2661

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A1: THERMAL PHYSICS

TRINITY TERM 2014

Wednesday, 18 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The chemical potential for a massless boson is 0 because you ~~ct~~ can put any # of them into the ground state → doesn't add any energy?

Section A

1. State what is meant by the following concepts in themodynamics giving an example in each case:

   (i) the zeroth law,
   (ii) a function of state,
   (iii) a reversible change. [6]

2. Derive the thermodynamic relation

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left\{T\left(\frac{\partial P}{\partial T}\right)_V - P\right\}.$$

Explain why, when a perfect gas expands into a vacuum, its temperature remains unchanged. [6]

3. Write down an expression relating the mean energy $\langle U \rangle$, where the angular brackets denote an average taken in the canonical ensemble, to the canonical partition function Z. Write down a similar expression for $\langle U^2 \rangle$. Hence, or otherwise, prove that

$$\langle (U - \langle U \rangle)^2 \rangle = k_B T^2 C_V$$

where T is the temperature and C_V is the specific heat at constant volume. [5]

4. Derive an expression for the dependence of the magnetization of a spin-$\frac{1}{2}$ paramagnet which has N localised spins, each of magnetic moment μ, on the magnetic field B and temperature T. Sketch the variation of the magnetization with field at a constant temperature. [5]

5. Calculate the Fermi energy of a highly relativistic gas of non-interacting neutrons at a density $\rho = 5 \times 10^{17}$ kg m^{-3}. [6]

?

6. The normal distribution may be written

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad -\infty \leq x \leq \infty.$$

Which properties of the distribution are described by the parameters μ and σ?
The z-component of the velocity of an ideal gas v_z is normally distributed. Write down μ and σ for $P(v_z)$ in terms of the properties of the gas and relate your answer to the equipartition theorem.
Briefly explain the importance of the normal distribution in describing the statistics of large data sets. [5]

7. Sketch, over the interval $(-2\pi, 2\pi)$, the function $f(x)$, of period 2π, that is equal to unity for $0 < x < \pi$ and can be expanded as a Fourier sine series. Find the Fourier series for $f(x)$, and hence show that

$$(1 - \frac{1}{3} + \frac{1}{5} - \ldots) = \frac{\pi}{4}.$$

[7]

Section B

8. Show that for an adiabatic expansion of a perfect gas PV^γ is constant where γ is the ratio of the specific heat at constant pressure C_P to the specific heat at constant volume C_V. [5]

The Joule cycle consists of the following steps:

$A \Rightarrow B$: adiabatic compression from a pressure P_1 to a pressure P_2.
$B \Rightarrow C$: expansion at a pressure P_2.
$C \Rightarrow D$: adiabatic expansion from a pressure P_2 to a pressure P_1.
$D \Rightarrow A$: compression at a pressure P_1.

7

Draw the cycle on a $P - V$ diagram, labeling the points A,B,C,D and indicating the steps during which heat enters and leaves the system. [3]

Assuming that the working substance is a perfect gas, with C_P independent of volume, obtain expressions for

(i) the work done by the gas along each of the 4 steps of the cycle,
(ii) the heat supplied to the gas,

in terms of P_1, P_2 and the volumes $V_A \ldots V_D$ at the state points $A \ldots D$. [7]

Hence show that the efficiency of the engine described by this cycle is

$$\eta = \frac{\text{total work done by the gas}}{\text{heat supplied to the gas}} = 1 - \left(\frac{P_1}{P_2}\right)^{(\gamma-1)/\gamma}.$$

[5]

7

9. A circular plate has mass m and radius a and is of negligible thickness. Prove that the moment of inertia about an axis passing through its centre perpendicular to the plane of the plate is $ma^2/2$. [3]

Use kinetic theory to show that the viscosity of a gas is

$$\eta = \alpha m n \langle c \rangle \lambda$$

where n is the number density, m is the mass, $\langle c \rangle$ is the mean speed, and λ is the mean free path of molecules in the gas, and α is a dimensionless constant of order unity. [7]

The circular plate is suspended on a wire attached to its centre in a container of nitrogen so that it lies parallel to, and a distance l from, the bottom surface of the container. It is set oscillating in rotational motion of small amplitude about a vertical axis through its centre. Assuming that the natural frequency of oscillation (in the absence of damping) is ω_0, and that the only contribution to damping is from the viscous interaction of the plate with the base of the container, find an approximate expression for the fractional change in amplitude $\delta\theta_0/\theta_0$ per cycle of the plate. [7]

If $l = 1$cm how will $\delta\theta_0/\theta_0$ depend on temperature at a pressure of (i) $P = 10^5$ Nm^{-2}, (ii) $P = 10^{-3}$ Nm^{-2}? $-\eta\frac{v}{d}$ [3]

$\frac{F}{A} = \Pi_{zx} = -\eta\frac{\partial u_x}{\partial z}$

$A\cos x + B\sin x = \theta_0 \cos(x+\phi)$

$= \theta_0(\cos x \cos\phi - \sin x \sin\phi)$

$\therefore \tan\phi = -\frac{B}{A}$ $\theta_0 =$

why is # of magnon not conserved ?

why is chemical potential of magnon /quasiparti /massless boson zero ?

10. At low temperatures magnons, the elementary excitations of a ferromagnetic ground state, can be treated as a gas of non-interacting, massless, spin-0 bosons with a dispersion relation $\omega = \alpha k^2$ where α is a constant.

Show that, in three dimensions, the magnon density of states is

$$g(\omega) \propto \omega^{1/2}$$

and obtain an expression for the constant of proportionality in terms of α and the volume V of the ferromagnet. [4]

Write down an expression for $E(\omega)\mathrm{d}\omega$, the total magnon energy in the frequency interval $(\omega, \omega + \mathrm{d}\omega)$, and show that $E(\omega)$ has a maximum at a frequency $\omega^* = ck_BT/\hbar$ where c is a constant that you should evaluate to 2 significant figures. [7]

Sketch $E(\omega)$ as a function of ω, comparing the behaviour for two different temperatures, $T_1 > T_2$, on the same graph. [3]

Show that the magnon specific heat at low temperature is proportional to $T^{3/2}$. Explain why the temperature dependence of the specific heat differs from that of a photon gas which is proportional to T^3. [6]

?

11. Derive the heat conduction equation

$$\frac{\partial T}{\partial t} = K\nabla^2 T$$

writing K in terms of physical constants which you should define. [5]

If the temperature on the plane $z = 0$ is

$$T(0,t) = T_0 + T_1 \cos\omega_1 t + T_2 \cos\omega_2 t$$

find an expression for the temperature $T(z,t)$ for $z > 0$ that decays to T_0 as $z \to \infty$. [5]

For $\omega_1 \ll \omega_2$ sketch $T(z,t)$ as a function of z for

(i) $T_1 = T_2$ at $t = 0$,
(ii) $T_2 = 0$ at $t = \pi/(2\omega_1)$. [5]

The average temperature in Yakutsk is 20°C in July and −40°C in January. Given that $K = 3 \times 10^{-7}\,\mathrm{m^2\,s^{-1}}$ estimate the depth below the surface of the top of the permafrost (permanently frozen ground). [5]

A1 2014 First Attempt.

1. (i) Zeroth Law : Two systems, each separatly in thermal ~~contact~~ equilibrium with the third, then they are in equilibrium with each other

(ii) A function of state is any physical quantity that has a well defined value for each equilibrium state of the system. Its differential is an exact differential and its change only depends on the initial and ~~fin~~ final states, not on the path taken

(iii) A reversible change is a change sufficiently slow that the ~~syse~~ system remains in equilibrium throughout the process. It's direction can be reversed and the entropy change is 0

2.

$$\left(\frac{\partial T}{\partial V}\right)_U = -\left(\frac{\partial T}{\partial U}\right)_V \left(\frac{\partial U}{\partial V}\right)_T$$

$$= -\frac{1}{C_V}\left(\frac{\partial U}{\partial V}\right)_T \qquad \left(C_V = \left(\frac{\partial U}{\partial T}\right)_V\right)$$

$$dU = TdS - PdV \rightarrow \left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P$$

maxwell relation : $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$

$$\therefore \left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]$$

QED.

perfect gas $\quad PV = Nk_BT$

$$\therefore \left(\frac{\partial P}{\partial T}\right)_V = \cancel{Nk_B}\ \frac{Nk_B}{V} = \frac{P}{T}$$

Expand into a vacuum :

No heat transfer : $đQ = 0$

No work done : $đW = 0$

$dU = đQ + đW = 0 \rightarrow U$ is constant

temperature change : $dT = \left(\frac{\partial T}{\partial V}\right)_U dV$

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]$$

$$= -\frac{1}{C_V}\left[\underbrace{T\left(\frac{P}{T}\right) - P}_{0}\right] = 0$$

$\rightarrow \quad dT = 0 \rightarrow$ temperature remains unchanged.

3. $\langle (U - \langle U\rangle)^2\rangle = \langle U^2 - 2U\langle U\rangle + \langle U\rangle^2\rangle$

$$= \langle U^2\rangle - \langle U\rangle^2$$

$$\langle U\rangle = -\frac{\partial \ln Z}{\partial \beta} = \sum_\alpha P_\alpha E_\alpha = \frac{1}{Z}\sum E_\alpha e^{-\beta E_\alpha} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$

$$\langle U^2\rangle = \sum_\alpha P_\alpha E_\alpha^2 = \frac{1}{Z}\sum E_\alpha^2 e^{-\beta E_\alpha}$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2}$$

$\beta = \frac{1}{k_BT}$

$T = \frac{1}{k_B\beta}$

$$C_V = \frac{\partial \langle U \rangle}{\partial T} = \frac{\partial \langle U \rangle}{\partial \beta}\frac{\partial \beta}{\partial T}$$

$$\therefore \quad \frac{\partial T}{\partial \beta} C_V = \frac{\partial \langle U \rangle}{\partial \beta} = -\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta}\right)$$

$$\cancel{= -\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\right)\frac{\partial^2 Z}{\partial \beta^2}}$$

$$= -\frac{\partial}{\partial \beta}\left(\frac{1}{Z}\right)\frac{\partial Z}{\partial \beta} - \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta}$$

$$= \cancel{\frac{1}{Z^2}\frac{\partial^2 Z}{\partial}} \frac{1}{Z^2}\left(\frac{\partial Z}{\partial \beta}\right)^2 - \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta}$$

$$= \langle U \rangle^2 - \langle U^2 \rangle$$

$$\frac{\partial T}{\partial \beta} C_V = -\frac{1}{k_B \beta^2} C_V = -\frac{k_B^2 T^2}{k_B} C_V = -k_B T^2 C_V$$

$$\rightarrow \quad \langle U^2 \rangle - \langle U \rangle^2 = k_B T^2 C_V$$

QED

4. spin-$\frac{1}{2}$ localised paramagnet: single particle ~~partition function~~ energy levels

$\cancel{Z}$ $\pm \mu B$

single particle partition function:

$$\cancel{Z_1 = e^{\beta \mu B} + e^{-\beta \mu B} = 2\sinh(\beta \mu B)}$$

~~Overall partition function: $Z = (2\sinh(\beta \mu B))^N$~~

~~$\left(\beta = \frac{1}{k_B T}\right)$~~

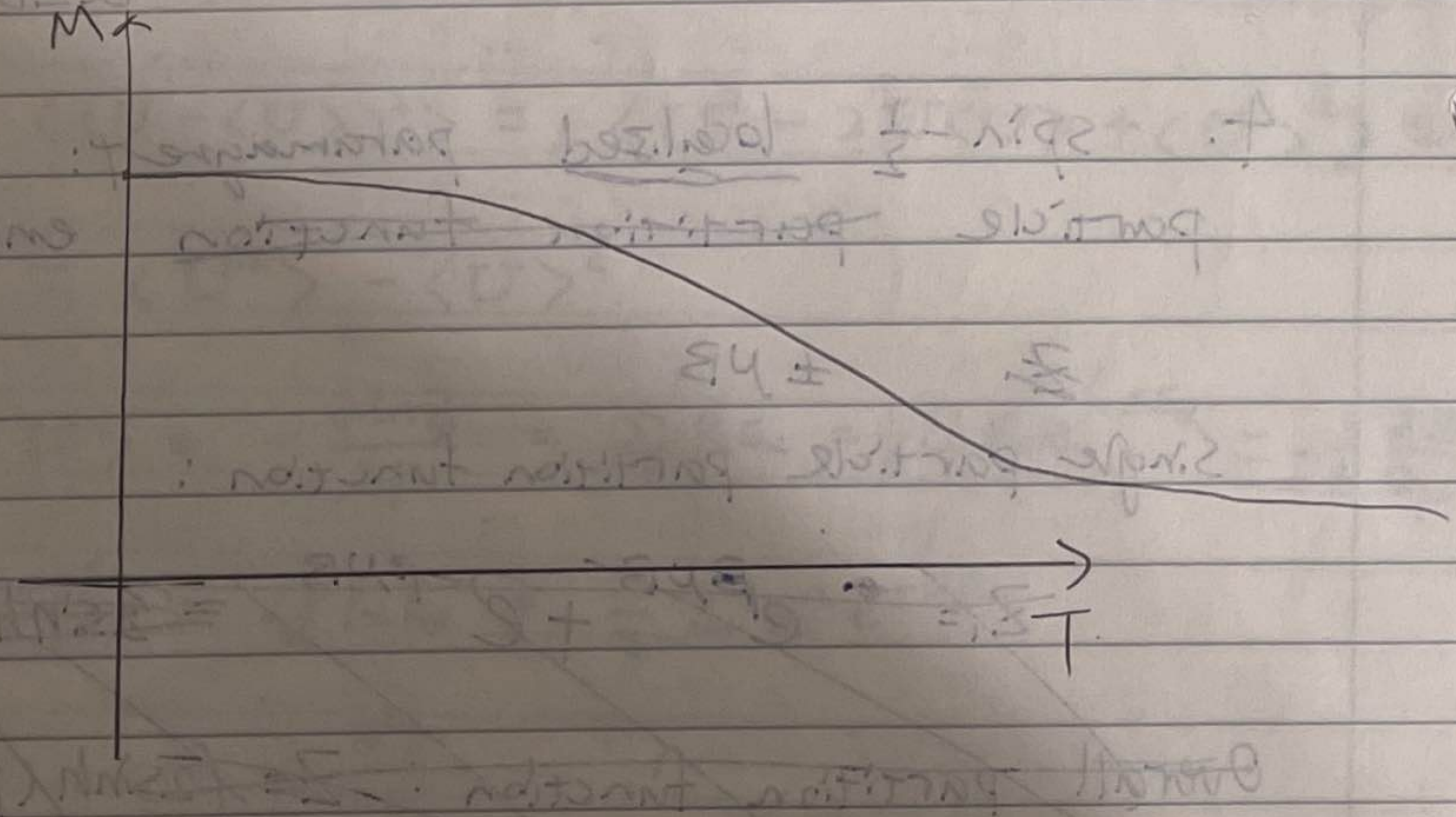
$$Z_1 = e^{\beta\mu B} + e^{-\beta\mu B} = 2\cosh(\beta\mu B)$$

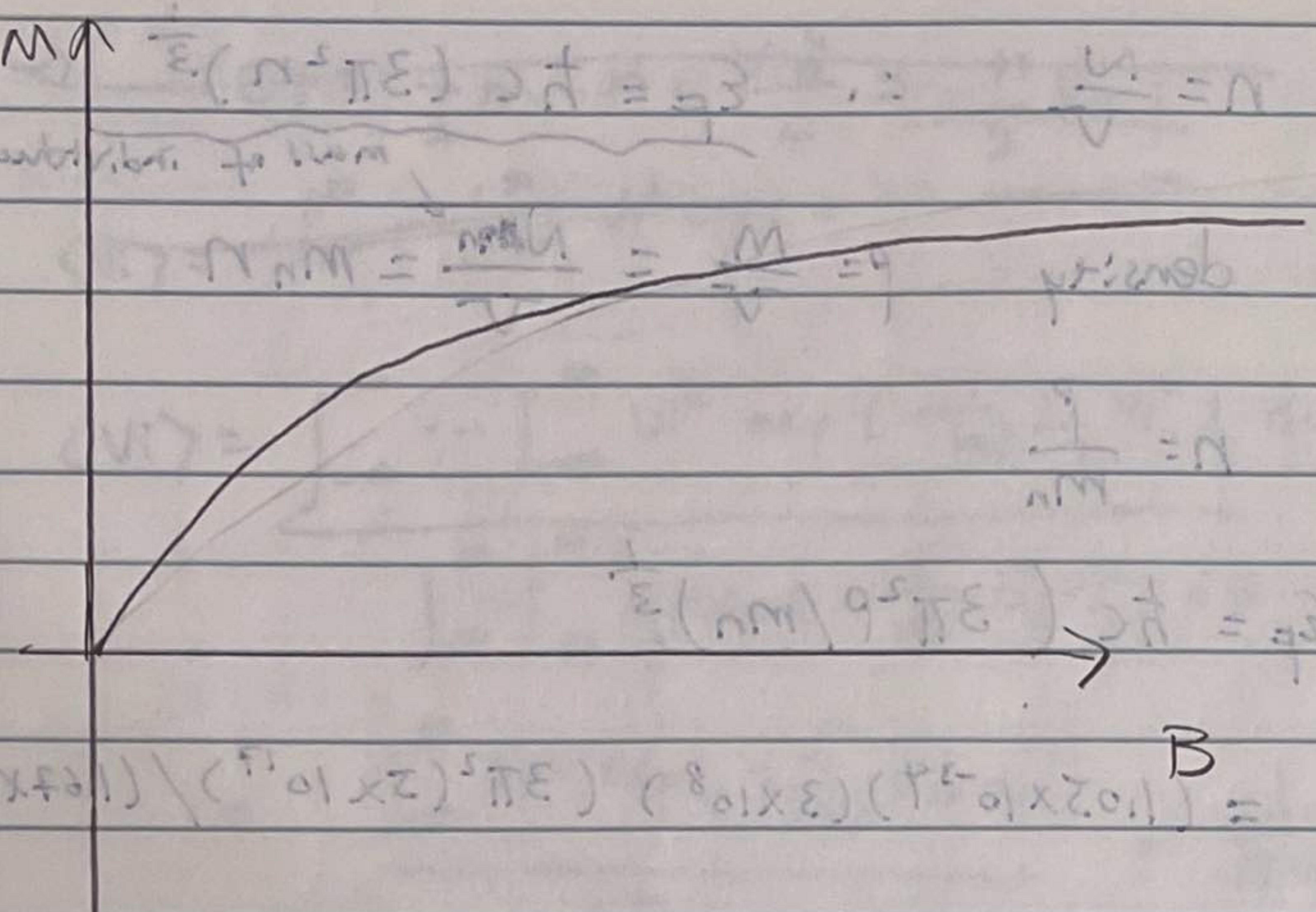
Overall $Z = Z_1^N = (2\cosh(\beta\mu B))^N$

magnetisation $M = -\frac{1}{V}\left(\frac{\partial F}{\partial B}\right)_T$

$$F = -k_B T \ln Z = \cancel{-k_B T \ln(2}$$
$$= -N k_B T \ln(2\cosh(\beta\mu B))$$

$$M = -\frac{1}{V}(-Nk_BT)\frac{\partial}{\partial B}\ln(2\cosh(\beta\mu B))$$
$$= \frac{Nk_BT}{V}\frac{1}{2\cosh(\beta\mu B)}(2\sinh(\beta\mu B))\cdot\beta\mu$$
$$= \frac{N\cancel{k_BT}}{V}\frac{\mu}{\cancel{k_BT}}\tanh(\beta\mu B)$$
$$= \frac{N\mu}{V}\tanh\left(\frac{\mu B}{k_BT}\right)$$





5. 3-D density of states : $g(k)\,dk = \frac{V}{2\pi^2} k^2 dk$

ultra-relativistic gas : $\varepsilon = \hbar k c$ $\therefore d\varepsilon = \hbar c\, dk$

$$\therefore g(\varepsilon) d\varepsilon = \frac{V}{2\pi^2} \left(\frac{\varepsilon}{\hbar c}\right)^2 \frac{1}{\hbar c} d\varepsilon \times (2s+1)$$

$$= \frac{V}{2\pi^2(\hbar c)^3} \varepsilon^2 d\varepsilon \times (2s+1)$$

At $T=0$, $\bar{n}(\varepsilon \le \varepsilon_F) = 1$, $\bar{n}(\varepsilon > \varepsilon_F) = 0$

$$\therefore N = \int_0^\infty d\varepsilon\, g(\varepsilon)\, \bar{n}(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, g(\varepsilon)$$

$$= \frac{V(2s+1)}{2\pi^2(\hbar c)^3} \int_0^{\varepsilon_F} d\varepsilon \cdot \varepsilon^2 = \frac{V}{2\pi^2(\hbar c)^3} \frac{\varepsilon_F^3}{3} (2s+1)$$

For neutron $s = \frac{1}{2} \rightarrow 2s+1 = 2$

$$\therefore N = \frac{V}{2\pi^2(\hbar c)^3} \frac{2}{3} \varepsilon_F^3$$

$\because \quad n = \frac{N}{V} \qquad \therefore \quad \underbrace{\varepsilon_F = \hbar c \, (3\pi^2 n)^{\frac{1}{3}}}$

mass of individual neutron

mass density $\quad \rho = \frac{M}{V} = \frac{N m_n}{V} = m_n n$

$\therefore \quad n = \frac{\rho}{m_n}$

$\therefore \quad \varepsilon_F = \hbar c \left(3\pi^2 \rho / m_n \right)^{\frac{1}{3}}$

$= (1.05 \times 10^{-34})(3 \times 10^{8}) \left(3\pi^2 (5 \times 10^{17}) / (1.67 \times 10^{-27}) \right)^{\frac{1}{3}}$

$= \underline{6.5 \times 10^{-11} \, J}$

6. $\quad p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(x-\mu)^2}{2\sigma^2} \right\}$

μ is the mean ; σ is the standard deviation

$\underline{P(v_z) = \frac{1}{\sqrt{\pi} v_{th}} \exp\left\{ - \frac{v_z^2}{v_{th}^2} \right\}} \qquad \left(v_{th} = \sqrt{\frac{2k_B T}{m}} \right)$

$\rightarrow \quad \underline{\mu = 0} \qquad 2\sigma^2 = v_{th}^2 \rightarrow \quad \underline{\sigma = \frac{v_{th}}{\sqrt{2}}}$

If a particle only has velocity in the z-direction
the average kinetic energy (one quadratic mode)

$\langle E \rangle = \frac{1}{2} m \langle v_z^2 \rangle = \frac{1}{2} m \frac{1}{\sqrt{\pi} v_{th}} \int_{-\infty}^{\infty} \cancel{v_{th}} \, v_z^2 \exp\left(- \frac{v_z^2}{v_{th}^2} \right) dv_z$

$= \frac{1}{2} m \frac{1}{\cancel{\sqrt{\pi}} \cancel{v_{th}}} \frac{v_{th}^2}{2} \cdot \cancel{\sqrt{\pi}} \cancel{v_{th}} = \frac{1}{4} m v_{th}^2 = \frac{1}{4} m \frac{2k_B T}{m} = \underline{\frac{1}{2} k_B T}$

If velocities has ~~A~~ n-components, then

$\langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \sum_{i=1}^{n} \langle v_i^2 \rangle$, $\cancel{\langle v_i^2 \rangle = \int_{-\infty}^{\infty} v_i^2 \exp\left(- \frac{v_i^2}{v_{th}^2} \right) dv_i}$

$= \frac{v_{th}^2}{2}$

$$\cancel{\rightarrow \langle E\rangle = \frac{1}{2} m \cdot n \cdot \frac{v_{th}^2}{2} = \frac{n}{2} k_B T}$$

$$\langle v_i \rangle = \cancel{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v_i^2 \exp(- = }$$

$$\langle v_i \rangle = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} v_i^2 \exp\left(-\frac{1}{v_{th}^2} \sum_j v_j^2\right) dv_1 \cdots dv_n}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(-\frac{1}{v_{th}^2} \sum_j v_j^2\right) dv_1 \cdots dv_n}.$$

$$= \frac{\int_{-\infty}^{\infty} v_i^2 \exp\left(-\frac{v_i^2}{v_{th}^2}\right) dv_i}{\int_{-\infty}^{\infty} \exp\left(-\frac{v_i^2}{v_{th}^2}\right) dv_i} = \frac{1}{\sqrt{\pi}\, v_{th}} \int_{-\infty}^{\infty} v_i^2 \exp\left(-\frac{v_i^2}{v_{th}^2}\right) dv_i$$

$$= \frac{v_{th}^2}{2}$$

$$\rightarrow \langle E \rangle = \frac{1}{2} m n \frac{v_{th}^2}{2} = \frac{n}{2} k_B T$$

Consistent with equipartition of energy

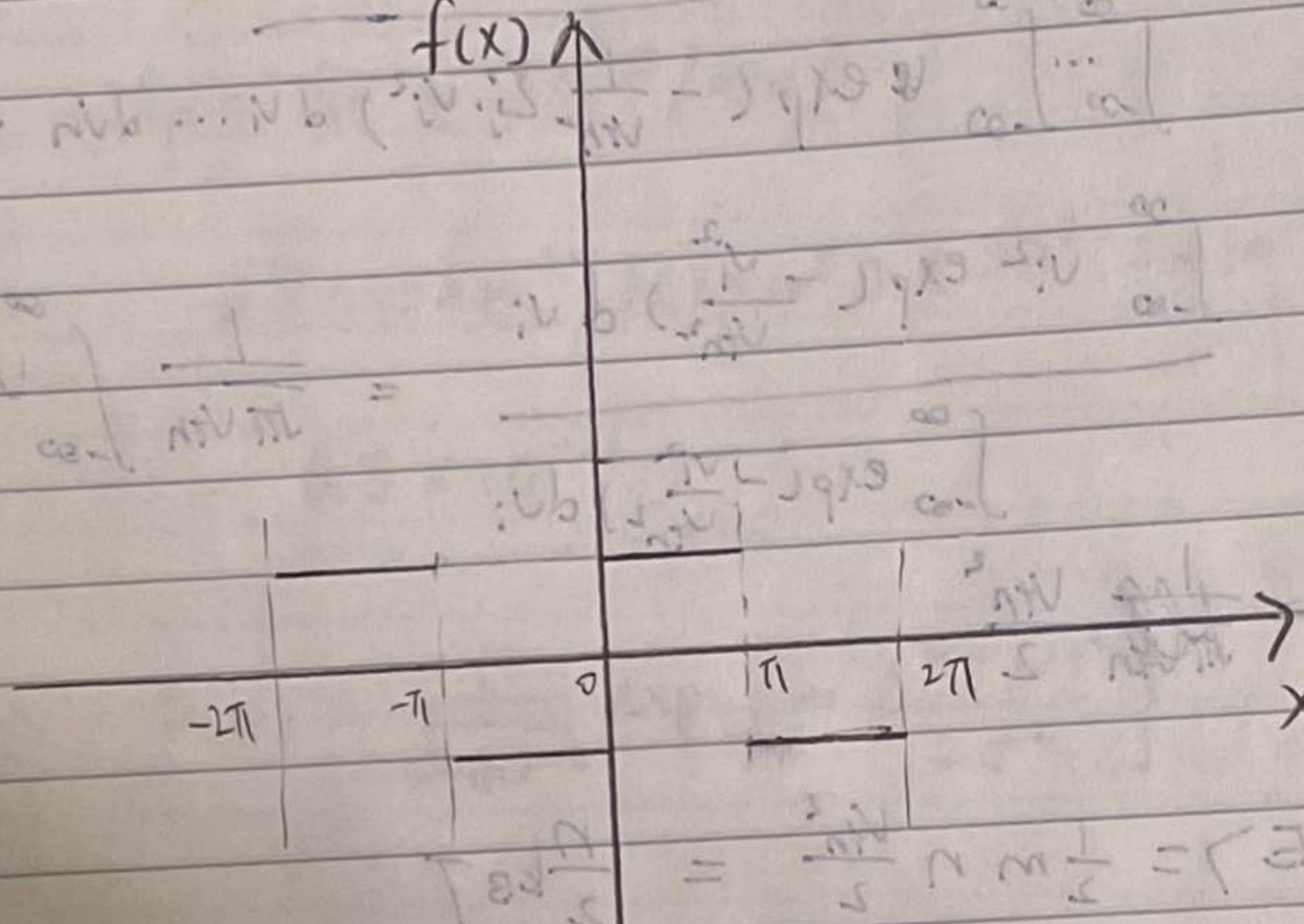
$$\langle E \rangle = \frac{\#\text{ of quadratic modes}}{2} \times k_B T.$$

→ When size of data sets gets large, the probability distribution in most cases tends to the normal distribution

So we can use normal distribution to model ~~many~~ many data if sample size is large.

7. $f(x) = 1$ for $0 < x < \pi$. $\cancel{\infty}$ period $= 2\pi$, and can be expressed as Fourier sine series $\rightarrow$ $f(x)$ is odd

$\therefore$ $f(x)$ is



fourier ~~cos~~ sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right) \qquad (L = \text{period} = 2\pi)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2\pi n}{L} x\right) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin\left(\frac{\cancel{2\pi} n}{\cancel{2\pi}} x\right) dx$$

$$= -\frac{1}{\pi} \int_{-\pi}^{0} \sin(nx)\, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin(nx)\, dx$$

$y = -x$

$$\hookrightarrow = -\frac{1}{\pi} \int_{\pi}^{0} -\sin(ny)(-dy) + \frac{1}{\pi} \int_{0}^{\pi} \sin(nx)\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} \sin(nx)\, dx = -\frac{2}{\pi n} \cos(nx) \Big|_{0}^{\pi}$$

$$= \frac{2}{n\pi}\left[1-(-1)\right] = \frac{4}{n\pi}$$

$$\therefore \quad f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$$

If $x = \frac{\pi}{2}$, then $f(x) = 1$

$$\sin(nx) = \begin{cases} \cancel{(-1)} & \text{if } n \text{ is odd} \quad (-1)^{\frac{n-1}{2}} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad 1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right].$$

$$\rightarrow \quad 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

QED.

8. ~~Adiabatic~~ $dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$

$$\cancel{\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P}$$

$$\cancel{\rightarrow \quad \frac{C_P}{T} - \frac{C_V}{T} = \left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P}$$

(maxwell's relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$)

~~perfect gas~~ $\cancel{PV = Nk_B T}$

$$\cancel{\left(\frac{\partial P}{\partial T}\right)_V = \frac{Nk_B}{V} = \frac{P}{T} \qquad \left(\frac{\partial V}{\partial T}\right)_P = \frac{Nk_B}{P} = \frac{V}{T}}$$

$$\cancel{\therefore \quad C_P - C_V = \frac{PV}{T} = Nk_B}$$

$C_P - C_V = Nk_B$!

~~Adiabatic~~ $\cancel{dS = \left(\frac{\partial S}{\partial T}\right)_V dT +}$

adiabatic $dS = \left(\frac{\partial S}{\partial V}\right)_P dV + \left(\frac{\partial S}{\partial P}\right)_V dP = 0$

$$\therefore \left(\frac{\partial S}{\partial T}\right)_p \left(\frac{\partial T}{\partial V}\right)_p dV + \left(\frac{\partial S}{\partial T}\right)_V \left(\frac{\partial T}{\partial p}\right)_V dp = 0$$

$\rightarrow$ perfect gas : $pV = Nk_B T$

$$\therefore \left(\frac{\partial T}{\partial V}\right)_p = \frac{p}{Nk_B} \qquad \left(\frac{\partial T}{\partial p}\right)_V = \frac{V}{Nk_B}$$

$$\therefore C_p = T\left(\frac{\partial S}{\partial T}\right)_p \qquad C_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

$$\therefore 0 = \frac{C_p}{T}\frac{p}{Nk_B}dV + \frac{C_V}{T}\frac{V}{Nk_B}dp$$

$$\Rightarrow 0 = C_p \frac{dV}{V} + C_V \frac{dp}{p}$$

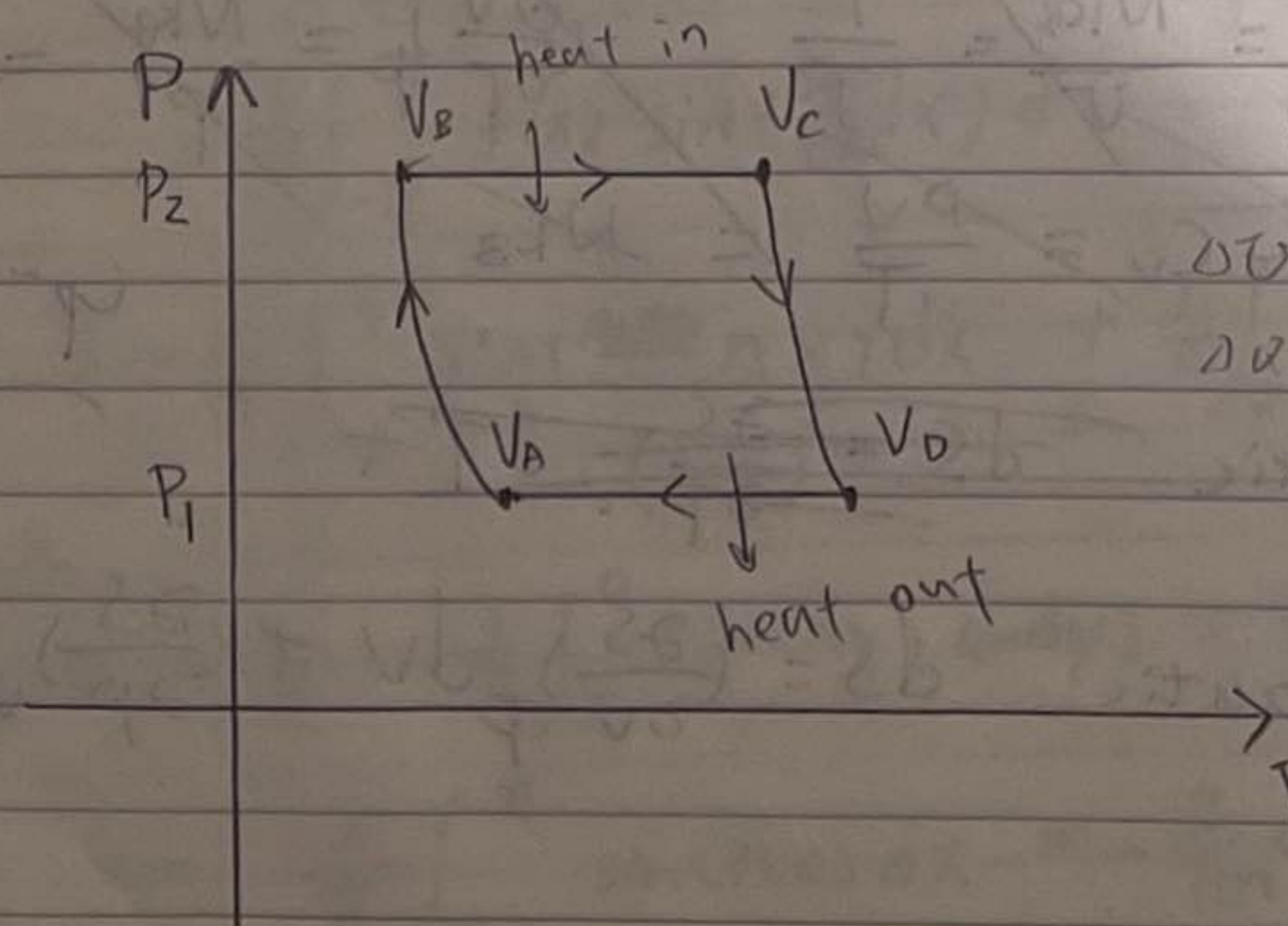
$$\therefore C_p \int_{V_1}^{V_2} \frac{dV}{V} + C_V \int_{p_1}^{p_2} \frac{dp}{p} = 0$$

$$\therefore C_p \ln\frac{V_2}{V_1} + C_V \ln\frac{p_2}{p_1} = 0$$

$$\rightarrow \frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^{C_p/C_V} = \left(\frac{V_2}{V_1}\right)^{\gamma}$$

$$\rightarrow p_1 V_1^{\gamma} = p_2 V_2^{\gamma}$$

$\therefore$ pV^{γ} is constant.



isobaric perfect gas

$\Delta U = \Delta Q + \Delta W$

$\Delta Q = \Delta U - \Delta W$ $\qquad \Delta W = -p\Delta V$

$\therefore \Delta Q = \Delta U + p\Delta V$

$B \rightarrow C$

$\Delta T > 0 \rightarrow \Delta U > 0$, $\Delta V > 0$

$\therefore \Delta Q > 0 \rightarrow$ heat in

$D \rightarrow A$

$\Delta T < 0 \rightarrow \Delta U < 0$

$\Delta V < 0 \rightarrow \Delta Q < 0$

heat out.

$A \Rightarrow B$

(i) work done by the gas

$$W = \int_{V_A}^{V_B} p\,dV = \int_{V_A}^{V_B} \frac{P_1 V_A^{\gamma}}{V^{\gamma}}\,dV =$$

$$= (P_1 V_A^{\gamma}) \left[- \frac{V^{-\gamma+1}}{\gamma - 1} \right]_{V_A}^{V_B}$$

$$= P_1 V_A^{\gamma} \frac{1}{1-\gamma} \left[V_B^{-\gamma+1} - V_A^{-\gamma+1} \right]$$

$$= \frac{1}{1-\gamma} \left[P_2 V_B^{\gamma} V_B^{-\gamma+1} - P_1 V_A^{\gamma} V_A^{-\gamma+1} \right]$$

$$= \frac{P_2 V_B - P_1 V_A}{1-\gamma}$$

~~⇒ No work~~

(ii) heat supplied $Q = 0$

$B \Rightarrow C$

(i)

$$W = P\Delta V = P_2 (V_C - V_B)$$

(ii) ~~$Q = \Delta U + W$~~ ~~by system~~

~~$\Delta U = \frac{3}{2} P\Delta V$~~ ~~$W = P\Delta V$~~

$\therefore$ ~~$Q = \frac{5}{2} P\Delta V = \frac{5}{2} P_2 (V_C - V_B)$~~

~~Q~~ isobaric : $Q = \int_{T_B}^{T_c} C_p \, dT$

$= C_p(T_c - T_B) = \frac{C_p}{nR}(P_2)(V_c - V_B)$

$= \frac{C_p P_2}{N k_B}(V_c - V_B)$

$C \Rightarrow D$

(i) similar to $A \Rightarrow B$

$W = \frac{P_1 V_D - P_2 V_c}{1-\gamma}$

(ii) $Q = 0$ $\because$ adiabatic

$D \Rightarrow A$

(i) similar to $B \Rightarrow C$

$W = P_1 (V_A - V_D)$

(ii) The heat "enters" the system

is $Q' = \int_{T_D}^{T_A} C_p \, dT = \frac{C_p P_1}{N k_B}(V_A - V_B)$

$\because C_p > 0 \quad \therefore \quad Q' < 0 \rightarrow$ heat leaves the system

$\therefore$ No heat is supplied

$\therefore \quad Q = 0$

Efficiency

$$\eta = \frac{\text{total } \cancel{\text{work}} \text{ work done}}{\cancel{\text{t}} \text{ heat supplied}}$$

$$= \frac{\left(\frac{1}{1-\gamma}\right)\left(P_2V_B - P_1V_A + P_1V_D - P_2V_C\right) + P_2(V_C - V_B) + P_1(V_A - V_D)}{\frac{C_P P_2 (V_C - V_B)}{Nk_B}}$$

$$= \frac{\frac{1}{1-\gamma}\left(P_2(V_B - V_C) + P_1(V_D - V_A)\right) + P_2(V_C - V_B) + P_1(V_A - V_D)}{\frac{C_P P_2}{Nk_B}(V_C - V_B)}$$

$$= \frac{P_2(V_C - V_B)\left[1 - \frac{1}{1-\gamma}\right] + P_1(V_A - V_D)\left[1 - \frac{1}{1-\gamma}\right]}{\frac{C_P P_2 (V_C - V_B)}{Nk_B}}$$

$$= \frac{\cancel{P_2 (V_C - V_B)} \left[\frac{\gamma}{\gamma - 1}\right] + P_1 (V_A - V_D) \left[\cancel{\frac{\gamma}{\gamma - 1}}\right]}{\cancel{\frac{\gamma}{\gamma-1}} \cancel{P_2 (V_C - V_B)}}$$

$Nk_B = C_P - C_V$

$$\frac{C_P}{C_P - C_V} = \frac{C_P/C_V}{C_P/C_V - 1} = \frac{\gamma}{\gamma - 1}$$

$$= 1 - \frac{P_1(V_D - V_A)}{P_2(V_C - V_B)}$$

$$V_C \cancel{V_D} = \left(\frac{P_1 V_D^{\gamma}}{P_2}\right)^{\frac{1}{\gamma}}, \quad V_B = \left(\frac{P_1 V_A^{\gamma}}{P_2}\right)^{\frac{1}{\gamma}}$$

$$\cancel{\therefore \eta = 1 - \left(\frac{P_1}{P_2}\right)\left(\frac{\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_C - V_A}{\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_C - \left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}} V_A}\right)}$$

$$\therefore \eta = 1 - \left(\frac{P_1}{P_2}\right)\left(\frac{\cancel{(V_D - V_A)}}{\left(\frac{P_1}{P_2}\right)^{\frac{1}{\gamma}}\cancel{(V_D - V_A)}}\right)$$

$$= 1 - \left(\frac{P_1}{P_2}\right)\left(\frac{P_1}{P_2}\right)^{-\frac{1}{\gamma}} = 1 - \left(\frac{P_1}{P_2}\right)^{\frac{\gamma-1}{\gamma}}$$

$P_1 V_1^{\gamma} = P_2 V_2^{\gamma}$ $\quad PV = nRT \quad V \propto \frac{T}{P}$

$$P_1 \frac{T_1^{\gamma}}{P_1^{\gamma}} = P_2 \frac{T_2^{\gamma}}{P_2^{\gamma}}$$

$$\therefore P_1^{1-\gamma} T_1^{\gamma} = P_2^{1-\gamma} T_2^{\gamma}$$

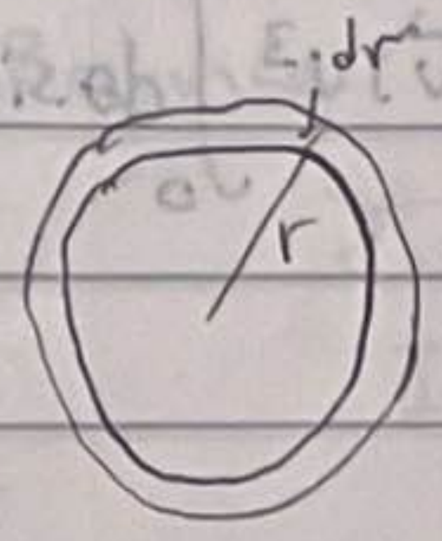
$$\therefore \left(\frac{P_1}{P_2}\right)^{1-\gamma} = \left(\frac{T_2}{T_1}\right)^{\gamma}$$

$$\left(\frac{P_1}{P_2}\right)^{\gamma-1} = \left(\frac{T_1}{T_2}\right)^{\gamma}$$

$$\therefore \frac{T_1}{T_2} = \left(\frac{P_1}{P_2}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\therefore \eta = 1 - \left(\frac{P_1}{P_2}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \frac{T_1}{T_2} \Rightarrow$$ reversible carnot engine

9.

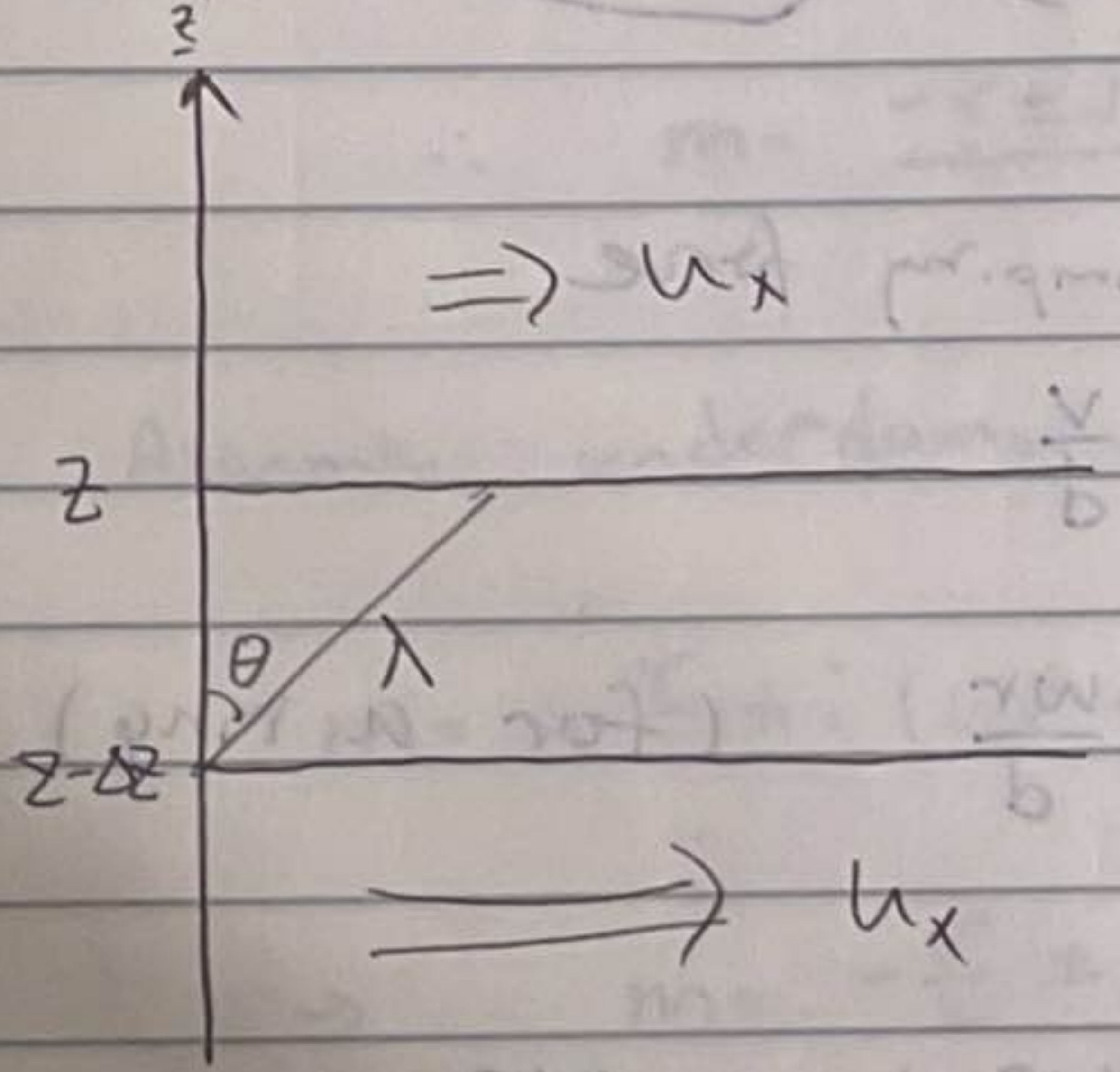


density $\sigma = \frac{m}{A} = \frac{m}{\pi a^2}$

mass of a ring $dm = \sigma\, 2\pi r\, dr$

$$I = \int r^2 dm = \int_0^a \sigma 2\pi r^3 dr = \frac{2m}{a^2} \cdot \frac{a^4}{4}$$

$$= \frac{1}{2} m a^2$$



Extra momentum brought by a particle last collided at $Z - \Delta Z$ to Z is

$$-\Delta p = m u_x(Z) - m u_x(Z - \Delta Z)$$
$$= \cancel{m u_x(Z)} - \cancel{m u_x(Z)} + m \frac{\partial u_x}{\partial Z} \Delta Z$$

$\because \Delta Z = \lambda \cos\theta \quad \therefore \quad \Delta p = -m\lambda \frac{\partial u_x}{\partial Z} \cos\theta$

particle flux in Z-direction

$$d\Phi(\vec{v}) = n v_z f(\vec{v}) d^3\vec{v} = n v^3 \cos\theta \sin\theta f(\vec{v})\, dv\, d\theta\, d\phi$$

$\therefore$ Total momentum flux is

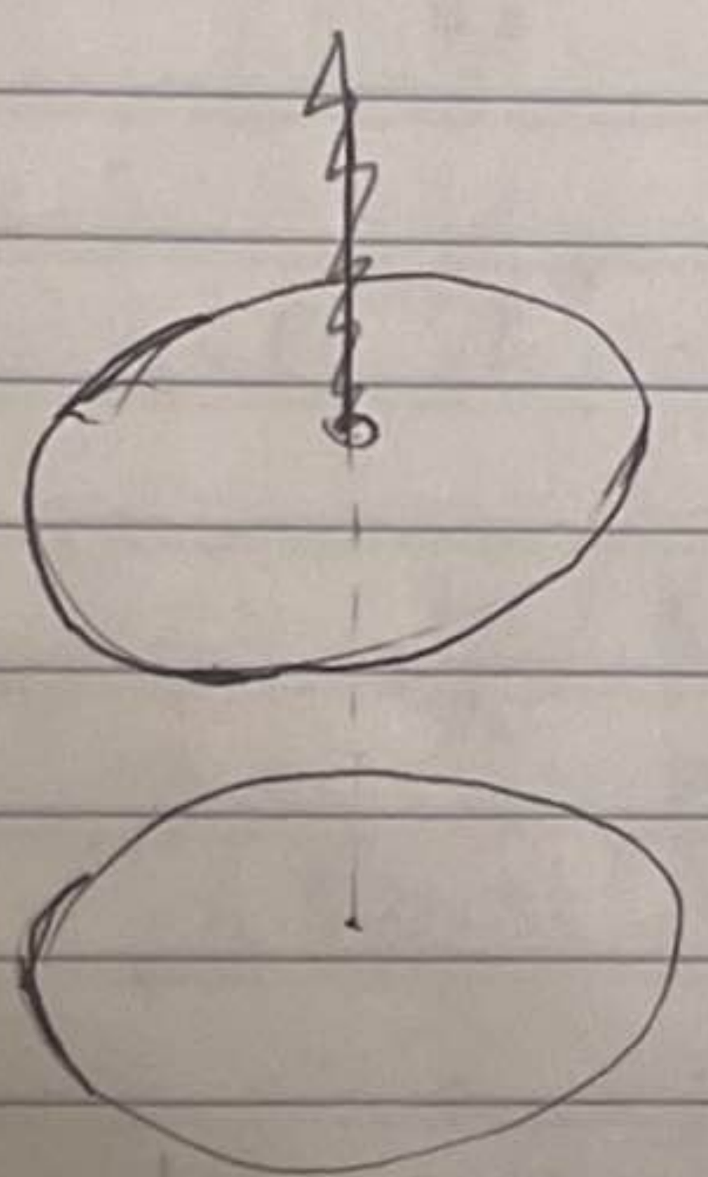
both direction

$$\Pi_{zx} = \int d\Phi(\vec{v})\, \Delta p = -mn\lambda \frac{\partial u_x}{\partial z} \underbrace{\int_0^\infty dv\, f(\vec{v}) v^3}_{\frac{1}{4\pi}\langle c\rangle} \underbrace{\int_0^\pi d\theta \sin\theta \cos^2\theta}_{2/3} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= -mn\lambda \frac{\partial u_x}{\partial z} \left(\frac{1}{4\pi}\langle c\rangle\right)\left(\frac{2}{3}\right)(2\pi)$$

$$\therefore = \cancel{\frac{1}{3} mn\lambda \frac{\partial u_x}{\partial z}} = -\frac{1}{3} mn\lambda \langle c\rangle \frac{\partial u_x}{\partial z} = \cancel{\eta} - \eta \frac{\partial u_x}{\partial z}$$

$$\therefore \eta = \underbrace{\frac{1}{3}}_{\alpha} mn\lambda\langle c\rangle = \alpha mn\lambda\langle c\rangle$$ QED.



damping force

$$\frac{F}{A} = \eta \frac{v}{d}$$

$$\frac{dF}{dA} = \eta \frac{wr}{d} \quad \text{(for a ring)}$$

$$\cancel{d\tau}\ dF = \eta \frac{wr}{d} dA = \eta \frac{wr}{d}(2\pi r\, dr)$$

→ force acting on a ring with thickness dr and radius r

Torque $d\tau = r\, dF = \frac{2\pi}{d}\eta w r^3 dr$

$$\therefore \text{total torque } \tau = \int d\tau = \int_0^a \frac{2\pi}{d}\eta w r^3 dr = \frac{2\pi}{d}\eta w \frac{a^4}{4}$$

$$= \frac{\pi\eta w a^4}{2d} = \underbrace{\left(\frac{\pi\eta a^4}{2d}\right)}_{=\gamma I} w$$

$$= \gamma w = \gamma\dot{\theta}$$

It is opposing direction of motion of plate

$\therefore \tau_\gamma = -\gamma\dot{\theta} I$ is the damping torque.

Balance the forces

$$I\ddot{\theta} = -I\gamma\dot{\theta} - \cancel{\tfrac{1}{4}}I\omega_0^2\theta .$$

$$\rightarrow \cancel{I\theta}\ \cancel{\ddot{\theta}+\gamma\theta+\omega_0^2\theta}$$

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2\theta = 0$$

try $\theta = e^{mt}$ we get $m^2 + \gamma m + \omega_0^2 = 0$

$$\therefore \quad m = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}$$

Assume underdamping $\gamma < 2\omega_0$, ~~then~~ ($\because$ oscillates), then

$$m = -\frac{\gamma}{2} \pm i\left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{\frac{1}{2}} \qquad \text{let } \omega_\gamma = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

$$\rightarrow \quad m = -\frac{\gamma}{2} \pm i\omega_\gamma$$

$\therefore$ General solution ~~$\theta(t) = e^{-\frac{\gamma}{2}t}(A\cos(\omega_\gamma t) + B(\omega_\gamma$~~

$$\theta(t) = e^{-\frac{\gamma}{2}t}\left(A\cos(\omega_\gamma t) + B\sin(\omega_\gamma t)\right)$$

$$= \theta_0 e^{-\frac{\gamma}{2}t}\cos(\omega_\gamma t + \phi)$$

(A, B constants. $\theta_0^2 = A^2 + B^2$, $\tan\phi = -\frac{B}{A}$)

period $T = \frac{2\pi}{\omega_\gamma}$, change in amplitude is ~~$\delta\theta_0$~~

$$\frac{\delta\theta_0}{\theta_0} = \frac{\theta_0\left(1 - e^{-\frac{\gamma}{2}T}\right)}{\theta_0} \approx 1 - \left(1 - \frac{\gamma}{2}T\right)$$

$$= \frac{\gamma T}{2} = \frac{\gamma\, 2\pi}{2\,\omega_\gamma} = \frac{\pi\gamma}{\omega_\gamma}$$

for small damping

Small damping $\omega_\gamma \simeq \omega_0$

$$\rightarrow \quad \frac{\delta\theta_0}{\theta_0} \simeq \frac{\pi\gamma}{\omega_0} = \frac{\pi}{\omega_0} \frac{\pi\eta a^4}{2dI} = \frac{\pi^2\eta a^{\cancel{4}2}}{2d\frac{1}{2}\cancel{ma^2}\omega_0}$$

$$= \cancel{\pi^2\eta a^4} = \frac{\pi^2 a^2}{md\omega_0}\eta \qquad (d=l)$$

$$\therefore \frac{\delta\theta_0}{\theta_0} = \frac{\pi^2 a^2}{ml\omega_0}\eta \qquad \eta 4\pi^2 \frac{1}{2l\omega_0 l}\frac{a^4}{4}$$

$\lambda = \frac{1}{\sqrt{2}\sigma n}$, assuming collisional cross-section $\sigma \simeq 10^{-20}\,m^2$, then

$\because p = nk_BT$

$$\therefore \lambda = \cancel{\frac{P}{\sqrt{2}\sigma k_BT}} \quad \frac{k_BT}{\sqrt{2}\sigma P}$$

$$\frac{k_B}{\sqrt{2}\sigma} \sim \frac{10^{-23}}{10^{-20}} \sim 10^{-3} \rightarrow \lambda \simeq (10^{-3})\frac{T}{P}$$

in S.I units

(i) If $P = 10^5\,Nm^{-2}$, then for $\lambda \simeq l = 10^{-2}m$

$$10^{-2} = 10^{-3}\frac{T}{10^5} \rightarrow T \simeq 10^6 K \quad \text{very large}$$

$\therefore$ Normally $\lambda \ll l$ so the mean free path will be defined as $\lambda = \frac{1}{\sqrt{2}n\sigma}$ and $\lambda \propto T$ for normal temperatures. ~~$\therefore \eta \propto \lambda \therefore \eta \propto T$~~

$$\cancel{\rightarrow \frac{\delta\theta_0}{\theta_0} \propto T}$$

~~(ii) If $P = 10^{-3}\,N/m^2$, then for $\lambda \simeq l = 10^{-2}m$,~~

~~$10^{-2} = 10^{-3}\frac{T}{10^{-3}} \rightarrow T \simeq 100K$ lower than room temperature~~

$n = \frac{P}{k_B T} \propto \frac{1}{T}$ at constant P

19a $\eta \propto n \lambda \langle c \rangle \propto \frac{1}{T} \cdot T \cdot v_{th} \propto \sqrt{T}$

$\rightarrow \frac{\delta\theta_0}{\theta_0} \propto \sqrt{T}$ ✓ $\left(v_{th} = \sqrt{\frac{2k_B T}{m}} \right)$

(ii) If $P = 10^{-3}\ N/m^2$ then for $\lambda \approx l = 10^{-2}\ m$

$10^{-2} = \cancel{10^{-3}} \frac{T}{\cancel{10^{-3}}} \rightarrow T \approx 10^{-2} K$ lower than room temperature $\rightarrow$ for room temperature $\lambda \gg l$ based on $\lambda = \frac{1}{\sqrt{2}\sigma n}$, but l is the size of the container so particles on average cannot travel longer than l before colliding with the wall (plates)

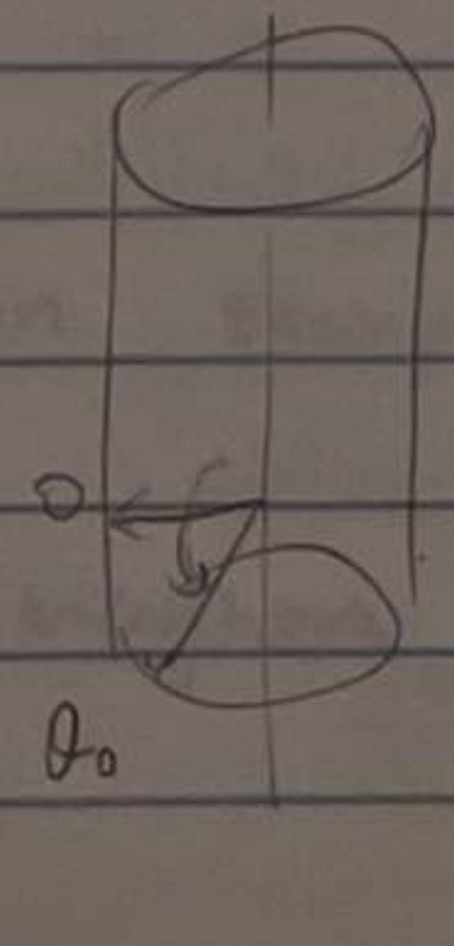
$\rightarrow$ we need to fix $\lambda = l$ $\therefore \lambda$ no longer depends on T

$\rightarrow \eta \propto n \langle c \rangle \propto \frac{1}{T} \cdot \sqrt{T} \propto \frac{1}{\sqrt{T}}$

$\rightarrow \frac{\delta\theta_0}{\theta_0} \propto \frac{1}{\sqrt{T}}$ ✓

Work out the amount of work being done to the disk per cycle

~~$2\theta_0 b$~~ $\cancel{2\int_{-\theta_0}^{\theta_0} d\theta\, \gamma \dot{\theta}}$ $\dot{\theta} = \frac{d\theta}{dt}$



10.

3-D density of states :

$$g(k)\,dk = \frac{V(2s+1)}{2\pi^2} k^2 dk \quad \because \text{ spin-0 bosons}$$

$s=0 \quad \therefore 2s+1 = 1 \quad \therefore g(k) = \frac{V}{2\pi^2} k^2 dk$

$\because \omega = \alpha k^2 \quad \therefore k = \left(\frac{\omega}{\alpha}\right)^{\frac{1}{2}} \quad \therefore dk = \frac{1}{2}\left(\frac{\omega}{\alpha}\right)^{-\frac{1}{2}}\left(\frac{1}{\alpha}\right) d\omega$

$\Rightarrow dk = \frac{1}{2\sqrt{\alpha}} \frac{d\omega}{\sqrt{\omega}} \qquad k^2 = \frac{\omega}{\alpha}$

$$\therefore g(k)\,dk = \frac{V}{2\pi^2} \cdot \frac{\omega}{\alpha} \cdot \frac{1}{2\sqrt{\alpha}} \cdot \frac{d\omega}{\sqrt{\omega}}$$

$$= \frac{V}{4\pi^2 \alpha^{3/2}} \sqrt{\omega}\, d\omega = g(\omega)\, d\omega$$

$$\therefore g(\omega) = \boxed{\frac{V}{4\pi^2 \alpha^{3/2}}} \omega^{1/2} \propto \omega^{1/2}$$

Q.E.D.

proportionality constant.

massless bosons : energy of each ~~pt~~ particle $\varepsilon = \hbar\omega$
(the energy levels)

chemical potential $= 0$

$\therefore$ mean occupation number $\bar{n}(\varepsilon) = \bar{n}(\omega) = \frac{1}{e^{\beta\hbar\omega} - 1}$

total magnon energy in $[\omega, \omega + d\omega]$ is

$(\beta = \frac{1}{k_B T})$

$$E(\omega)d\omega = \bar{n}(\omega)\, g(\omega)\, d\omega\, (\hbar\omega)$$

$$= \frac{V}{4\pi^2 \alpha^{3/2}} \frac{\omega^{1/2} \cdot \hbar\omega}{e^{\frac{\hbar\omega}{k_BT}} - 1} d\omega = \frac{V\hbar}{4\pi^2\alpha^{3/2}} \frac{\omega^{3/2}}{e^{\hbar\omega/k_BT} - 1} d\omega$$

finding maximum $(\omega = \omega^*)$. set $\frac{dE(\omega)}{d\omega} = 0$

then $0 = \frac{d}{d\omega}\left(\frac{\omega^{3/2}}{e^{\hbar\omega/k_BT} - 1}\right)$

$$= \frac{\frac{3}{2}\omega^{+\frac{1}{2}}(e^{\hbar\omega/k_BT} - 1) - \omega^{3/2}\left(\frac{\hbar}{k_BT} e^{\hbar\omega/k_BT}\right)}{(e^{\hbar\omega/k_BT} - 1)^2}$$

let $\frac{\hbar\omega^*}{k_BT} = c$, then

$\therefore \cancel{\frac{1}{2}\, e^{\hbar\omega^*/k_BT} - 1 = 2\omega^*\left(\frac{\hbar}{k_BT} e^{\hbar\omega^*/k_BT}\right)}$

$\cancel{\therefore \text{let } x = \frac{\hbar\omega}{k_BT} \text{ then } \omega =}$

$\therefore$ ~~let~~ $\cancel{c = \frac{\hbar\omega^*}{k_BT}}$, then

$\cancel{e^c - 1 = 2ce^c}$

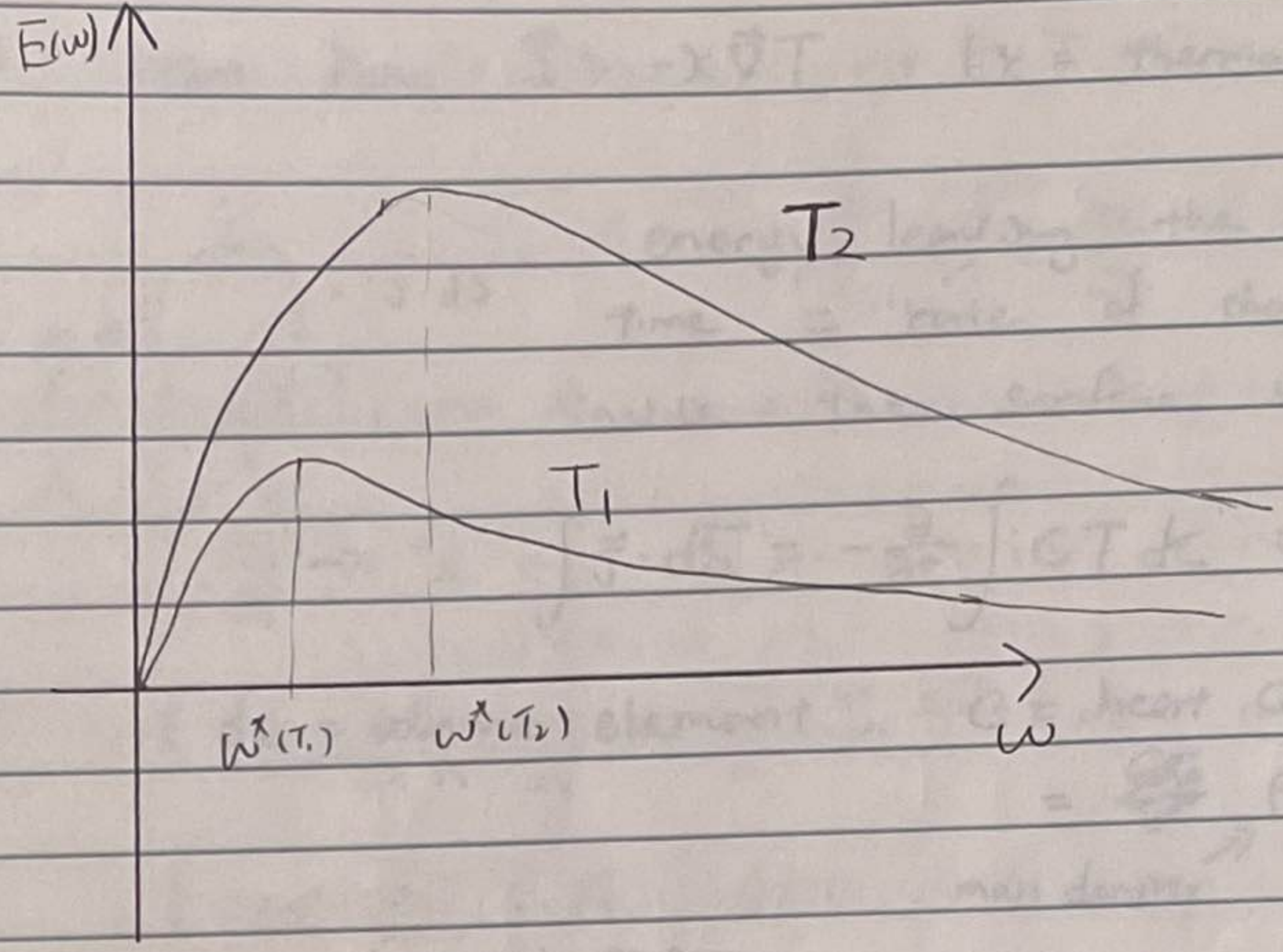
$\therefore \cancel{2c - 1 + e^{-c} = 0}$

$$\frac{3}{2}(e^c - 1) = ce^c$$

$$\therefore\ c - \frac{3}{2} + \frac{3}{2}e^{-c} = 0$$

$$\rightarrow \underline{c = 0.87}$$

$$\therefore \underline{\omega^* = 0.87 \frac{k_BT}{\hbar}}$$



internal energy let $x = \frac{\hbar\omega}{k_B T}$, $\omega = \frac{k_B T}{\hbar} x$, $d\omega = \frac{k_B T}{\hbar} dx$

$$U = \int d\omega E(\omega) = \frac{V\hbar}{4\pi^2\alpha^{3/2}} \int_0^\infty \frac{\omega^{3/2}}{e^x - 1} d\omega$$

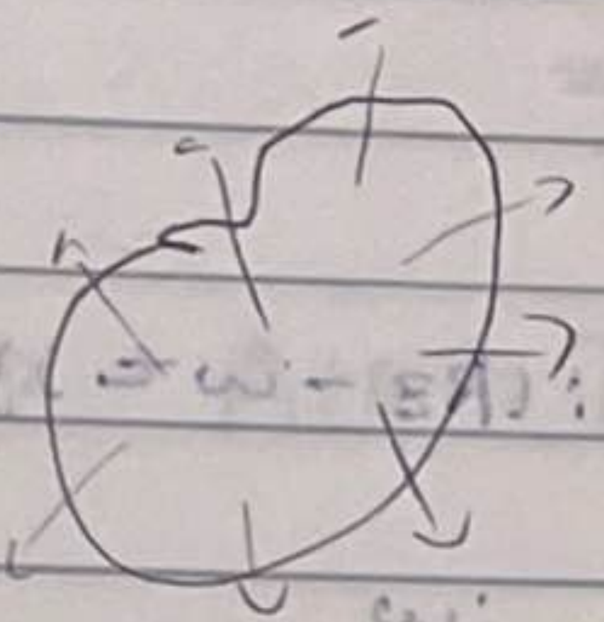
$$= \frac{V\hbar}{4\pi^2\alpha^{3/2}} \left(\frac{k_B T}{\hbar}\right)^{3/2} \cdot \left(\frac{k_B T}{\hbar}\right) \int_0^\infty \frac{x^{3/2}}{e^x - 1} dx$$

$$= \frac{V k_B^{5/2}}{4\pi^2\alpha^{3/2}\hbar^{3/2}} \left[\int_0^\infty \frac{x^{3/2}}{e^x - 1} dx\right] T^{5/2} \quad \propto T^{5/2}$$

$$\therefore C_V = \left(\frac{\partial U}{\partial T}\right)_V \propto T^{3/2}$$ QED.

The specific heats differ because photon gas has dispersion relation $\omega = ck \propto k^1$, ~~whereas~~ whereas for magnon gas dispersion relation is $\omega = \alpha k^2$

11. heat flux $\vec{J} = -\chi \vec{\nabla} T$ (χ = thermal conductivity)



$\vec{J} \cdot d\vec{S}$

energy leaving the surface per time = rate of change of energy inside the surface with a "$-$" sign

$$\rightarrow \int \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int CT \, d\tau = -\frac{\partial}{\partial t} \int \rho c \, d\tau$$

($d\tau$ = volume element, C = heat capacity per volume $= \rho c$)

ρ: mass density; c: specific heat capacity ~~per particle~~

CT = internal energy

~~Divergence theorem~~ For ideal gas:

$$\int \vec{J} \cdot d\vec{S} = -\frac{\partial}{\partial t} \int \frac{3}{2} n k_B T \, d\tau$$

Divergence theorem: $\int \vec{J} \cdot d\vec{S} = \int \vec{\nabla} \cdot \vec{J} \, d\tau$

$$\therefore \int \vec{\nabla} \cdot \vec{J} \, d\tau = \int -\frac{\partial}{\partial t} \left(\frac{3}{2} n k_B T\right) d\tau \quad \text{for any surface}$$

$$\rightarrow \vec{\nabla} \cdot \vec{J} = -\frac{\partial}{\partial t} \left(\frac{3}{2} n k_B T\right)$$

isotropic distribution assuming $\frac{\partial n}{\partial t} = 0$, ~~and~~

$$\therefore -\chi \vec{\nabla} \cdot (\vec{\nabla} T) = -\frac{3}{2} n k_B \frac{\partial T}{\partial t}$$

$$\rightarrow \frac{\partial T}{\partial t} = \underbrace{\left(\frac{\chi}{\frac{3}{2} n k_B}\right)}_{K} \nabla^2 T \rightarrow \frac{\partial T}{\partial t} = K \nabla^2 T$$

QED

If $T = T(z,t)$ independent of x, y, then

$$\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial z^2}$$

trial solution $T(z,t) \propto \exp(i(kz - \omega t))$

then $-i\omega = -Kk^2 \Rightarrow k^2 = \frac{i\omega}{K}$

$\rightarrow k = \pm\sqrt{\frac{\omega}{K}}\sqrt{i} = \pm\frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\omega}{K}} = \pm(1+i)\sqrt{\frac{\omega}{2K}}$

If $T(z,t) \propto \exp(i((-1-i)\sqrt{\frac{\omega}{2K}}z - \omega t)$

$$= \exp(\sqrt{\tfrac{\omega}{2K}}z)\exp(-i(\sqrt{\tfrac{\omega}{2K}}z + \omega t))$$

blows up at $z \rightarrow +\infty \rightarrow$ neglect this

$\therefore k = (1+i)\sqrt{\frac{\omega}{2K}}$

$T(z,t) \propto \exp(i((1+i)\sqrt{\frac{\omega}{2K}}z - \omega t))$

$$= \exp(-\sqrt{\tfrac{\omega}{2K}}z)\exp(i(\sqrt{\tfrac{\omega}{2K}}z - \omega t))$$

(if $\omega \neq 0$ this vanishes at $+\infty \rightarrow$ ✓)

linear equation $\rightarrow$ superposition principle applies.

$\rightarrow$ let $\underbrace{\delta_\omega = \sqrt{\frac{2K}{\omega}}}$ = skin depth, then general solution

$$T(z,t) = \sum_\omega A(\omega)\exp(-i\omega t)\exp\left[(i-1)\frac{z}{\delta_\omega}\right]$$

The boundary conditions :

$$T(z=0,t) = T_0 + T_1\cos\omega_1 t + T_2\cos\omega_2 t$$

$$= T_0 + \frac{T_1}{2}e^{i\omega_1 t} + \frac{T_1}{2}e^{-i\omega_1 t} + \frac{T_2}{2}e^{i\omega_2 t} + \frac{T_2}{2}e^{-i\omega_2 t}$$

$$\rightarrow \quad A(0) = T_0, \quad A(\omega_1) = A(-\omega_1) = \frac{T_1}{2}$$

$$A(\omega_2) = A(-\omega_2) = \frac{T_2}{2}$$

$$A(\omega \neq 0, \omega_1, \omega_2, -\omega_1, -\omega_2) = 0$$

$$\rightarrow \cancel{T(z,t) = T_0 + \frac{T_1}{2}}$$

$$\because T(z,t) = \sum_{\omega} A(\omega)\exp(-i\omega t)\exp\left[(i-1)\sqrt{\frac{\omega}{2K}}z\right]$$

$$\therefore T(z,t) = T_0 + \frac{T_1}{2}\exp(-i\omega_1 t)\exp\left((i-1)\sqrt{\frac{\omega_1}{2K}}z\right)$$

$$+ \frac{T_1}{2}\exp(+i\omega_1 t)\exp\left((-i-1)\sqrt{\frac{\omega_1}{2K}}z\right)$$

$$+ \frac{T_2}{2}\exp(-i\omega_2 t)\exp\left((i-1)\sqrt{\frac{\omega_2}{2K}}z\right)$$

$$+ \frac{T_2}{2}\exp(+i\omega_2 t)\exp\left((-i-1)\sqrt{\frac{\omega_2}{2K}}z\right)$$

$$= T_0 + T_1\exp\left(-\sqrt{\frac{\omega_1}{2K}}z\right)\left[\frac{\exp\left(-i\left(\omega_1 t - \sqrt{\frac{\omega_1}{2K}}z\right)\right) + e^{i\left(\omega_1 t - \sqrt{\frac{\omega_1}{2K}}z\right)}}{2}\right]$$

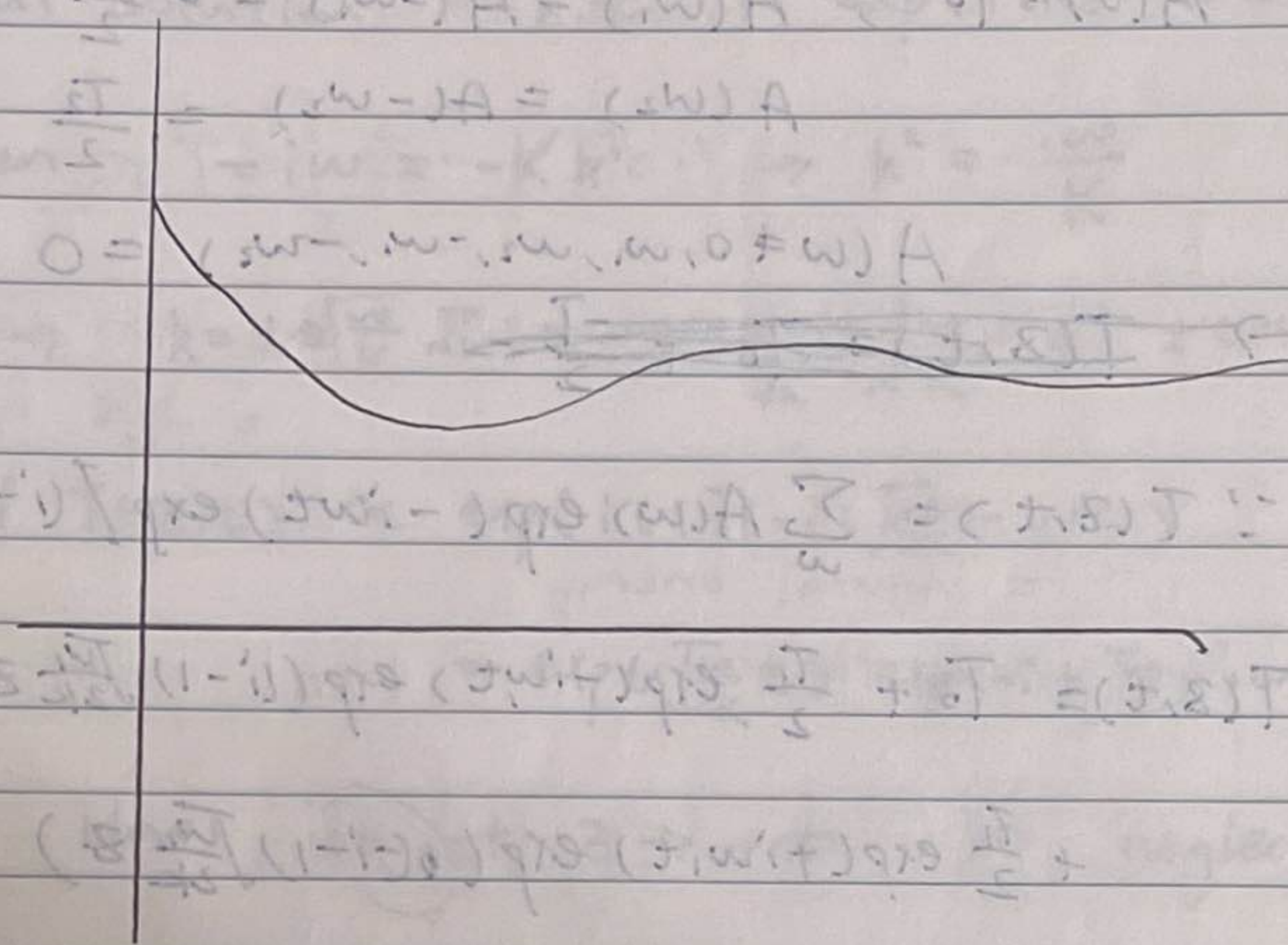
$$+ T_2\exp\left(-\sqrt{\frac{\omega_2}{2K}}z\right)\left[\frac{e^{\left(-i\left(\omega_2 t - \sqrt{\frac{\omega_2}{2K}}z\right)\right)} + e^{i\left(\omega_2 t - \sqrt{\frac{\omega_2}{2K}}z\right)}}{2}\right]$$

$$\rightarrow \quad T(z,t) = T_0 + T_1 e^{-\sqrt{\frac{\omega_1}{2K}}z}\cos\left(\omega_1 t - \sqrt{\frac{\omega_1}{2K}}z\right)$$

$$+ T_2 e^{-\sqrt{\frac{\omega_2}{2K}}z}\cos\left(\omega_2 t - \sqrt{\frac{\omega_2}{2K}}z\right)$$

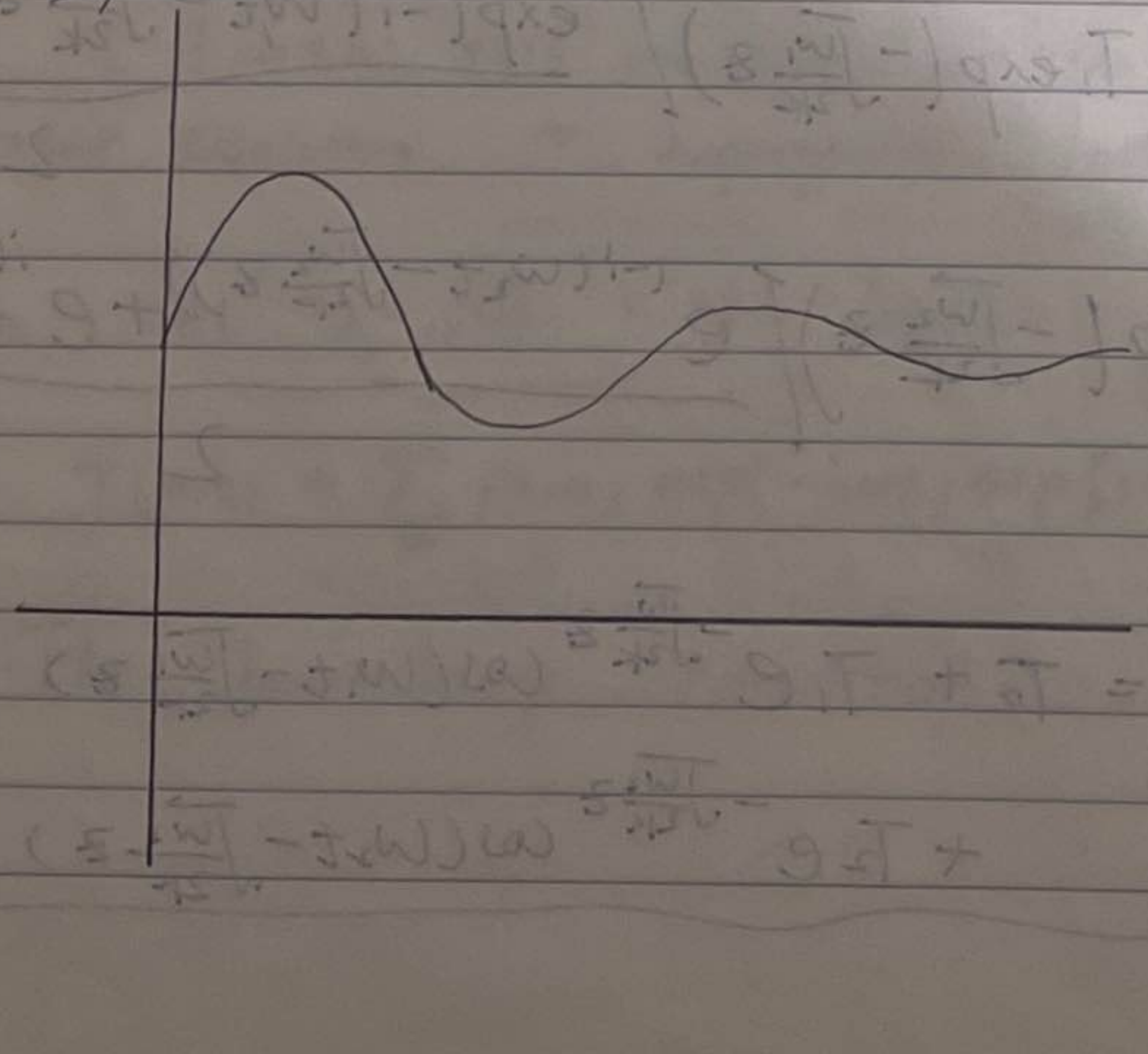
$\omega_1 << \omega_2$

(i) $T_1 = T_2$, $t=0$, then the T_2 part decays away very quickly compare to T_1 part. so we ignore the T_2 part.



(ii) $T_2 = 0$, $t = \frac{\pi}{2\omega_1}$ $\therefore \cos(\omega_1 t - \sqrt{\frac{\omega_1}{2k}} z)$

$= \cos(\frac{\pi}{2} - \sqrt{\frac{\omega_1}{2k}} z) = \sin(\sqrt{\frac{\omega_1}{2k}} z)$



Boundary condition $z=0 \to$ water surface

$\omega_1 << \omega_2$: $\omega_2 \to$ frequency of daily variation

$\omega_1 \to$ frequency of annual variation

ignore ω_2

$$\to \cancel{T} \; T(z,t) = T_0 + T_1 \cos\left(\omega_1 t - \sqrt{\tfrac{\omega_1}{2K}}\, z\right) e^{-\sqrt{\frac{\omega_1}{2K}} z}$$

$$T(0,t) = T_0 + T_1 \cos(\omega_1 t).$$

$$\to T_0 = \frac{\cancel{20} \; 20°C - 40°C}{2} = -10°C = 263K$$

$$T_1 = 30K$$

For permanently frozen T always $< 0°C = 273K$

$$\therefore 263K + (30K) e^{-\sqrt{\frac{\omega_1}{2K}} z} = 273K$$

$$\therefore \sqrt{\frac{\omega_1}{2K}}\, z = \ln 3.$$

$$\therefore z = \ln 3 \left(\frac{2K}{\omega_1}\right)^{\frac{1}{2}} = (\sqrt{2})(\ln 3)\sqrt{\frac{K}{\omega_1}}$$

$$\omega_1 = \frac{2\pi}{1\text{year}} = \frac{2\pi}{365 \times 24 \times 60 \times 60 s} = 1.99 \times 10^{-7} s^{-1}$$

$$K = 3 \times 10^{-7} m^2 s^{-1}$$

$$\therefore z = \underline{1.9m}$$