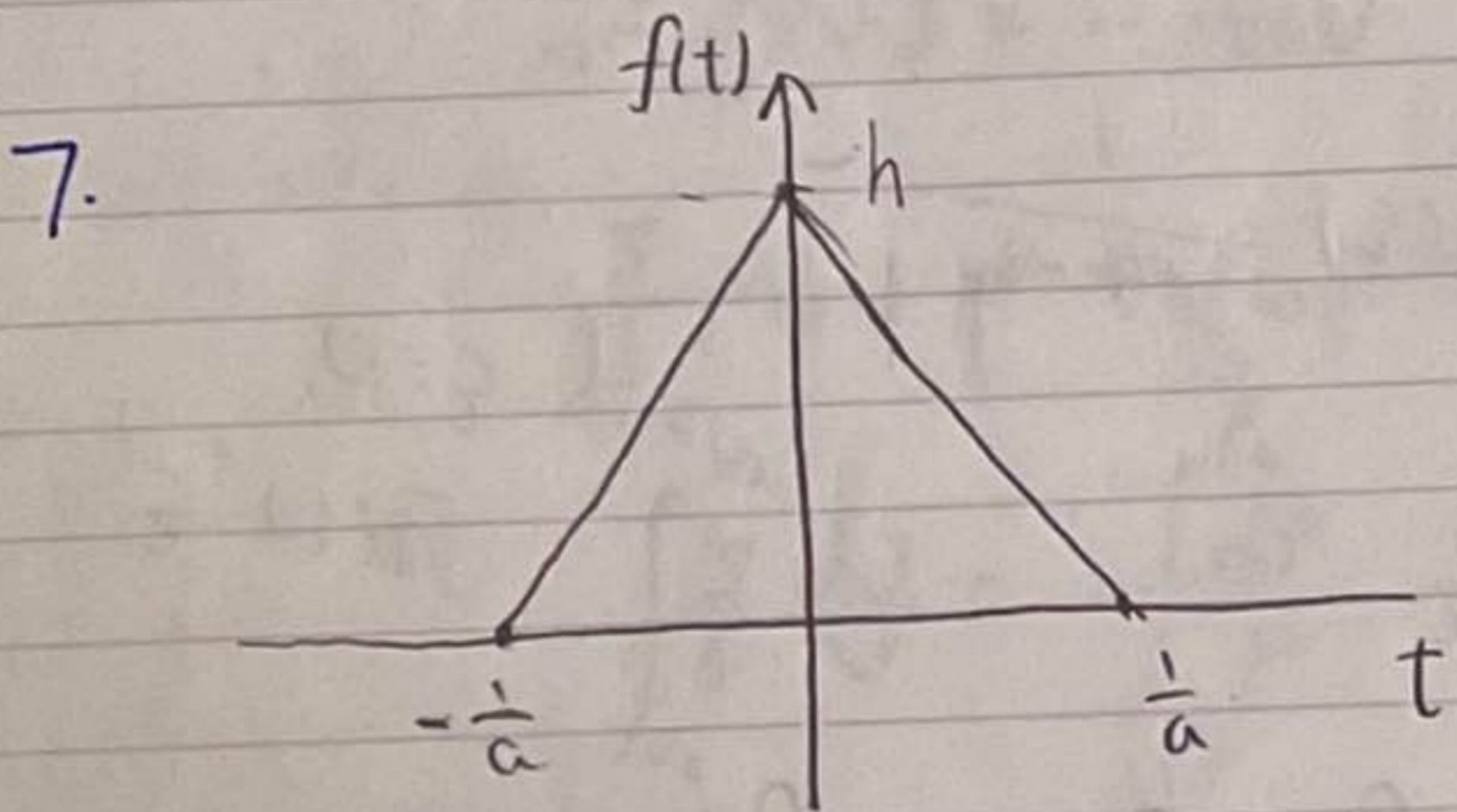
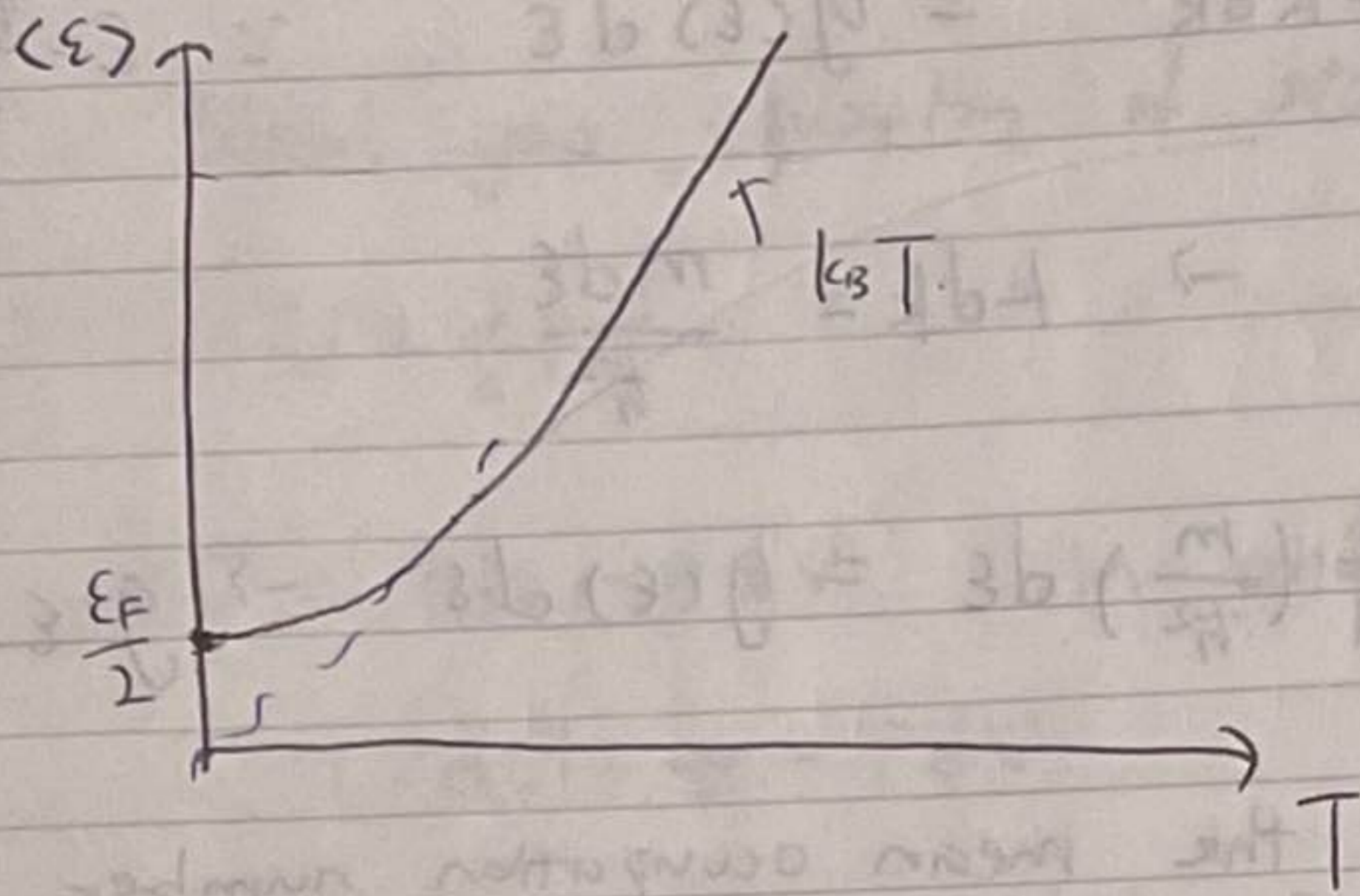
$$= \frac{mA}{2\pi \hbar^2} (\epsilon_F^2)$$

Average energy per particle is

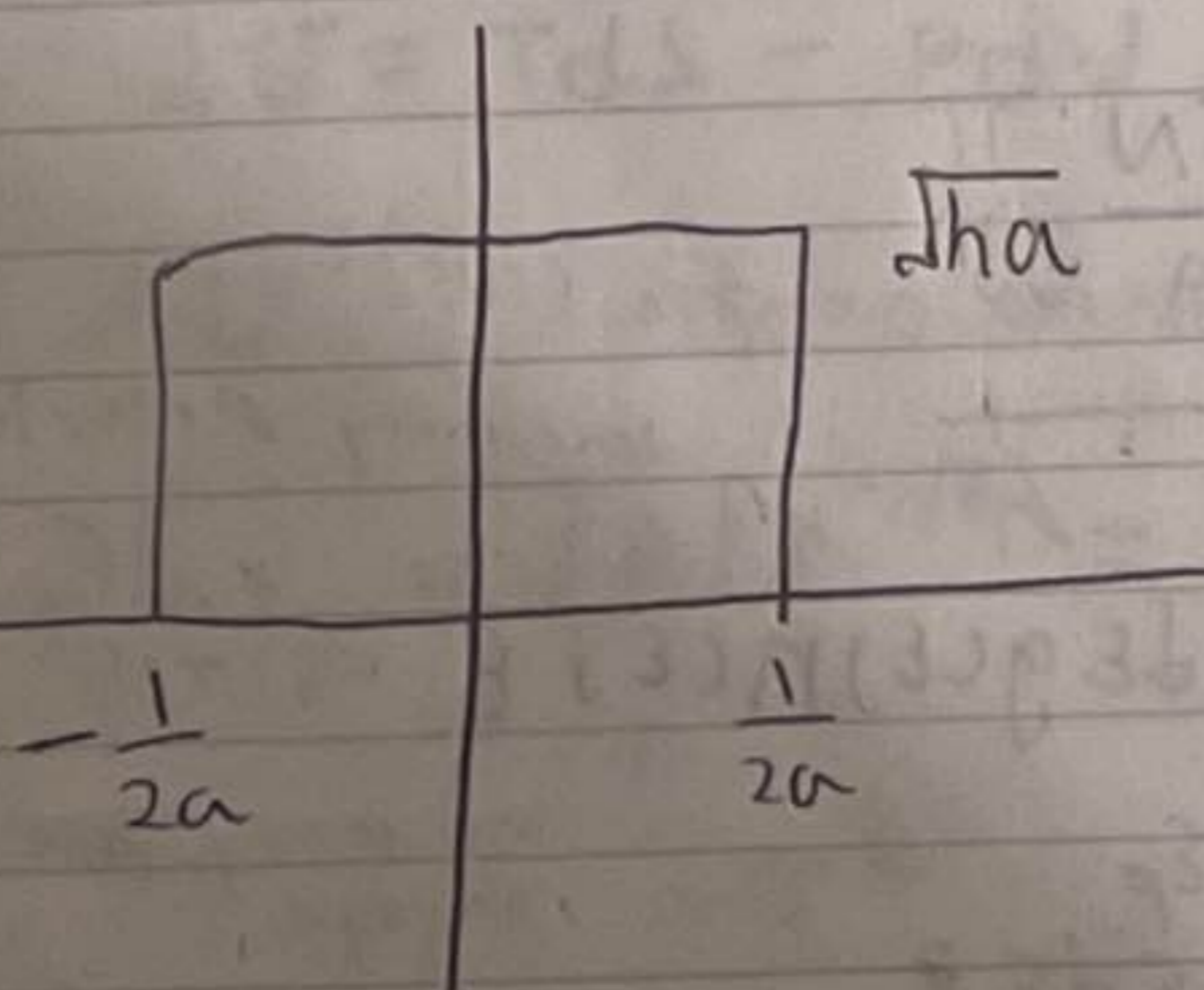
$$\langle \epsilon \rangle = \frac{U}{N} = \frac{\frac{mA}{2\pi \hbar^2} \epsilon_F^2}{\frac{mA}{\pi \hbar^2} \epsilon_F} = \frac{\epsilon_F}{2} = \frac{\pi \hbar^2 N}{2mA}$$

At high temperature each particle has degree of freedom = 2 (2 translational directions of motion)

$\therefore \langle \epsilon \rangle = \frac{\text{d.o.f}}{2} k_B T = k_B T$  by equipartition theorem



Recognise that  $f(t)$  is the convolution of function  $u(t)$  with itself,  $u(t)$  looks like



$$\left( \int_{-\infty}^{\infty} u(t-t') u(t') dt' = \frac{1}{a} \right)$$

By convolution theorem:  $\therefore f(t) = u(t) \otimes u(t)$

$$g(\omega) = V(\omega) \times V(\omega) = \frac{2}{\sqrt{2\pi}} \times 2\pi = 2\pi V^2(\omega)$$

$(V(\omega))$  is the Fourier transform of  $u(x)$

$$V(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} u(t) dt$$

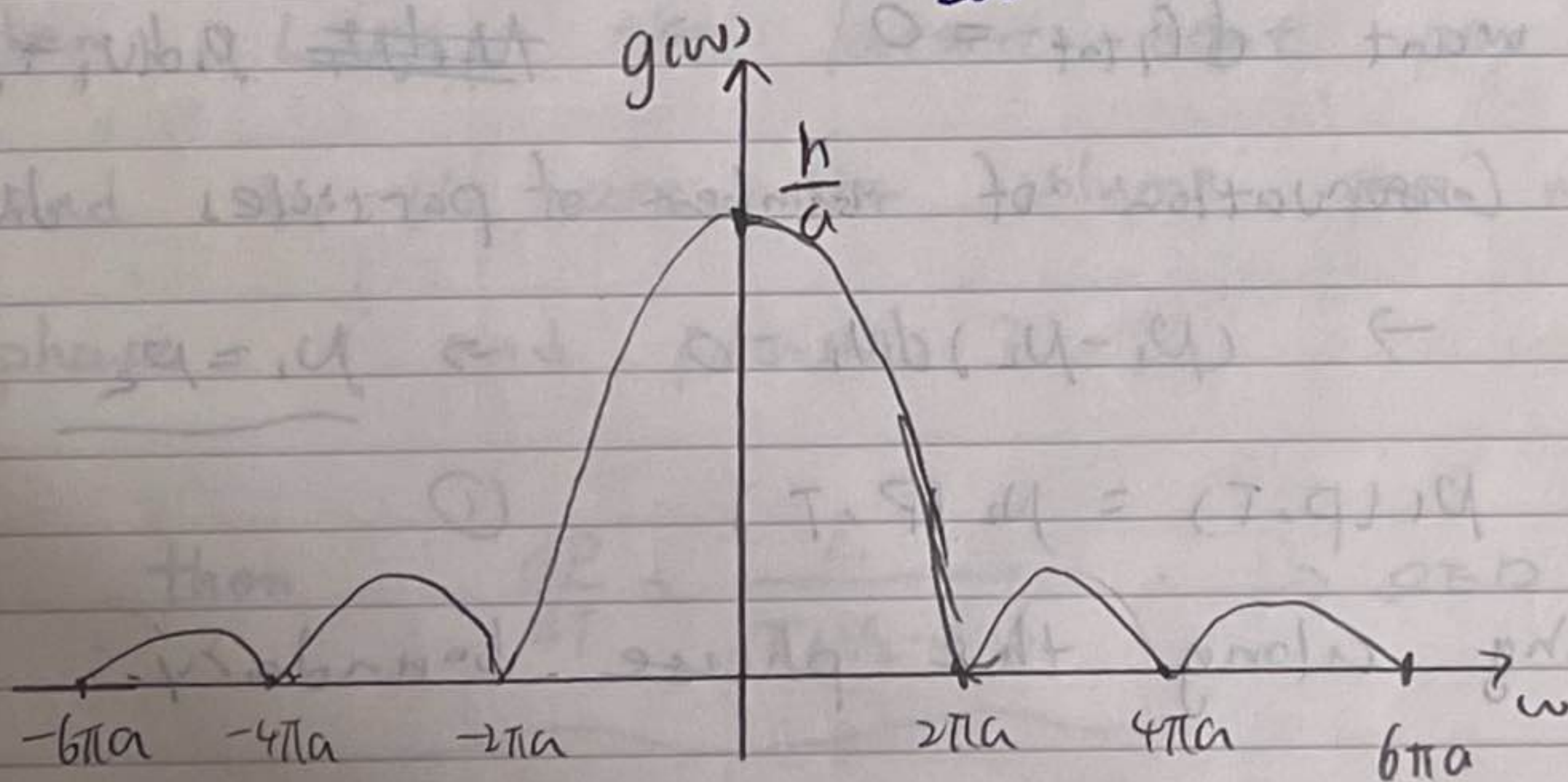
$$= \frac{\sqrt{ha}}{\sqrt{2\pi}} \int_{-\frac{1}{2a}}^{\frac{1}{2a}} e^{-i\omega t} dt$$

$$= \frac{\sqrt{ha}}{\sqrt{2\pi}} \left( \frac{1}{-i\omega} \right) \left[ e^{-i\frac{\omega}{2a}} - e^{i\frac{\omega}{2a}} \right]$$

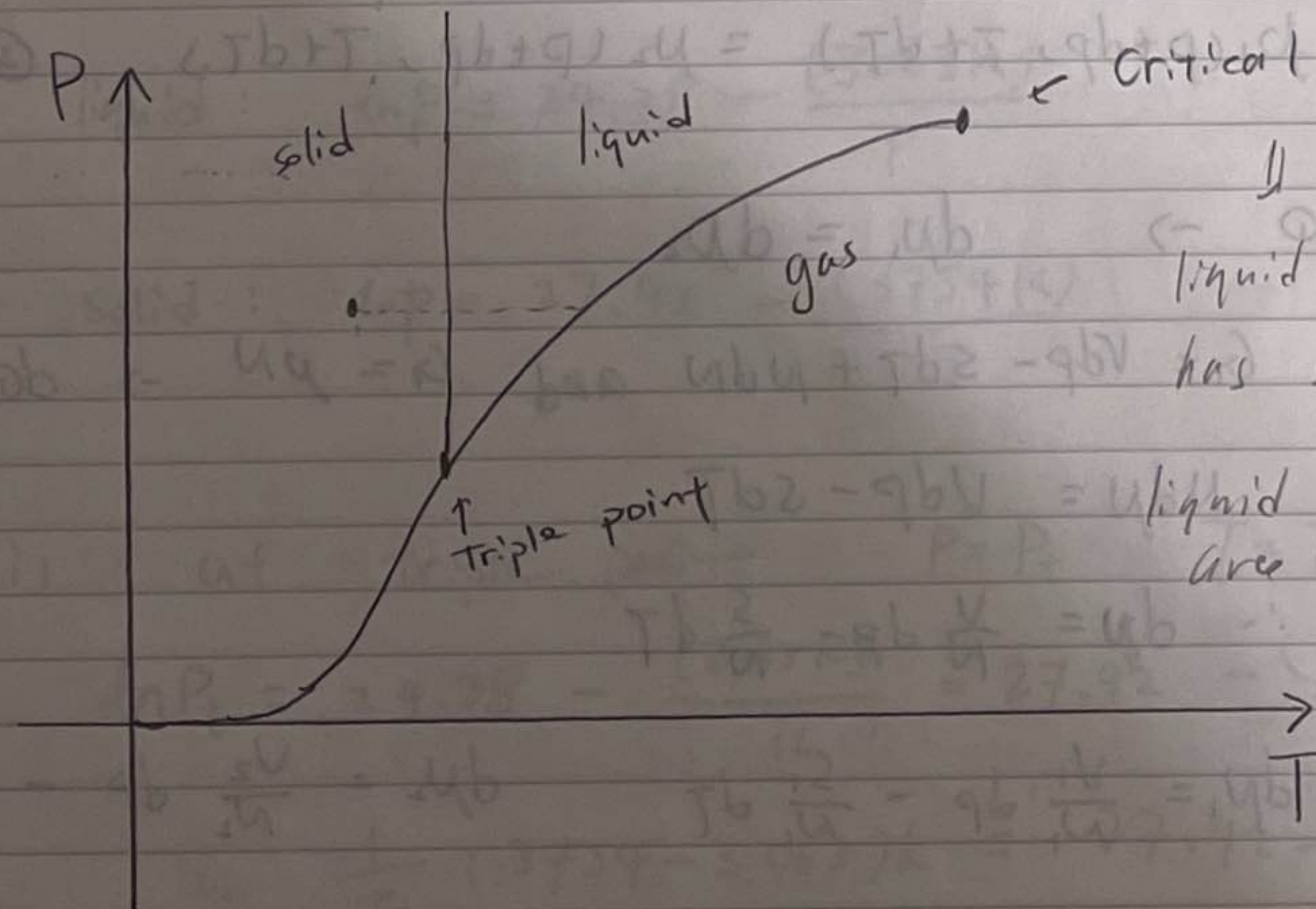
$$= \frac{\sqrt{ha}}{\sqrt{2\pi}} \frac{-2i \sin\left(\frac{\omega}{2a}\right)}{-i\omega} = \frac{\sqrt{ha}}{a\sqrt{2\pi}} \frac{\sin\left(\frac{\omega}{2a}\right)}{\left(\frac{\omega}{2a}\right)}$$

$$g(\omega) = V^2(\omega) \times 2\pi = \frac{h}{a^2} \frac{\sin^2\left(\frac{\omega}{2a}\right)}{\left(\frac{\omega}{2a}\right)^2}$$

→ minima  $\omega \neq 0, \frac{\omega}{2a} = m\pi \rightarrow \omega = 2m\pi a$



8. P ↑



← Critical point  
 ↓  
 liquid & gas  
 has same density  
 liquid & gas  
 are indistinguishable

The triple point is where the gas, liquid, and solid phase coexist.

The critical point is where the coexistence curves of phases terminate. Above the critical pressure and above the critical temperature of the liquid-gas critical point the substance becomes a supercritical fluid.

→ When system is held at constant pressure and temperature, we need to minimize the Gibbs function (the corresponding availability) to find the equilibrium condition

$$\rightarrow dG = 0$$

$$(dG = Vdp - SdT + \sum_i N_i dN_i)$$

Consider 2 phases  $\therefore G = \mu N \therefore$  for 2 phases

$$G_{\text{tot}} = N_1 \mu_1 + N_2 \mu_2$$

$$\text{We want } dG_{\text{tot}} = 0 \rightarrow \mu_1 dN_1 + \mu_2 dN_2 = 0$$

$$\text{Conservation of number of particles } dN_1 + dN_2 = 0$$

$$\rightarrow (\mu_1 - \mu_2) dN_1 = 0 \rightarrow \underline{\mu_1 = \mu_2}$$

$$\rightarrow \mu_1(P, T) = \mu_2(P, T) \quad (1)$$

moving along the phase boundary:

$$\mu_1(P+dp, T+dT) = \mu_2(P+dp, T+dT) \quad (2)$$

$$(2) - (1) \rightarrow d\mu_1 = d\mu_2$$

$$\therefore G = Vdp - SdT + Nd\mu \text{ and } G = \mu N \rightarrow dG = \mu dN + Nd\mu$$

$$\rightarrow Nd\mu = Vdp - SdT$$

$$\therefore d\mu = \frac{V}{N} dp - \frac{S}{N} dT$$

$$\therefore d\mu_1 = \frac{V_1}{N_1} dp - \frac{S_1}{N_1} dT \quad d\mu_2 = \frac{V_2}{N_2} dp - \frac{S_2}{N_2} dT$$

$$\text{let } s_{1,2} = \frac{S_{1,2}}{N_{1,2}} \quad \text{and } v_{1,2} = \frac{V_{1,2}}{N_{1,2}}$$

$$\text{then } -s_1 dT + v_1 dp = -s_2 dT + v_2 dp$$

$$\rightarrow \frac{dp}{dT} = \frac{s_2 - s_1}{v_2 - v_1}$$

( $s, v$  are entropy and volume per particle)

Define latent heat per particle to be

$$l \equiv T \Delta s = T(s_2 - s_1) \quad \text{then}$$

$$\frac{dp}{dT} = \frac{l}{T(v_2 - v_1)}$$

$$\text{let } L = l N_A, \quad v_2 = N_A v_2, \quad v_1 = N_A v_1$$

then  $L$  is the latent heat per mole

and  $v_1, v_2$  are the volume per mole of phase 1 and phase 2

$$\text{then } \frac{dp}{dT} = \frac{L}{T(v_2 - v_1)} \quad \rightarrow \text{Clausius-Clapeyron equation}$$

$$\text{liquid: } \ln p = 24.38 - \frac{(3063 \text{ K})}{T}$$

$$\text{solid: } \ln p = 27.92 - \frac{(3754 \text{ K})}{T}$$

(i) at triple point:  $P = P_t, T = T_t$

$$\ln P_t = 24.38 - \frac{(3063 \text{ K})}{T_t} = 27.92 - \frac{(3754 \text{ K})}{T_t}$$

$$\rightarrow \frac{1}{T_t} (3754 - 3063) \text{ K} = 27.92 - 24.38$$

$$\therefore \bar{T}_t = \frac{3754 - 3063}{27.92 - 24.38} \text{ K} = 195.2 \text{ K} \quad \checkmark$$

$$\ln P_t = 24.38 - \frac{3063}{195.2} = 8.688$$

$$\rightarrow P_t = e^{8.688} \text{ Pa} = 5931 \text{ Pa} \quad \checkmark$$

(ii)  $\rightarrow$  Assuming ~~latent heat~~ latent heat  $L$  is independent of temperature

$\rightarrow$  Assuming  $V_{\text{vapour}} \gg V_{\text{liquid}}$  and  $V_{\text{solid}}$

~~then~~  $\rightarrow$  Assuming there is a boundary between solid and gas near the triple point

For both liquid and solid.

let  $V_{\text{vapour}} = V$ , then  $\Delta V \approx V$

$$\therefore \frac{dP}{dT} \approx \frac{L}{TV} \quad \because PV = RT \text{ for 1 mole}$$

$$\therefore \frac{dP}{dT} = \frac{LP}{RT^2}$$

$$\rightarrow \frac{dP}{P} = \frac{LdT}{RT^2}$$

~~$$\rightarrow \ln P = \text{const} + \frac{L}{RT}$$~~

$$\rightarrow \int \frac{dP}{P} = \frac{L}{R} \int \frac{dT}{T^2} \rightarrow \ln P = \text{const} - \frac{(L/R)}{T}$$

$$\therefore \text{liquid-gas} : \frac{L_{\text{vap}}}{R} = 3063 \text{ K}$$

$$\rightarrow L_{\text{vap}} = (3063 \text{ K}) \times (8.314 \text{ J/mol} \cdot \text{K})$$

$$= \underline{\underline{2.55 \times 10^4 \text{ J/mol}}} \quad \checkmark$$



~~$L_{\text{melt}}$~~   $\frac{L_{\text{sub}}}{R} = 3754 \text{ K}$

$$\rightarrow L_{\text{sub}} = (3754 \text{ K}) \times (8.314 \text{ J/mol}\cdot\text{K})$$
$$= \underline{3.12 \times 10^4 \text{ J/mol}} \quad \checkmark$$

near the triple point,  $L_{\text{sub}} = L_{\text{vap}} + L_{\text{melt}}$

$$\rightarrow L_{\text{melt}} = L_{\text{sub}} - L_{\text{vap}}$$
$$= \underline{5.7 \times 10^3 \text{ J/mol}} \quad \checkmark$$

(iii) <sup>if</sup> ~~we~~ need a ~~gas~~ <sup>Solid  $\rightarrow$  gas</sup> to Solid to gas  
phase transition

$$\therefore \ln(5 \times 10^4 \text{ Pa}) = 27.92 - \frac{3754}{T}$$

$$\rightarrow T = 219.5 \text{ K} > 195.2 \text{ K}$$

Not this

$$\rightarrow \Delta T = T - T_0 = \cancel{219.5 \text{ K}} - 165 \text{ K}$$
$$= \underline{54.5 \text{ K}}$$

$\rightarrow$  Solid  $\rightarrow$  liquid  $\rightarrow$  gas.

$$\ln(5 \times 10^4 \text{ Pa}) = \cancel{3063} 24.38 - \frac{3063}{T}$$

$$\rightarrow T = 225.8 \text{ K}$$

$$\therefore \Delta T = T - T_0 = \cancel{2} 225.8 \text{ K} - 165 \text{ K}$$
$$= \underline{60.8 \text{ K}}$$



9. The chemical potential is the energy cost of adding a particle into the system

$$dU = TdS - pdV + \mu dN$$

$$\therefore \mu = \left( \frac{\partial U}{\partial N} \right)_{S, V}$$

$$\mu = \frac{G}{N}$$

$$\mu = \frac{G}{N}$$

where  $G =$  ~~Gibbs~~ Gibbs function

If particles were distinguishable then the overall partition function is

$Z_N = Z_1^N$ , but since particles are indistinguishable, we need to divide this by the number of permutations that gives the same state overall of single states

→ for ideal classical gas neglecting the possibility that a single particle state can be occupied by more than 1 particle

we divide by the permutation of  $N$  particles  $N!$

$$\rightarrow Z_N = \frac{Z_1^N}{N!}$$

Grand partition function  $Z'$

$$Z' = \sum_{\alpha} e^{-\beta(E_{\alpha} - \mu N_{\alpha})}$$

States

for all possible

# of particles

$$\sum_{\alpha} e^{\beta \mu N_{\alpha}} \sum_N e^{\beta \mu N} \sum_{\alpha_N} e^{-\beta E_{\alpha_N}}$$

states for  $N$  particles

$$= \sum_N e^{\beta \mu N} Z_N$$

$$= \sum_N e^{\beta \mu N} \frac{Z_1^N}{N!} = \sum_N \frac{(e^{\beta \mu} Z_1)^N}{N!} = e^{Z_1 e^{\beta \mu}}$$

Grand partition function

$$\Phi = -k_B T \ln Z' = -k_B T Z_1 e^{N/k_B T}$$

$$\bar{N} = - \left( \frac{\partial \Phi}{\partial N} \right)_{T, V} = - \left( \frac{\partial \Phi}{\partial N} \right)_{T, Z_1}$$

$$\therefore Z_1 \propto V$$

$$= Z_1 e^{N/k_B T}$$

$$\therefore N = -k_B T \ln \frac{Z_1}{N}$$

~~energy of a~~

For the absorbed atoms, they effectively have a potential energy  $= -\epsilon_0$

$$\therefore \text{their energies } \epsilon = \frac{1}{2} m v^2 - \epsilon_0 = \frac{\hbar^2 k^2}{2m} - \epsilon_0$$

$$Z_{1,a} = \sum_{\alpha} e^{-\beta \epsilon_{\alpha}} = \int d\epsilon g(\epsilon) e^{-\beta \epsilon}$$

2-D density of states  $g(k) dk = \frac{A k}{2\pi} dk$  (Ar atom spins)  $2s+1=1$

$$\therefore \epsilon = \frac{\hbar^2 k^2}{2m} - \epsilon_0 \quad \therefore d\epsilon = \frac{\hbar^2 k}{m} dk$$

$$\therefore g(\epsilon) d\epsilon = g(k) dk \rightarrow g(\epsilon) \frac{\hbar^2 k}{m} dk = \frac{A k}{2\pi} dk$$

$$\therefore g(\epsilon) = \frac{m A}{2\pi \hbar^2}$$

$$\therefore Z_{1,a} = \int_{-\epsilon_0}^{\infty} d\epsilon \frac{m A}{2\pi \hbar^2} e^{-\beta \epsilon}$$

$$= \frac{m A}{2\pi \hbar^2} \left( -\frac{1}{\beta} \right) \left[ e^{-\beta \epsilon} \right]_{-\epsilon_0}^{\infty}$$

$$= \left( \frac{m A k_B}{2\pi \hbar^2} \right) T e^{\frac{\epsilon_0/k_B}{T}} = a A T e^{b/T}$$

$$a = \frac{mk_B}{2\pi\hbar^2} \quad b = \frac{\epsilon_0}{k_B}$$

For the atoms in the gas:

$$Z_{1,g} = \int d\epsilon g(\epsilon) e^{-\beta\epsilon}$$

3-D density of states with spin-0 Ar atoms:

$$g(k) = \frac{V}{2\pi^2} k^2 \quad \epsilon = \frac{\hbar^2 k^2}{2m} \quad d\epsilon = \frac{\hbar^2 k}{m} dk$$

$$g(\epsilon) d\epsilon = g(k) dk \Rightarrow$$

$$g(\epsilon) \frac{\hbar^2 k}{m} dk = \frac{V}{2\pi^2} k^2 dk$$

$$\rightarrow g(\epsilon) = \frac{mV}{2\pi^2 \hbar^2} k$$

$$\because \epsilon = \frac{\hbar^2 k^2}{2m} \quad \therefore k = \sqrt{\frac{2m}{\hbar^2}} \sqrt{\epsilon}$$

$$\therefore g(\epsilon) = \frac{mV}{2\pi^2 \hbar^2} \sqrt{\frac{2m}{\hbar^2}} \sqrt{\epsilon}$$

$$\therefore Z_{1,g} = \frac{mV}{2\pi^2 \hbar^3} \sqrt{2m} \int_0^\infty d\epsilon \sqrt{\epsilon} e^{-\beta\epsilon}$$

$$Z_{1,g} = \int_0^\infty dk g(k) e^{-\beta \frac{\hbar^2 k^2}{2m}} = \frac{V}{2\pi^2} \int_0^\infty e^{-\frac{\beta \hbar^2}{2m} k^2} k^2 dk$$

$$\text{let } x = \sqrt{\frac{\beta \hbar^2}{2m}} k, \text{ then } k^2 = \frac{2m}{\beta \hbar^2} x^2 \quad dk = \sqrt{\frac{2m}{\beta \hbar^2}} dx$$

$$\therefore Z_{1,g} = \frac{V}{2\pi^2} \left( \frac{2m}{\beta \hbar^2} \right)^{3/2} \int_0^\infty x^2 e^{-x^2} dx$$

$$= \frac{V}{4^{3/2}} \frac{1}{\pi^{3/2}} \left( \frac{2mk_B T}{\hbar^2} \right)^{3/2} = \frac{V}{\hbar^3} \left( \frac{mk_B T}{2\pi} \right)^{3/2}$$

$$= C V T^{3/2}$$

$$\rightarrow C = \frac{1}{\hbar^3} \left( \frac{mk_B}{2\pi} \right)^{3/2}$$

Equilibrium between 2 systems, chemical potentials are equal, temperatures equal

$$\rightarrow N_a = N_g = N, T_a = T_g = T$$

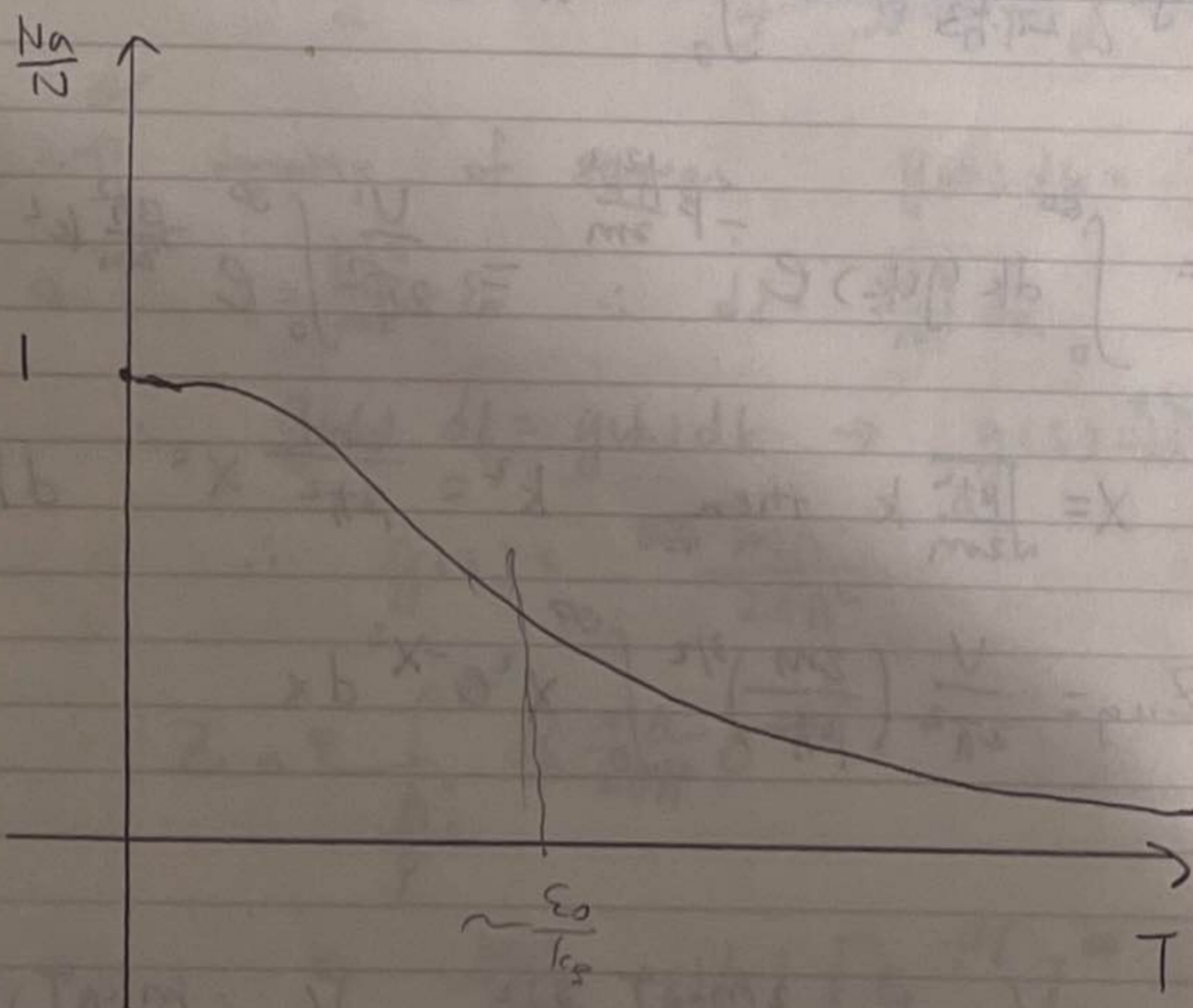
$$\therefore -k_B T \ln\left(\frac{Z_{1,a}}{N_a}\right) = -k_B T \ln\left(\frac{Z_{1,g}}{N_g}\right) = \mu$$

$$\rightarrow \frac{Z_{1,a}}{N_a} = \frac{Z_{1,g}}{N_g}$$

$$\therefore \frac{N_a}{N_g} = \frac{Z_{1,a}}{Z_{1,g}} = \frac{a A T e^{b/T}}{c V T^{3/2}}$$

$$= A \frac{a A}{c V} \frac{1}{\sqrt{T}} e^{b/T}$$

$$\frac{N_a}{N} = \frac{N_a}{N_a + N_g} = \frac{\frac{N_a}{N_g}}{\frac{N_a}{N_g} + 1} = \frac{\frac{a A}{c V} \frac{1}{\sqrt{T}} e^{b/T}}{\frac{a A}{c V} \frac{1}{\sqrt{T}} e^{b/T} + 1}$$



- At ~~low~~ low temperatures  $T \rightarrow 0$ , All atoms are absorbed and ~~none~~ none have enough energy to escape  $\therefore \frac{N_a}{N} \rightarrow 1$ .
- At high temperatures  $T \rightarrow \infty$ , All ~~atoms are~~ average energy per particle  $\gg E_0$   $\therefore$  All atoms can escape, none absorbed  $\therefore \frac{N_a}{N} \rightarrow 0$ .

$$10. \quad dU = Tds + f dL \quad \checkmark$$

$$(i) \quad \left(\frac{\partial S}{\partial L}\right)_T = ?$$

$$\text{Define } F = U - TS \quad \therefore dF = dU - Tds - SdT$$

$$\rightarrow dF = Tds + f dL - Tds - SdT = f dL - SdT$$

$$\text{At } \left(\frac{\partial F}{\partial L}\right)_T = f, \quad -S = \left(\frac{\partial F}{\partial T}\right)_L$$

$$\therefore \frac{\partial^2 F}{\partial L \partial T} = \frac{\partial^2 F}{\partial T \partial L} \quad \therefore \left(\frac{\partial f}{\partial T}\right)_L = -\left(\frac{\partial S}{\partial L}\right)_T$$

$$\rightarrow \left(\frac{\partial S}{\partial L}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_L \quad \therefore f = aT^2(L-L_0)$$

$$\therefore \left(\frac{\partial S}{\partial L}\right)_T = -\frac{\partial}{\partial T} (aT^2(L-L_0))$$

$$= -2aT(L-L_0)$$

$$T > 0 \quad \therefore \text{stretching} \quad \therefore L - L_0 > 0$$

$$\therefore \left(\frac{\partial S}{\partial L}\right)_T < 0 \rightarrow \text{entropy decrease}$$

$$(ii) \quad \left(\frac{\partial T}{\partial L}\right)_S$$

$$\left(\frac{\partial T}{\partial L}\right)_S > 0 ?$$

Adiabatic

$$ds = 0$$

$$\therefore dU = f dL = aT^2(L-L_0) dL$$

$$\therefore \text{stretching} \quad \therefore aT^2(L-L_0) > 0 \quad \therefore \text{if } dL > 0, \quad \rightarrow dU > 0 \rightarrow dT > 0 \quad \therefore \text{temperature increases}$$

$$C_L = T \left(\frac{\partial S}{\partial T}\right)_L = \left(\frac{\partial U}{\partial T}\right)_L$$

$$\therefore dU = \left(\frac{\partial U}{\partial T}\right)_L dT + \left(\frac{\partial U}{\partial L}\right)_T dL$$

$$= C_L dT + T \left(\frac{\partial S}{\partial L}\right)_T dL$$

$$\therefore dU = Tds + f dL \quad \therefore \left(\frac{\partial U}{\partial L}\right)_T = T \left(\frac{\partial S}{\partial L}\right)_T + f$$

$$= -2\alpha T^2(L-L_0) + f = -2f + f = f$$

$$\therefore dU = C_L dT - f dL \quad \rightarrow \text{general}$$

$$dU = f dL \quad \rightarrow \text{adiabatic}$$

$$\rightarrow \text{adiabatically} \quad C_L dT - f dL = f dL$$

$$\rightarrow C_L dT = 2f dL$$

$$\rightarrow \left(\frac{\partial T}{\partial L}\right)_S = \frac{C_L}{2f}$$

$$\because f > 0 \quad \therefore \text{if } C_L > 0,$$

$\therefore$  Under adiabatic stretching,

rubber warms if  $C_L > 0$

rubber cools if  $C_L < 0$

Physically:

(i) rubber band is made of long molecular chains. If stretched isothermally the molecules align better  $\rightarrow$  ~~more~~ less disorder  $\rightarrow$  entropy decreases.

(ii) stretched adiabatically then molecules align better. So to keep  $S$  constant we need to adjust temperature to increase entropy back to original value.

$\rightarrow$  If  $C_L > 0$  then  $\frac{\partial S}{\partial T}$  temperature needs to increase to get the system back to the same  $S$ .

$\rightarrow$  If  $C_L < 0$ , temperature needs to decrease to get back to same  $S$ .

$$\therefore \left(\frac{\partial S}{\partial L}\right)_T = -2aT(L-L_0) \quad \therefore \left(\frac{\partial S}{\partial(L-L_0)}\right)_T = -2aT(L-L_0)$$

$$\therefore S = -aT(L-L_0)^2 + S'(T)$$

where  ~~$S'(L)$~~   $S'(T)$  is a function of

$T$  only

$$\text{At } L=L_0 \quad S(L_0, T) = S'(T)$$

$$C_L = T \left(\frac{\partial S}{\partial T}\right)_L = T \frac{dS'}{dT} = bT$$

$$\rightarrow \frac{dS'}{dT} = b = \text{const} \quad \rightarrow S' = bT + S_0$$

$$\therefore \underline{S(L, T) = [b - a(L-L_0)^2]T + S_0} \rightarrow \text{Q.E.D.}$$

$$C_L = T \left(\frac{\partial S}{\partial T}\right)_L = \underline{[b - a(L-L_0)^2]T}$$



1 → 2 and 3 → 4.

iso thermal process :  $dT = 0$

$$dU = C_L dT - f dL = dQ + f dL$$
$$= -f dL$$

$$\rightarrow dQ = -2f dL$$

iso-length process :  $dL = 0$

$$dU = C_L dT - f dL = dQ + f dL$$
$$= C_L dT = dQ$$

$$\rightarrow dQ = C_L dT$$

2 → 3 ~~1 → 2~~

$$Q_{23} = \int_{T_1}^{T_h} [b - a(L_2 - L_0)^2] T dT$$

$$= \frac{1}{2} [b - a(L_2 - L_0)^2] [T_h^2 - T_1^2]$$

1 → 2 ~~2 → 3~~

$$Q_{12} = - \int_{L_1}^{L_2} 2f dL = -2aT_1^2 \int_{L_1 - L_0}^{L_2 - L_0} (L - L_0) d(L - L_0)$$

$$= -aT_1^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

3 → 4

$$Q_{34} = - \int_{L_2}^{L_1} 2f dL = aT_h^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

4 → 1

$$Q_{41} = \int_{T_h}^{T_1} [b - a(L_1 - L_0)^2] T dT$$

$$= \frac{1}{2} [b - a(L_1 - L_0)^2] [T_h^2 - T_1^2]$$

If  $L > 0$  :

Heat absorbed

$$Q_h = Q_{23} + Q_{34}$$

$$= \frac{1}{2} [b - a(L_2 - L_0)^2] [T_h^2 - T_c^2]$$

$$+ aT_h^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

Heat gives away:

$$Q_c = -(Q_{12} + Q_{41})$$

$$= \frac{1}{2} [b - a(L_1 - L_0)^2] [T_h^2 - T_c^2]$$

$$+ aT_c^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

work done

$$W = Q_h - Q_c = -\frac{1}{2} a [T_h^2 - T_c^2] [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$+ a [T_h^2 - T_c^2] [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$= \frac{1}{2} a [T_h^2 - T_c^2] [(L_1 + L_2 - 2L_0)(L_2 - L_1)]$$

$$\therefore \eta = \frac{W}{Q_h} = \frac{\frac{1}{2} a [T_h^2 - T_c^2] [(L_1 + L_2 - 2L_0)(L_2 - L_1)]}{\frac{1}{2} [b - a(L_2 - L_0)^2] [T_h^2 - T_c^2] + aT_h^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]}$$

$$\text{If } C_L < 0 \quad b - a(L^2 - L_0)^2 < 0$$

→

→

$$Q_h = Q_{12} + Q_{34}$$

$$= -\frac{1}{2}(b - a(L_1 - L_0)^2) [T_h^2 - T_1^2]$$

$$+ a T_h^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$Q_r = -(Q_{23} + Q_{41}) = -\frac{1}{2} [b - a(L_2 - L_0)^2] [T_h^2 - T_1^2]$$

$$+ a T_1^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$\therefore W = -\frac{1}{2} a [(L_2 - L_0)^2 - (L_1 - L_0)^2] (T_h^2 - T_1^2)$$

$$+ a (T_h^2 - T_1^2) [(L_2 - L_0)^2 - (L_1 - L_0)^2]$$

$$= \frac{1}{2} a (T_h^2 - T_1^2) (L_1 + L_2 - 2L_0) (L_2 - L_1)$$

$$\therefore \eta = \frac{W}{Q_h} = \frac{\frac{1}{2} a (T_h^2 - T_1^2) (L_1 + L_2 - 2L_0) (L_2 - L_1)}{-\frac{1}{2} (b - a(L_1 - L_0)^2) (T_h^2 - T_1^2) + a T_h^2 [(L_2 - L_0)^2 - (L_1 - L_0)^2]}$$

If  $C_L(L_1) > 0$   $C_L(L_2) < 0$ , then:

11. 3-D density of states for photons

$$g(k)dk = \frac{V}{2\pi^2} k^2 \times 2 dk = \frac{V}{\pi^2} k^2 dk$$

2 linearly independent polarisation states

photons:  $E_N = N\hbar\omega$ ,  $\omega = ck$ ,  $dk = \frac{1}{c}d\omega$   
 $\uparrow$   
 # of photons,  $k = \frac{\omega}{c}$

$$\therefore g(\omega)d\omega = g(k)dk = \frac{V}{\pi^2} \left(\frac{\omega}{c}\right)^2 \frac{d\omega}{c} = \frac{V\omega^2}{\pi^2 c^3} d\omega$$

$$\rightarrow g(\omega) = \frac{V}{\pi^2 c^3} \omega^2 d\omega, \text{ (frequency is quantized)}$$

Single frequency partition function

$$Z_\omega = \sum_{N=0}^{\infty} e^{-\beta N\hbar\omega} = 1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots$$

$$= \frac{1}{1 - e^{-\beta\hbar\omega}} \rightarrow \ln Z_\omega = -\ln(1 - e^{-\beta\hbar\omega})$$

overall partition function  $Z = \prod_{\omega} Z_\omega$

$$\therefore \ln Z = \sum_{\omega} \ln Z_\omega = \int_0^{\infty} d\omega g(\omega) [-\ln(1 - e^{-\beta\hbar\omega})]$$

$$= -\frac{V}{\pi^2 c^3} \int_0^{\infty} d\omega \omega^2 \ln(1 - e^{-\beta\hbar\omega})$$

total energy  $U = -\frac{\partial \ln Z}{\partial \beta} = \frac{V}{\pi^2 c^3} \int_0^{\infty} \omega^2 \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} d\omega$

$$= \frac{V}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^2 d\omega}{e^{\beta\hbar\omega} - 1} (\hbar\omega) \quad (1)$$

If  $n(\omega) d\omega$  is the # of photons in between  $[\omega, \omega + d\omega]$ , then another expression for internal energy is

$$U = \int_0^{\infty} \hbar\omega n(\omega) d\omega \quad (2)$$

Compare (1) and (2)  $\rightarrow n(\omega) d\omega = \frac{V}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/k_B T} - 1}$   
 (with  $\beta = \frac{1}{k_B T}$ )

CEP

energy density  $u = \frac{U}{V}$

$$u = \frac{1}{\pi^2 c^3} \int_0^{\infty} \frac{\omega^2 d\omega \cdot \hbar\omega}{e^{\hbar\omega/k_B T} - 1}$$

let  $x = \frac{\hbar\omega}{k_B T}$   
 $\omega = \frac{k_B T}{\hbar} x$   
 $d\omega = \frac{k_B T}{\hbar} dx$

$$\therefore u = \frac{\hbar}{\pi^2 c^3} \int_0^{\infty} \frac{\left(\frac{k_B T}{\hbar}\right)^3 x^3 dx}{e^x - 1}$$

$$= \frac{k_B^4 T^4}{\pi^2 (c\hbar)^3} \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

$$= \left( \frac{\pi^2 k_B^4}{15 (c\hbar)^3} \right) T^4 = AT^4$$

$$= A$$

$P =$  power per unit area  $=$  energy flux

$$= \int \left( \frac{u \cancel{V}}{N} \right) \times \left( \frac{N}{\cancel{V}} \right) \times (\cos \theta) (A) (dt) (\sin \theta d\theta d\phi)$$

$\frac{u \cancel{V}}{N}$ : energy per particle  
 $\frac{N}{\cancel{V}}$ : particle density  
 $(\cos \theta)$ : perpendicular velocity  
 $(A)$ : area  
 $(dt)$ : time interval  
 $(\sin \theta d\theta d\phi)$ : solid angle element  
 $A dt \int d\Omega$ : area normalise with respect to full solid angle

$$= \frac{u c}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin \theta \cos \theta$$

$\int_0^{\pi/2}$ : only forward moving photons hits the ~~cavity~~ wall  
 $\int_0^{2\pi}$ : solid angle element

$$= \frac{1}{4} c u = \frac{\pi^2 k_B^4}{60 \hbar^3 c^3} \left( \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \right) T^4 = \sigma T^4$$

$\sigma = \frac{\pi^5 k_B^4}{15 \hbar^3 c^3}$

$Q_h =$  heat ~~extra~~ extracted from nuclear ~~pt~~ reactor

$Q_c =$  heat released from the fins

$W =$  work done

Carnot engine

$$Q_h : Q_c : W = T_h : T_c : (T_h - T_c)$$

~~$$= T_0 : T : (T_0 - T)$$~~

$$= T_0 : T : (T_0 - T)$$

true for ~~an~~ any instant of time

$$\frac{dW}{dt} : \frac{dQ_r}{dt} : \frac{dQ_c}{dt} = T_0 : T : (T_0 - T)$$

$$\text{fin area} = S \quad \text{time } \Delta t$$

$$\frac{\text{Work done}}{\text{Heat released}} = \frac{dW}{dQ_r} = \frac{T_0 - T}{T} = \frac{W}{Q_r} = \frac{T_0 - T}{T}$$

$$Q_r = (\text{power radiated by fins / area})$$

$$\times S \times \Delta t$$

$$= \sigma T^4 S \Delta t$$

$$W = \frac{T_0 - T}{T} \times \sigma T^4 S \Delta t$$

$$= \sigma S \Delta t (T_0 - T) T^3$$

$\therefore$  maximising  $W \rightarrow$  maximising  $T^3(T_0 - T)$

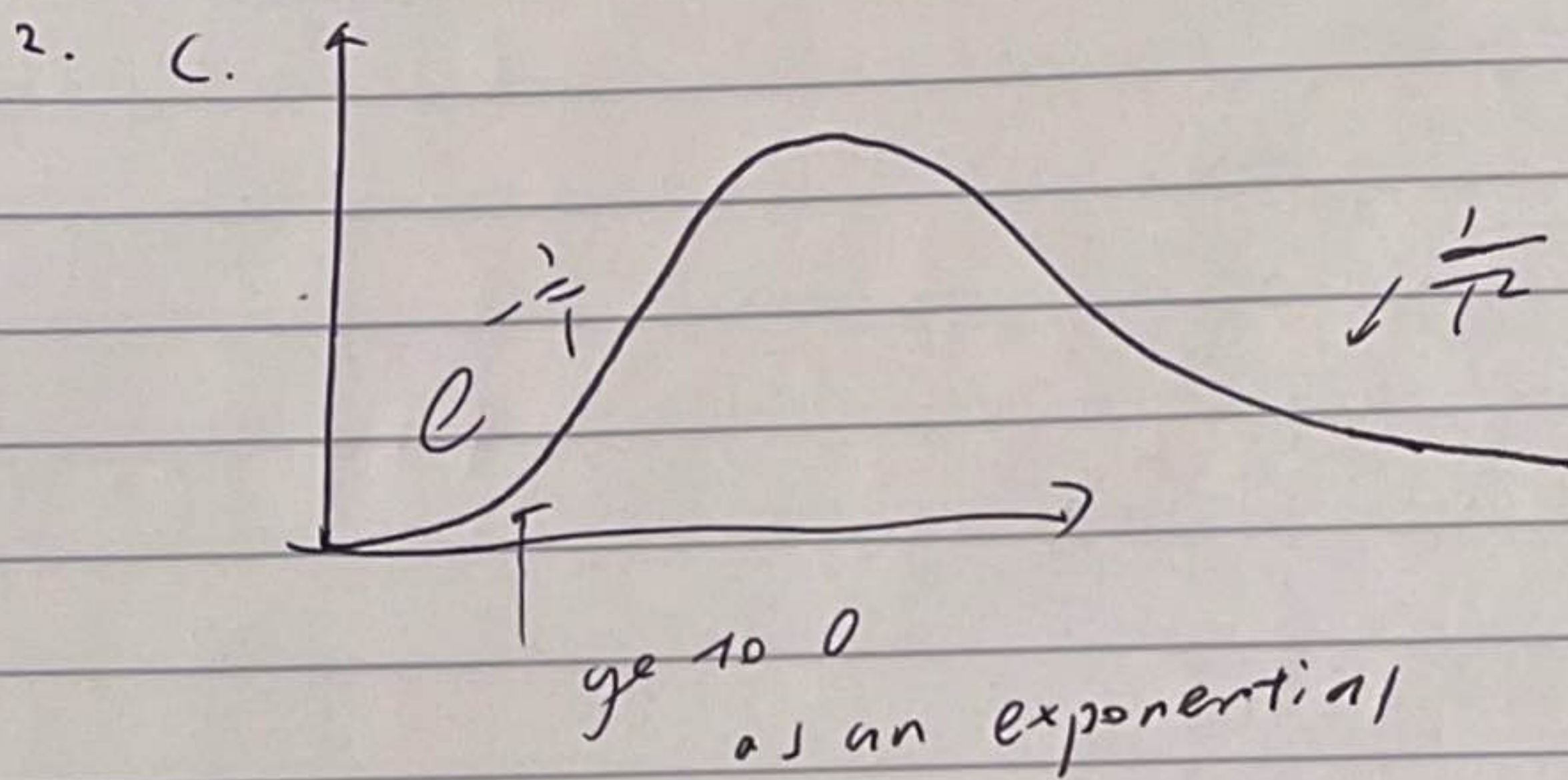
$$0 = \frac{d}{dT} [T^3(T_0 - T)] = 3T^2 T_0 - 4T^3$$

$$\rightarrow 4T = 3T_0 \rightarrow T = \frac{3}{4} T_0 \rightarrow \text{Optimum temperature}$$

$$\text{efficiency } \eta = \frac{T_0 - T}{T} = 1 - \frac{T}{T_0} = 1 - \frac{3}{4} = \frac{1}{4}$$

# AI 2015 Tutorial Notes

1. ✓ ✓ ✗

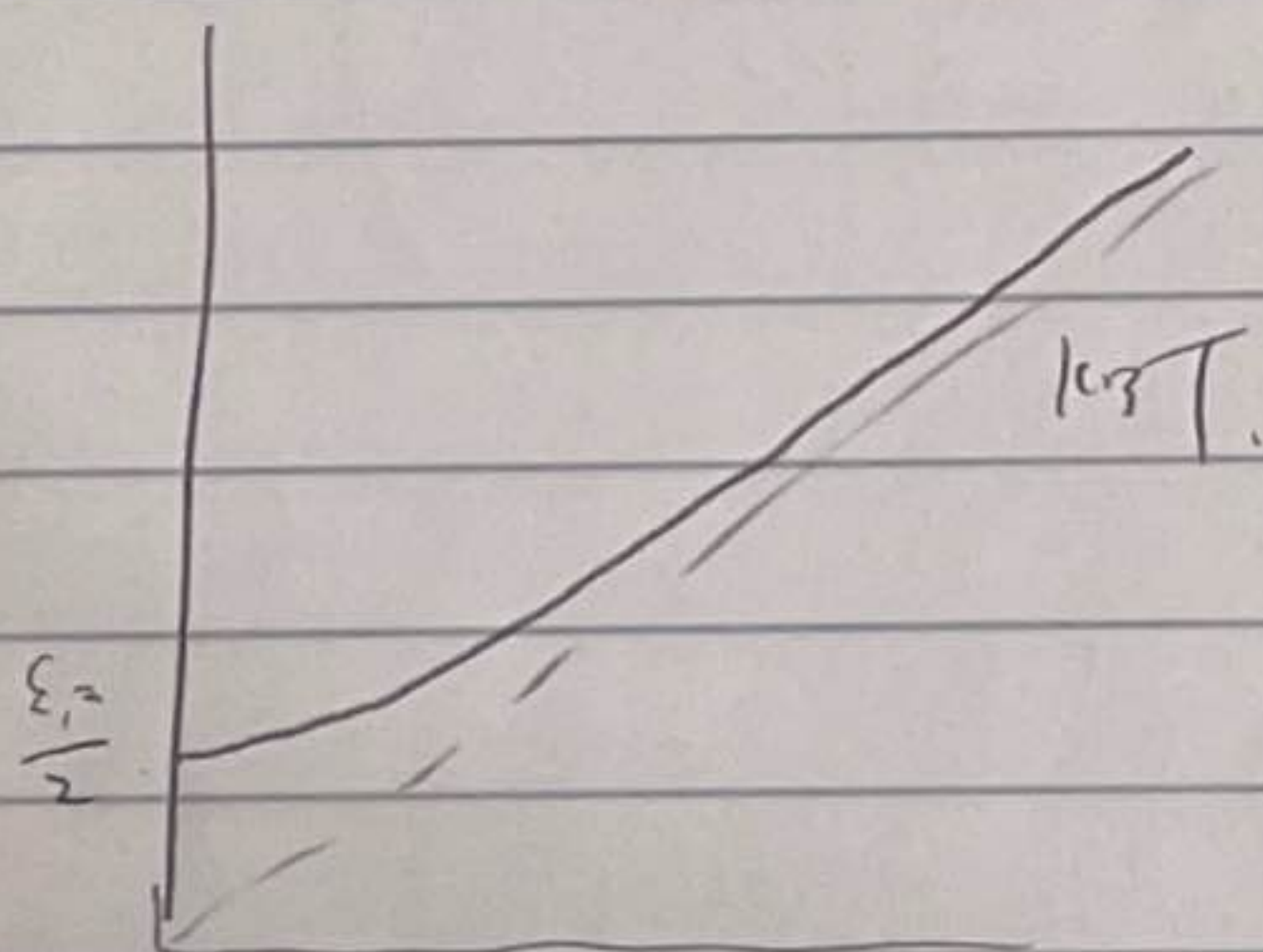


3

4

5

6.  $g(k)dk = g(\epsilon)d\epsilon = \frac{2\pi k dk}{(\frac{2\pi}{L})^2}$





A1 2015 Q10

$$dU = Tds + fdL$$

$$f = aT^2(L - L_0)$$

(i)  $dT = 0$

$$F = U - TS$$

$$dF = dU - Tds - sdT = fdL - sdT$$

$$= \left(\frac{\partial F}{\partial L}\right)_T dL + \left(\frac{\partial F}{\partial T}\right)_L dT$$

$$\left(\frac{\partial F}{\partial L}\right)_T = f \quad \left(\frac{\partial F}{\partial T}\right)_L = -s \Rightarrow \left(\frac{\partial f}{\partial T}\right)_L = -\left(\frac{\partial s}{\partial L}\right)_T$$

$$\rightarrow -\frac{2f}{T} = \left(\frac{\partial s}{\partial L}\right)_T < 0$$

$\rightarrow$  entropy decreases

(ii)  $ds = 0$

$$dU = \left(\frac{\partial U}{\partial L}\right)_S dL + \left(\frac{\partial U}{\partial S}\right)_L dS$$

$$\rightarrow \left(\frac{\partial f}{\partial S}\right)_L = \left(\frac{\partial T}{\partial L}\right)_S = \left(\frac{\partial f}{\partial T}\right)_L \left(\frac{\partial T}{\partial S}\right)_L$$

↑ chain rule (not cyclic!)

$$T \left(\frac{\partial s}{\partial T}\right)_L = C_L$$

$$\rightarrow \left(\frac{\partial f}{\partial T}\right)_L \frac{T}{C_L} = \left(\frac{\partial T}{\partial L}\right)_S$$

$$\frac{2f}{T} \cdot \frac{T}{C_L} = \left(\frac{\partial T}{\partial L}\right)_S > 0 \rightarrow \text{the rubber heats up}$$

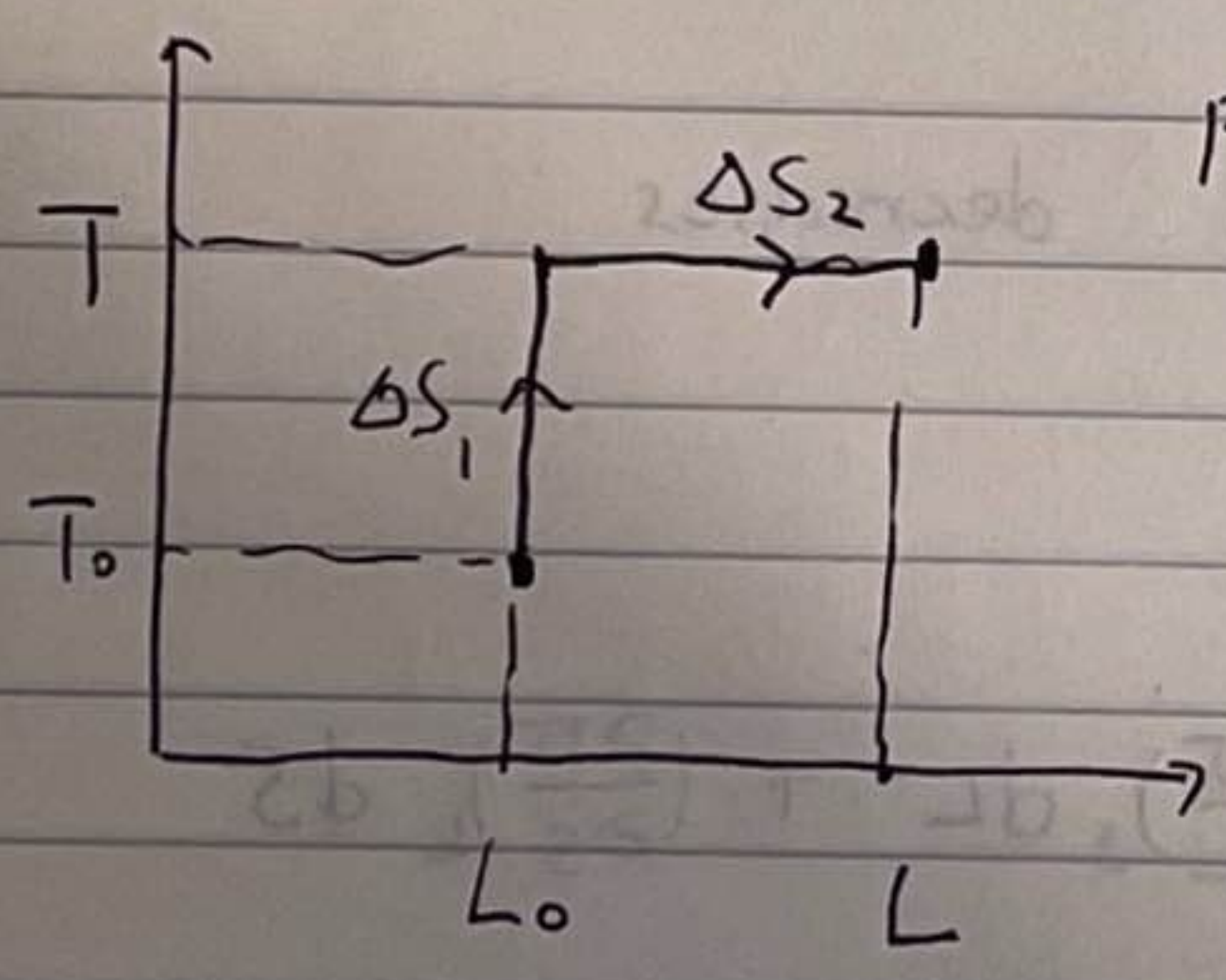
(i) as rubber stretched, molecules are better aligned  $\rightarrow S \downarrow$  if  $T -$

(ii) as rubber stretched, ~~the~~ molecule . . .  
 $\rightarrow$  if  $S -$   $T \uparrow$

$$C_L(L_0, T) = bT$$

$$\rightarrow S(L, T) = [b - a(L - L_0)^2] T + S_0 \quad ?$$

$$dS = \left(\frac{\partial S}{\partial L}\right)_T dL + \left(\frac{\partial S}{\partial T}\right)_L dT$$



path doesn't matter

$$S(L, T) = S(L_0, T_0) + \Delta S_1 + \Delta S_2$$

$$\Delta S_1 = \int_{T_0}^T dT \left(\frac{\partial S}{\partial T}\right)_L = \int_{T_0}^T dT \frac{C_L(L_0, T)}{T}$$

$$= b(T - T_0)$$

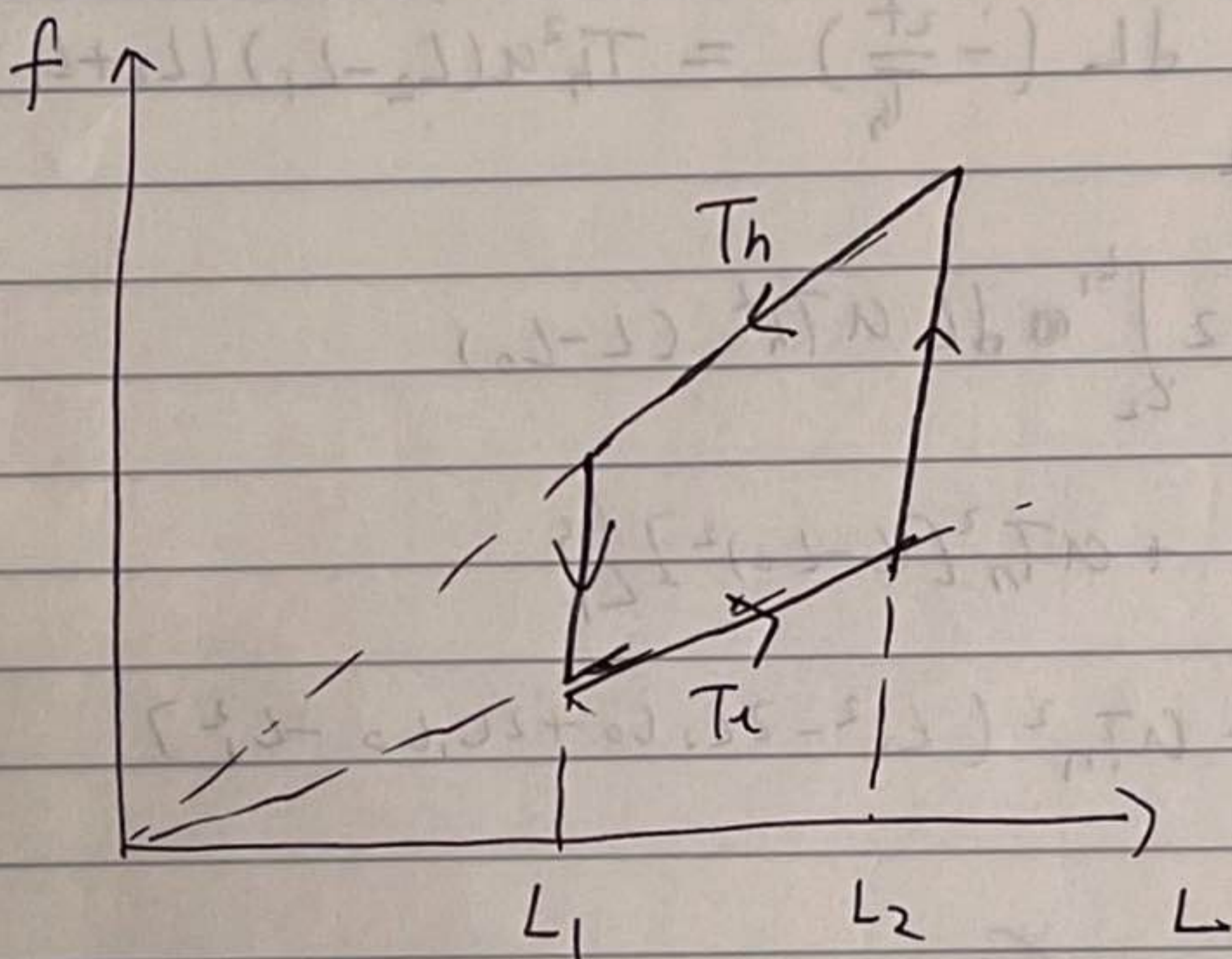
$$\Delta S_2 = \int_{L_0}^L dL \left(\frac{\partial S}{\partial L}\right)_T = - \int_{L_0}^L dL \left(\frac{df}{dL}\right)_T = -aT(L - L_0)^2$$

$$S(L, T) = S(L_0, T_0) + b(T - T_0) - aT(L - L_0)^2$$

$$\text{if } S_0 = S(L_0, T_0) - bT_0$$

$$\text{then } S(L, T) = [b - a(L - L_0)^2]T + S_0$$

$$C_L = T \left( \frac{\partial S}{\partial T} \right)_L = [b - a(L - L_0)^2]T$$



work done by the system

$$-\oint f dL > 0$$

Work done  $> 0$

$\rightarrow$  counter clockwise.

$$Q_{\text{abs}} = Q_{23} + Q_{34}$$

↑  
increasing  $T$   
fixed  $L$

↑  
isothermal  
entropy increases.

$$Q_{23} = \int_{T_c}^{T_h} dT (L) = [b - a(L - L_0)^2] \left[ \frac{T_h^2 - T_c^2}{2} \right]$$

$$Q_{34} = T_h \int_{L_2}^{L_1} dL \left( -\frac{2a}{T_h} \right) = T_h^2 a (L_2 - L_1) (L_2 + L_1 - 2L_0)$$

$$= -2 \int_{L_2}^{L_1} a dL a T_h^2 (L - L_0)$$

$$= + a T_h^2 [(L - L_0)^2]_{L_1}^{L_2}$$

$$= a T_h^2 (L_2^2 - 2L_2 L_0 + 2L_1 L_0 - L_1^2)$$

$$\eta = 1 - \frac{Q_{\text{out}}}{Q_{\text{in}}}$$