2661

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A1: THERMAL PHYSICS

TRINITY TERM 2012

Wednesday, 13 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

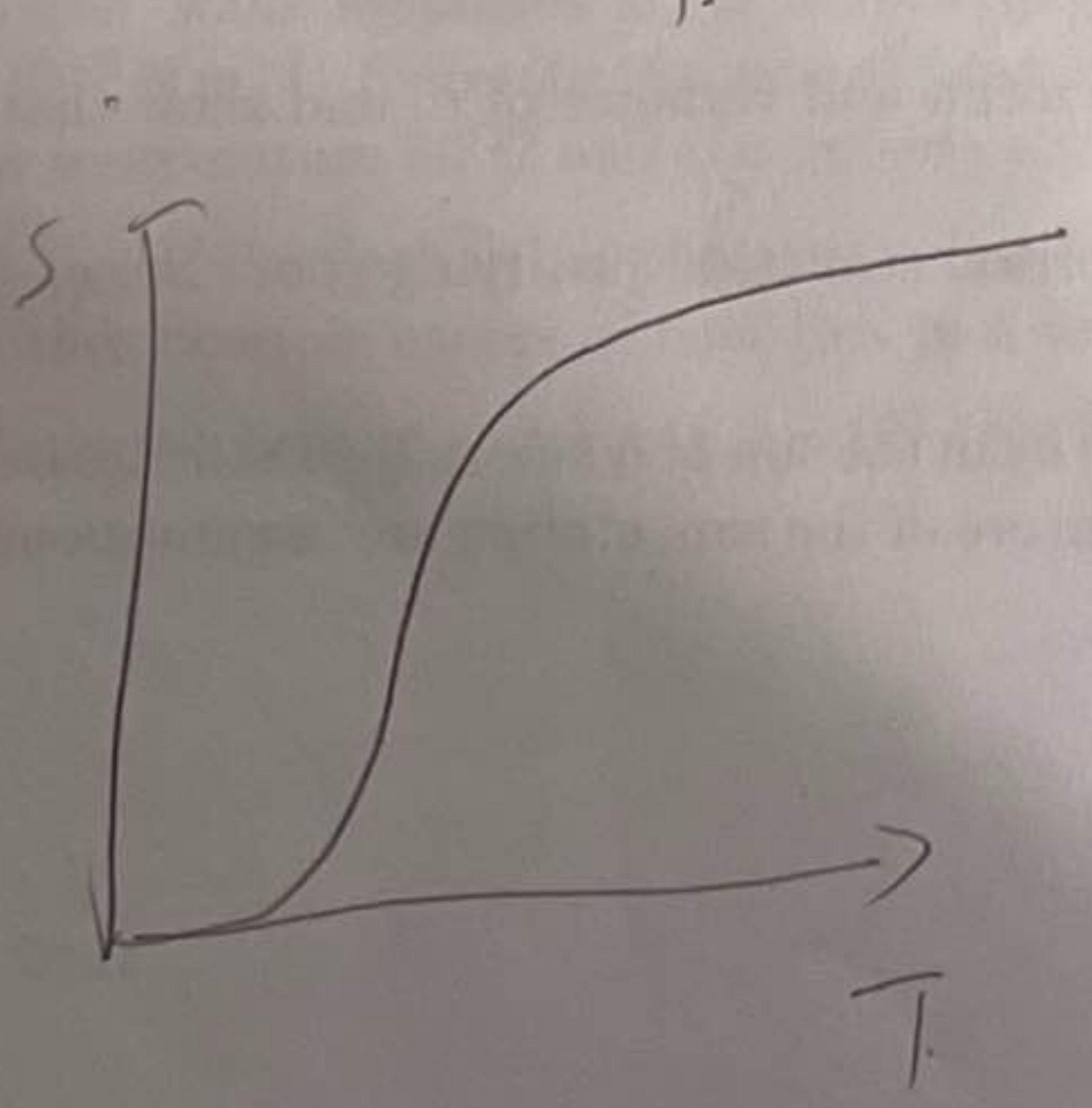
A list of physical constants and conversion factors accompanies this paper.

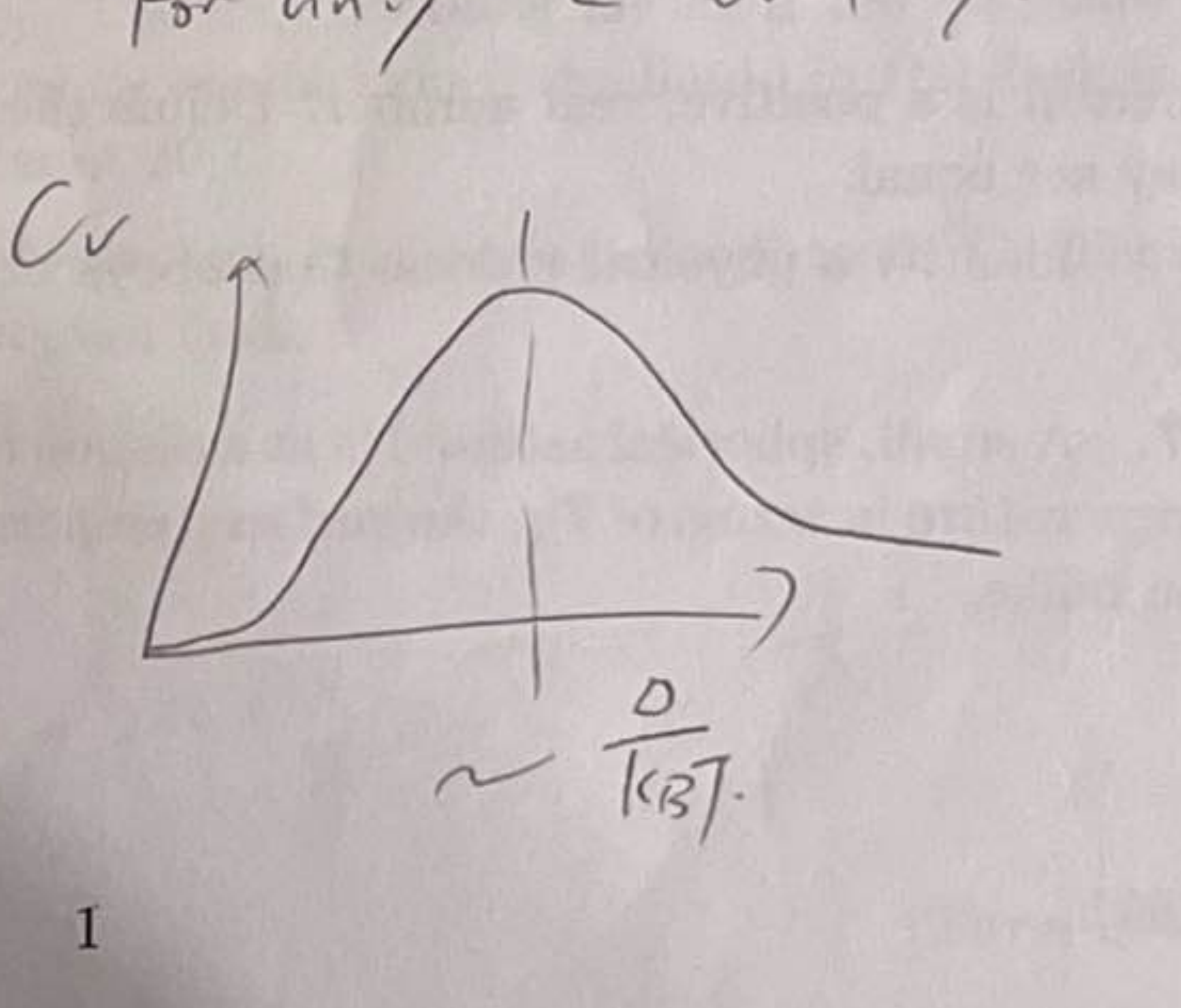
The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

$C_V = T \frac{\partial S}{\partial T}.$

For any 2-level system





Section A

1. Draw a labelled diagram showing how the specific heat per mole of a heteronuclear, diatomic gas varies with temperature. Give expressions for the temperatures at which the magnitude of the specific heat changes in terms of molecular properties. [6]

2. Prove the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V .$$

Calculate the entropy change when one mole of a perfect gas undergoes a Joule expansion from a volume V to a volume $2V$. [6]

3. Two bodies of heat capacity C, which are initially at temperatures T_A and $T_B < T_A$, are brought to thermal equilibrium by the operation of a reversible heat engine. Calculate their final temperature. [4]

4. Estimate the change in the boiling point of water between sea level and a height of $10^3\,\mathrm{m}$. You may assume that the density of air is $1\,\mathrm{kg\,m^{-3}}$. [6]

[The latent heat of vaporization of water $= 4.0 \times 10^4\,\mathrm{J\,mole^{-1}}$.]

5. The partition function of a *perfect*, *classical* gas of N *indistinguishable* particles of mass m occupying a volume V at temperature T is

$$Z_N = \frac{1}{N!}\left\{\int_0^\infty \frac{V4\pi p^2}{h^3}\exp\left(-\frac{p^2}{2mk_\mathrm{B}T}\right)\mathrm{d}p\right\}^N = \frac{V^N}{N!}\left(\frac{2\pi mk_\mathrm{B}T}{h^2}\right)^{3N/2} .$$

Explain the words in italics, and their relevance to the form of the partition function.

Estimate the number density at which the gas can no longer be treated as classical in terms of m and T. [6]

6. The Poisson distribution for a non-negative integer variable n is

$$P(n \mid \mu) = \frac{\mu^n \mathrm{e}^{-\mu}}{n!}$$

where μ is a positive, real number. Define the mean and variance of n, and show that they are equal.

Identify a physical process that obeys Poisson statistics, justifying your choice. [7]

7. A small, spherical asteroid is at a distance from the sun of α sun radii. Estimate its temperature in terms of T_S, the surface temperature of the sun, stating any assumptions you make. [5]

$dU = TdS - m\,dB$ $\therefore\ m = \left(\frac{\partial F}{\partial B}\right)_T$

$F = -SdT - m\,dB$

Section B

$F = -k_B T \ln Z$

8. A paramagnetic solid consists of a large number N of non-interacting, spin-$\frac{1}{2}$ particles, each of magnetic moment μ, on fixed lattice sites. The solid is placed in a uniform magnetic field B at a temperature T.

Write down an expression for the partition function of the solid (neglecting lattice vibrations). Show that the entropy of the solid can be written as

$$S = Nk_B\{C \ln(\cosh x) + Dx \tanh x + E\} ,$$

where $x = (\mu B/k_B T)$, and give values for C, D and E. [7]

The paramagnetic solid is taken slowly around the following cycle:

$e^{x} + e^{-x}$

(1) isothermal magnetization from (B_1, T_i) to (B_2, T_i), where $B_2 > B_1$,
(2) adiabatic demagnetization from (B_2, T_i) to (B_1, T_f), where $T_i > T_f$,
(3) demagnetization at constant applied field from (B_1, T_f) to (B_1, T_i).

$0 \quad 0 \quad \ln(4)$

Plot the entropy as a function of temperature for fields B_1 and B_2, and indicate the steps of the cycle on your diagram. Find an expression for the temperature T_f in terms of T_i, B_1 and B_2. [7]

Describe and explain the changes in the relative occupation numbers of the magnetic energy levels during each of the three steps of the cycle. [6]

$\frac{e^{x}+e^{-x}}{2}$ $\quad \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

9. Use kinetic theory to show that the thermal conductivity of a gas is

$$\kappa = \alpha n\, C_{\text{mol}}\, \lambda \langle c \rangle \, , \qquad \frac{\kappa}{\lambda}$$

where C_{mol} is the specific heat at constant volume of a molecule, n is the number density, λ the mean free path and $\langle c \rangle$ the mean speed of the molecules of the gas, and α is a dimensionless constant of order unity. State the assumptions that underlie this derivation. [8]

A vacuum flask of radius 5 cm and length 20 cm consists of concentric inner and outer walls separated by a narrow gap. The gap contains air at a low pressure of $10^{-2}\,\text{N}\,\text{m}^{-2}$. Estimate the rate of heat loss by conduction if the liquid in the flask is at a temperature 60°C and the outside air is at 20°C. [7]

If the flask is full of water, estimate how long it will take to cool to 40°C. Discuss other possible causes of heat loss in a vacuum flask. [5]

[Specific heat of water = $4.2 \times 10^3\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$.]

$\frac{Q}{t} = -2\pi r x \frac{\Delta T}{\lambda}$ $\quad \frac{\kappa}{\lambda} = \alpha n C_{mol} \sqrt{\frac{8 k_B T}{\pi m}}$

$= -a$

$Q = -2\pi r l\, \Delta T \left(\frac{\kappa}{\lambda}\right)$

$(2\pi 2\pi \bar{\nu} x)$ $2\pi\bar{\lambda}$ $2\pi\bar{\nu}x)$

$\frac{f}{c}$

10. Show that, for a two-dimensional gas of free electrons of mass m, occupying an area A, the density of states D as a function of energy ϵ is given by

$$D(\epsilon)\,\mathrm{d}\epsilon = \left(\frac{Am}{\hbar^2\pi}\right)\mathrm{d}\epsilon\,.$$

Write down the number of electrons $n(\epsilon)\,\mathrm{d}\epsilon$ in the energy range ϵ to $\epsilon + \mathrm{d}\epsilon$ for a two-dimensional electron gas, defining any new symbols you introduce. [6]

Define the Fermi temperature, T_f, and sketch the dependence of $n(\epsilon)$ on ϵ for (a) $T = 0$, (b) $T \ll T_f$ and (c) $T \gg T_f$. Show that, at $T = 0$, the mean energy per particle is equal to $k_{\mathrm{B}}T_f/2$. [8]

Obtain the temperature dependence of the heat capacity (numerical prefactors are not required). Explain why the magnitude of the heat capacity is much smaller than that of a classical, two-dimensional gas. [6]

11. The diffusion constant D is defined by

$$\mathbf{j} = -D\nabla n$$

where $\mathbf{j}$ is the number of particles crossing unit area in unit time and n is the number density of the particles. Assuming that the number of particles is conserved, derive the diffusion equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n\,.$$ [4]

A long, hollow, rigid tube, of length L and constant cross section is initially filled with water. It is placed in a solution of colloidal particles, with number density n_0, dissolved in water. The tube is impermeable to the particles. At time $t = 0$ the end of the tube at $x = 0$ is opened to allow the particles to enter. Solving the diffusion equation using the method of separation of variables (including the solution that corresponds to separation constant zero), or otherwise, show that the number density of colloids in the tube is

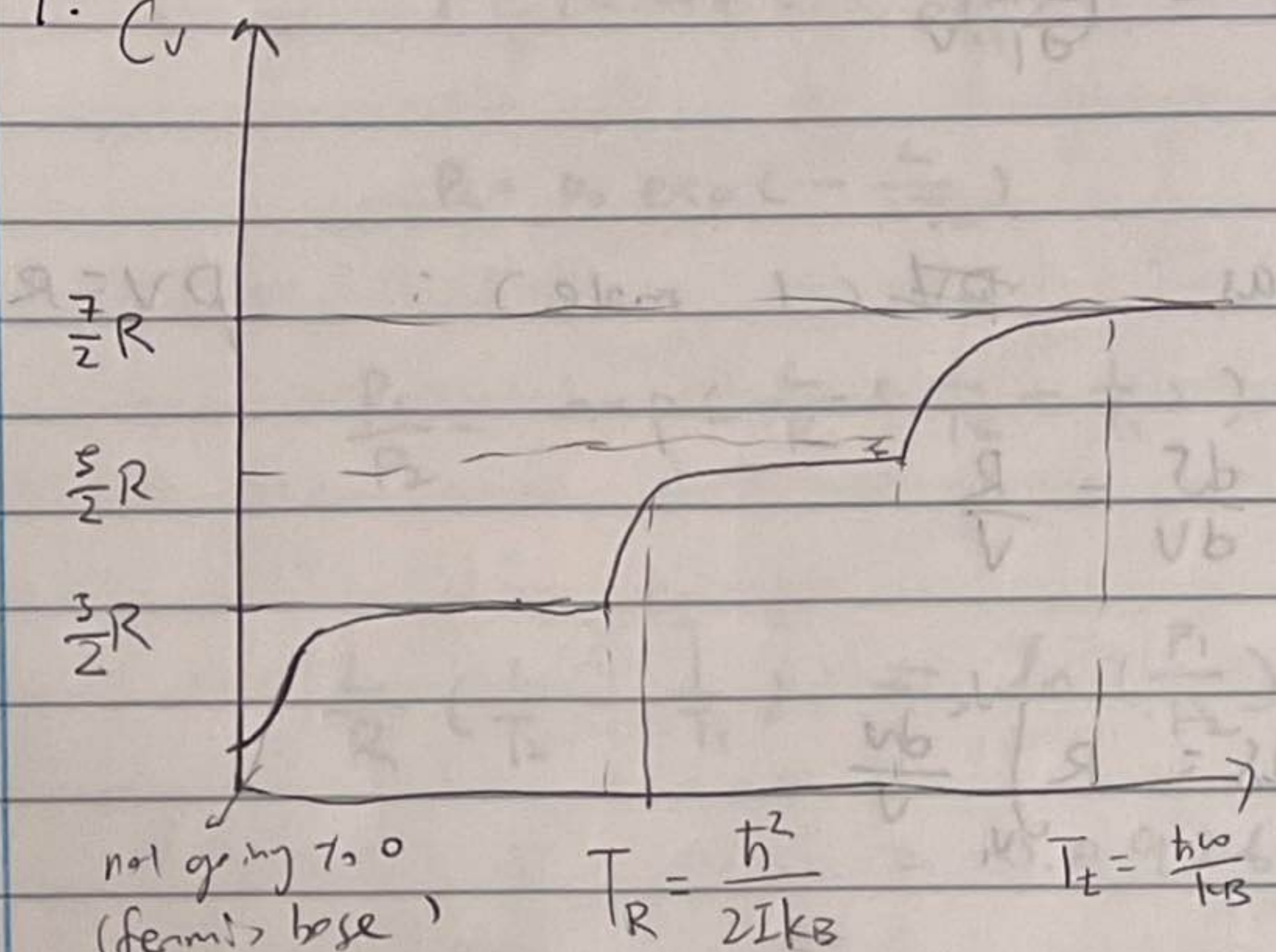
$$\frac{n(x,t)}{n_0} = 1 - \sum_{p=0}^{\infty}\frac{4}{(2p+1)\pi}\sin\left(\frac{(2p+1)\pi x}{2L}\right)\exp\left\{-D\left(\frac{(2p+1)\pi}{2L}\right)^2 t\right\}.$$

[12]

Estimate the time scale over which the average colloid number density inside the tube becomes of the same order as that in the outside solution if $L = 50\,\mathrm{cm}$ and $D = 2.5\times10^{-12}\,\mathrm{m^2\,s^{-1}}$. [4]

A1 2012 First Attempt

1.



→ Rotational degrees of freedom excited when $T \sim T_R = \frac{\hbar^2}{2Ik_B}$. C_v goes from $\frac{3}{2}R$ to $\frac{5}{2}R$

(I = moment of inertia of the diatomic molecule)

→ Vibrational degrees of freedom excited when $T \sim T_t = \frac{\hbar\omega}{k_B}$ (ω is the angular frequency of vibration)

2. $dU = TdS - pdV$ define $F = U - TS$

$$dF = dU - TdS - SdT = -SdT - pdV$$

$$\therefore S = -\left(\frac{\partial F}{\partial T}\right)_V \quad p = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$\therefore -\frac{\partial^2 F}{\partial T \partial V} = -\frac{\partial^2 F}{\partial V \partial T} \quad \therefore \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

QED

Joule expansion → constant T $\therefore$ ~~$\frac{dS}{dT} = \left(\frac{\partial S}{\partial T}\right)$~~

~~$\frac{dS}{dT} = \left(\frac{\partial S}{\partial V}\right)_T$~~

$$\frac{dS}{dV} = \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

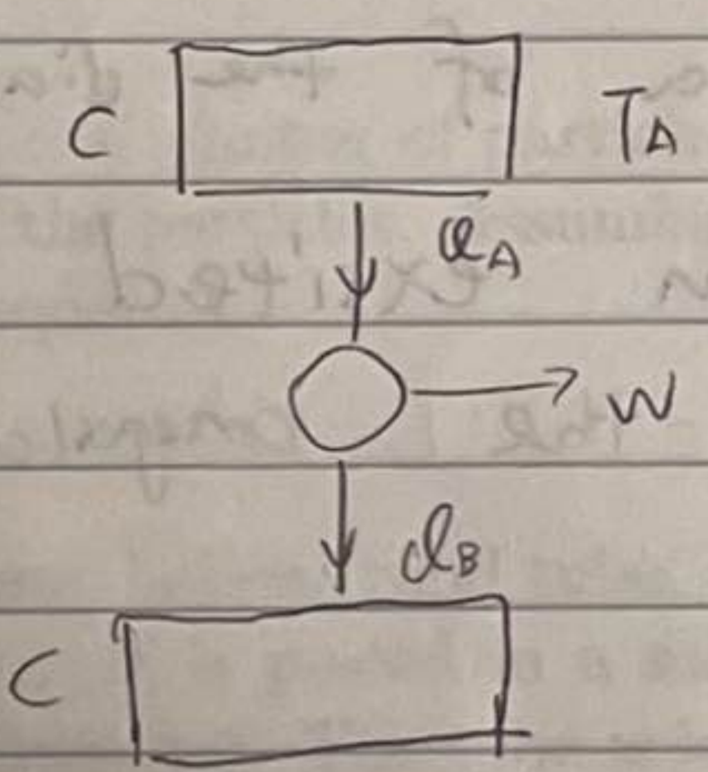
For ideal gas ~~PV~~ (1 mole) : $PV = RT$

$$\therefore \cancel{\frac{dS}{dT}} \quad \frac{dS}{dV} = \frac{R}{V}$$

$$\therefore \int_{S_1}^{S_2} dS = R\int_{V_1}^{V_2} \frac{dV}{V}$$

$$\rightarrow \Delta S = R\ln\frac{V_2}{V_1} = \underline{R\ln 2}$$

3.



No heat transferred to the surrounding $\rightarrow \Delta S_r = 0$

Reversible engine $\rightarrow$ ~~ΔS_u not~~ entropy of universe unchanged $\rightarrow \Delta S_u = 0$

$\therefore$ ~~ΔS~~ System $\Delta S_s = \Delta S_u - \Delta S_r = 0$

$$0 = dS = \frac{dQ_A}{T_A} + \frac{dQ_B}{T_B} = C\left[\frac{dT_A}{T_A} + \frac{dT_B}{T_B}\right]$$

$$\rightarrow \int_{T_A}^{T_f}\frac{dT_A}{T_A} = \int_{T_B}^{T_f}\frac{dT_B}{T_B} = \ln\frac{T_f}{T_A} + \ln\frac{T_f}{T_B} = 0$$

$$\therefore \frac{T_f^2}{T_A T_B} = 1 \quad \therefore \underline{T_f = \sqrt{T_A T_B}}$$

4. $P_1 = P_0 \exp(-\frac{L}{RT_1})$

$P_2 = P_0 \exp(-\frac{L}{RT_2})$

$\frac{P_1}{P_2} = \exp(\frac{L}{R}(\frac{1}{T_2} - \frac{1}{T_1}))$

$\frac{L}{R}(\frac{1}{T_2} - \frac{1}{T_1}) = \ln(\frac{P_1}{P_2}) = \ln\left(\frac{1.01\times10^5 + (1)(10^3)(9.8)}{1.01\times10^5}\right)$

$= 0.0926$

$\frac{1}{T_2} - \frac{1}{T_1} = 0.0926 \frac{(8.314)}{(4.0\times10^4)} = 1.92\times10^{-5}$

$\cancel{\frac{1}{T_1} = \frac{1}{T_2} -}$ $\frac{1}{T_2} = \frac{1}{T_1} + 1.92\times10^{-5}$

$= \frac{1}{373K} + 1.92\times10^{-5}$

$= 2.7\times10^{-3}$

$\rightarrow T_2 = 370.3\ K$

$\therefore \Delta T = T_2 - T_1 = -2.7\ K$

5. perfect $\rightarrow$ no intermolecular forces
no volume of particles

classical $\rightarrow$ hot $(T\rightarrow\infty)$ and ~~dilute~~
dilute $(n\rightarrow 0)$

perfect → only kinetic energy in the exponential of partition

classical → the $\frac{1}{N!}$ term corrects the indistinguish-ability because number of particles << number of available states.

let $\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mk_BT}} = \sqrt{\frac{h^2}{2\pi m k_B T}}$. then

$Z_N = \frac{1}{N!}\left(\frac{V}{\lambda_{th}^3}\right)^N \rightarrow Z_1 = \frac{V}{\lambda_{th}^3}$

$Z_1 \sim$ # of available ~~microstate~~ single particle states

Classical $Z_1 >> N \rightarrow \frac{V}{\lambda_{th}^3} >> N \rightarrow n << \frac{1}{\lambda_{th}^3}$

$\rightarrow$ ~~$n <<$~~ $n << \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}$

6. $P(n|\nu) = \frac{\nu^n e^{-\nu}}{n!}$

mean $\langle n\rangle = \sum_{n=0}^{\infty} P(n|\nu)\, n$

variance $= \langle n^2\rangle - \langle n\rangle^2$

$\langle n\rangle = \sum_{n=0}^{\infty} \frac{\nu^n e^{-\nu}}{n!} n = e^{-\nu}\sum_{n=0}^{\infty} n\frac{\nu^n}{n!}$

$= e^{-\nu}\sum_{n=1}^{\infty} n\frac{\nu^n}{n!} = e^{-\nu}\underbrace{\sum_{n=1}^{\infty}\frac{\nu^{n-1}}{(n-1)!}}_{e^{\nu}}\nu$

$= \nu e^{-\nu}e^{\nu} = \underline{\nu}$

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} \frac{\nu^n e^{-\nu}}{n!} n^2$$

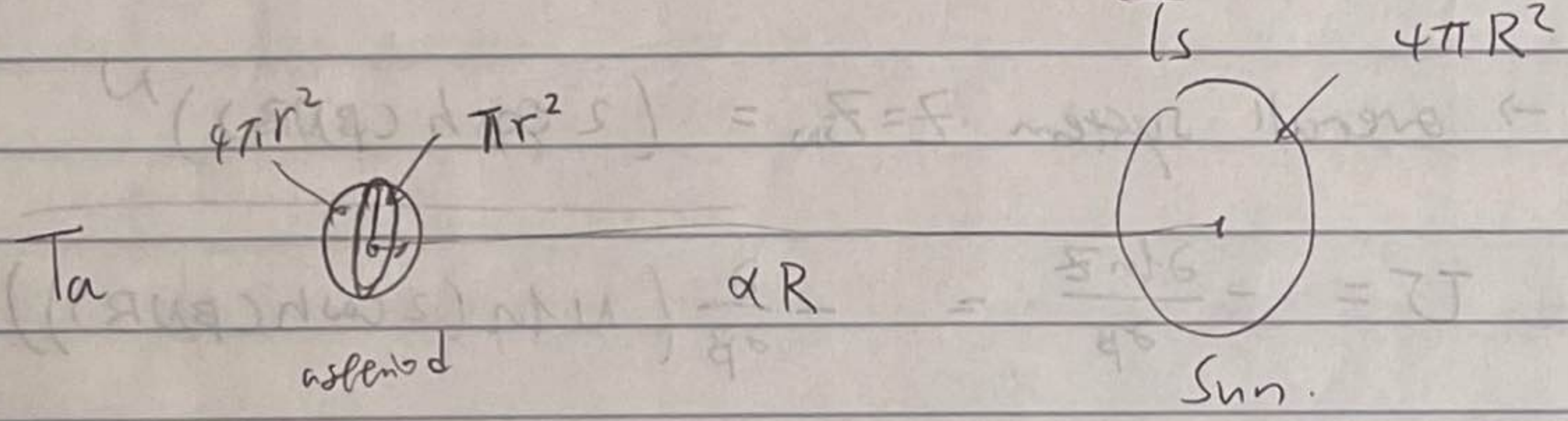
$$= e^{-\nu} \sum_{n=1}^{\infty} \frac{\nu^n}{(n-1)!}(n-1+1)$$

$$= e^{-\nu}\left[\left(\nu \underbrace{\sum_{n=1}^{\infty} \frac{\nu^{n-1}}{(n-1)!}}_{\nu e^{\nu}}\right) + \nu^2 \underbrace{\sum_{n=2}^{\infty} \frac{\nu^{n-2}}{(n-2)!}}_{\nu^2 e^{\nu}}\right]$$

$$= \nu^2 + \nu$$

Number of people kicked to death by horses every year obeys poisson statistics.

7. Assume Sun and asteriod are perfect black bodies.



power ~~per sur~~ emitted by the Sun

$$P_1 = 4\pi R^2 \sigma T_s^4$$

power received by asteriod

$$P_2 = \cancel{4\pi R^2} \sigma T_s^4 \frac{\pi r^2}{\cancel{4\pi} \alpha^2 \cancel{R^2}} = \sigma T_s^4 \frac{\pi r^2}{\alpha^2}$$

power emitted by asteriod

$$P_3 = \sigma T_a^4 \, 4\pi r^2$$

perfect black body

$$P_2 = P_3 \rightarrow \cancel{\sigma} T_a^4 \, 4\cancel{\pi r^2} = \cancel{\sigma} T_s^4 \frac{\cancel{\pi r^2}}{\alpha^2}$$

$$\therefore \quad T_a^4 = T_s^4 \left(\frac{1}{2\alpha}\right)^2$$

$$\therefore \quad T_a = \frac{T_s}{\sqrt{2\alpha}}$$

8. energy levels $E_1 = -\mu B$ $\quad E_2 = \mu B$

single particle partition function

$$Z_1 = e^{+\beta\mu B} + e^{-\beta\mu B} = 2\cosh(\beta\mu B)$$

$\rightarrow$ overall system $Z = Z_N = \left(2\cosh(\beta\mu B)\right)^N$

$$U = -\frac{\partial \ln Z}{\partial \beta} = -\frac{\partial}{\partial \beta}\left(N \ln(2\cosh(\beta\mu B))\right)$$

$$= -N \frac{1}{\cancel{2}\cosh(\beta\mu B)} \cancel{2}\sinh(\beta\mu B)\,\mu B$$

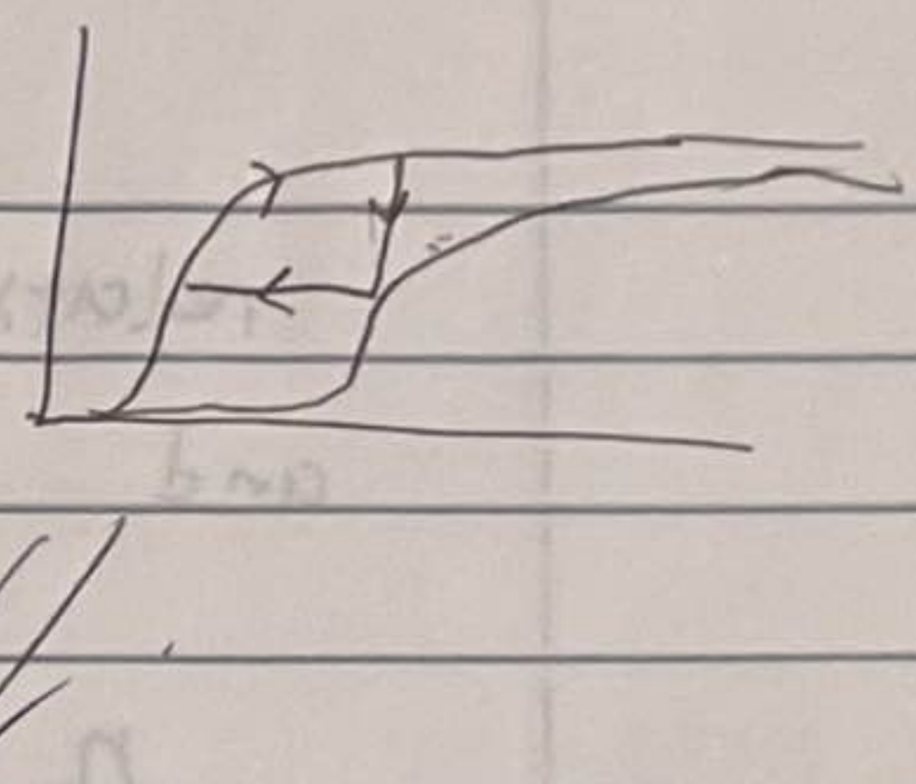
$$= -N\mu B \tanh(\beta\mu B)$$

$$S = \frac{U}{T} + k_B \ln Z$$

$$= -Nk_B \underbrace{\left(\frac{\mu B}{k_B T}\right)}_{x} \tanh\Big(\underbrace{\frac{\mu B}{k_B T}}_{x}\Big) + Nk_B \ln\Big(2\cosh\big(\underbrace{\frac{\mu B}{k_B T}}_{x}\big)\Big)$$

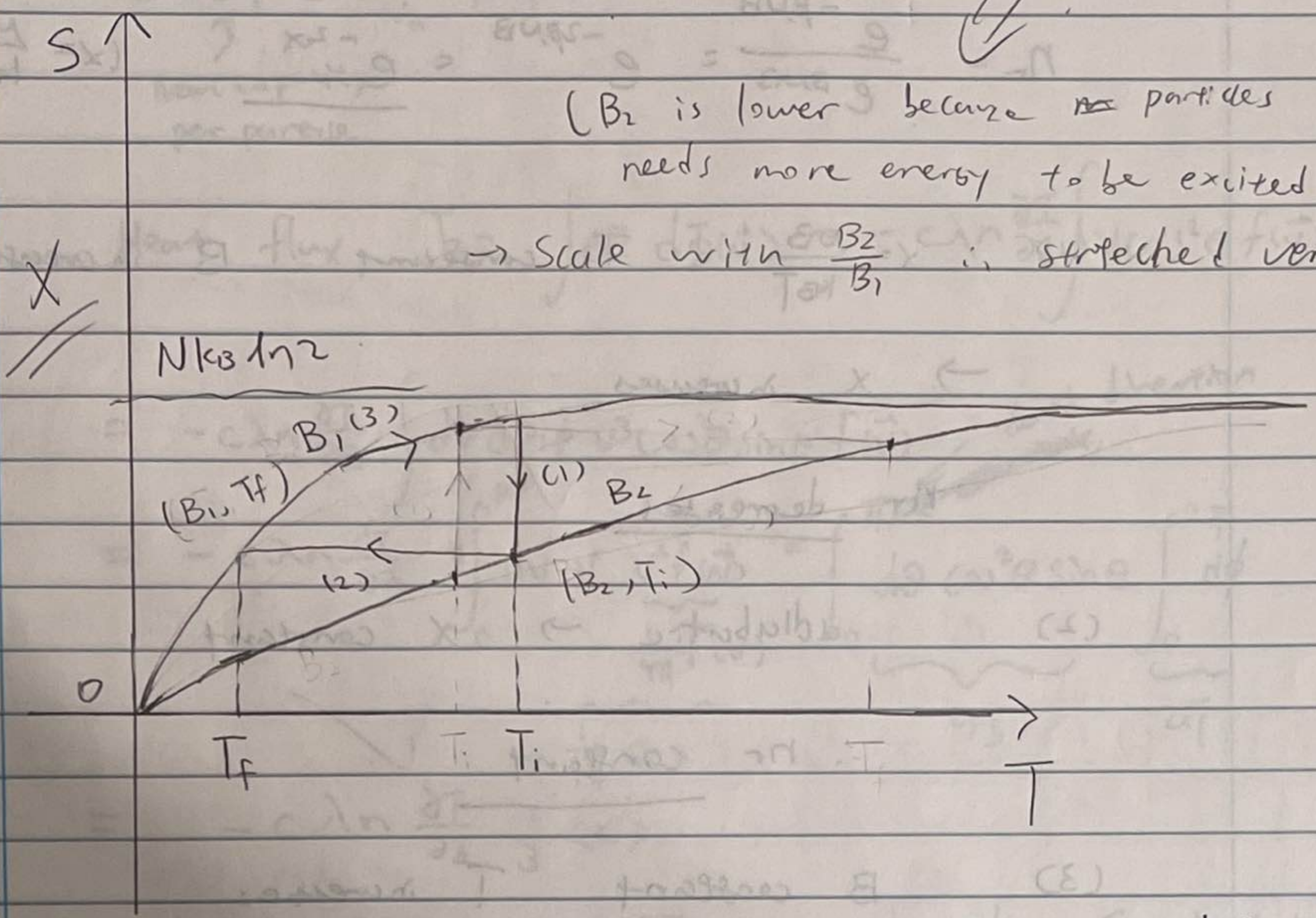
$$= Nk_B \left[\ln(\cosh(x)) - x\tanh(x) + \ln 2\right]$$

$\rightarrow$ $\underline{C = 1}$ $\underline{D = -1}$ $\underline{E = \ln 2}$



(B₂ is lower because ~~..~~ particles needs more energy to be excited)

$\rightarrow$ Scale with $\frac{B_2}{B_1}$ $\therefore$ stretched version



$\because$ (2) $(B_2, T_i) \rightarrow (B_1, T_f)$ is adiabatic

$\therefore$ $S(B_2, T_i) = S(B_1, T_f)$

$\because$ $x = \frac{\mu B}{k_B T}$ and S is a function only of x $\therefore$ ~~x~~ $\frac{\mu B_2}{k_B T_i} = \frac{\mu B_1}{k_B T_f}$

$\rightarrow$ $\underline{T_f = \frac{B_1}{B_2} T_i}$

$U = nrg$ $\quad p = \frac{}{z}$

$= n.$

relative occupation number is between ~~lower~~ upper and ~~upper~~ lower states is

$$n_r = \frac{e^{-\beta \mu B}}{e^{\beta \mu B}} = e^{-2\beta \mu B} = \underline{e^{-2x}} \qquad \left(x = \frac{\mu B}{k_B T}\right)$$

(1) $x = \frac{\mu B}{k_B T}$ $\quad T$ constant $\quad B$ increases

$\rightarrow$ x increases

n_r <u>decreases</u>

(2) adiabatic $\rightarrow$ x constant

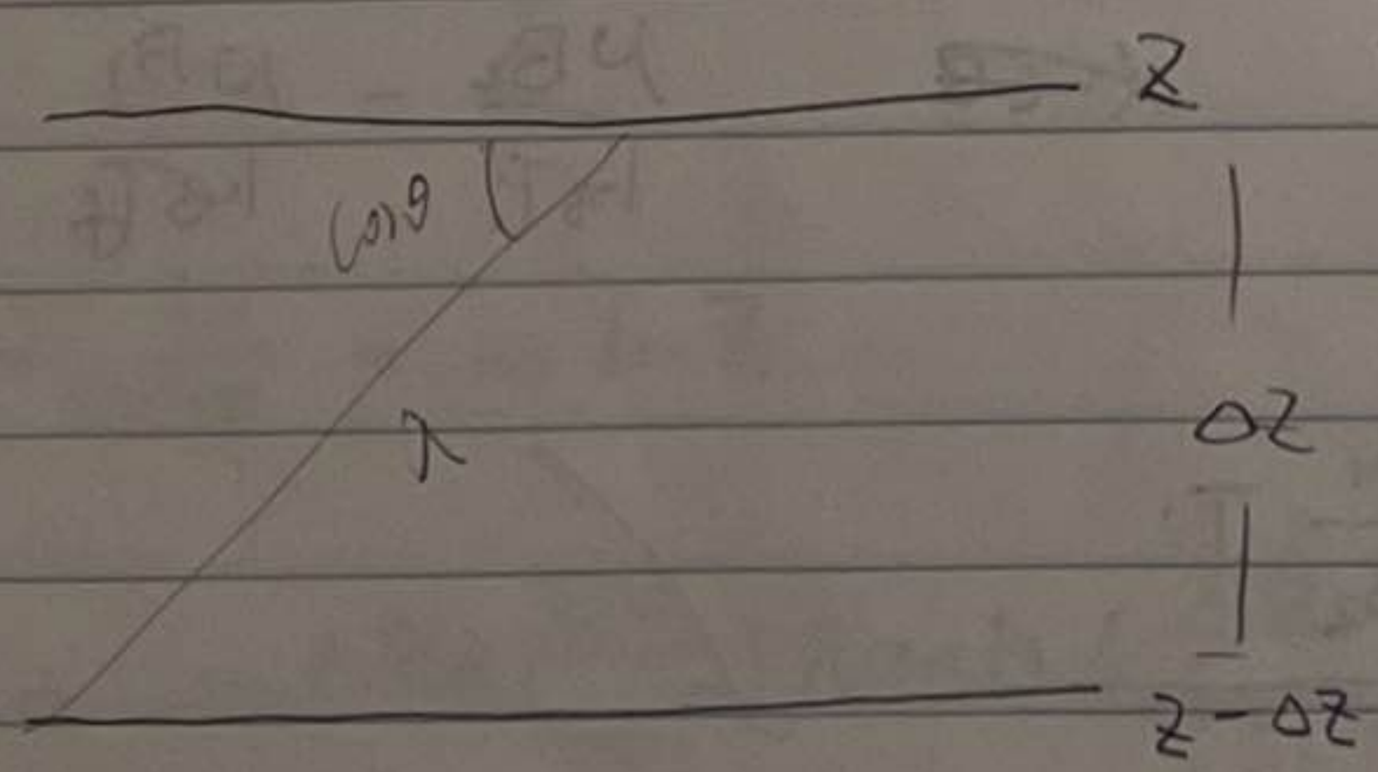
$\therefore$ n_r <u>constant</u>

(3) B constant $\quad T$ increases

$\rightarrow$ x decreases

n_r <u>increases</u>

9.



particle flux ~~is~~ (per area per time)

$$d\Phi(\vec{v}) = n v_z f(\vec{v}) d^3\vec{v}$$

Extra thermal energy per particle

$$\Delta E = E(z-\Delta z) - E(z) = \cancel{E(z)} - \frac{\partial E}{\partial z}\Delta z - \cancel{E(z)}$$

$$= -\frac{\partial E}{\partial z}\underbrace{\Delta z}_{=\lambda\cos\theta} = -C\lambda\cos\theta\frac{\partial T}{\partial z}$$

$$(E = CT = \frac{C_{mol}}{N_A} T)$$

C: heat capacity per particle; C_{mol}: molar heat capacity

Heat flux $$J_z = \int \Delta E \, d\Phi(\vec{v}) = -C\lambda n\frac{\partial T}{\partial z}\int v\cos^2\theta f(\vec{v})\, d^3\vec{v}$$

$$= -C\lambda n\frac{\partial T}{\partial z}\int dv\, d\theta\, d\phi\; v^3\cos^2\theta\sin\theta f(\vec{v})$$

$$= -C\lambda n\frac{\partial T}{\partial z}\cdot\underbrace{\int_0^\infty dv\, v\, \underbrace{v^2 f(\vec{v})}_{\frac{1}{4\pi}f(v)}}_{\frac{1}{4\pi}\langle c\rangle}\cdot\underbrace{\int_0^{\pi} d\theta\cos^2\theta\sin\theta}_{2/3}\underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

both direction ~~$\pi/2$~~

$$= -C\lambda n\frac{\partial T}{\partial z}\frac{1}{3}\langle c\rangle$$

$$= -\left(\frac{1}{3}nC\lambda\langle c\rangle\right)\frac{\partial T}{\partial z} = -\left(\frac{1}{3N_A}nC_{mol}\lambda\langle c\rangle\right)\frac{\partial T}{\partial z}$$

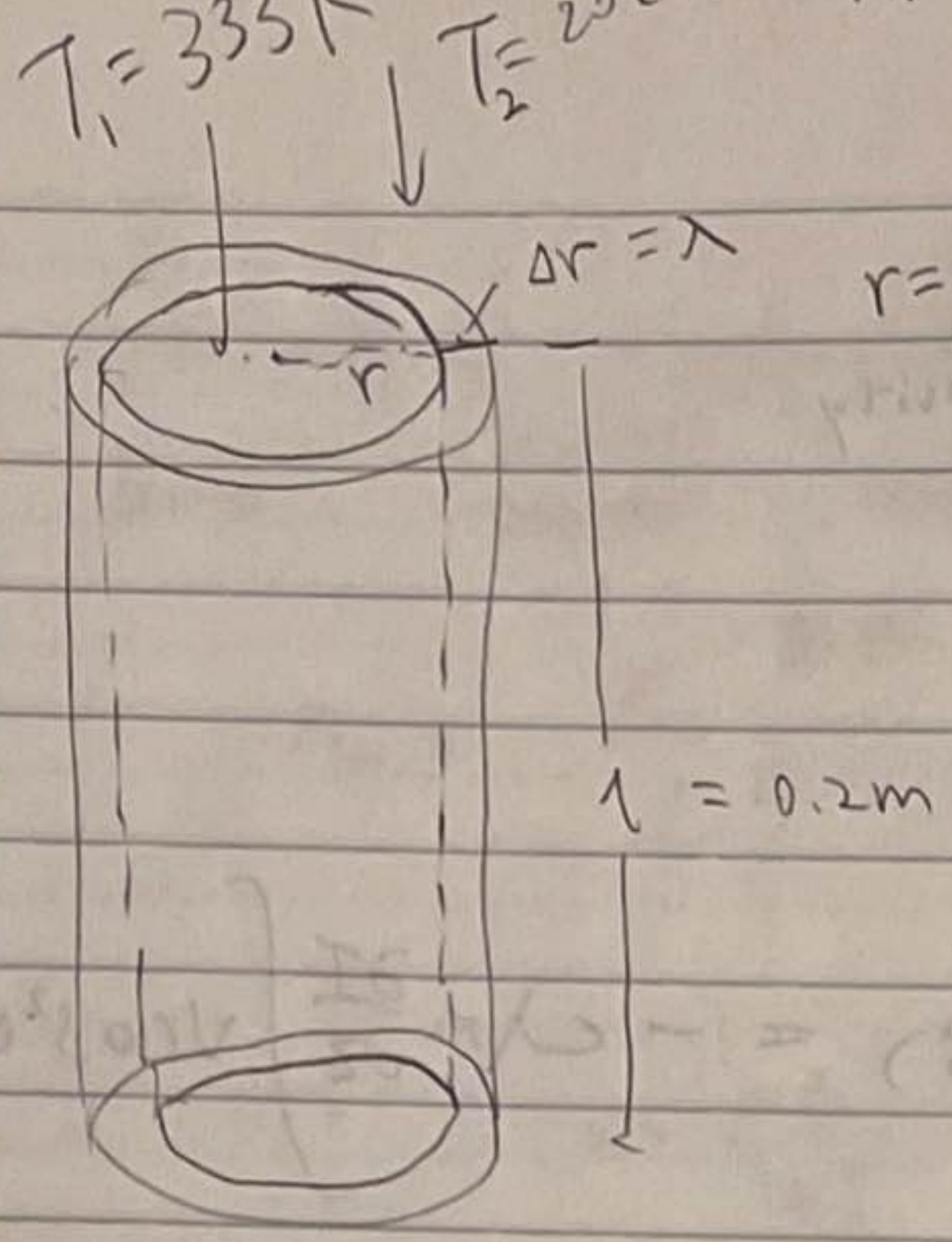
$$= -\kappa\frac{\partial T}{\partial z}$$

$$\rightarrow \kappa = \frac{1}{3N_A}nC_{mol}\lambda\langle c\rangle$$

α

Assumptions: all particles ~~have~~ have travelled by ~~a~~ mean free path since their last collision.

maxwellian / isotropic / no forces except collisions



$\tilde{T} = 40°C = 313K$

$r = 0.05m$

$\Delta r \approx \lambda$ for low pressure

$P = 10^{-2}$ Pa

Rate of heat loss

$$Q = 2\pi r l J_r$$

$\uparrow$ radial heat flux

$$\underline{J} = -\kappa \underline{\nabla} T \qquad J_r = \underline{J}\cdot\hat{\underline{r}} = -\kappa \underbrace{\underline{\nabla} T\cdot\hat{\underline{r}}}_{\frac{\partial T}{\partial r}} = -\kappa \frac{\partial T}{\partial r}$$

$$\therefore \quad Q = -2\pi r l \kappa \frac{\partial T}{\partial r} \approx -2\pi r l \kappa \frac{\Delta T}{\Delta r} \approx -2\pi r l \left(\frac{\kappa}{\lambda}\right)\Delta T$$

$$\kappa = \alpha n C_{mol} \lambda \langle c \rangle \rightarrow \frac{\kappa}{\lambda} = \alpha n C_{mol} \langle c \rangle$$

$$= \frac{1}{3} n c \langle c \rangle$$

$c \rightarrow \frac{5}{2} k_B$ for Nitrogen, $\quad \langle c \rangle \rightarrow \sqrt{\frac{8 k_B T}{\pi m}}$

$$\therefore \quad Q = -2\pi r l \cdot \frac{1}{3} n \frac{5}{2} k_B \sqrt{\frac{8 k_B \tilde{T}}{\pi m}} \Delta T$$

$n \rightarrow \frac{P}{k_B \tilde{T}}$, $\quad m \rightarrow 28 m_p$

$$= \cancel{-2\pi r l} \; -\frac{5}{3}\sqrt{\pi}(2\sqrt{2}) r l P \sqrt{\frac{k_B}{\tilde{T}(28 m_p)}} \Delta T$$

$$= -\frac{5}{3}\sqrt{\pi}(2\sqrt{2})(0.05)(0.2)(10^{-2})\sqrt{\frac{1.38\times10^{-23}}{313(28\times1.67\times10^{-27})}}(-40)$$

$$= \underline{0.032 \text{ W}}$$

$\cancel{Qt = C_w(60°C - 40°C)}$

$\cancel{Q =}$ $Q(T_1 = 60°C) = 0.032\,W$

$$Q \propto \frac{\Delta T}{\sqrt{\bar{T}}} = 0.014 \frac{\Delta T}{\sqrt{\bar{T}}} = 0.02 \frac{T_1 - 293K}{\sqrt{T_1 + 293K}}$$

$$\bar{T} = \frac{T_1 + 293}{2}$$

$\therefore\ Q(T)\,dt = \cancel{C_w(T_1 - 313K)}\ C_w\,dT_1$

$\therefore\ \cancel{dt = C_w}$

$$\therefore\ dt = C_w \frac{dT}{Q(T)} =$$

$$\rightarrow T = \int_0^T dt = C_w (0.02)^{-1} \int_{333}^{313} \frac{\sqrt{T_1 + 293}}{T_1 - 293}\, dT$$

$$= (4.2 \times 10^3)(0.02)^{-1}(17.19)$$

$$= \cancel{1444\,s}\ \underline{3.61 \times 10^6\,s}$$

or $\quad T = \frac{C_w(60 - 40)}{0.032} = \cancel{2.6}\ \underline{2.63 \times 10^6\,s}$

Other possible ways: convection
radiation

10. 2-D density of states $D(k) = \frac{A}{2\pi} k\,dk\ (2s+1)$

classical ~~speeds~~ → $\varepsilon = \frac{\hbar^2 k^2}{2m}$ → $d\varepsilon = \frac{\hbar^2}{m} k\,dk$

$\rightarrow k\,dk = \frac{m\,d\varepsilon}{\hbar^2}$

$\therefore$ ~~D(ε)dε~~ $D(\varepsilon)\,d\varepsilon = (2s+1)\frac{A}{2\pi}\frac{m}{\hbar^2}\,d\varepsilon = \left(\frac{Am}{\hbar^2\pi}\right)\frac{2s+1}{2}\,d\varepsilon$

$\because$ electrons $s = \frac{1}{2}$ $\therefore$ $2s+1 = 2$

$\rightarrow$ $D(\varepsilon) = \left(\frac{Am}{\hbar^2\pi}\right) d\varepsilon$

mean occupation numbers for fermions

$\bar{n}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$ μ is chemical potential

$\beta = \frac{1}{k_B T}$

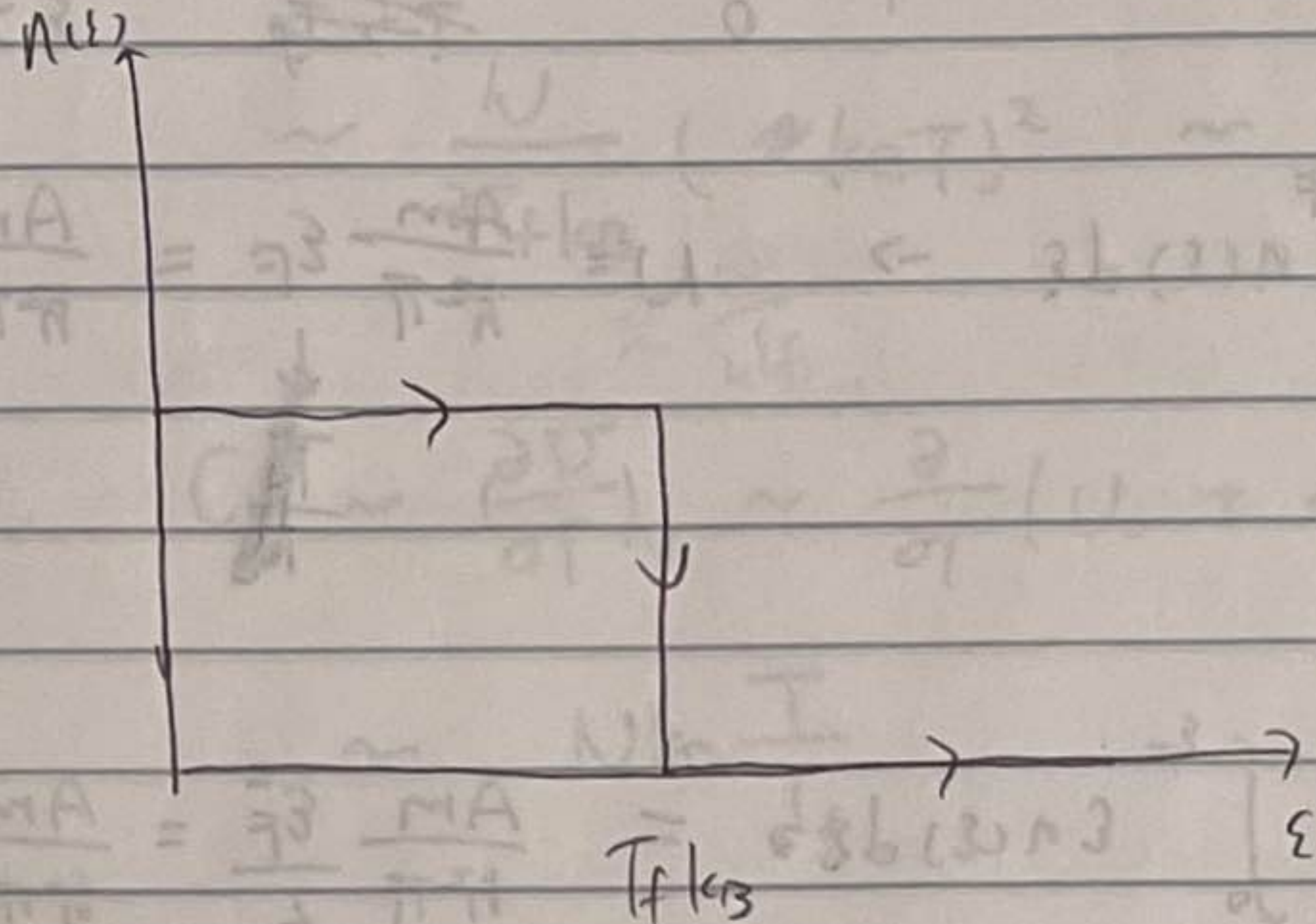
$n(\varepsilon)\,d\varepsilon = \bar{n}(\varepsilon)\,D(\varepsilon)\,d\varepsilon$

$= \frac{Am}{\hbar^2\pi}\left(\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right) d\varepsilon$

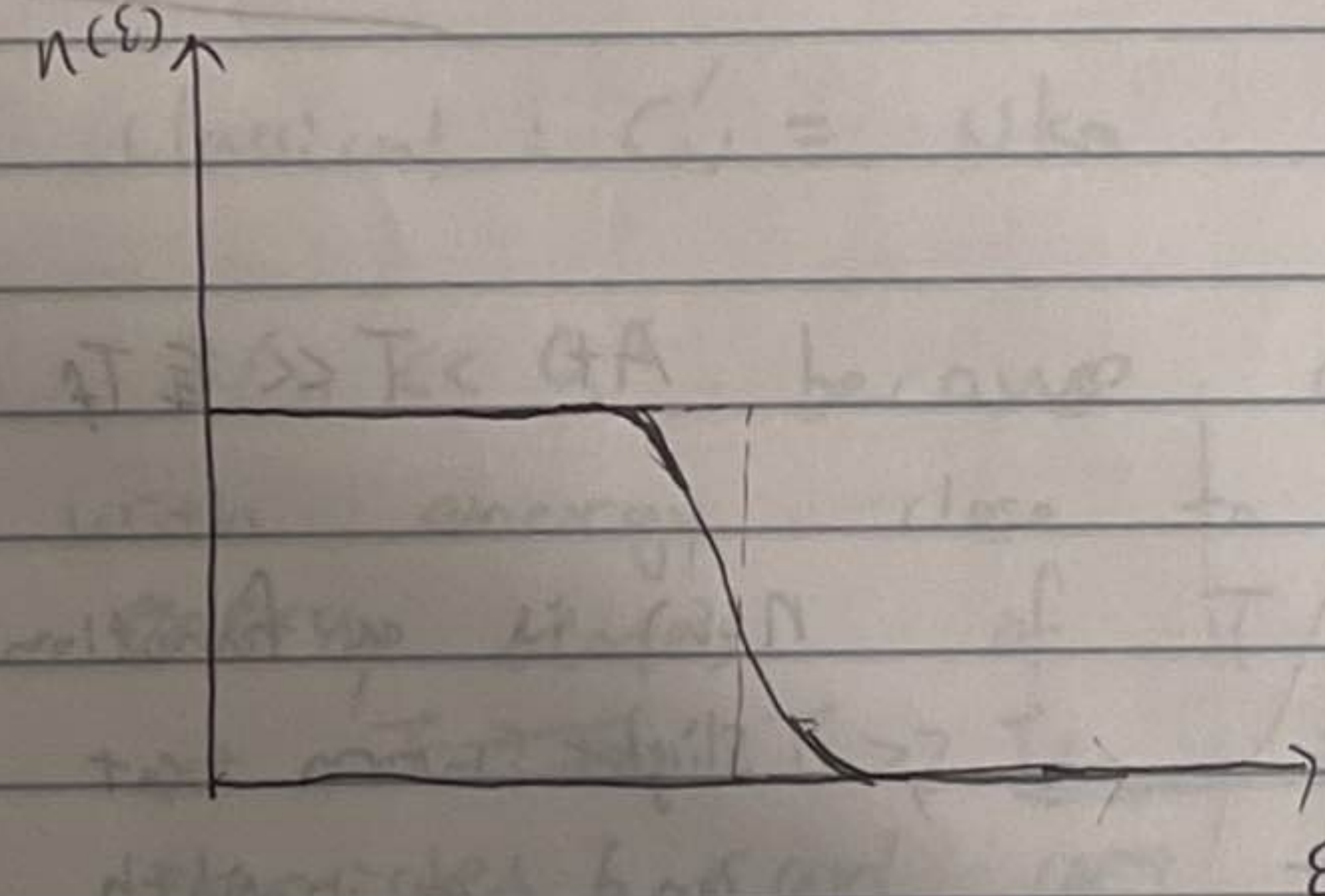
Fermi energy ε_F is the chemical potential of degenerate Fermi Gas at $T=0$

Fermi temperature $T_F = \frac{\varepsilon_F}{k_B}$

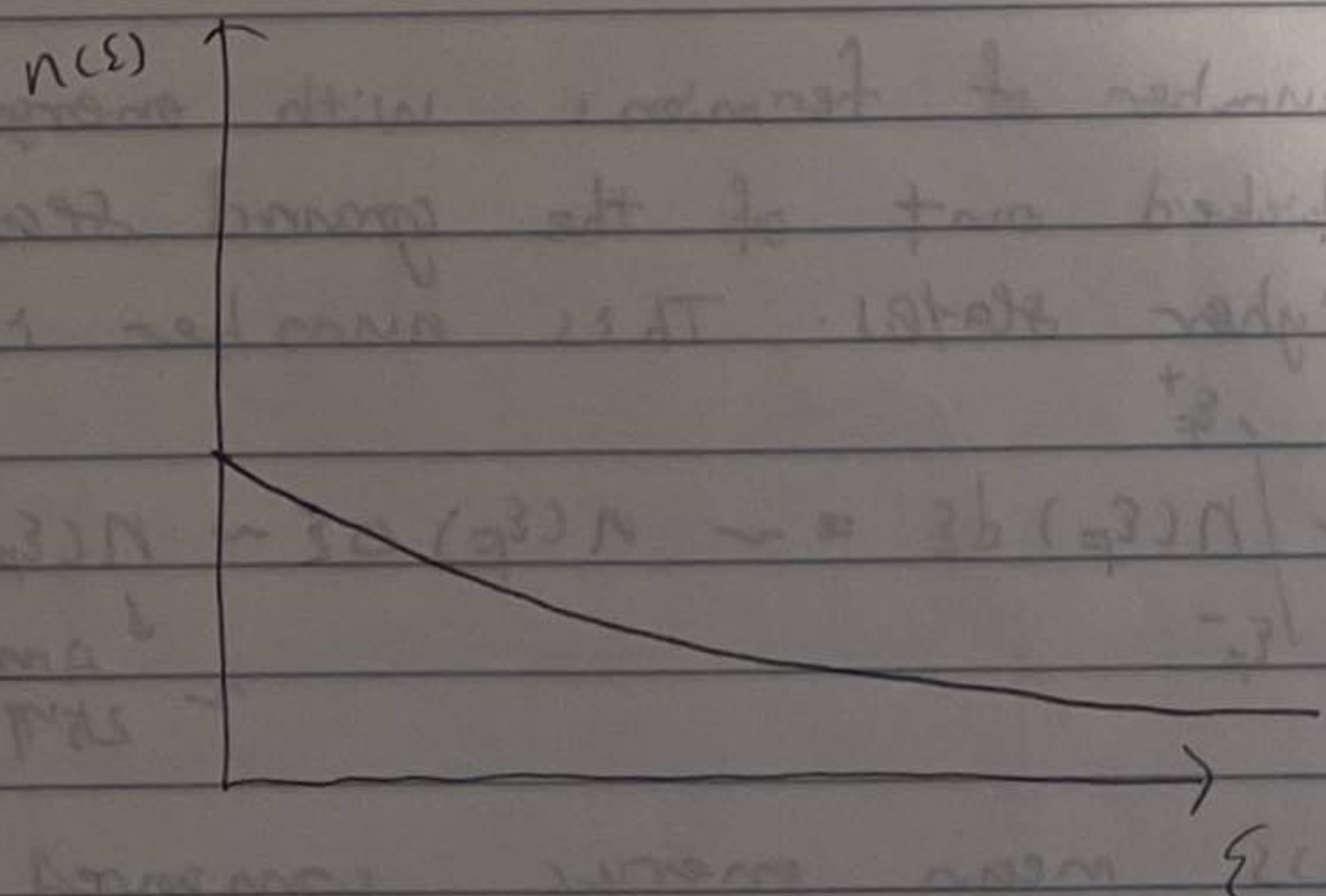
(a) $T=0$



(b) $T \ll T_f$



(c)



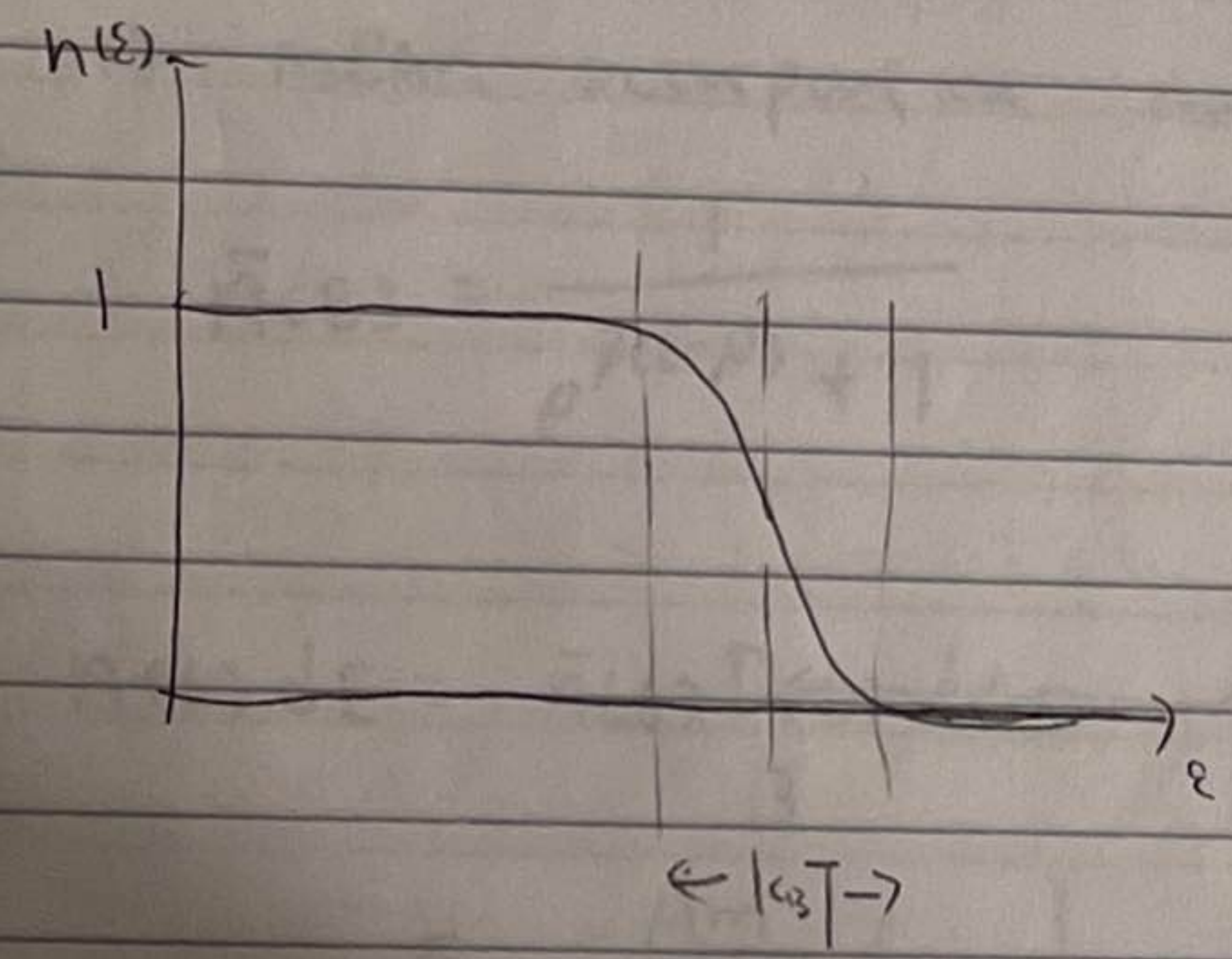
At $T=0$ $\quad n(\varepsilon) = \begin{cases} \dfrac{Am}{\hbar^2\pi} & 0<\varepsilon<\varepsilon_F \\ 0 & \varepsilon>\varepsilon_F \end{cases}$

$\therefore \quad N = \int_0^{\varepsilon_F} n(\varepsilon)\, d\varepsilon \rightarrow \quad N = \frac{Am}{\hbar^2\pi}\varepsilon_F = \frac{Am}{\hbar^2\pi} T_f k_B$

internal energy

$$U = \int_0^{\varepsilon_F} \varepsilon\, n(\varepsilon)\, d\varepsilon = \frac{Am}{\hbar^2\pi}\frac{\varepsilon_F^2}{2} = \frac{Am}{\hbar^2\pi}\left(\frac{1}{2}T_f k_B\right)T_f k_B$$

energy per particle $\quad E = \frac{U}{N} = \frac{1}{2}k_B T_f$



At $T \ll T_f$

$n(\varepsilon)$ is a function that is slight worn at $\varepsilon = \varepsilon_F$ and a width of order $\Delta\varepsilon \sim k_B T$

→ A small number of fermions with energy $\sim \varepsilon_F$ can be kicked out of the ground state to slightly higher states. This number is

$$\Delta N \sim \int_{\varepsilon_F^-}^{\varepsilon_F^+} n(\varepsilon_F)\, d\varepsilon \sim n(\varepsilon_F)\Delta\varepsilon \sim \underbrace{n(\varepsilon_F)}_{\sim \frac{Am}{2\hbar^2\pi}} k_B T$$

→ excess mean energy compared with $T=0$ is

$$\delta U(T) \sim \Delta N \, \Delta \varepsilon \sim n(\varepsilon_F)(k_B T)^2 \sim \frac{Am}{2\hbar^2 \pi}(k_B T)^2$$

$$\sim \underbrace{\frac{N}{2T_F k_B}}_{\approx \frac{N}{2\varepsilon_F}} (\cancel{2} k_B T)^2 \sim \cancel{\frac{A}{2}} \frac{N k_B}{\cancel{2} T_F} T^2$$

$$\therefore \quad C_V \sim \left.\frac{\partial U}{\partial T}\right|_V \sim \frac{\partial}{\partial T}(U_0 + \delta U)$$

$$\sim N k_B \frac{T}{T_F}$$

or estimate $n(\varepsilon_F)$ to be $\left(\frac{N}{\varepsilon_F}\right)$

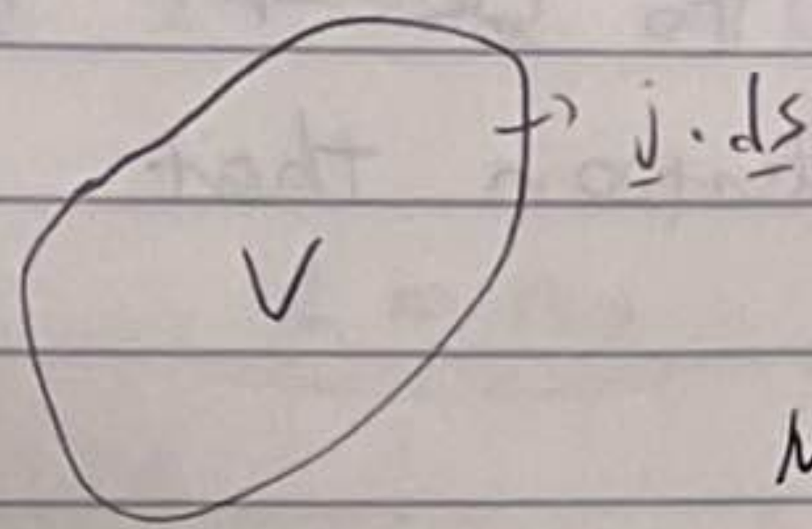
$\therefore N = \int_0^{\varepsilon_F} n(\varepsilon)\, d\varepsilon$

$$\rightarrow \quad \underline{C_V = \text{const} \times N k_B \frac{T}{T_F}}$$

classical : $C_V' = N k_B$ for 2-D gas (equipartition)

$C_V \ll C_V'$ because at $T \ll T_F$, only particles with energy close to ε_F can get to higher energy states if T is increased, whereas at ~~$T \to \infty$~~ $T \gg T_F$, if T is increased all particles can get to higher energy states.

11.



$\underline{j} = -D\underline{\nabla} n$

Number of particles going out of the surface per unit time is the rate of change of the number of particles inside the volume (decreases)

$$\therefore \int \underline{j} \cdot d\underline{s} = -\frac{\partial}{\partial t} \int n \, dV$$

$\rightarrow$ Divergence theorem $\int \underline{\nabla} \cdot \underline{j} \, dV = -\frac{\partial}{\partial t} \int n \, dV$

$$\therefore \frac{\partial n}{\partial t} = -\underline{\nabla} \cdot \underline{j} = -\underline{\nabla} \cdot (-D\underline{\nabla} n)$$

$$\rightarrow \frac{\partial n}{\partial t} = D\nabla^2 n$$

in 1-D $n = n(x,t)$ $\therefore \nabla^2 n = \frac{\partial^2 n}{\partial x^2}$

$$\rightarrow \frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}$$

separation of variables : $n(x,t) = X(x) T(t)$

$$\therefore X \frac{dT}{dt} = DT \frac{dX^2}{dx^2} \rightarrow \frac{1}{T}\frac{dT}{dt} = D \frac{1}{X} \frac{dX^2}{dx^2} = -m^2 D$$

(A function only of t = A function only of x $\Rightarrow$ they must both equal to a constant)

~~If m~~

$$\therefore \frac{1}{T}\frac{dT}{dt} = -m^2 \rightarrow \frac{dT}{dt} = -m^2 T D$$

(choose this constant to be $-m^2$ because we want a ~~so~~ solution that becomes steady as $t \to \infty$)

If $m \neq 0$

$\to \frac{dT}{T} = -m^2 dt D \quad \therefore \quad \ln T = -m^2 tD + A''$

$\to \quad T(t) = A'_m e^{-m^2 tD}$

$\frac{d^2X}{dx^2} + m^2 x = 0 \quad \to \quad X(x) = \cancel{A} B'_m \sin(mx) + C_m \cos(mx)$

$\to$ The general solution (let $A'_m B_m = A_m$, $A'_m C_m = B_m$

$$n(x,t) = \sum_m \left(A_m \sin(mx) + B_m \cos(mx)\right) e^{-m^2 tD}$$

If $m = 0$, then $\frac{dT}{dt} = 0 \quad \to T(t) = \cancel{\alpha t + \beta}\ \alpha$

$\frac{dX^2}{dx^2} = 0 \quad \to \quad X(x) = \beta x + \gamma$

$\because$ the βx term explodes at $x \to \infty$

$\therefore \quad \beta = 0 \quad$ let $K = \alpha\gamma$

then the general solution is

$$n(x,t) = K + \sum_m \left(A_m \sin(mx) + B_m \cos(mx)\right) e^{-m^2 tD} \quad ①$$

~~Initial conditions~~

Initial / Final / Boundary Conditions :

$\rightarrow$ As $t \to \infty$ $\quad n(x,t) \to n_0$

$\therefore$ $\underline{K = n_0}$

$\rightarrow$ At $x = L$ $\quad \frac{\partial n}{\partial x} \to 0$ $\quad$ (no particle flux at $x = L$

$\rightarrow \nabla n = 0$)

$\rightarrow \frac{\partial n}{\partial x} = 0$

$$\therefore \quad 0 = \left.\frac{\partial n}{\partial x}\right|_{x=L} = \sum_m e^{-m^2 tD}(m A_m \cos(mx) - m B_m \sin(mx))$$

②

$\rightarrow$ At $x = 0$ $\quad n = n_0$

$$\therefore \quad n_0 + \sum_m B_m \underbrace{\cos(\overset{0}{m x})}_{1} e^{-m^2 tD} = 0 \quad \rightarrow \underline{B_m = 0}$$

(true for all t

$$\therefore ② \rightarrow \quad 0 = \sum_m e^{-m^2 tD}(m A_m \cos(mx))\Big|_{x=L}$$

$\therefore$ ~~$\cos(mx)$~~ $\therefore$ $\cos(mL) = 0$

$\therefore$ $mL = \frac{2p+1}{2}\pi$ $\quad$ ($p = 0, 1, 2, 3, \cdots$)

$\rightarrow$ $m = \frac{(2p+1)\pi}{2L}$ $\quad$ ($p = 0, 1, 2, 3, \cdots$)

Final condition: $\quad$ At $t = 0$, $n(x,t) = 0$ except at $x = 0$

$$\therefore \quad 0 = n_0 + \sum_{p=0}^{\infty} A_p \sin\left(\frac{(2p+1)\pi x}{2L}\right) \underbrace{\exp\left(-D\left(\frac{(2p+1)\pi}{2L}\right)^2 t\right)}_{=1}$$

for $t = 0$

use the orthogonality of sine functions

$$\frac{2}{a}\int_{0}^{a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right) = \delta_{mn} \qquad (m, n \text{ integers})$$

$\therefore$ let $n = 2p+1$ $\quad m = 2q+1$ $\quad a = 2L$

then

$$-\frac{n_0}{\cancel{2}L}\int_{0}^{2L} \sin\left(\frac{(2p+1)\pi x}{2L}\right) dx$$

not 2L

$$= \sum_{q=0}^{\infty} \frac{1}{L}\underbrace{\int_{0}^{2L} \sin\left(\frac{(2p+1)\pi x}{2L}\right)\sin\left(\frac{(2q+1)\pi x}{2L}\right)}_{\delta_{pq}} A_q$$

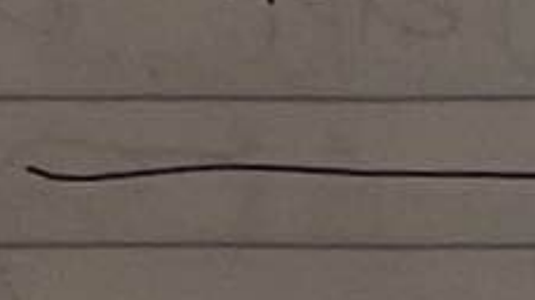
$$= A_p$$

$$\rightarrow A_p = \frac{n_0}{L}\underbrace{\cos\left(\frac{(2p+1)\pi x}{2L}\right)\Bigg|_{0}^{2L}}_{=-1-1} \times \frac{2L}{(2p+1)\pi}$$

$$= -\frac{2n_0}{L} \times \frac{2L}{(2p+1)\pi} = -\frac{4n_0}{(2p+1)\pi}$$

$$\therefore n(x,t) = n_0 - \sum_{p=0}^{\infty} \frac{4n_0}{(2p+1)\pi}\sin\left(\frac{(2p+1)\pi x}{2L}\right)\exp\left(-D\left(\frac{(2p+1)\pi}{2L}\right)^2 t\right)$$

$$\rightarrow \frac{n(x,t)}{n_0} = 1 - \sum_{p=0}^{\infty} \frac{4}{(2p+1)\pi}\sin\left(\frac{(2p+1)\pi x}{2L}\right)\exp\left(-D\left(\frac{(2p+1)\pi}{2L}\right)^2 t\right)$$



need an appendix to explain

since $A_p = \frac{4}{(2p+1)\pi}$ term decreases with increasing p

significantly, we ignore all terms except $p=0$

$(A_0 = \frac{4}{\pi})$ for large t.

$$\therefore \frac{n(x,t)}{n_0} \simeq 1 - \frac{4}{\pi} \sin\left(\frac{\pi x}{2L}\right) \exp\left(-D\left(\frac{\pi}{2L}\right)^2 t\right)$$

$$\left\langle \frac{n(x,t)}{n_0} \right\rangle \simeq 1 - \frac{4}{\pi}\left[\frac{1}{L}\int_0^L \sin\left(\frac{\pi x}{2L}\right) dx\right] e^{-\frac{\pi^2 D}{4L^2}t}$$

$$= 1 - \frac{4}{\pi \cancel{L}} \cdot \frac{2\cancel{L}}{\pi}\underbrace{\left[-\cos\left(\frac{\pi x}{2L}\right)\right]_0^L}_{1}$$

$$= 1 - \frac{8}{\pi^2} e^{-\frac{\pi^2 D}{4L^2}t}$$

If $n(x,t) \sim n_0$, then

$$\left\langle \frac{n(x,t)}{n_0} \right\rangle \sim 1$$

$$\therefore \frac{8}{\pi^2} e^{-\frac{\pi^2 D}{4L^2}t} \approx 0.1$$

$$\therefore e^{-\frac{\pi^2 D}{4L^2}t} \approx \frac{\pi^2}{80}$$

$$\therefore \ln\left(\frac{80}{\pi^2}\right) \approx \frac{\pi^2 D}{4L^2} t$$

$$\therefore t \approx \frac{4L^2}{\pi^2 D} \ln\left(\frac{80}{\pi^2}\right) \approx \underline{8.5 \times 10^{10}\,s}$$

$$\approx \underline{2700 \text{ years}}$$

→ (or use 0.01 to get 5650 years)