SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part A

A1: Thermal Physics

Wednesday, 18 June 2003, 9.30 am – 12.30 pm

Answer all of Section A and three questions from Section B.

Start the answer to each question on a fresh page.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight which the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. What is the maximum possible efficiency of an engine operating between two thermal reservoirs, one at 100°C and the other at 0°C? [5]

2. A 10 Ω resistor is held at a temperature of 300 K. A current of 5 A is passed through the resistor for 2 minutes. Ignoring changes in the source of current, what is the change of entropy in (a) the resistor and (b) the Universe? [7]

3. The heat capacity at constant volume of helium (He), measured at room temperature, is $12.5\,\mathrm{J\,K^{-1}\,mol^{-1}}$, while that of nitrogen (N_2) is $20.7\,\mathrm{J\,K^{-1}\,mol^{-1}}$. Explain these values. [7]

4. Estimate the rms velocity of a helium atom (mass $\sim 4m_p$) in a gas at room temperature. Estimate the rms velocity of a typical electron in a metal with Fermi energy 5 eV. Comment on any difference. [7]

5. The surface temperature of the Sun is 5700 K, and the spectrum of radiation which it emits has a maximum at a wavelength of 510 nm. Estimate the surface temperature of the North Star, for which the corresponding maximum is at 350 nm. [5]

6. The density of ice is $0.92 \times 10^3\,\mathrm{kg\,m^{-3}}$ and its latent heat of fusion is $3.3 \times 10^5\,\mathrm{J\,kg^{-1}}$. Estimate the melting temperature of the ice at the bottom of a glacier which is 100 m deep, stating any assumptions that you make. [9]

Section B

7. What is meant by a *function of state*? [5]

Consider n moles of a gas, initially confined within a volume V and held at temperature T. The gas is expanded to a total volume αV, where α is a constant, by (a) a reversible isothermal expansion and (b) removing a partition and allowing a free expansion into the vacuum. Both cases are illustrated below.

(a)

(b)

partition

V gas	$(\alpha\text{-}1)V$

V gas	$(\alpha\text{-}1)V$ vacuum
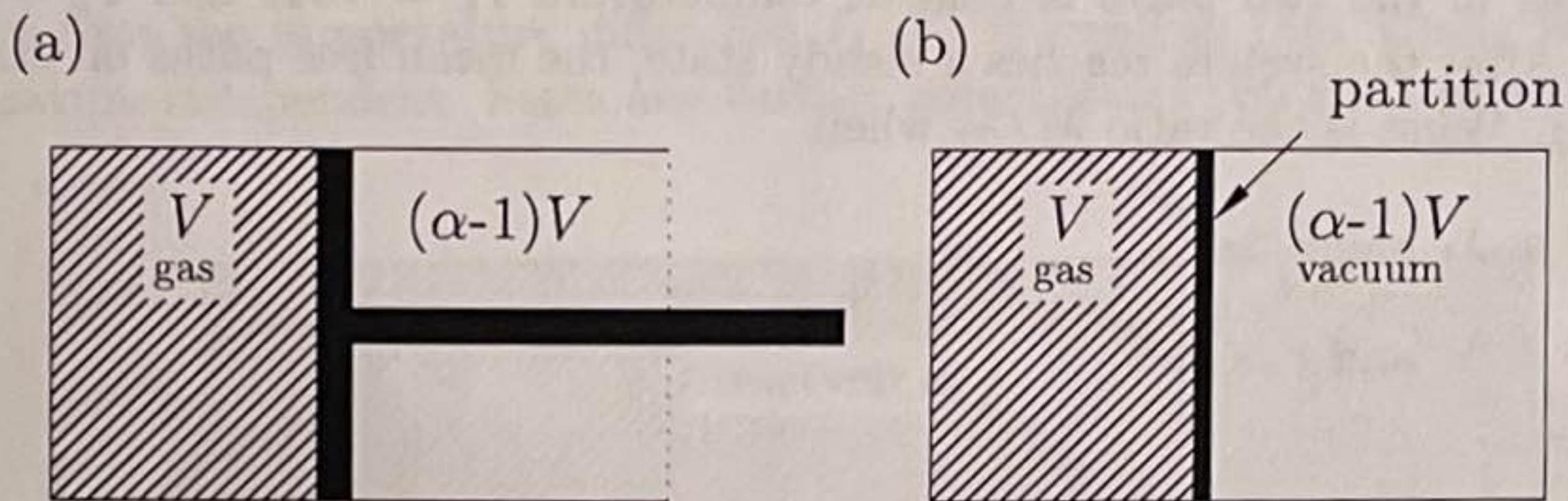
Assuming the gas is ideal, derive an expression for the change of entropy of the gas in each case. [7]

Repeat this calculation for case (a), assuming that the gas obeys the van der Waals equation of state

$$\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT.$$

Show further that for case (b) the temperature of the van der Waals gas falls by an amount proportional to $(\alpha - 1)/\alpha$. [8]

$d\bar{U} = \frac{an^2}{v^2}\, dv$

$\frac{an^2}{v^2}\, dv = Tds - pdv$

$Tds = \left(p + \frac{an^2}{v^2}\right)dv$

$= \frac{nRT}{v - nb}\, dv$

8. Show that the number of molecules in a gas that hit the walls of its container per second per unit area is

$$\frac{1}{4}n\langle c\rangle,$$

where n is the number of molecules per unit volume and $\langle c\rangle$ is the mean molecular speed.

[8]

A container is divided into two parts by a partition containing a hole of radius r. Helium gas in the two parts is held at temperature $T_1 = 75\,\text{K}$ and $T_2 = 300\,\text{K}$ respectively. After the system reaches a steady state, the mean free paths on each side are λ_1 and λ_2. What is the ratio λ_1/λ_2 when

(a) $r \gg \lambda_1$ and $r \gg \lambda_2$,

(b) $r \ll \lambda_1$ and $r \ll \lambda_2$?

[12]

9. Write down an expression for the partition function of a single-particle system. Show that the mean energy $\langle E\rangle$ is given by

$$\langle E\rangle = -\frac{d\ln Z}{d\beta},$$

where Z is the partition function and $\beta = 1/(k_BT)$.

[6]

A one-dimensional quantum harmonic oscillator is in thermal equilibrium with a heat bath at temperature T. It has energy levels given by

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \qquad n = 0,1,2,\dots$$

Show that its partition function is given by

$$Z = \frac{1}{2\sinh(\hbar\omega\beta/2)}.$$

[6]

Hence derive expressions for $\langle E\rangle$ and the entropy. How do these behave in the limits $k_BT \ll \hbar\omega$ and $k_BT \gg \hbar\omega$? Give a physical interpretation of your results.

[8]

10. In a measurement of the thermal conductivity of a gas, the gas is placed between a reservoir of large thermal capacity, held at a low temperature T_0, and a metal slab of very high thermal conductivity. The metal slab is warmed using a heater which supplies a power per unit area Q. The temperature of the metal slab, T_1, is monitored using a thermocouple. The apparatus is thermally isolated from its surroundings by insulation, so that the heat generated in the metal slab can only pass to the cooler reservoir by conduction through the gas. Assuming steady state conditions, derive an expression for the thermal conductivity of the gas, κ, in terms of T_0, T_1, d and Q. Assume that the temperature difference $T_1 - T_0$ is small so that κ may be taken to be temperature-independent. State any further assumptions you need to make. [8]
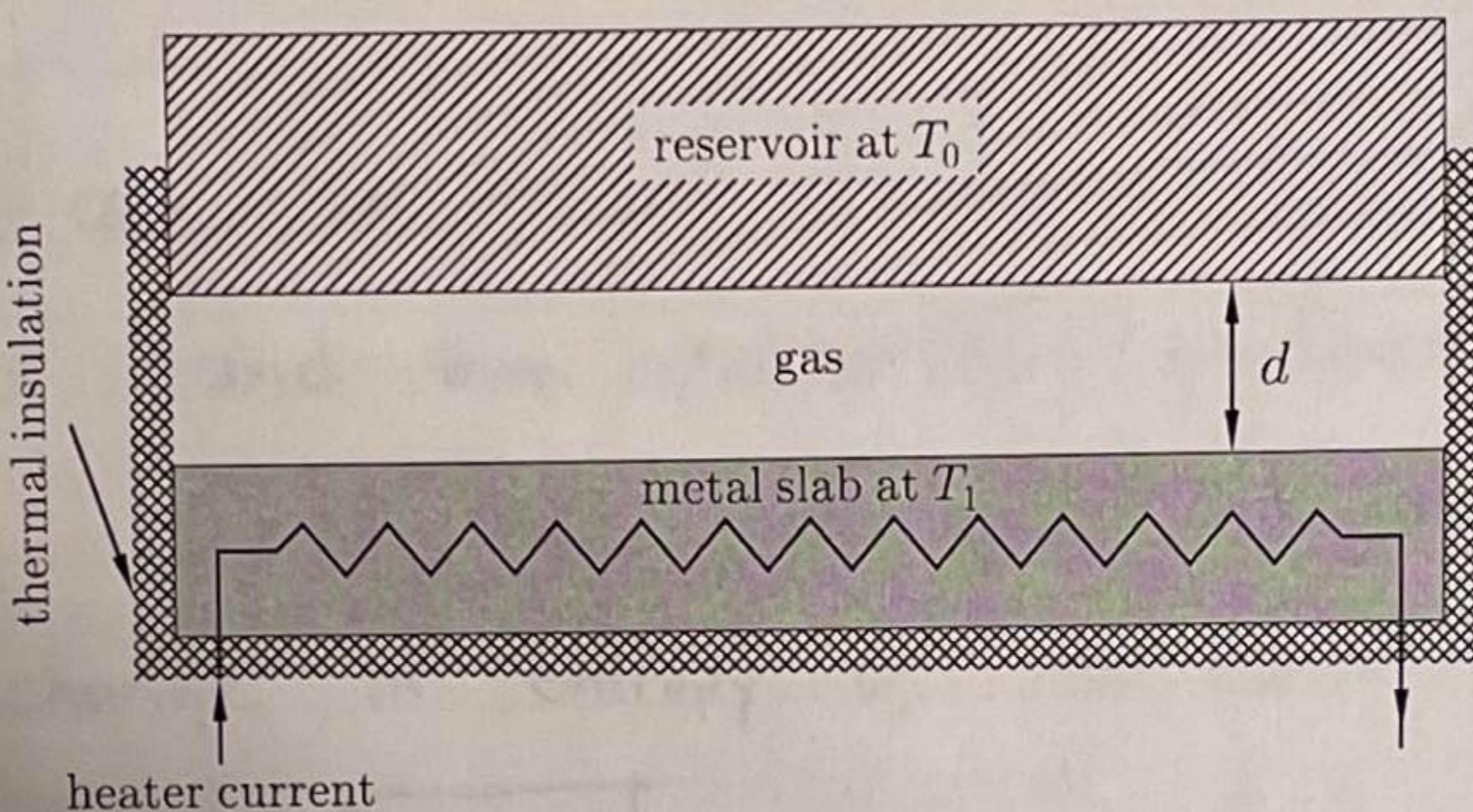
A series of experiments are conducted using helium gas (assumed ideal) at different pressures. In the experiments, $d = 1\,\text{mm}$, $T_0 = 77\,\text{K}$ and $Q = 100\,\text{W m}^{-2}$. The collision cross section of a helium atom is $0.21\,\text{nm}^2$. It is found that κ is independent of pressure p until p is lowered below a value p_1. Explain this observation and provide a rough estimate for (a) p_1 and (b) the value of T_1 which would be measured for p well above p_1. [12]

$$\lambda = \frac{1}{\hbar n \sigma}$$

$$\lambda = \frac{1}{\sqrt{2}\, n \sigma}$$

$$\frac{1}{\sqrt{2}\, n \sigma} = d^{-}$$

$$\boxed{\kappa = \frac{Qd}{T_1 - T_0}}$$

$$v_{th} = \sqrt{\frac{2 k_B T}{m}}$$

$$\kappa = \frac{1}{3} n \langle \lambda \rangle \sqrt{d p} \langle v \rangle$$

$$p = n k_B T$$

$$n = \frac{p}{k_B T}$$

$$= \frac{1}{3} \left(\frac{p}{k_B T}\right) \frac{3}{2} k_B$$

2661   5   [End of paper]

1.

The maximum possible efficiency is the Carnot efficiency:

$$\eta_{max} = 1 - \frac{T_c}{T_h} = 1 - \frac{T_c}{T_h}$$

$$= 1 - \frac{273 + 0 \text{ K}}{273 + 100 \text{ K}} = \frac{100}{373} = \boxed{26.8\%}$$

2. (a) power is supplied to the resistor and then dissipated as heat

→ No net heat goes into the resistor

change in entropy of the resistor

$$\boxed{\Delta S_r = 0}$$

(b)   The heat supplied to the surrounding

$$\Delta Q_s = I^2 R t = (5A)^2 (10\Omega)(2 \times 60s) = 3 \times 10^4 J$$

change in entropy of the surroundings

$$\Delta S_s = \frac{\Delta Q_s}{T} = \frac{3.0 \times 10^4 J}{300 K} = \boxed{100 \text{ J/K}}$$

→ change in entropy of the universe

$$\Delta S_u = \Delta S_r + \Delta S_s = \boxed{100 \text{ J/K}}$$

3.   The gas constant $R = 8.31 \; JK^{-1}mol^{-1}$

molar heat capacity of an ideal gas is

$$C_V = \frac{n}{2} R \; ; \; \text{where } n \text{ is the number}$$

of degrees of freedom excited.

" This result is a consequence of the
  "equipartition theorem" of energy.

He is a monatomic gas , whereas N_2

is a ~~drag~~ diatomic gas. Assuming temperature

is not ~~so~~ high enough for the vibration

degree of freedom to be excited for N_2,

then

He   has   3   degrees of freedom

(3 translational)

N_2 has   5 degrees of freedom

(3 translational + 2 rotational)

→ $C_V (He) = 12.5 \; J \cdot k^{-1} mol^{-1} \simeq \frac{3}{2} R$ ✗

$C_V (N_2) = 20.7 \; J \cdot k^{-1} mol^{-1} \simeq \frac{5}{2} R$

4. (a) Helium atom at room temperature ~300

(~ 300K), the gas is ideal and follows

the maxwellian distribution for speed is

$$f(v) = \frac{4\pi v^2}{(\sqrt{\pi} v_{th})^3} \exp\left(-\frac{v^2}{v_{th}^2}\right) \qquad \left(v_{th} = \sqrt{\frac{2k_B T}{m}}\right)$$

$$V_{rms} = \langle v^2 \rangle^{\frac{1}{2}} = \left(\int_0^\infty v^2 f(v)\, dv\right)^{\frac{1}{2}}$$

$$= \left(\frac{3}{2} v_{th}^2\right)^{\frac{1}{2}} = \sqrt{\frac{3 k_B T}{m}}$$

For Helium $m = 4m_p$:

$$V_{rms}\,(He) \approx \sqrt{\frac{3(1.38\times 10^{-23})(300)}{4(1.67\times 10^{-27})}} = \boxed{1.36\times 10^3\, m/s}$$

---

a2 internal energy for ideal gas is

$$U = \frac{3}{2} N k_B T$$

→ Kinetic energy per particle is $\frac{3}{2} k_B T$

→ $\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T$

→ $V_{rms} = \langle v^2 \rangle^{\frac{1}{2}} = \sqrt{\frac{3 k_B T}{m}} = 1.36\times 10^3\, m/s$

(b) Internal energy for degenerate

Fermi gas is

$$U = \frac{3}{5} \varepsilon_F \qquad (\varepsilon_F \text{ is the Fermi energy})$$

$$\rightarrow \frac{1}{2} m \langle v^2 \rangle = \frac{3}{5} \varepsilon_F$$

$$\rightarrow \langle v^2 \rangle = \frac{6 \varepsilon_F}{5m}$$

$$V_{rms} = \langle v^2 \rangle^{\frac{1}{2}} = \left(\frac{6 \varepsilon_F}{5m} \right)^{\frac{1}{2}} = \left(\frac{6 \times 0.5 \times 1.6 \times 10^{-19} J}{5 \times 9.11 \times 10^{-31} kg} \right)^{\frac{1}{2}}$$

$$= \boxed{3.2 \times 10^5 \, m/s}$$

Comments:

For an ideal gas the ~~aoe~~ rms speed is proportional to $\sqrt{T}$, as temperature increases or decreases so does the temperature. But for ~~when te~~ Fermi gas such as electrons, when temperature gets low and density gets high, the gas becomes degenerate. Degenerate Fermi gases has finite rms speed even if $T \to 0$.

This is because the Pauli exclusion principle states that ~~enen~~ even at $T < 0$ ~~the~~ only not all ~~and~~ particles can occupy the state with lowest energy ($\varepsilon = 0$). Hence total energy is

finite

non-zero and there must be ~~v~~ average speed.

5. By Wien's Law,

$$\lambda_1 T_1 = \lambda_2 T_2$$

where λ is the wavelength at max. emission, T is surface temperature, 1 is sun, 2 is the North star

$$\Rightarrow \quad T_2 = \frac{\lambda_1 T_1}{\lambda_2} = \frac{510}{350} \times 5700 \, K$$

$$\approx \boxed{8300 \, K}$$

6. The Clausius – Clapeyron equation

$$dp = \frac{L \, dT}{T \Delta V} \quad , \quad \text{assume} \quad L, \Delta V \text{ independent of } T$$

$$\int_{P_0}^{P} dp = \frac{L}{\Delta V} \int_{T_0}^{T} \frac{dT}{T}$$

$$\Rightarrow \quad P = P_0 + \frac{L}{\Delta V} \ln\left(\frac{T}{T_0}\right) \quad \text{is the } P\text{-}T$$

curve of phase boundary.

$$\Rightarrow \quad P - P_0 = \frac{L}{V_2 - V_1} \ln\left(\frac{T}{T_0}\right)$$

$$\Rightarrow \quad P - P_0 = \frac{L}{\frac{m}{P_2} - \frac{m}{P_1}} \ln\left(\frac{T}{T_0}\right)$$

$$\Rightarrow \quad P - P_0 = \frac{(L/m)}{\frac{1}{P_2} - \frac{1}{P_1}} \ln\left(\frac{T}{T_0}\right)$$

$$\Rightarrow \quad \frac{(P - P_0) \, L\left(\frac{1}{P_2} - \frac{1}{P_1}\right)}{(L/m)} = \ln\frac{T}{T_0}$$

(P_0, T_0) is on the phase boundary. We can use $P_0 = 1 \text{ atm} = 1.01 \times 10^5 \text{ Pa}$ and $T_0 = \cancel{\text{273}} \cancel{\text{0°C = 37°C}}$ $0°C = 273 \text{ K}$

$$P - P_0 = \rho g H = (1.0 \times 10^3 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(100 \text{ m}) \cancel{+1\text{atm}}$$

$$= 9.8 \times 10^5 \text{ Pa}$$

$$\Rightarrow \quad T = \cancel{t}(273 \text{k}) \exp\left[\frac{(9.8 \times 10^5)\left(\frac{1}{1.0 \times 10^3} - \frac{1}{0.92 \times 10^3}\right)}{(3.3 \times 10^5)}\right]$$

$$= 272.93 \text{ K} = \boxed{-0.07°C}$$

Assumptions: $\cancel{L \text{ and }}$ L remains constant as temperature changes,

density of water remains constant ~~as~~ across the depth of water body.

7. A function of state is a function of the system that only depends on the current equilibrium state of the system. It does not depend on the path by which the system arrives at the current state.

Change of ~~ener~~ entropy :

(a) ideal gas $pV = nRT$, process is reversible

so $\Delta S = \int dS = \int \frac{d\theta}{T}$

~~System thermally isolated~~

~~Expand~~ Expansion is isothermal

$\rightarrow$ T is constant

∴ Ideal gas ∴ U is function of T only

(in fact $U = \frac{3}{2} nRT$)

$\rightarrow$ ~~δU~~ $dU = 0$

∴ $d\theta - pdV = 0 \rightarrow d\theta = pdV$

∴ $\Delta S = \frac{1}{T} \int pdV$ ∵ $p = \frac{nRT}{V}$

∴ $\Delta S_a = nR \int_a^{\alpha v} \frac{dV}{V} = \boxed{nR \ln \alpha}$

(b) ∵ entropy is a function of state, and the initial and ~~the~~ final states of (a) and (b) are ~~iden~~ identical ∴ we also have

$\boxed{\Delta S_b = nR \ln \alpha}$

For van der waal gas, $(P + \frac{n^2 a}{V^2})(V - nb) = nRT$

$$U = \frac{3}{2} nRT - \frac{an^2}{V}$$

when $dT = 0$, $dU = \frac{an^2}{V^2} dV = Tds - pdv$

$\rightarrow \bar{d}l_{rev} = \bar{d}l = Tds = (P + \frac{an^2}{V^2})dv = \frac{nRT}{V - nb} dv$

$\therefore \Delta S_a = \frac{1}{T} \int Tds = \int_{V_i}^{\alpha V} \frac{nRdV}{V - nb} = \boxed{nR \ln\left(\frac{\alpha V - nb}{V - nb}\right)}$

In expansion in (b) $dQ = 0$, $dw = 0$ ($\bar{d}l \neq Tds$ irreversible)

$\rightarrow dU = 0$

The Joule coefficient

$$N_J = \left(\frac{\partial T}{\partial V}\right)_U = -\left(\frac{\partial T}{\partial U}\right)_V \left(\frac{\partial U}{\partial V}\right)_T = -\frac{1}{C_V}\left(\frac{\partial U}{\partial V}\right)_T$$

$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - P = \underbrace{T\left(\frac{\partial P}{\partial T}\right)_V - P}_{\text{maxwell relation}}$

$\rightarrow \left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left[T\left(\frac{\partial P}{\partial T}\right)_V - P\right]$

For van der waal gas $\boxed{C_V = \frac{3}{2} R \quad \frac{3}{2} nR.}$

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2}$$

$\left(\frac{\partial P}{\partial T}\right)_V = \frac{nR}{V - nb}$

$\therefore \left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left[\frac{nRT}{V - nb} - P\right]$

$= -\frac{1}{C_V}\left[\frac{nRT}{V - nb} - \left(\frac{nRT}{V - nb} - \frac{an^2}{V^2}\right)\right]$

$$= -\frac{1}{C_V}\left(\frac{an^2}{V^2}\right)$$

$$\Delta T = \text{~~~~}$$

→ for the free expansion in (b)

$$\Delta T = -\frac{an^2}{C_V}\int_V^{dV}\frac{1}{V^2}\,dV$$

$$= \left.\frac{an^2}{C_V}\frac{1}{V}\right|_V^{\alpha V} = \frac{an^2}{C_V}\left(\frac{1}{\alpha V}-\frac{1}{V}\right)$$

$$= \frac{an^2}{V C_V}\left(\frac{1}{\alpha}-1\right)$$

$$= -\frac{an^2}{C_V V}\left(1-\frac{1}{\alpha}\right)$$

$$\boxed{\Delta T = -\frac{an^2}{C_V V^2}\left(\frac{\alpha-1}{\alpha}\right)}$$

→ Temperature falls by ΔT ~~pepper~~ proportional

to $\dfrac{\alpha-1}{\alpha}$

Number of particles hitting the 𝓊 area A of the wall during time t with velocity $[\vec{v}, \vec{v}+d^3\vec{v}]$ is
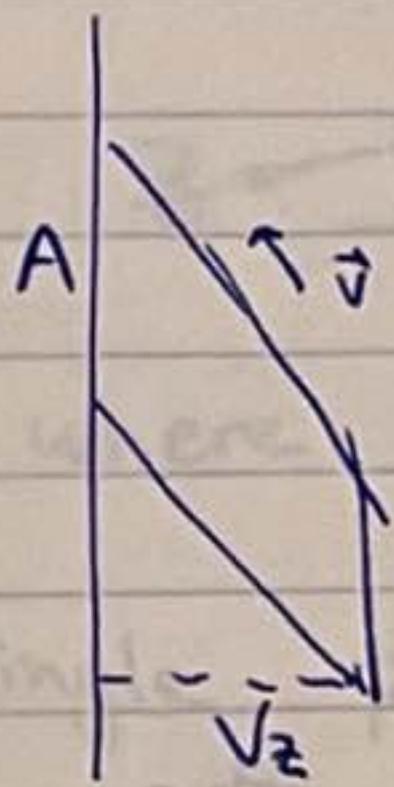
$$A v_z t \, n \, f(\vec{v}) \, d^3\vec{v}$$

The flux of particles hitting the wall with velocities $(\vec{v}, \vec{v}+d^3\vec{v})$ is

$$d\Phi(\vec{v}) = \frac{A v_z t \, n \, f(\vec{v}) \, d^3\vec{v}}{A t} = n v_z f(\vec{v}) d^3\vec{v}$$

isotropic distribution of velocities $f(\vec{v}) = f(v)$

change to spherical coordinates $d^3\vec{v} = v^2 \sin\theta \, dv \, d\theta \, d\phi$

$$v_z = V \cos\theta$$

→ Total flux of particles, i.e. number of molecules hit the walls per area per time is

$$\Phi = \int d\Phi(\vec{v}) = \int n v_z f(v) \, d^3\vec{v}$$

$$= n \int_0^\infty dv \, v^3 f(v) \underbrace{\int_0^\pi d\theta \sin\theta \cos\theta}_{\frac{1}{2}} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}$$

$$= \frac{n\pi}{4\pi} \int dv \, v \, \underbrace{(4\pi v^2 f(v))}_{\tilde{f}(v) \;\to\; \text{the speed distribution}}$$

$$\cancel{\Phi} \to \quad \Phi = \frac{1}{4} n \int dv \cdot v \cdot \hat{f}(v)$$

$$\underbrace{\phantom{dv \cdot v \cdot \hat{f}(v)}}_{\langle c \rangle, \text{ the average } \cancel{\text{velocity}} \text{ speed}}$$

$$\to \quad \underline{\Phi = \frac{1}{4} n \langle c \rangle} \qquad QED$$
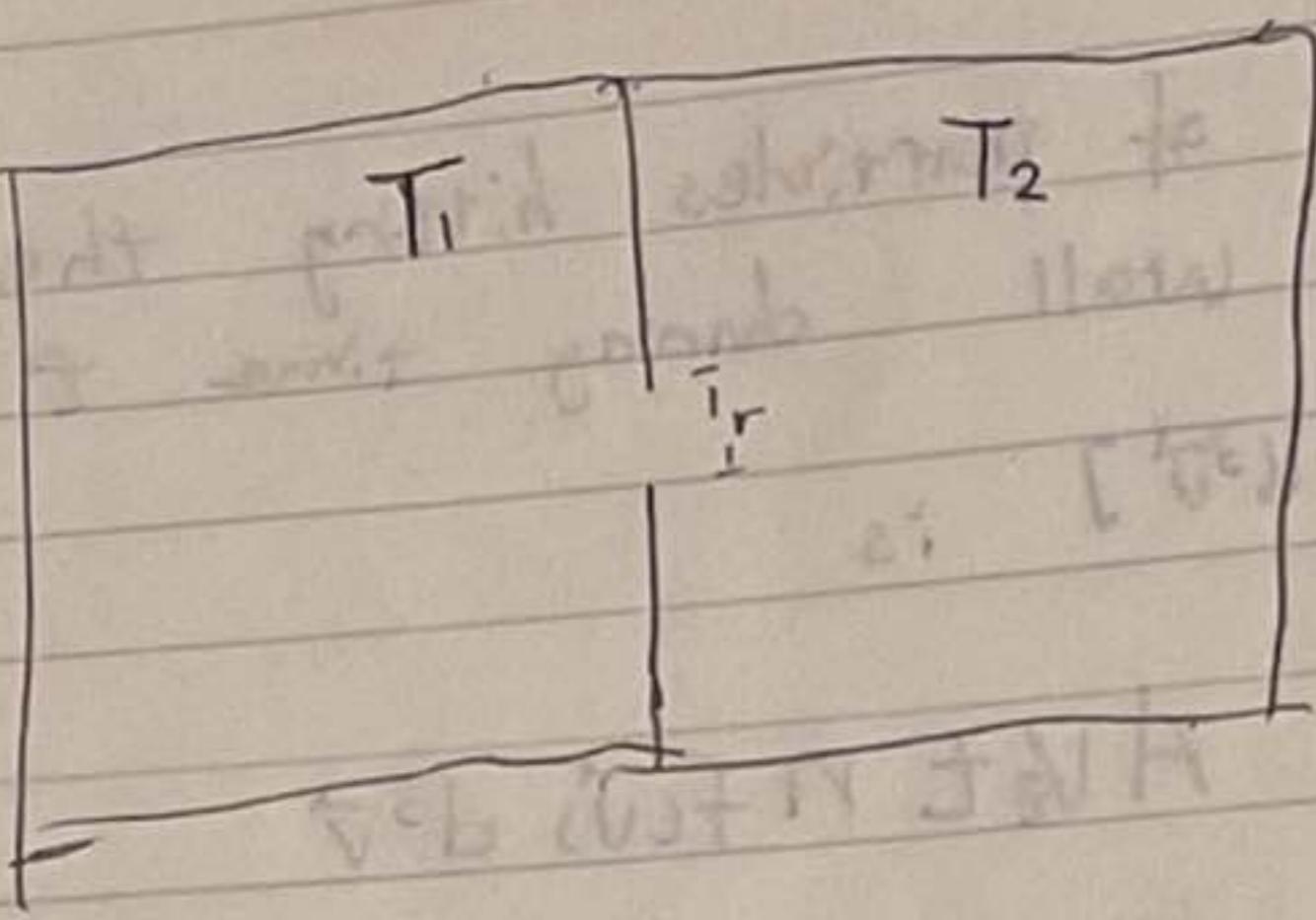
(a) if $r \gg \lambda_1, \lambda_2$, the condition for equilibrium is @ equal pressure $P_1 = P_2$ because there is mean the flow of gas.

σ is collisional cross-area, n_1, n_2 are number densities of 1, 2. then.

$$\lambda_1 \propto \frac{1}{n_1 \sigma} \qquad \lambda_2 \propto \frac{1}{n_2 \sigma}$$

$$PV = nRT \rightarrow P_1 V = n_1 R T_1, \qquad P_2 V = n_2 R T_2$$

$$\rightarrow \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} = \frac{P_2/T_2}{P_1/T_1} = \frac{T_1}{T_2} = \frac{75}{300} = \boxed{0.25}$$

$$\underbrace{\qquad}_{P_1 = P_2}$$

(b) If $r \ll \lambda_1, \lambda_2$, the exchange of gas is due to effusion, and the equilibrium condition is the flux of effusion equal for both sides.

$$\rightarrow \quad \frac{1}{4} n_1 \langle c_1 \rangle = \frac{1}{4} n_2 \langle c_2 \rangle$$

$$\langle c \rangle = \frac{P}{\sqrt{2\pi m k_B T}} \rightarrow \frac{n_1 P_1}{\sqrt{T_1}} = \frac{n_2 P_2}{\sqrt{T_2}}$$

$$\langle c_1 \rangle = \sqrt{\frac{8 k_B T_1}{\pi m}} \qquad \text{\&} \qquad \langle c_2 \rangle = \sqrt{\frac{8 k_B T_2}{\pi m}}$$

$$n_1 \sqrt{T_1} = n_2 \sqrt{T_2}$$

$$\therefore \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} = \frac{\sqrt{T_1}}{\sqrt{T_2}} = \sqrt{\frac{75}{300}} = \sqrt{0.25} = \boxed{0.5}$$

9.　　　Single particle system.　Partition function

$$Z = \sum_\alpha P_\alpha E \qquad Z = \sum_\alpha e^{-\beta E_\alpha}$$

where $\beta = \frac{1}{k_B T}$, and E_α is the energy of each

single particle state.

$$\langle E \rangle = \sum_\alpha P_\alpha E_\alpha \qquad \because P_\alpha = \frac{e^{-\beta E_\alpha}}{Z}$$

$$\therefore \langle E \rangle = \sum_\alpha \frac{1}{Z} E_\alpha e^{-\beta E_\alpha} = -\frac{1}{Z}\sum_\alpha (-E_\alpha) e^{-\beta E_\alpha}$$

$$= -\frac{1}{Z}\frac{\partial}{\partial \beta}\left(\sum_\alpha e^{-\beta E_\alpha}\right) = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$$

$$= -\frac{\partial \ln Z}{\partial \beta}$$

$$E_n = (n + \tfrac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \cdots$$

→　　partition function

$$Z = \sum_n e^{-\beta E_n}$$

$$= \sum_n \exp\left(-(n+\tfrac{1}{2})\beta\hbar\omega\right)$$

$$= \exp(-\tfrac{1}{2}\beta\hbar\omega) \cdot \exp$$

$$= \exp(-\tfrac{1}{2}\beta\hbar\omega)\left(\cancel{\exp} 1 + \exp(-\beta\hbar\omega) + \exp(-2\beta\hbar\omega) + \cdots\right)$$

$$= \exp(-\tfrac{1}{2}\beta\hbar\omega)\,\frac{1}{1-\exp(-\beta\hbar\omega)}$$

$$= \frac{1}{\exp(\tfrac{1}{2}\beta\hbar\omega) - \exp(-\tfrac{1}{2}\beta\hbar\omega)} = \frac{1}{2\sinh(\hbar\omega\beta/2)} \qquad QED$$

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}$$

$$= -\frac{\partial}{\partial \beta} \left(\ln \left(\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} \right) \right)$$

$$= \frac{\partial}{\partial \beta} \ln \left(2\sinh\left(\frac{\beta\hbar\omega}{2}\right) \right)$$

$$= \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} \frac{\partial}{\partial \beta} \left(2\sinh\left(\frac{\beta\hbar\omega}{2}\right) \right)$$

$$= \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} \cdot 2\cosh\left(\frac{\beta\hbar\omega}{2}\right) \cdot \frac{1}{2}\hbar\omega$$

$$= \frac{1}{2}\hbar\omega \tanh\left(\frac{\beta\hbar\omega}{2}\right) \quad \frac{1}{2}\hbar\omega \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

Entropy :

$$S = \frac{\langle E \rangle}{T} + k_B \ln Z$$

$$= \frac{1}{2}\frac{\hbar\omega}{T} \coth\left(\frac{\beta\hbar\omega}{2}\right) + k_B \ln \left(\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} \right)$$

$$= \frac{1}{2} k_B \beta\hbar\omega \coth\left(\frac{\beta\hbar\omega}{2}\right) - k_B \ln\left(2\sinh\left(\frac{\beta\hbar\omega}{2}\right)\right)$$

When $k_B T \ll \hbar\omega$, $\beta\hbar\omega \gg 1$

$$\therefore \tanh\left(\frac{\beta\hbar\omega}{2}\right) \to 1 \quad \coth(\beta\hbar\omega) \to 1$$

$$\therefore \langle E \rangle \to \frac{1}{2}\hbar\omega$$

$$2\sinh\left(\frac{\beta\hbar\omega}{2}\right) = \exp\left(\frac{1}{2}\beta\hbar\omega\right) - \exp\left(-\frac{1}{2}\beta\hbar\omega\right)$$

$$\to \exp\left(\frac{1}{2}\beta\hbar\omega\right)$$

$$\therefore \ln\left(2\sinh\left(\tfrac{1}{2}\beta\hbar\omega\right)\right) \rightarrow \ln\left(\exp\left(\tfrac{1}{2}\beta\hbar\omega\right)\right) = \tfrac{1}{2}\beta\hbar\omega$$

$$S = \tfrac{1}{2}k_B \qquad S \rightarrow \tfrac{1}{2}k_B\beta\hbar\omega - \tfrac{1}{2}k_B\beta\hbar\omega = 0$$

→ the lowest energy state is certainly occupied (probability = 1)

No uncertainty and disorder so entropy is 0

When $k_B T \gg \hbar\omega$, $\beta\hbar\omega \ll 1$

$$\tanh\left(\tfrac{\beta\hbar\omega}{2}\right) \rightarrow \tfrac{1}{2}\beta\hbar\omega$$

$$\therefore \cosh\left(\tfrac{\beta\hbar\omega}{2}\right) = \frac{1}{\tanh\left(\tfrac{\beta\hbar\omega}{2}\right)} \approx \frac{2}{\beta\hbar\omega}$$

$$\rightarrow \langle E \rangle = \tfrac{1}{2}\hbar\omega \frac{2}{\beta\hbar\omega} = \frac{1}{\beta} = k_B T.$$

$$S \Rightarrow \sinh\left(\tfrac{\beta\hbar\omega}{2}\right) \rightarrow 0$$

$$\therefore \ln\left(2\sinh\left(\tfrac{\beta\hbar\omega}{2}\right)\right) \rightarrow -\infty$$

$$\therefore -\ln\left(2\sinh\left(\tfrac{\beta\hbar\omega}{2}\right)\right) \rightarrow +\infty$$

And as $x \rightarrow 0$ $\quad x\cosh\quad x\coth(x) = x\dfrac{x}{\tanh(x)} \sim \dfrac{x}{x} = 1$

is finite

$$\therefore \quad S \rightarrow +\infty$$

$\langle E \rangle = k_B T$ means system approaches the classical limit and mean energy tends to that predicted by the equipartition theorem. $S \rightarrow +\infty$ means system tends to maximum disorder

10.

Transport equation:

$$\frac{3}{2}nk_B \frac{\partial T}{\partial t} = -\frac{\partial J_z}{\partial z} = x\frac{\partial^2 T}{\partial z^2} \quad \text{(there is no source or sink inside the gas)}$$

$$(J_z = -x\frac{\partial T}{\partial z})$$

At steady state: $\frac{\partial T}{\partial t} = 0$

$$\to \frac{\partial^2 T}{\partial z^2} = 0 \quad \to \quad T(z) = Az + B$$

Boundary conditions $T(z=0) = T_1$, $T(z=d) = T_0$

$$\to B = T_1 , \quad Ad + T_1 = T_0 , \quad A = \frac{T_0 - T_1}{d}$$

$$\to T(z) = T_1 + (T_0 - T_1)\frac{z}{d} \quad \text{①}$$

On the other hand: $J_z(z=0) = Q$

$$\because \frac{\partial T}{\partial t} = 0 \quad \therefore \frac{\partial J_z}{\partial z} = 0 \quad \text{in steady state}$$

$$\to J_z = Q \quad \text{for} \quad @ \quad 0 \le z \le d$$

$$\to Q = -x\frac{\partial T}{\partial z} \to T = -\frac{Q}{x}z + C \quad \xleftarrow{\text{we assumed } x \text{ is independent of } T}$$

⑥ Boundary Condition:

$$T(z=d) = T_0 \quad \therefore T_0 = -\frac{Q}{x}d + C$$

$$C = T_0 + \frac{Q}{x}d$$

$$T(z) = T_0 + \frac{Q}{x}(d-z) \quad \text{②}$$

Hence From ① and ② we have

$$T_1 + (T_0 - T_1)\frac{z}{d} = T_0 + \frac{Q}{x}(d-z)$$

$$\therefore \left(\frac{T_0 - T_1}{d} + \frac{Q}{x}\right)z + \left[(T_1 - T_0) - \frac{Q}{x}d\right] = 0 \quad \text{③}$$

③ is true for all z

$\therefore \dfrac{T_1 - T_0}{d} = \dfrac{Q}{\chi}$ $\implies$ $\chi = \dfrac{Qd}{T_1 - T_0}$

OR $(T_1 - T_0) - \dfrac{Q}{\chi}d = 0$ $\implies$ $\chi = \dfrac{Qd}{T_1 - T_0}$

$\implies$ ~~Overall~~ consistent

overall, we have $\chi = \dfrac{Qd}{T_1 - T_0}$

Further assumption:

$\quad T_1$ is held constant

Theoretically $\quad \chi \cancel{\leq} \cancel{\dfrac{k_B}{\sigma}}$ $\quad \chi = \dfrac{1}{\sqrt{\pi}} n k_B \lambda_{mfp} v_{th}$

$\quad$ Normally $\quad$ ~~mfp~~ ~~mfp~~ $\quad \lambda_{mfp} = \dfrac{1}{n\sigma}$

So $\quad \chi = \dfrac{1}{\sqrt{\pi}} \dfrac{k_B}{\sigma} v_{th}$ $\quad$ independent of n

$\quad$ Ideal gas $\quad P = n k_B T$.

$\quad$ At fixed T, P and n are proportional

$\therefore \chi$ independent of n ~~or~~ $\therefore \chi$ independent of P

$\mathcal{B}$ As we decrease P, we decrease n.

$\therefore \lambda_{mfp} = \dfrac{1}{n\sigma}$ $\therefore$ smaller n, larger λ_{mfp}.

But λ_{mfp} cannot be larger than d, the smallest dimension of the container of the gas.

$\therefore \lambda_{mfp} \leq d$. If $\lambda_{mfp} = d$ at $P = P_0$,

Then $\qquad \frac{1}{n\sigma d} = 1 \qquad \frac{1}{n\sigma} = d \qquad \rightarrow \quad n = \frac{1}{\sigma d}$

$$\rightarrow \quad \frac{P_0}{k_B T_0} = \frac{1}{\sigma d} \qquad \rightarrow \quad \boxed{P_0 = \frac{k_B T_0}{\sigma d}}$$

If $P < P_0$, then λ_{mfp} is fixed to d.

$$\chi = \frac{1}{3\pi} n k_B d \, v_{th} = \frac{1}{3\pi} \frac{P}{k_B T} k_B v_{th} d$$

which depends on P.

Estimate $P_0 \approx \dfrac{(1.38 \times 10^{-23})(77)}{(0.21 \times 10^{-18})(1 \times 10^{-3})}$

$$\approx \underline{5.06 \; Pa}$$

For P well above P_0. $\quad \lambda_{mfp} = \frac{1}{n\sigma}$

$$\chi = \frac{1}{3\pi} n k_B \frac{1}{n\sigma} \sqrt{\frac{2 k_B T_0}{m}} \qquad \text{For He} \quad m = 4 m_p$$

$$\rightarrow \quad \chi = \frac{k_B}{\sigma} \sqrt{\frac{2 k_B T_0}{4\pi m_p}} = \frac{k_B}{\sigma} \sqrt{\frac{k_B T_0}{2\pi m_p}} = \frac{Q d}{T_1 - T_0}$$

$$\rightarrow \quad \frac{Q d}{T_1 - T_0} = \frac{(1.38 \times 10^{-23})}{(0.21 \times 10^{-18})} \left(\frac{(1.38 \times 10^{-23})(77)}{(2\pi)(1.67 \times 10^{-27})} \right)^{\frac{1}{2}}$$

$$= 0.021$$

$$\rightarrow \qquad T_1 = T_0 + \frac{Q d}{0.021}$$

$$= 77k + \frac{(100)(1 \times 10^{-3})}{0.021}$$

$$= 77k + 4.76k$$

$$= \boxed{81.76k}$$