XPHC 2661
XPHA 2661

# SECOND PUBLIC EXAMINATION

**Honour School of Physics Part A: 3 and 4 Year Courses**

**Honour School of Physics and Philosophy Part A**

A1: THERMAL PHYSICS

**Wednesday, 17 June 2009, 9.30 am – 12.30 pm**

TRINITY 2009

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*Answer* **all** *questions of Section A and* **three** *questions from Section B.*

*Start the answer to each question on a fresh page.*

*A list of physical constants and conversion factors accompanies this paper.*

*The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.*

**Do NOT turn over until told that you may do so.**

## Section A

**1.** The speed $v$ of molecules in an ideal gas is given by the Maxwell distribution

$$P(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 e^{-mv^2/2k_B T},$$

where the symbols have their normal meanings. Estimate

(a) the most probable speed, average speed and rms speed for nitrogen molecules ($N_2$) at 300 K.

(b) the temperture of a gas comprised of nitrogen molecues that have a mean translational kinetic energy of $6.07 \times 10^{-21}$ J.

[You may take the molar mass of a nitrogen atom (N) to be 14 g mol$^{-1}$.] [6]

**2.** An Otto cycle consists of four stages: (a) an adiabatic compression from $V_1$ to $V_2$, (b) an isochoric pressure increase; (c) an adiabatic expansion from $V_2$ to $V_1$; and (d) an isochoric pressure decrease. Sketch the $p$–$V$ diagram indicating where heat enters and leaves the system. Assuming that the gas behaves like an ideal gas with a constant heat capacity, show that the efficiency is

$$\eta = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1},$$

where $\gamma$ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. [8]

**3.** A person (mass 57.8 kg) tries to penetrate a garden fork (mass 1 kg) into a melting block of ice by standing on it with all his/her weight. Assume that no heat flows from the fork to the ice. The fork has four prongs, and each prong has a square cross section of area 1 mm$^2$. By how much must the temperature of the ice be lowered to resist penetration?
[You may take the density of water and ice at 0 °C to be 1000 kg m$^{-3}$ and 916.7 kg m$^{-3}$, respectively. The latent heat of fusion of ice is $333 \times 10^3$ J kg$^{-1}$.] [6]

**4.** By considering entropy $S$ to be a function of temperature $T$ and volume $V$ show that

$$C_p - C_V = \frac{VT\beta_p^2}{\kappa_T},$$

where $C_p$ and $C_V$ are the heat capacities at constant pressure and volume, respectively, $\beta_p$ is the isobaric expansivity defined by $\beta_p = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$, and $\kappa_T$ is the isothermal compressibility defined by $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$. [7]

**5.** Write down the fundamental postulate of statistical mechanics and use it to show that, for an isolated ensemble of weakly-interacting distinguishable systems of the same type, the most likely population distribution is the Boltzmann distribution.

[You may assume that, for an ensemble of $N$ distinguishable systems of the same type, the number of ways of assigning systems to states with occupation numbers $n_i$ is given by $W = \dfrac{N!}{n_1!n_2!n_3!\cdots}$.] [8]

**6.** In the classical limit, the partition function for $N$ molecules of an ideal gas can be conveniently written $Z = (Vz)^N/N!$ where $z$ depends only on the temperature $T$, some internal properties of a molecule and fundamental constants. Find an expression for the chemical potential in terms of $k_B T$, $N/V$ and $z$. [5]

$f(\vec{v})\, d^3\vec{v} = \frac{\langle n_k \rangle}{N} \underbrace{\frac{V}{(2\pi)^3}}_{1} d^3k = \frac{\langle n_k \rangle}{\cancel{N}} \left(\frac{m}{2\pi\hbar}\right)^3 d^3\vec{v}$

$\Rightarrow f(\vec{v}) = \left(\frac{m}{2\pi\hbar}\right)^3 \cancel{\frac{\langle n_k \rangle}{N}} \frac{\langle n_k \rangle}{n}.$

$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} \pm 1} \Rightarrow e^{-\beta(\varepsilon - \mu)}$

$= e^{\beta\mu} \cdot e^{-\beta\varepsilon}$

$u = m^2$

$du = 2m\,dm$

$\mu = k_B T \ln(n\lambda_{th}^3)$

$\pm$

$\int \frac{m}{1-m^2}\, dm$

$e^{\beta\mu} = \cancel{\ } \; n\lambda_{th}^3$

$= \int \frac{1}{1-u}\left(\frac{1}{2} du\right).$

$-\frac{1}{2}\ln(1-m^2)$

$= -\frac{1}{2}\ln(1-u)$



$-\frac{1}{1-m^2}(-2m)\cdot\frac{1}{2}$

## Section B

The heat diffusion equation in cylindrical coordinates is:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{C_V}\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{H}{C_V}.$$

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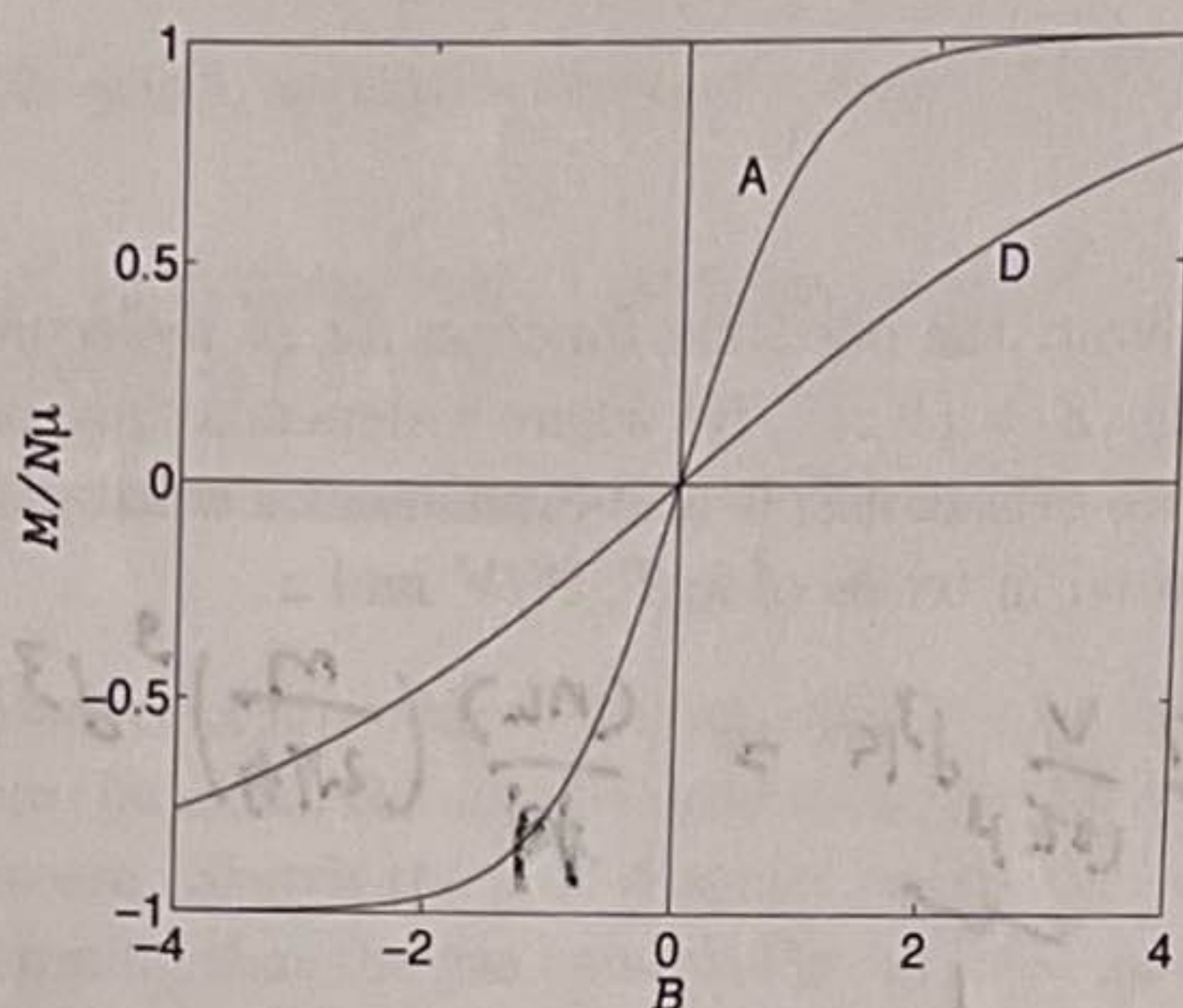
**7.**




A paramagnetic solid object is placed inside a solenoid in vacuo. The solid can be modelled as an ensemble of weakly interacting spin-half systems, each with magnetic dipole moment $\mu$. The diagram shows the magnetic dipole moment $M$ of the object as a function of applied magnetic field $B$, with two isotherms plotted.

(a) State, with reasons, which isotherm (A or D) corresponds to the higher temperature.

(b) Write down an expression for $M$ in terms of the populations of the two spin states and other relevant quantities. Give an argument to show that $M$ is constant if the entropy is constant.

(c) Explain why the spins do not all relax or 'flip' to their lowest energy state.

(d) Define a *Carnot cycle*. Copy the diagram and indicate on your diagram an example Carnot cycle. [9]

Write down the partition function for the system and use it to obtain the equation of state relating $M$, $B$ and $T$. Show that

$$B = \frac{k_B T}{2\mu}\ln\left(\frac{1+m}{1-m}\right),$$

where $m = M/(N\mu)$, and find $(\partial B/\partial m)_T$. Show that the magnetic work required to magnetise the object isothermally from $M = 0$ has the form $a(T)\ln(1-m^2)$ and obtain the proportionality constant $a(T)$. [6]

Write down expressions for the work done in each of the four parts of a Carnot cycle between limits $m_1, m_2$ and $T_1, T_2$, where $m_1 < m_2$ and $T_1 < T_2$. Hence, show that the net work done per cycle is proportional to $T_2 - T_1$.
[The magnetic work done during a small change is $dW = -M dB$.] [5]

**8.** Show that $|AB| = |A||B|$ for arbitrary $2\times 2$ matrices $A$ and $B$, where $|M|$ signifies the determinant of $M$.

A three-component real vector $\mathbf{r} = (x, y, z)$ is related to a complex $2\times 2$ matrix $S$ by

$$S = \begin{pmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{pmatrix}.$$

Let $S' = USU^\dagger$ be similarly related to $\mathbf{r}' = (x', y', z')$, where $U$ is a unitary matrix of determinant 1. Find $|S|$, and show that $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$. [6]

The matrix exponential $\exp(M)$ is defined by

$$\exp(M) = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \ldots .$$

Show that $\exp(\mathrm{i}\theta\sigma_x) = \cos(\theta)\, I + \mathrm{i}\sin(\theta)\,\sigma_x$. By relating $S$ to the Pauli spin matrices, or otherwise, show that, when $S$ is transformed to $S' = USU^\dagger$ by $U = \exp(\mathrm{i}\theta\sigma_x)$, the associated vector $\mathbf{r}'$ is related to $\mathbf{r}$ by a rotation through $2\theta$ about the $x$ axis. [11]

Show that any $2\times 2$ unitary matrix with unit determinant can be written as

$$\begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

for some pair of complex numbers $a$, $b$. [3]

[The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\ ]$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$ $\quad (ad)^* - (bc)^* = 1.$

$ad - bc = 1.$

$ad =$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \Big( a^*a + c^*c \quad a^*b$

**9.** What is meant by a quasistatic, adiabatic process? Starting from the First Law of Thermodynamics, show that, for an ideal gas at temperature $T$ occupying volume $V$,

$$TV^{\gamma-1} = \text{constant},$$

where $\gamma$ is the ratio of the specific heat at constant pressure to the specific heat at constant volume. [7]

The hydrostatic equation expresses the change in pressure $\mathrm{d}p$ due to a layer of atmosphere of thickness $\mathrm{d}z$ as

$$\mathrm{d}p = -\rho g\, \mathrm{d}z,$$

where $\rho$ is the density of air and $g$ is the acceleration due to gravity. Using this expression, show that the change in temperature with height for a parcel of air that rises adiabatically in the atmosphere can be expressed as

$$-\frac{(\gamma-1)}{\gamma}\frac{mg}{k_{\mathrm{B}}},$$

where $m$ is the molecular mass of dry air. [8]

Hence estimate the change in temperature with height for an ideal parcel of dry air as it rises adiabatically in the Earth's lower atmosphere. Repeat the calculation for a gas parcel in a planetary atmosphere composed of a mixture of monatomic gases whose effective molar mass is $28.96\times10^{-3}$ kg for Earth-like conditions, i.e. where $g = 9.8\,\mathrm{m\,s^{-2}}$. Comment on your result. [5]

$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)$

$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}$

**10.** The heat flux $J_z$ in the z-direction is given by

$$J_z = -\kappa \frac{\partial T}{\partial z},$$

where $\kappa$ is the thermal conductivity. Explain the necessity for a negative sign on the right-hand-side of this equation. Using simple kinetic theory, show that $\kappa$ is

$$\kappa = \tfrac{1}{3} C_V \lambda \langle v \rangle,$$

where $C_V$ is the heat capacity per unit volume, $\lambda$ is the mean free path, and $\langle v \rangle$ is the mean molecular velocity. [5]

Consider a cylinder of radius $a$ with thermal conductivity $\kappa_1$ with uniformly distributed heat sources providing heat per unit volume $H$ ($\mathrm{W\,m^{-3}}$). This cylinder is surrounded by a second cylinder of outer radius $b$ with thermal conductivity $\kappa_2$. If the cylinder is sufficiently long so that the temperature may be considered as a function of radius only, show that the temperature in the cylinder at equilibrium can be described by

$$T(r) = \begin{cases} T_a + \dfrac{H}{4\kappa_1}(a^2 - r^2) + \dfrac{Ha^2}{2\kappa_2}\ln(b/a), & \text{if } 0 \le r \le a, \\ T_a + \dfrac{Ha^2}{2\kappa_2}\ln(b/r), & \text{if } a \le r \le b, \end{cases}$$

where $T_a$ is the temperature of the cylinder at radius $b$. [9]

A stainless-steel wire is 0.1 mm in diameter and 1 m long. If the outside of the wire is held fixed at 20°C, estimate the steady-state current passing through the wire when the stainless steel at the centre of the wire begins to melt. A second wire similar to the first is encased in glass 2 mm thick. If the outside of the glass is held fixed at 20°C, estimate the current passing through the wire when the stainless steel at the centre of the wire begins to melt.

[For stainless steel, the melting point is 1400°C, $\kappa = 19\,\mathrm{W\,m^{-1}K^{-1}}$, and the resistivity is $70\,\mu\Omega\,\mathrm{cm}$. For glass $\kappa = 1.3\,\mathrm{W\,m^{-1}K^{-1}}$.] [6]

$\frac{d}{dr}\left(\frac{1}{r^2}\frac{dT}{dr}\right)$

$70\times10^{-6}\,\Omega/\mathrm{cm}$

$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)$

$70\times10^{-4}\,\Omega/\mathrm{m}$

$= \frac{1}{r^2}\left(r^2\frac{d^2T}{dr^2} + \right.$

A1 2009 First Attempt.

1. $P(v) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} 4\pi v^2 e^{-mv^2/2k_B T}$.

(a) most probable $\Rightarrow$ $v_{th} = \sqrt{\frac{2k_B T}{m}}$

$m = 14\ g\ mol^{-1} = 14 \times 10^{-3} / (6.02 \times 10^{23})$.

$= 2.33 \times 10^{-26}$ kg per particle.

$T = 300K$

$\rightarrow$ ~~Vth~~ $v_{prob} = v_{th} = \sqrt{\frac{2(1.38\times10^{-23})(300)}{(2.33\times10^{-26})}}$

$= \cancel{600\ m/s}$ $\underline{596K} \times \frac{1}{\sqrt{2}} \Rightarrow \underline{421}$ m/s

average $\langle v \rangle = \frac{2}{\sqrt{\pi}} v_{th} = \underline{672K} \frac{1}{\sqrt{2}} \Rightarrow \underline{476}$ m/s

~~rt~~ mean square speed $\langle v^2 \rangle^{\frac{1}{2}} = \left(\frac{3}{2} v_{th}^2\right)^{\frac{1}{2}}$ ~~= 894K (594K)²~~

$= \sqrt{\frac{3}{2}(596)^2} = \underline{730K}$

$\Rightarrow \frac{1}{\sqrt{2}} \Rightarrow$

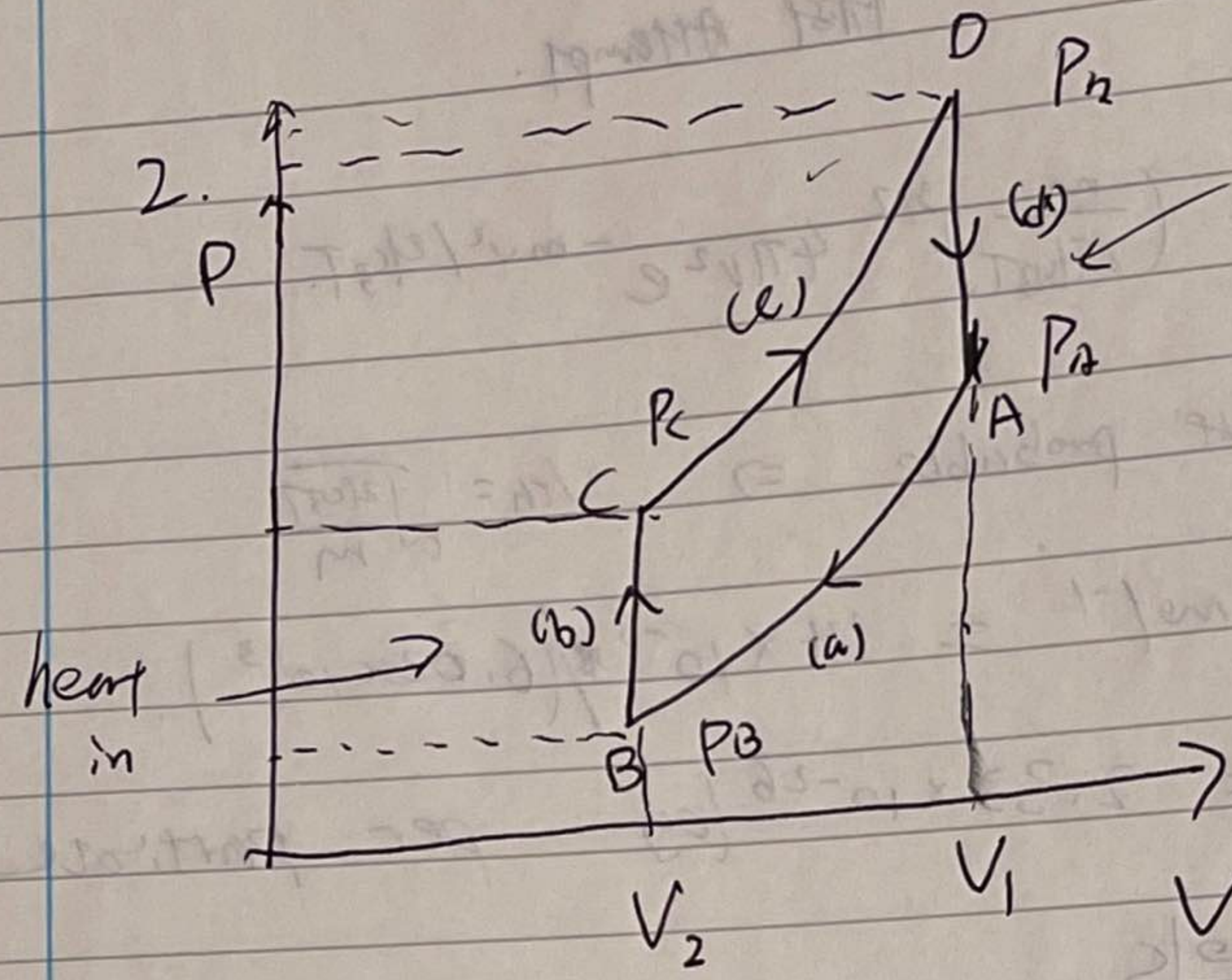
$\underline{516\ m/s}$

$\frac{1}{2} m \langle v^2 \rangle = \varepsilon \Rightarrow \frac{1}{2} m \frac{3}{2} v_{th}^2 = \varepsilon$

$\rightarrow \varepsilon = \frac{1}{2} m \frac{3}{2} \frac{2 k_B T}{m} = \frac{3}{2} k_B T$.

$\rightarrow T = \frac{2}{3} \frac{\varepsilon}{k_B} = \underline{293K}$ ✓

2.




$$Q_{in} = Q_{BC} = C_V \Delta T = \cancel{C_V} \frac{C_V}{nR} V_2 (P_C - P_B)$$

$$Q_{out} = Q_{DA} = C_V \Delta T = \frac{C_V}{nR} V_1 (P_D - P_A).$$

$$\eta = 1 - \frac{Q_{out}}{Q_{in}} = 1 - \frac{V_2}{V_1} \frac{P_C - P_B}{P_D - P_A}$$

$\gamma = \frac{C_p}{C_V}$ adiabatic $PV^\gamma = \text{const.}$

$\rightarrow P_A V_1^\gamma = P_B V_2^\gamma \qquad P_D V_1^\gamma = P_C V_2^\gamma$

$$\therefore \frac{P_C}{P_D} = \frac{P_B}{P_A} = \left(\frac{V_1}{V_2}\right)^\gamma$$

$$\rightarrow \eta = 1 - \left(\frac{V_1}{V_2}\right)^{-1} \left(\frac{V_1}{V_2}\right)^\gamma$$

$$= 1 - \left(\frac{V_1}{V_2}\right)^{\gamma - 1}$$

3. pressure increase

$$\Delta p = \frac{(57.8kg)(9.8N/kg)}{4\times(1\times10^{-6}m^2)} = \underline{1.42\times10^{8}\ Pa}$$

Clausius - Clapeyron equation : $\frac{dP}{dT} = \frac{L}{T\Delta V}$

$$\Rightarrow dp = \frac{L}{\Delta V}\frac{dT}{T}$$

$$\Rightarrow \Delta p = \frac{L}{\Delta V}\ln\left(\frac{T}{T_0}\right)$$

Assuming ~~mat~~ m= 1kg. $L = 333\times10^3 J.$

$$\Delta V = \left(\frac{1}{1000} - \frac{1}{916.7}\right) m^3$$

$$= -9.09\times10^{-5} m^3$$

$$\therefore \frac{\Delta P\,\Delta V}{L} = \frac{(1.42\times10^8)(-9.09\times10^{-5})}{333\times10^3}$$

$$= 0.0388$$

$$\frac{T}{T_0} = e^{-0.0388} = 0.96$$

$$\therefore \Delta T = -0.04T_0 = -0.04\times273 = \underline{\underline{-11K}}$$

4. $S = S(T, V)$

$$dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV$$

$$\left(\frac{\partial S}{\partial T}\right)_P = \left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

$$T\left(\frac{\partial S}{\partial T}\right)_P - T\left(\frac{\partial S}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P$$

$$\rightarrow C_p - C_V = T\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P$$

$$\cancel{-\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial T}\right)_P = \left(\frac{\partial P}{\partial V}\right)}$$

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial T}{\partial V}\right)_P = -1$$

$$\rightarrow \left(\frac{\partial P}{\partial T}\right)_V = \frac{-(\partial V/\partial T)_P}{(\partial V/\partial P)_T}$$

$$\therefore C_p - C_V = -T\,\frac{\left(\frac{\partial V}{\partial T}\right)_P^2}{(\partial V/\partial P)_T} \left(\frac{1}{V^2}\right)\left(\frac{V}{1}\right) V$$

$$= VT\,\frac{\left(\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P\right)^2}{\left(-\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T\right)} = \frac{VT\beta_P^2}{\kappa_T}$$

5.

$$W = \frac{N!}{n_1!\, n_2!\, n_3! \cdots}$$

Sterling's formula.

$$\ln N! = N\ln N - N$$

$$\ln W = N\ln N - N - \sum_i (n_i \ln n_i - n_i)$$

$$= N\ln N \cancel{- N} - \sum_i n_i \ln n_i + \underbrace{\cancel{\sum_i n_i}}_{N}$$

$$= N\ln N - \sum_i n_i \ln n_i$$

$$S_G = \frac{\ln W}{N} = \ln N - \sum_i \frac{n_i}{N} \ln n_i$$

$$= \ln N - \sum_i \frac{n_i}{N}\left(\ln \frac{n_i}{N} + \ln N\right).$$

$$= \ln N - \left(\sum_i \cancel{n_i}\right)\frac{\ln N}{\cancel{N}} - \sum_i \frac{n_i}{N} \ln \frac{n_i}{N}.$$

$$= - \sum_i \frac{n_i}{N} \ln \frac{n_i}{N}$$ Let $p_i = \frac{n_i}{N}$.

$$S_G = -\sum_i p_i \ln p_i$$

Two constraints : $\sum_i p_i - 1 = 0.$ ①

$\sum_i p_i E_i - U = 0.$ ②

maximise $S_G$ ~~⇒ ma~~ subject to ①, ② ⇒ maximise $S_G' = S_G \mp \alpha ① \mp \beta ②$

$$dS_G = -\sum_i p_i \, d(\ln p_i) + dp_i \ln p_i$$

$$= -\sum_i p_i \frac{1}{p_i} dp_i + \ln p_i \, dp_i$$

$$= -\sum_i (\ln p_i + 1) \, dp_i$$

$dS_G' = 0$ $\quad$ $$S_G' = -\sum_i p_i \ln p_i - \lambda\left(\sum_i p_i - 1\right) - \beta\left(\sum_i p_i E_i - U\right)$$

$$\Rightarrow 0 = -\sum_i (\ln p_i + 1) \, dp_i - \lambda \sum_i dp_i - \beta \cancel{E_i} \sum_i E_i \, dp_i$$

$$- \left(\sum_i p_i - 1\right) d\lambda - \left(\sum_i p_i E_i - U\right) d\beta.$$

$$\rightarrow \quad \ln p_i + 1 + \lambda + \beta E_i = 0.$$

$$\ln p_i = -(1 + \lambda + \beta E_i)$$

$$\rightarrow \quad p_i = \underbrace{e^{-1-\lambda}}_{\frac{1}{Z}} e^{-\beta E_i} \Rightarrow \quad p_i = \frac{e^{-\beta E_i}}{Z}.$$

Boltzman

6. $$Z = \frac{(Vz)^N}{N!}$$

$$F = -k_B T \ln Z = -k_B T \ln\left(\frac{(Vz)^N}{N!}\right)$$

$$= -k_B T N \ln(Vz) + k_B T \ln N!$$

$$= -k_B T N \ln(Vz) + k_B T (N \ln N - N).$$

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T} = -k_B T \ln(Vz)$$

$$+ k_B T \left(\cancel{-1} + \cancel{N \frac{1}{N}} + \ln N\right)$$

$$= -k_B T \ln\left(\frac{Vz}{N}\right).$$

7. (a) D corresponds to higher temperatures because higher temperature means the particles spins are more likely to be ~~re randomly~~ oriented in both directions. i.e. the higher energy level is more accessible

→ low magnetisation.

(b) $M = \mu(N_1 - N_2)$

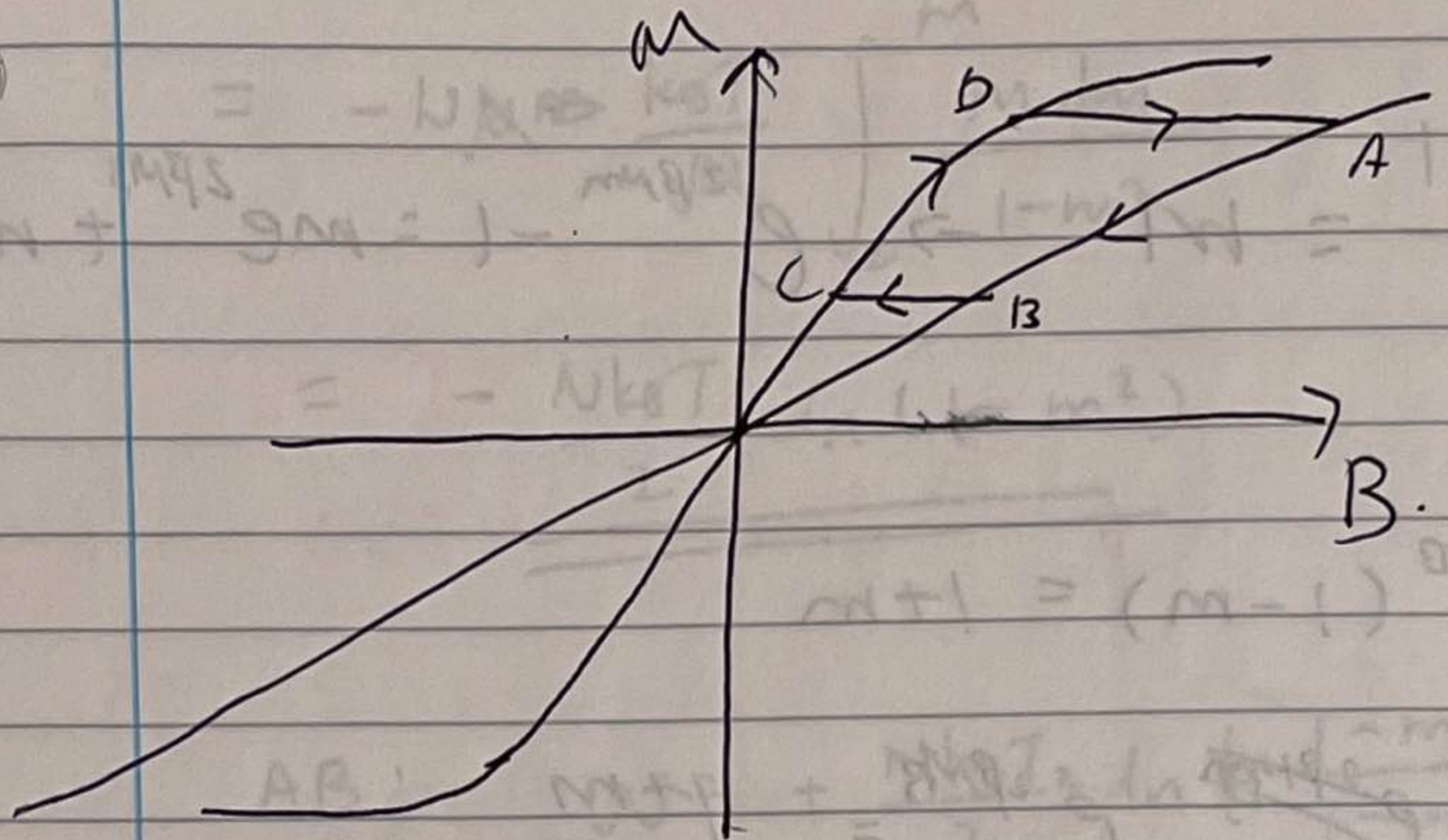
# spin up    # spin down..

If entropy is constant, ~~relat~~ occupation # of particles in each ~~state~~ should not change

→ $M$ → constant ∵ $N_1$, $N_2$ constant.

(c) ~~The~~ for $T > 0$ the thermal ~~random~~ energy ~~motion~~ of the particle excites part ~~of~~ of the particles to the upper level.

(d) Carnot cycle : A cycle consists of ~~2~~ reversible isothermal expansion / compression and adiabatic expansion / compressions




$$Z = \left(e^{\beta\mu B} + e^{-\beta\mu B}\right)^N = \left(2\cosh(\beta\mu B)\right)^N$$

$$dU = TdS - MdB \qquad dF = d(U - TS) = -SdT - MdB$$

$$\rightarrow \quad M = -\frac{\partial F}{\partial B}\Big|_{T,N}$$

$$F = -k_B T \ln Z$$

$$\cancel{= -k_B T N (\cos}$$

$$= -k_B T N \ln(2\cosh(\beta\mu B)).$$

$$M = -\frac{\partial F}{\partial B}\Big|_{T,N} = \cancel{k_B T} N \frac{(2\sinh(\beta\mu B))}{2\cosh(\beta\mu B)} \beta\mu$$

$$= \mu N \tanh(\beta\mu B)$$

$$\therefore \frac{e^{\beta\mu B} - e^{-\beta\mu B}}{e^{\beta\mu B} + e^{-\beta\mu B}} = +\frac{M}{\mu N} = +m.$$

$$\frac{e^{2\beta\mu B}-1}{e^{2\beta\mu B}+1} = m \rightarrow e^{2\beta\mu m} - 1 = me^{2\beta\mu B} + m$$

$$e^{2\beta\mu B}(1-m) = 1+m$$

$$\rightarrow \cancel{e^{2\beta\mu B}}\ e^{2\beta\mu B} = \frac{1+m}{1-m}$$

$$2\beta\mu B = \ln\left(\frac{1+m}{1-m}\right).$$

$$\rightarrow \underline{\underline{B = \frac{k_B T}{2\mu} \ln\left(\frac{1+m}{1-m}\right).}}$$

$$\left(\frac{\partial B}{\partial m}\right)_T = \frac{k_B T}{2\mu} \frac{\partial}{\partial m}\left(\ln(1+m) - \ln(1-m)\right).$$

$$= \frac{k_B T}{2\mu}\left(\frac{1}{1+m} + \frac{1}{1-m}\right)$$

$$= \frac{k_B T}{\cancel{2}\mu} \frac{\cancel{2}}{1-m^2} = \frac{k_B T}{\mu} \frac{1}{1-m^2}$$

$$dW = -M dB = \cancel{-\frac{1}{N\mu} m dB} - m\mu N dB.$$

constant $T \rightarrow dW = -\left(\frac{1}{N\mu}\right)^{-1} m \left(\frac{\partial B}{\partial m}\right)_T dm.$

$$= -N\cancel{\mu} \frac{k_B T}{\cancel{\mu}} \int_0^m \frac{m\,dm}{1-m^2}$$

$$= -\frac{N k_B T}{2} \ln(1-m^2)$$

AB: $W = +\frac{N k_B T_2}{2} \ln\left(\frac{1-m_2^2}{1-m_1^2}\right).$

BC: $W = -N\mu m_2 (B_B - B_C)$

$$= -\cancel{\mu} N m_{\cancel{1}} \frac{k_B}{2\cancel{\mu}} \ln\left(\frac{1+m_{\cancel{1}}}{1-m_{\cancel{1}}}\right)(T_2 - T_1).$$

CD: $W = \frac{-N k_B T_1}{2} \ln\left(\frac{1-m_2^2}{1-m_1^2}\right).$

DA: $W = -N\mu m_{\cancel{2}} (B_D - B_A)$

$$= -N\cancel{\mu} m_2 \ln\left(\frac{1+m_2}{1-m_2}\right) \frac{k_B}{2\cancel{\mu}} (T_1 - T_2).$$

$$\boxed{\therefore \sum W = (T_2 - T_1)\left[\frac{N k_B}{2} \ln\left(\frac{1-m_2^2}{1-m_1^2}\right) + \frac{N k_B}{2}\left(m_2 \ln\left(\frac{1+m_2}{1-m_2}\right) - m_1 \ln\left(\frac{1+m_1}{1-m_1}\right)\right)\right]}$$

8 $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$\cancel{de}\ |A| = ad - bc \qquad |B| = \alpha\delta - \gamma\beta$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

$$|AB| = \cancel{\alpha\delta}\ (a\alpha + b\gamma)(c\beta + d\delta) - (c\alpha + d\gamma)(a\beta + b\delta)$$

$$= \cancel{a\alpha + 2b\gamma}\ \cancel{a\alpha c\beta} + b\gamma c\beta + \cancel{a\alpha d\gamma}\ a\alpha d\delta + \cancel{b\gamma d\delta}$$

$$- \cancel{c\alpha a\beta} - c\alpha b\delta - d\gamma a\beta - \cancel{d\gamma b\gamma}$$

$$= \cancel{ad}\ ad(\alpha\delta - \beta\gamma) - bc(\alpha\delta - \gamma\beta)$$

$$= (ad - bc)(\alpha\delta - \gamma\beta) = |A||B|$$

$$S = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \qquad S' = \begin{pmatrix} z' & x' - iy' \\ x' + iy' & -z' \end{pmatrix}$$

$\because S' = USU^{\dagger} \quad \therefore |S'| = |US||U^{\dagger}| = |USU^{\dagger}|$

$$= \underbrace{|U^{\dagger}US|}_{U^{\dagger}U = I \ \because\ U \text{ is unitary.}} = |S|$$

$\therefore \rightarrow x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 = -|S| = -|S'|$

$$\sigma_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

$$\exp(i\theta\sigma_x) = \cancel{I + i\theta\sigma_x + \frac{1}{2!}\theta}$$

$$= I + i\theta\sigma_x + \frac{1}{2!}(i\theta\sigma_x)^2 + \frac{1}{3!}(i\theta\sigma_x)^3 \cdots$$

$$= I + i\theta\sigma_x - \frac{1}{2!}\theta^2 I - \frac{i\theta^3\sigma_x}{3!} + \frac{1}{4!}\theta^4 I + \cdots$$

$$= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\sigma_x\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \underline{\underline{\cos(\theta) I + i\sin(\theta)\sigma_x}}$$

$$S = x\sigma_x + y\sigma_y + z\sigma_z$$

$$U = \exp(i\theta\sigma_x) \qquad U^\dagger = \exp(-i\theta\sigma_x)$$
$$= \cos\theta I + i\sin\theta\sigma_x \qquad = \cos\theta I - i\sin\theta\sigma_x.$$

$\therefore$

$$S' = (\cos\theta I - i\sin\theta\sigma_x)(x\sigma_x + y\sigma_y + z\sigma_z)(\cos\theta I + i\sin\theta\sigma_x)$$

$$= (\cos\theta I - i\sin\theta\sigma_x)(x\cos\theta\sigma_x + y\cos\theta\sigma_y + z\cos\theta\sigma_z$$
$$+ ix\sin\theta I + iy\sin\theta(-i\sigma_z) + iz\sin\theta(i\sigma_y))$$

$$= x\cos^2\theta\sigma_x + y\cos^2\theta\sigma_y + z\cos^2\theta\sigma_z$$
$$+ ix\sin\theta\cos\theta I + y\sin\theta\cos\theta\sigma_z - z\sin\theta\cos\theta\sigma_y$$
$$- ix\sin\theta\cos\theta I. - iy\sin\theta\cos\theta(i\sigma_z) + iz\sin\theta\cos\theta(i\sigma_y).$$
$$+ x\sin^2 I - iy\sin^2\theta(-i\sigma_y) + iz\sin^2\theta(i\sigma_z)$$

$$= x\,\sigma_x + \left([\cos^2\theta - \sin^2\theta]\,y - [2\sin\theta\cos\theta]\,z\right)\sigma_y$$

$$+ \left([2\sin\theta\cos\theta]\,\cancel{\sigma_z}\,y + [\cos^2\theta - \sin^2\theta]\,z\right)\sigma_z.$$

$$= x\,\sigma_x + (\cos 2\theta\, y - \sin 2\theta\, z)\,\sigma_y$$

$$+ (\sin 2\theta\, y + \cos 2\theta)\,\sigma_z.$$

$$\underline{r}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix} \underline{r}$$

$\rightarrow$ rotation by $\underline{\underline{2\theta}}$

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $\cancel{A^T}$ $A^\dagger A = I$
and $\det(A) = 1$.

$\rightarrow$ $\underline{ad - bc = 1}$

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\rightarrow$ $a^*a + c^*c = 1$
$a^*b + c^*d = 0$
$b^*a + d^*c = 0$
$b^*b + d^*d = 1$

$(a^* - d)a - (b + c^*)c = 0$ ③
$-(c + b^*)b + (a - d^*)d = 0$
$-(c^* + b)b^* + (a^* - d)d^* = 0$
$(a^* - d)ab^* - (b + c^*)cb^* = 0$ ①
$-(c^* + b)cb^* + (a^* - d)cd^* = 0$ ②

$\Rightarrow$ ① $-$ ② $\Rightarrow (a^* - d)(\underbrace{ab^* - cd^*}) = 0$

$\rightarrow (a^* - d)\,2ab^* = 0.$

$\rightarrow a^* = d.$

If $a = 0 \rightarrow c \neq 0 \rightarrow d = 0$

If $b^* = 0 \rightarrow d \neq 0 \rightarrow c = 0 \rightarrow a \neq 0$

$③ \Rightarrow (a^* - d) a = 0 \rightarrow a^* = d$

or simply $a^* = d$

$\Rightarrow a^* = d$ ~~atw~~ always

So $(b + c^*) c = 0$

If $c = 0 \rightarrow a \neq 0. \rightarrow b = 0 \rightarrow b = -c^*.$

or simply $b = -c^*.$

$$\therefore A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}$$

a. quasistatic → system remains in equilibrium throughout the process.

adiabatic → process is reversible and no heat exchange with surroundings

$$dQ = 0$$

First law: $dU = dQ - pdV$ ∵ $dQ = 0$. $dU = C_v dT$

∴ $C_v dT + pdV = 0$ ∵ $p = \frac{nRT}{V}$

$nR = C_p - C_v$

∴ $C_v dT + \frac{(C_p - C_v)T}{V} dV = 0$

∴ $\frac{dT}{T} + \frac{C_p - C_v}{C_v} \frac{dV}{V} = 0$

→ $\ln\left(\frac{T_2}{T_1}\right) + (\gamma - 1)\ln\left(\frac{V_2}{V_1}\right) = 0$

∴ $\ln\left(\frac{T_2 V_2^{\gamma-1}}{T_1 V_1^{\gamma-1}}\right) = 0$

→ $T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$ ⇒ $TV^{\gamma-1} = \text{const}$

$\frac{dT}{dz}$ $PV = Nk_BT \rightarrow \cancel{P = \frac{Nk_BT}{V}}$ $V = \frac{Nk_BT}{P}$

$TV^{\gamma-1} = \text{const.}$

$\therefore \cancel{PV^{\gamma} = \text{const.}}$

$\rightarrow T\left(\frac{T}{P}\right)^{\gamma-1} = \text{const.}$

$\therefore T^{\gamma}P^{1-\gamma} = \text{const.}$

$\therefore d(T^{\gamma}P^{1-\gamma}) = 0$

$\rightarrow (P^{1-\gamma})(\gamma T^{\gamma-1})\,dT + T^{\gamma}(1-\gamma)P^{-\gamma}\,dP = 0$

$$\therefore \frac{dT}{dP} = -\frac{T^{\gamma}(1-\gamma)P^{-\gamma}}{P^{1-\gamma}\gamma T^{\gamma-1}} = \frac{\gamma-1}{\gamma}\frac{T}{P}$$

$dP = -\rho g\,dz \rightarrow \frac{dP}{dz} = -\rho g = -mng$

$$\frac{dT}{dz} = \frac{dT}{dP}\frac{dP}{dz} = -m\left(\frac{P}{k_BT}\right)\left(\frac{\gamma-1}{\gamma}\right)\frac{T}{P}g$$

$$= -\frac{\gamma-1}{\gamma}\frac{mg}{k_B}$$

$m = 4.65\times10^{-26}$ kg

molecular mass of dry air ~~$m \approx 28\times1.66\times10^{-27}$ kg. $= 4.65\times10^{-26}$ kg~~

$\gamma = \frac{5}{3}$

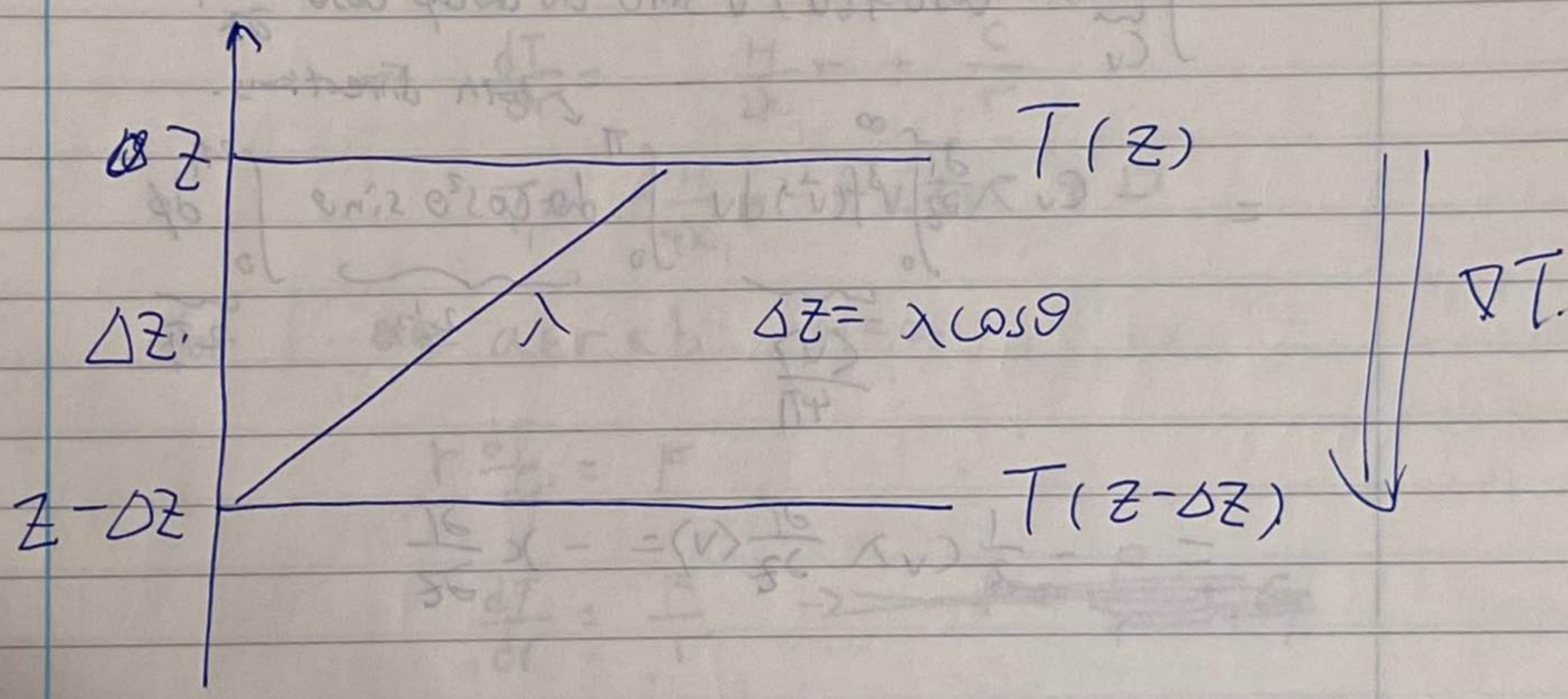
monatomic gas $\rightarrow$ $$\frac{\Delta T}{\Delta z} = -\frac{\frac{5}{3}-1}{\frac{5}{3}}\frac{(4.8\times10^{-26})(9.8)}{(1.38\times10^{-23})} = -0.013\text{ K/m}$$

if $m = 28.96\times10^{-3}/(6.02\times10^{23}) = 4.81\times10^{-26}$ kg

dry air diatomic gas $\rightarrow$ $$\frac{\Delta T}{\Delta z} = -\frac{\frac{7}{5}-1}{\frac{7}{5}}\frac{(4.65\times10^{-26}\text{ kg})(9.8)}{1.38\times10^{-23}} = -0.0094\text{ K/m}$$

10. $J_z = -\kappa \frac{\partial T}{\partial z}$

Heat flow from high temperature to low temperature so if $\kappa > 0$ there must be a negative sign.




Flux of particles $d\Phi(\vec{v}) = n v_z f(\vec{v}) d^3\vec{v}$

$\rightarrow \quad d\Phi(\vec{v}) = n v \cos\theta f(\vec{v}) v^2 \sin\theta \, dv \, d\theta \, d\phi.$

Each particle ~~per~~ brings extra thermal energy from $z-\Delta z$ to $z$ (let $c$ be heat capacity per particle, $C_V = nc$) ($\lambda$ = mean free path)

$$\Delta E = \cancel{cT(z)} \cancel{- cT} \; cT(z-\Delta z) - cT(z)$$

$$= c\left(T(z) - \frac{\partial T}{\partial z}\Delta z\right) - cT(z) = -c\frac{\partial T}{\partial z}\Delta z.$$

$$= -c\lambda\cos\theta\frac{\partial T}{\partial z}$$

total heat flux

$J_z = \int \Delta E \, d\Phi(\vec{v})$

$= -\int \underbrace{cn}_{C_v} \lambda \cos\theta \, f(\vec{v}) \, v^2 \sin\theta \, dv \, d\theta \, d\phi \, \cos\theta \, \frac{\partial T}{\partial z}$

both directions.

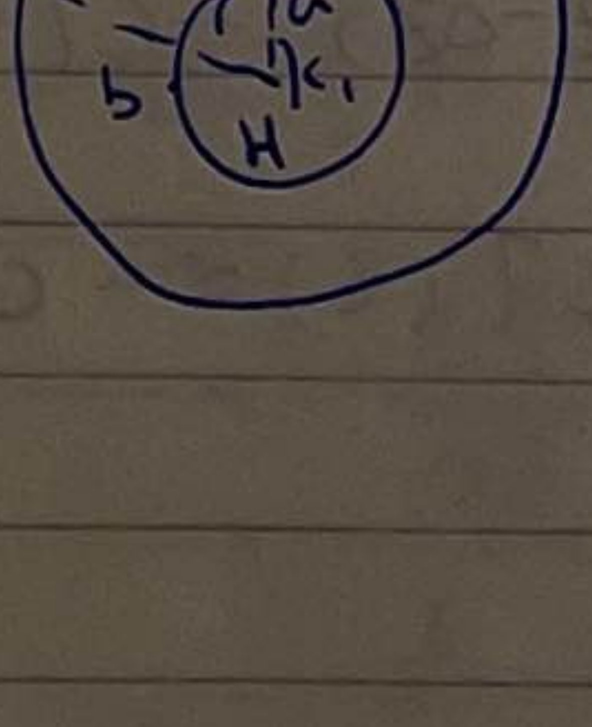
$= -C_v \lambda \frac{\partial T}{\partial z} \underbrace{\int_0^\infty v^3 f(\vec{v}) \, dv}_{\frac{\langle v \rangle}{4\pi}} \underbrace{\int_0^\pi d\theta \cos^2\theta \sin\theta}_{2/3} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$

$= -\frac{1}{3} C_v \lambda \frac{\partial T}{\partial z} \langle v \rangle = -\kappa \frac{\partial T}{\partial z}$

$\Rightarrow \kappa = -\frac{1}{3} C_v \lambda \langle v \rangle$

Steady state $H + \kappa \nabla^2 T = 0$ ∵ $T = T(r)$

∴ $\kappa\left(\frac{d^2 T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) + H = 0$




$\rightarrow \kappa \frac{1}{r} \frac{d}{dr}\left(r \frac{dT}{dr}\right) + H = 0$

$\rightarrow \frac{d}{dr}\left(r \frac{dT}{dr}\right) = -\frac{H}{k} r$

for $0 \le r \le a \quad H = H$

$$\rightarrow \frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{H}{K_1} r.$$

$$r\frac{dT}{dr} = -\frac{H}{2K_1}r^2 + C$$

$$\rightarrow \frac{dT}{dr} = -\frac{H}{2K_1}r + \frac{C}{r}$$

$$\therefore \quad T_1 = -\frac{H}{4K_1}r^2 + C\ln r + D$$

for $\cancel{0 \le}\ a \le r \le b \quad H = 0.$

$$r\frac{dT}{dr} = F$$

$$\frac{dT}{dr} = \frac{F}{r} \quad \rightarrow \cancel{T = \frac{F}{r} + G}.$$

$$T_2 = F\ln(r) + G$$

Boundary Conditions: ~~at~~

at $r = b \quad T_2 = T_a \Rightarrow \cancel{T_2 = F}\ T_a = F\ln b + G.$

~~at $r=b \quad \frac{\partial T}{\partial r} = 0 \quad \frac{\partial T}{\partial r} = 0$~~

~~$\Rightarrow$~~

At $r = 0 \quad T_1$ is finite $\Rightarrow \quad C = 0.$

At $r = a$. temperature continuous

$$\cancel{\frac{A}{4}} \quad -\frac{Ha^2}{4k_1} + D = F \ln a + G.$$

Heat flux continuous:

$$-k_1 \frac{dT_1}{dr} = -k_2 \frac{dT_2}{dr}.$$

$$\Rightarrow \cancel{k_1} \frac{Ha}{2\cancel{k_1}} = -k_2 \frac{F}{a} \quad \rightarrow \quad F = -\frac{Ha^2}{2k_2}$$

$$\rightarrow \quad \cancel{T_2} = T_a = -\frac{Ha^2}{2k_2} \ln b + G$$

$$\rightarrow \quad G = \frac{Ha^2}{2k_2} \ln b + T_a.$$

$$\therefore \quad T_2 = \frac{Ha^2}{2k_2} \ln b + T_a - \frac{Ha^2}{2k_2} \ln(r)$$

$$= T_a + \frac{Ha^2}{2k_2} \ln(b/r). \qquad a \leq r \leq b.$$

$$D = F \ln a + G + \frac{Ha^2}{4k_1}$$

$$= -\frac{Ha^2}{2k_2} \ln a + \frac{Ha^2}{2k_2} \ln b + T_a + \frac{Ha^2}{4k_1}.$$

$$= \frac{Ha^2}{2k_2} \ln(b/a) + \frac{Ha^2}{4k_1}.$$

$$\rightarrow T_1 = T_a + \frac{H}{4k_1}(a^2 - r^2) + \frac{Ha^2}{2k_2} \ln(b/a) \qquad 0 \leq r \leq a$$

→ single wire

$$T(r) = T_a + \frac{H}{4k}(a^2 - r^2)$$

$a = 0.05 \times 10^{-3}$ m.
$T_a = 20°C = 293K.$
$T(0) = 1400°C = 1673K.$
$k = 1.3\ Wm^{-1}K^{-1}.$

$$\therefore T(0) - T_a = \frac{H}{4k} a^2.$$

$$H = \frac{4k(T_0 - T_a)}{a^2} = \frac{I^2 R}{(\frac{\pi}{4}a^2)L} = \frac{I^2 \rho \frac{L}{(\frac{\pi}{4}a^2)}}{(\frac{\pi}{4}a^2)L}$$

$$\rightarrow H = \cancel{2.87 \times 10^{12}\ W/m^3}\ 4.2 \times 10^{13}\ W/m^3$$

$\rho = 70 \times 10^{-8}\ \Omega m$

$$\left(\frac{(\frac{\pi}{4}a^2)^2 H}{\rho}\right)^{\frac{1}{2}} = I \Rightarrow \cancel{I = 1.58\ mA}$$

~~I = 0.152A~~ $I = 60.8A$

with the glass.

$$T_0 - T_a = a^2\left(\frac{1}{4k_1} + \frac{\ln(b/a)}{2k_2}\right)H.$$

$$\Rightarrow \cancel{H = 10}\ H = 3.85 \times 10^{11}\ W/m^3$$

$$\rightarrow I = \left(\frac{H(\pi a^2)}{\rho}\right)^{\frac{1}{2}} = 5.83A$$