A10430W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part A: 3 and 4 Year Courses

A2: ELECTROMAGNETISM AND OPTICS

TRINITY TERM 2015

Saturday, 20 June, 9.30 am – 12.30 pm

Answer **all** *of Section A and* **three** *questions from Section B.*

For Section A start the answer to each question on a **fresh page.**
For Section B start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Derive the conditions that relate the **B** and **H** fields in two linear, isotropic, homogeneous magnetic media of relative permeabilities μ_1 and μ_2, separated by a common boundary. [4]

Define the *magnetic susceptibility* and *magnetization density*, and briefly discuss the distinction between paramagnetic and ferromagnetic materials. [3]

2. Write down the Lorentz transformations for an electric field $\mathbf{E}_\perp$ and magnetic field $\mathbf{B}_\perp$ connecting a stationary frame and one moving at velocity $\mathbf{v}$ perpendicular to both $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$. Show that $\mathbf{E}_\perp \cdot \mathbf{B}_\perp$ is invariant in these two frames. [4]

[You may assume $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$.]

3. Starting from Maxwell's equations for an isotropic conducting medium with no free charges, show that the current density **J** satisfies the equation

$$\nabla^2 \mathbf{J} = \mu_0 \mu_r \sigma \frac{\partial \mathbf{J}}{\partial t}$$

provided the conductivity σ is sufficiently large, where μ_r is the relative permeability and μ_0 the permeability of vacuum. [5]

By considering plane wave solutions propagating in the z-direction of the form

$$J_y = J_0 \exp\left[\mathrm{j}(\omega t - kz)\right] ,$$

where J_0 is a constant and ω and k have their usual meaning, show that the amplitude of **J** decays exponentially with z with a 'skin depth'

$$\delta = \sqrt{\frac{2}{\sigma \omega \mu_0 \mu_r}} .$$

[4]

4. Two plane polarized beams of light of equal intensity with polarizations that are mutually orthogonal are produced from a gas discharge lamp emitting nearly monochromatically at mean wavenumber ν and with bandwidth $\Delta\nu$ using a polarizing beamsplitter. Suggest how these two beams may be combined to produce

(a) a circularly polarized beam (of either parity),

(b) an unpolarized beam.

In each case state what additional optical components might be needed to obtain the required result. Indicate how the intensity of the resulting beam in each case compares with that of the original beams? [6]

5. Explain with the aid of a ray diagram the operation of a simple astronomical telescope and obtain an expression for its angular magnification. What limits the angular resolution of such an instrument? [4]

An astronomer wishes to design a telescope that can separately image a star and a planet orbiting the star. The star lies 20 parsecs from the Sun and the planet is orbiting at a distance of 10 AU from its star. Estimate the minimum size of telescope that can obtain separate images of the planet and star. What other factors should be allowed for in designing such observations? [4]

6. Electromagnetic waves of wavelength λ in air are incident normally onto a glass surface of refractive index n_1, covered by a thin coating of a transparent dielectric material of refractive index n_2 and thickness $\lambda/(4n_2)$. Determine the value of n_2 in terms of n_1 that reduces the reflected intensity to zero. Such an arrangement is used in 'non-reflective' coatings for spectacles. Suggest reasons for why such coatings might not be completely effective. [6]

Section B

7. A dielectric sphere of radius r_0 and relative permittivity ϵ_r in air (with $\epsilon_r = 1$) is placed in a uniform, vertical electric field $\mathbf{E}_0$. Show that the electric potential must satisfy Laplace's equation and that the potentials

$$\phi_1 = -A_1 r \cos\theta \;\; (r \leq r_0)$$

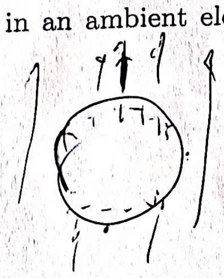
and

$$\phi_2 = -A_2 r \cos\theta + \frac{A_3}{r^2}\cos\theta \;\; (r > r_0)$$

represent the electric field inside and outside the sphere, respectively, where r is the radial distance measured from the centre of the sphere, θ is an angle measured from the direction of $\mathbf{E}_0$, and A_1, A_2 and A_3 are constants. [7]

Determine the constants A_1, A_2 and A_3 in terms of $E_0 \equiv |\mathbf{E}_0|$ and ϵ_r, and hence obtain an expression for the electric field inside the sphere and the polarization $\mathbf{P}$. [7]

Raindrops may be idealized as spheres of water (for which $\epsilon_r \simeq 80$). Estimate by how much an ambient electric field may be locally enhanced near high voltage gridlines when it rains. Hence or otherwise, estimate the maximum force between two raindrops of radius 1 mm, whose centres are separated by 5 mm in an ambient electric field of 1 kV m^{-1}. [6]



8. Give a brief outline of the derivation of the telegraph equations below for a loss-less transmission line:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} \; ; \quad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} \; ,$$

where L and C are respectively the inductance and capacitance per unit length. Obtain expressions for the characteristic impedance and propagation speed of waves along the line. [6]

An air-filled coaxial transmission line has an inner radius a and an outer radius b. Given that the inductance per unit length is

$$L = \frac{\mu_0}{2\pi}\ln(b/a) \; ,$$

where μ_0 is the permeability of vacuum, show that the characteristic impedance of the line is

$$Z_0 = \frac{1}{2\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\ln\left(\frac{b}{a}\right) \; ,$$

where ϵ_0 is the permittivity of vacuum. Hence determine the ratio b/a for a 50 Ω air-filled coaxial line. [5]

Sketch the electric and magnetic fields associated with an electromagnetic wave propagating inside a coaxial transmission line and obtain an expression for the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ inside the line in terms of the rms voltage V_{rms}. Verify that the total power $P = V_{\mathrm{rms}}^2/Z_0$ transmitted down an infinite, loss-less coaxial line is equal to the time-averaged integral of $\mathbf{S}$ over its cross-sectional area. [6]

Given that the breakdown electric field for air is 3 MV m^{-1}, estimate the maximum continuous power that can be transmitted without arcing down a 50 Ω air-filled coaxial line of outer radius 1 cm. [3]

9. Define the Fraunhofer approximation and state the conditions under which it is valid in terms of aperture size D and distance r separating object and observer. [3]

A linear array of 10 radio antennae is set up as a radio telescope to view the sky at a wavelength of $\lambda = 21$ cm. Each antenna receives radiation uniformly from all directions towards the sky. Adjacent antennae are a distance d apart and feed signals to the same receiver via a mixer and identical cables of equal length.

Using the Fraunhofer approximation, obtain an expression for the angular distribution of the intensity, $I(\theta)$, received from a distant source of radiation whose line of sight is perpendicular to the array, where θ is an angle measured with respect to the normal to the array from its centre. [5]

The array is designed to receive radiation via a single primary maximum in $I(\theta)$ with no first- or higher-order maximum present, such that the main reception beam is as narrow as possible. Suggest with reasons a suitable value for d to achieve this and make a labelled sketch of the reception pattern $I(\theta)$. Calculate the angular width of the resulting pattern (between the first zeroes of intensity on either side of the principal maximum of intensity). How could the principal direction of reception be changed without moving the array? [6]

Discuss how the reception pattern of the array would change if each antenna were fitted with reflectors that effectively turned each element into an aperture of width $b(\leq d)$ that receives radiation at uniform amplitude and phase across its width. Obtain an expression for $I(\theta)$ in this case and hence suggest a value of b with $d = 1$ m that would give a narrow main beam but with no first order beam. Determine the angular width of the resulting beam and suggest ways in which the design of the array could be improved to obtain much higher angular resolution without introducing strong secondary sidelobes. [6]

10. Sketch the components of a Michelson interferometer, explaining the purpose of each. Show that the intensity I at the centre of the field of view depends on the physical mirror separation x (so that the optical path difference is $2x$) as

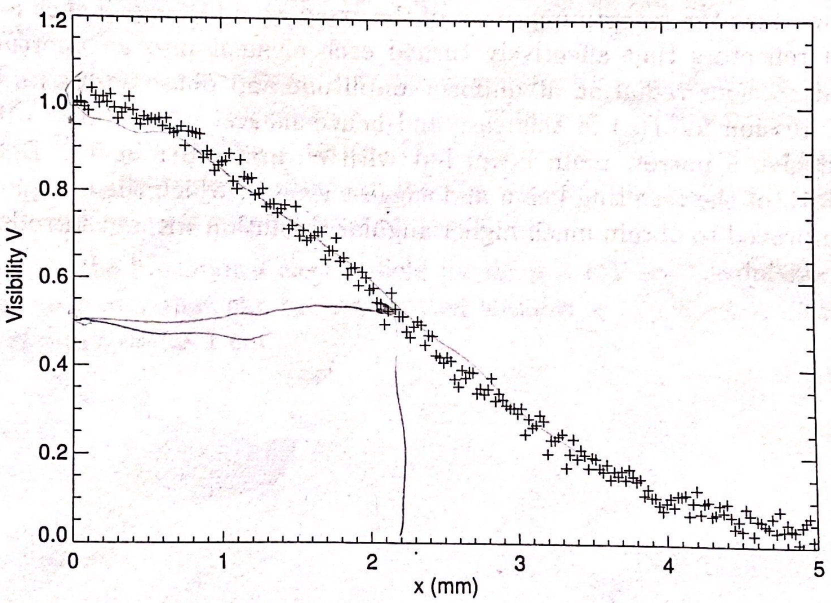
$$I(x) = I_0 \left[1 + \cos(4\pi\overline{\nu}x)\right] ,$$

where I_0 is the rms intensity, $\overline{\nu} = 1/\lambda$ and λ is the wavelength. Hence or otherwise, explain how the Michelson interferometer can be used as a Fourier-transform spectrometer. [7]

The visibility V of a fringe pattern is defined to be

$$V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} ,$$

where $I_{\max}$ is the light intensity at a bright fringe and $I_{\min}$ the intensity at an adjacent dark fringe. A Michelson interferometer is illuminated by a source emitting light with equal intensity at two wave numbers $\overline{\nu_0}$ and $\overline{\nu_0} + \delta\nu$. Obtain an expression for the visibility as a function of the mirror separation x. [4]



An astronomer uses a Michelson interferometer to observe the Hα line in emission from a star at a wavelength of 656 nm. The diagram above shows the visibility curve obtained from these observations. Hence, estimate the temperature of the stellar photosphere assuming that the Hα line is Doppler broadened. [9]

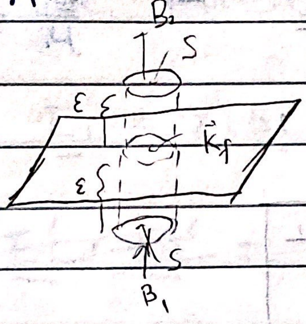
[You may use $\int_{-\infty}^{\infty} \exp(-a^2x^2)\cos(bx+c)\,\mathrm{d}x = \frac{\sqrt{\pi}}{a}\exp\left(\frac{-b^2}{4a^2}\right)\cos(c)$, where a, b and c are constants.]

To: Caroline Terquem

A2 2015 1st Attempt Ziyan Li

Part A:

1\.



$\perp$ = perpendicular to the boundary

$//$ = parallel to the boundary

Gauss's law for magnetism:

$$\oint \underline{B} \cdot d\underline{S} = 0$$

$$\rightarrow \quad B_2^{\perp} S - B_1^{\perp} S = 0 \qquad \text{as } \varepsilon \to 0$$

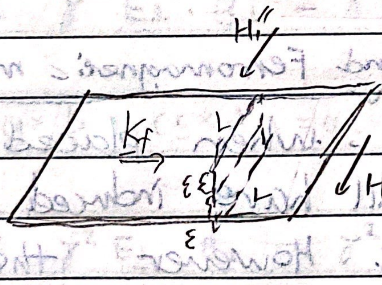
$$\rightarrow \quad \underline{B}_2^{\perp} = \underline{B}_1^{\perp}$$

$\because$ ~~Ma~~ Media are liner $\therefore$ $\underline{B}_1 = \mu_1 \underline{H}_1$ $\quad \underline{B}_2 = \mu_2 \underline{H}_2$

$$\underline{B}_1^{\perp} = \underline{B}_2^{\perp} \rightarrow \mu_1 \underline{H}_1^{\perp} = \mu_2 \underline{H}_2^{\perp}$$

This is the perpendicular component.

For the parallel component



K_f is the free current density (linear density)

Use the Ampere's Law for $\underline{H}$

We have $\oint \underline{H} \cdot d\underline{\ell} = I_{free}$

Consider the loop shown.

$$H_2'' L - H_1'' L = K_f L$$

And in vector form $\quad \underline{H}_2'' - \underline{H}_1'' = \underline{K}_f \times \hat{\underline{n}}$

If there is no free current ~~a~~ on the boundary

, then we have $H_1'' = H_2''$ $\rightarrow$ $\frac{B_1''}{\mu_1} = \frac{B_2''}{\mu_2}$

So the conditions are:

$$\boxed{\begin{array}{ll} B_1^{\perp} = B_2^{\perp} & \mu_1 H_1^{\perp} = \mu_2 H_2^{\perp} \\ \frac{B_1''}{\mu_1} = \frac{B_2''}{\mu_2} & H_1'' = H_2'' \end{array}}$$

→ The magnetic susceptibility χ_m is given by

$\underline{M} = \chi_m \underline{H}$ where $\underline{M}$ is the magnetization density, and $\underline{H}$ is the auxillary field.

→ The magnetization density is the magnetic dipole moment per unit volume.

→ Both Paramagnetic materials and Ferromagnetic materials, when placed in an external magnetic field, will have induced magnetization along the imposed field. However, the distinction is that for paramagnetic materials the induced magnetisation is present only when the external magnetic field is present, whereas for ferromagnetic materials the magnetization can be present even after the external field has been removed.

2. The Lorentz transformations for "$\perp$" component:

$$\underline{E}_\perp' = \gamma(\underline{E}_\perp + \underline{v}\times\underline{B}_\perp) \qquad \underline{B}_\perp' = \gamma(\underline{B}_\perp - \frac{1}{c^2}\underline{v}\times\underline{E}_\perp).$$

$\cancel{\underline{E}_\perp' \cdot \underline{B}_\perp' =}$ where $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

We have

$$\underline{E}_\perp' \cdot \underline{B}_\perp' = \gamma(\underline{E}_\perp + \underline{v}\times\underline{B}_\perp)\cdot\gamma(\underline{B}_\perp - \frac{1}{c^2}\underline{v}\times\underline{E}_\perp)$$

$$= \gamma^2\underline{E}_\perp\cdot\underline{B}_\perp - \gamma^2\frac{1}{c^2}\underbrace{\underline{E}_\perp\cdot(\underline{v}\times\underline{E}_\perp)}_{=0} + \gamma^2\underbrace{\underline{B}_\perp\cdot(\underline{v}\times\underline{B}_\perp)}_{=0}$$

$$- \gamma^2\frac{1}{c}(\underline{v}\times\underline{B}_\perp)\cdot(\underline{v}\times\underline{E}_\perp)$$

since $\underline{a}\cdot(\underline{b}\times\underline{a}) = 0$

$$= \gamma^2\underline{E}_\perp\cdot\underline{B}_\perp - \gamma^2\frac{1}{c}[(\underline{v}\cdot\underline{v})(\underline{E}_\perp\cdot\underline{B}_\perp) - (\underline{B}_\perp\cdot\underline{v})(\underline{E}_\perp\cdot\underline{v})]$$

$(\underline{B}_\perp\cdot\underline{v}) = 0$, $(\underline{E}_\perp\cdot\underline{v}) = 0$

use $(\underline{a}\times\underline{b})\cdot(\underline{c}\times\underline{d}) = (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d}) - (\underline{b}\cdot\underline{c})(\underline{a}\cdot\underline{d})$

($\underline{B}_\perp\cdot\underline{v} = 0$, $\underline{E}_\perp\cdot\underline{v} = 0$ because $\underline{v} \perp \underline{B}_\perp$ and $\underline{E}_\perp$)

$$= \gamma^2\underline{E}_\perp\cdot\underline{B}_\perp - \gamma^2\frac{v^2}{c}\underline{E}_\perp\cdot\underline{B}_\perp$$

$$= \frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}\underline{E}_\perp\cdot\underline{B}_\perp = \underline{E}_\perp\cdot\underline{B}_\perp$$

$\rightarrow$ $\underline{E}_\perp\cdot\underline{B}_\perp$ is Lorentz invariant.

3. Consider ohm's Law : $\underline{J_f} = \sigma \underline{E}$

$\underline{J_f}$ is the free current

let $\mu = \mu_0\mu_r$ be the magnetic permeability and

ε be the electric permitivity

The maxwell's equation in matter is

$$\nabla \cdot \underline{D} = \rho_f \quad \nabla \cdot \underline{B} = 0, \quad \nabla \times \underline{E} = -\frac{\partial B}{\partial t}, \quad \nabla \times H = J_f + \frac{\partial D}{\partial t}$$

and ~~since~~ $D = \cancel{\varepsilon E}\ \varepsilon E$, $\underline{B} = \mu \underline{H}$, $\underline{J_f} = \sigma \underline{E}$

$\because$ No free charge $\therefore \rho_f = 0$

$$\therefore \nabla \cdot \underline{D} = 0 \rightarrow \nabla \cdot (\varepsilon \underline{E}) = 0 \rightarrow \frac{1}{\varepsilon\sigma} \nabla \cdot \underline{J_f} = 0 \rightarrow \nabla \cdot \underline{J_f} = 0$$

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \rightarrow \frac{1}{\sigma}(\nabla \times \underline{J_f}) = -\mu \frac{\partial H}{\partial t} \rightarrow \nabla \times J_f = -\sigma\mu \frac{\partial H}{\partial t}$$

$$\nabla \times \underline{H} = \underline{J_f} + \frac{\partial D}{\partial t} = \underline{J_f} + \varepsilon \frac{\partial E}{\partial t} = J_f + \frac{\varepsilon}{\sigma}\frac{\partial J_f}{\partial t}$$

$\because \sigma$ is sufficiently large

$$\therefore \nabla \times \underline{H} \simeq \underline{J_f}$$

Consider $\nabla \times (\nabla \times \underline{J_f}) = \nabla(\nabla \cdot \underline{J_f}) - \nabla^2 \underline{J_f}$

$$= 0 - \nabla^2 \underline{J_f} = -\nabla^2 \underline{J_f}$$

Also $\nabla \times (\nabla \times \underline{J_f}) = -\sigma\mu \frac{\partial}{\partial t}(\nabla \times H) \simeq -\sigma\mu \frac{\partial J_f}{\partial t}$

$$\rightarrow \nabla^2 \underline{J_f} = \sigma\mu \frac{\partial J_f}{\partial t} = \mu_0\mu_r\sigma \frac{\partial J_f}{\partial t}$$

Also consider $\nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \mu_0 \underline{J} + \frac{\mu_0\varepsilon_0}{\sigma}\frac{\partial J_f}{\partial t}$

$\because \sigma$ very large $\therefore \nabla \times B \simeq \mu_0 J$

$$J_f \simeq \nabla \times \underline{H} = \nabla \times \frac{B}{\mu_0\mu_r} = \frac{1}{\mu_0\mu_r}(\nabla \times B) = \frac{1}{\mu_r}\left(\frac{1}{\mu_0}(\nabla \times \underline{B})\right)$$

$$\simeq \frac{1}{\mu_r} \underline{J}$$

$\therefore$ $\underline{J}$ and $\underline{J_f}$ are approximately proportional

$\because$ $\nabla^2 \underline{J_f} = \mu_0 \mu_r \sigma \frac{\partial \underline{J_f}}{\partial t}$ $\therefore$ we have

$$\nabla^2 \underline{J} = \mu_0 \mu_r \sigma \frac{\partial \underline{J}}{\partial t}$$

Substitute the y-component of $\underline{J}$ into this equation

$$J_y = J_0 \exp[j(\omega t - kz)]$$

$$\nabla^2 J_y = \mu_0 \mu_r \sigma \frac{\partial J_y}{\partial t}$$

$$\rightarrow (-jk)^2 J_0 \exp[j(\omega t - kz)]$$

$$= \mu_0 \mu_r \sigma (j\omega) J_0 \exp[j(\omega t - kz)]$$

$$\rightarrow \cancel{-k^2 \mu_0 \mu_r \sigma} \quad -k^2 = j \sigma \omega \mu_0 \mu_r$$

$$\rightarrow k = \pm \sqrt{-j} \left(\sqrt{\sigma \omega \mu_0 \mu_r}\right) = \pm \frac{1}{\sqrt{2}} (1-j) \sqrt{\sigma \omega \mu_0 \mu_r}$$

$$= \pm (1-j) \sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}}$$

$$J_y = J_0 \exp[j(\omega t - kz)] = J_0 \exp(j\omega t) \exp\left((-j)(\pm(1-j)) \sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}} z\right)$$

$$\rightarrow J_y = J_0 \exp(j\omega t) \exp\left[(-j-1) \sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}} z\right]$$

$$= J_0 \exp\left(-\sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}} z\right) \exp\left[j\left(\omega t - \sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}} z\right)\right]$$

(we choose $\sqrt{-j}$ to be $1-j$ instead of $j-1$ because we want forward traveling wave)

The exponential decay factor is $\exp\left(-\sqrt{\frac{\sigma \omega \mu_0 \mu_r}{2}} z\right)$

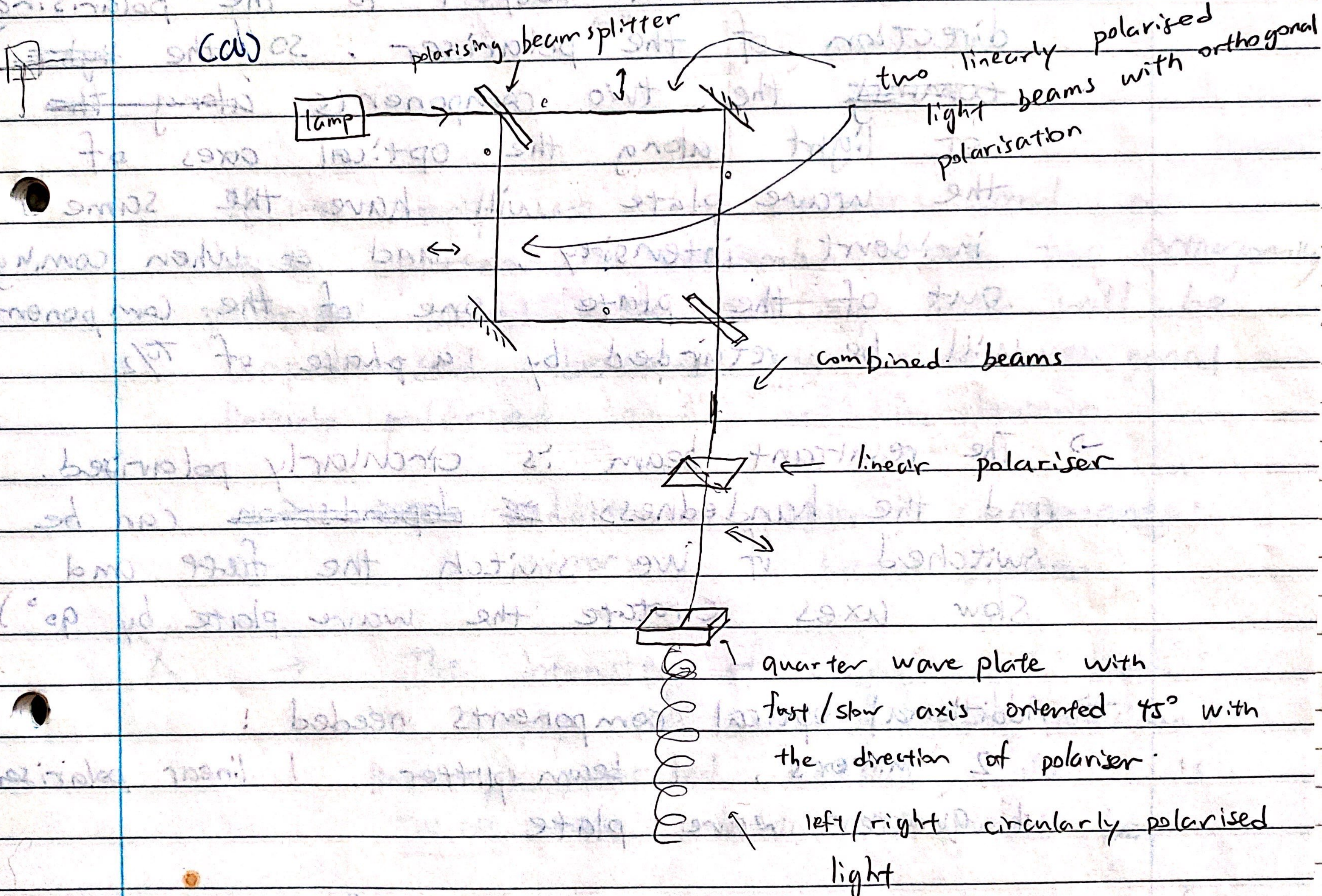
$$= \exp\left(-z \Big/ \sqrt{\frac{2}{\sigma \omega \mu_0 \mu_r}}\right) \equiv \exp\left(-\frac{z}{\delta}\right)$$

$\rightarrow$ skin depth $\boxed{\delta = \sqrt{\frac{2}{\sigma \omega \mu_0 \mu_r}}}$

4. ∵ the light ~~emitting~~ emitted by the lamp has a bandwidth $\Delta\nu$

∴ the two linearly polarised beams produced using the beam-splitter are not coherent

(a)



→ As shown in the diagram, we can use two ~~mirror~~ mirrors and another beam splitter to recombine the two ~~beam~~ beams

→ Then we ~~pass~~ let the combined beam pass through ~~the~~ a linear polariser to produce a single coherent light beam, ~~Then we~~ ~~use~~ which is linearly polarised in the direc set by the linear polariser.

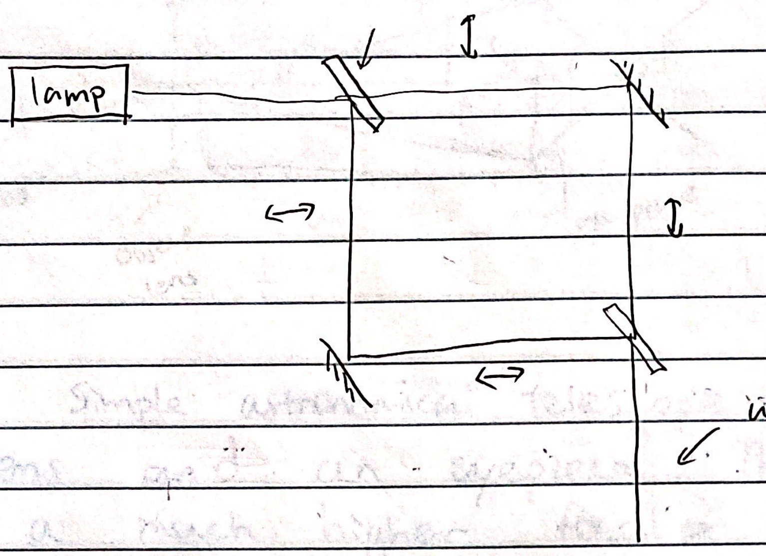
→ Then we use ~~put~~ a quarter-wave plate ~~to~~ such that the optical axes of the plate is in 45° with respect to the polarising direction of the polariser, so the ~~light~~ ~~coming~~ the two components ~~along the~~ of light along the optical axes of the wave plate will have the same incident intensity, and ~~at~~ when coming out of the plate one of the components will be retarded by a phase of $\pi/2$

→ The resultant beam is circularly polarised, and the handedness ~~is~~ ~~depends on~~ can be switched if we switch the fast and slow axes (rotate the wave plate by 90°)

→ Additional optical components needed:
2 mirrors, 1 beam splitter, 1 linear polariser, 1 quarter wave plate

→ ~~The light beam coming out of~~
The recombined beam is unpolarised before entering the linear polariser, so the linear polariser reduces the intensity by $\frac{1}{2}$, which becomes the same as each of the original beams. wave plate gives no change in intensity.
→ Hence intensity is the same as each individual original beam.

(b)



If we simply use two mirrors and a beam splitter to recombine the two orthogonally polarised beams, the resulting beam will be unpolarised because the two orthogonal linearly polarised beams are incoherent.

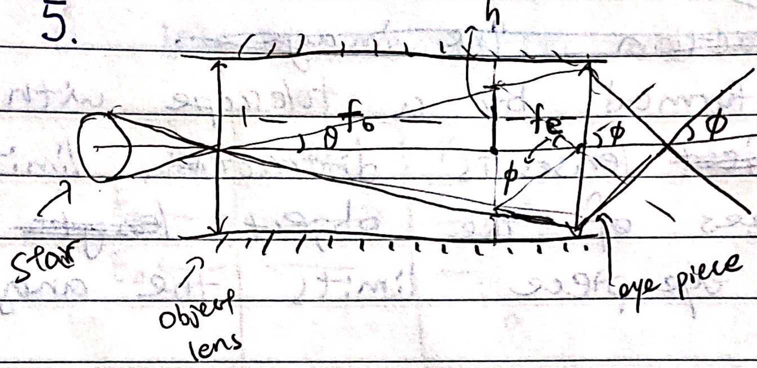
→ ~~inten~~ additional optical components:
2 mirrors, 1 beam splitter

X → The intensity of unpolarised beam is simply the sum of two individual linearly polarised beams, so it is <u>twice</u> the individual original beam.

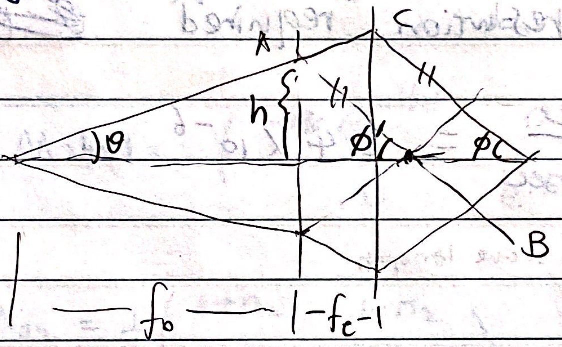
~~wait Combine the beams and wait for it to travel a distance longer than the coherence length, then the beam is unpolarised.~~

wait for the ~~two~~ optical path diffrence of the two component to be longer than the coherence length of the emitted light, then combine the two components to get unpolarised light.

5.



Simple astronomical telescopes consist of an object lens and an eyepiece. The object lens has a much higher focal ~~s~~ lens than the eye piece so the object ~~go~~ becomes angularly magnified and looks larger.



$\theta = \frac{h}{f_o}$, $\phi' = \frac{h}{f_e}$ $\because AB // CD \therefore \phi = \phi'$

$\therefore \phi = \frac{h}{f_e}$

Angular magnification $= \frac{2\phi}{2\theta} = \frac{\phi}{\theta} = \frac{h/f_e}{h/f_o} = \boxed{\frac{f_o}{f_e}}$

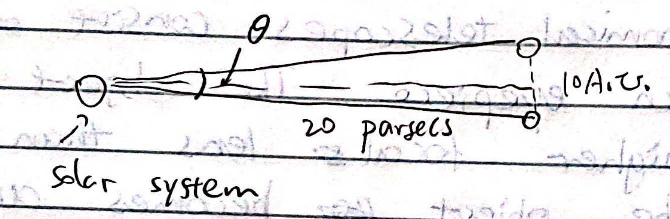
Where f_o = focal length of object lens

f_e = focal length of eye piece

→ ~~Other factors: Aberr Aberration~~ ~~k~~ may be a problem if we want to make the eye piece too powerful ~~k~~, i.e. f_e ~~too small~~.

~~Due to diffraction, the image~~.

The resolution of the image formed by a telescope with finite size of ~~object~~ lenses is diffraction limited. So the sizes of the object ~~length~~ lens and the eye piece limits the angular resolution.



Angular resolution required ~~θ=~~

$$\theta = \frac{10\,A.U.}{20\,parsecs} = 2.4 \times 10^{-6}\ rad$$

$$\theta = 1.22 \frac{\lambda}{D}$$

wave length

size of telescope

Assume $\lambda = 500\ nm$ for typical visible light, then

$$\cancel{D = \frac{\theta}{1.22\lambda} = \frac{2.4\times10^{-6}}{1.22\,(500)\times10^{-9}} =}$$

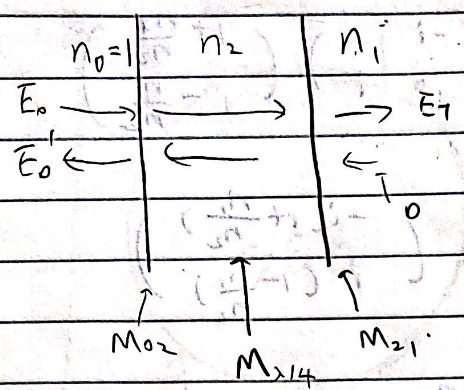
$$D = \frac{1.22\,\lambda}{\theta} = \frac{1.22 \times 500 \times 10^{-9}}{2.4\times10^{-6}} = \boxed{0.25\,m}$$

is the minimum size of telescope.

→ other factors need to consider:

Aberration may be a problem if we want to make the eye piece too powerful, i.e. the value of f_e too small.

6. Use the matrix method :



the phase through the coating is

$$kd = \frac{2\pi n_2}{\lambda} \cdot \frac{\lambda}{4n_2} = \frac{\pi}{2}$$

$$\therefore M_{\lambda/4} = \begin{pmatrix} e^{-ikd} & 0 \\ 0 & e^{ikd} \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$M_{02} = \frac{1}{2}\begin{pmatrix} 1+n_2 & 1-n_2 \\ 1-n_2 & 1+n_2 \end{pmatrix}$$

$$M_{21} = \frac{1}{2}\begin{pmatrix} 1+\frac{n_1}{n_2} & 1-\frac{n_1}{n_2} \\ 1-\frac{n_1}{n_2} & 1+\frac{n_1}{n_2} \end{pmatrix}$$

$$\begin{pmatrix} E_0 \\ E_0' \end{pmatrix} = M_{02} M_{\lambda/4} M_{21} \begin{pmatrix} E_T \\ 0 \end{pmatrix}$$

$$\cancel{M_{02} M_{\lambda/4} M_{21}} = \frac{1}{2}\begin{pmatrix} 1+n_2 & 1-n_2 \\ 1-n_2 & 1+n_2 \end{pmatrix}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1+\frac{n_1}{n_2} & 1-\frac{n_1}{n_2} \\ 1-\frac{n_1}{n_2} & 1+\frac{n_1}{n_2} \end{pmatrix}\begin{pmatrix} E_T \\ 0 \end{pmatrix}$$

$$\cancel{= \frac{1}{4}\begin{pmatrix} 1+n_2 & 1-n_2 \\ 1-n_2 & 1+n_2 \end{pmatrix}\begin{pmatrix} -i(1+\frac{n_1}{n_2}) & -i(1-\frac{n_1}{n_2}) \\ i(1-\frac{n_1}{n_2}) & i(1+\frac{n_1}{n_2}) \end{pmatrix}}$$

$$\cancel{= \frac{1}{4}\Big((1+n_2)(-i)(1+\frac{n_1}{n_2})}$$

$$\cancel{= \frac{E_T}{4} (1 + n_2)}$$

$$= \frac{E_T}{4} \begin{pmatrix} 1+n_2 & 1-n_2 \\ 1-n_2 & 1+n_2 \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1+\frac{n_1}{n_2} \\ 1-\frac{n_1}{n_2} \end{pmatrix}$$

$$= \frac{E_T}{4} \begin{pmatrix} 1+n_2 & 1-n_2 \\ 1-n_2 & 1+n_2 \end{pmatrix} \begin{pmatrix} -i(1+\frac{n_1}{n_2}) \\ i(1-\frac{n_1}{n_2}) \end{pmatrix}$$

$$= \frac{E_T}{4} \begin{pmatrix} -i(1+n_2)(1+\frac{n_1}{n_2}) + i(1-n_2)(1-\frac{n_1}{n_2}) \\ -i(1-n_2)(1+\frac{n_1}{n_2}) + i(1+n_2)(1-\frac{n_1}{n_2}) \end{pmatrix}$$

$\rightarrow$ reflectivity $r = \frac{E_0'}{E_0}$

$$\therefore r = \frac{-\cancel{i}(\cancel{1} - n_2 + \frac{n_1}{n_2} - \cancel{n_1}) + \cancel{i}(\cancel{1} + n_2 - \frac{n_1}{n_2} - \cancel{n_1})}{-\cancel{i}(\cancel{1} + n_2 + \frac{n_1}{n_2} + \cancel{n_1}) + \cancel{i}(\cancel{1} - n_2 - \frac{n_1}{n_2} + \cancel{n_1})}$$

$$= \frac{n_2 - \frac{n_1}{n_2}}{-n_2 - \frac{n_1}{n_2}} = \frac{n_1 - n_2^2}{n_1 + n_2^2}$$

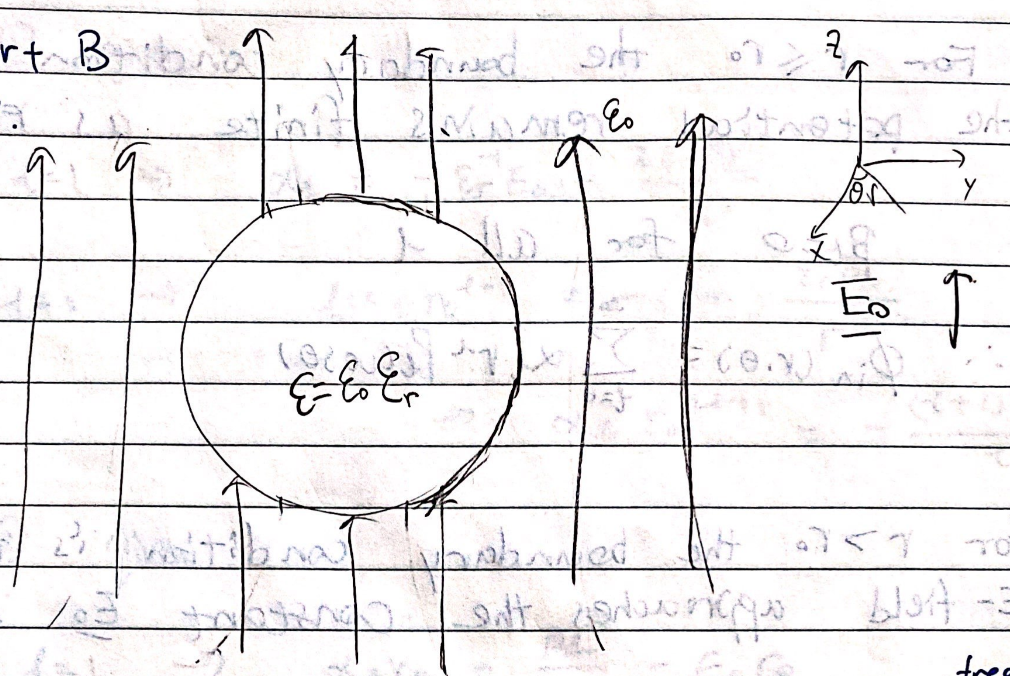
for non-reflective coating, $r = 0$

$\Rightarrow$ $n_1 - n_2^2 = 0$ $\rightarrow$ $\boxed{n_1 = n_2^2}$ is the required condition.

The reasons for why such coating is not completely reflective is that practically light have a band-width in ~~wa~~ wavelength. Light can never be perfectly monochromatic. But the anti-reflection condition requires the thickness of coating to be $\frac{\lambda}{4n_2}$ which depends strictly on λ. So this arrangement cannot ~~reflect~~ ~~re~~ reflect light perfectly in practice.

Part B

7\.



/ free charge density

Maxwell's equations : $\nabla \cdot D = \rho_f$

$\cancel{D = \varepsilon}$ $\quad D = \varepsilon E$

$\rightarrow \nabla \cdot D = \nabla \cdot \varepsilon E = \varepsilon \nabla \cdot E = \rho_f = 0 \quad \because$ No free ~~char~~ charge

$\rightarrow \nabla \cdot E = 0$ both inside and outside sphere

$\nabla \times E = -\frac{\partial B}{\partial t} = 0 \quad \because$ No magnetic field

$\therefore$ E can be written as $E = -\nabla\phi$

$\because$ $\cancel{E}$ $\nabla \cdot E = 0 \quad \therefore \quad \nabla \cdot (-\nabla\phi) = 0 \rightarrow \nabla^2\phi = 0$

$\therefore$ The electric potential ϕ satisfies the Laplaces equation ~~for~~ both inside and outside the sphere.

The system has azimuthal symmetry so the solution to laplace's equation is

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \frac{\beta_l}{r^{l+1}}\right) P_l(\cos\theta)$$

↑ Legendre polynomial

For $r \leq r_0$ the boundary condition is that the potential remains finite as ~~r→~~ $r \to 0$

$\therefore \quad \beta_l = 0$ for all l

$$\therefore \phi_{in}(r,\theta) = \sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos\theta)$$

For $r > r_0$ the boundary condition is that the E-field approaches the constant $\underline{E_0}$ as $r \to \infty$

$$\therefore \phi_{out}(r,\theta) \to -E_0 z = -E_0 r\cos\theta \quad \text{as } r \to \infty$$

$\because P_1(\cos\theta) = \cos\theta$ and Legendre polynomials are orthogonal

~~$\therefore \phi_{out} = \alpha_1 r \cos\theta$~~

$$\therefore \phi_{out} = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{\beta_l}{r^{l+1}} P_l(\cos\theta)$$

At the surface $r = r_0$, No free charge $\therefore$ D-field is continuous

$$\therefore \frac{\varepsilon_0 \varepsilon_r E_{in}}{\cancel{\varepsilon_0}} = \frac{E_{out}\varepsilon_0}{\cancel{\varepsilon_0}} \Rightarrow \frac{E_{in}}{\cancel{\varepsilon_0}} = E_{out}\varepsilon_r$$

$$\therefore -\frac{\partial\phi_{in}}{\partial r}\bigg|_{r=r_0} = \frac{1}{\varepsilon_r}\left(-\frac{\partial\phi_{out}}{\partial r}\right)\bigg|_{r=r_0}$$

$$① \quad \therefore \sum_{l=0}^{\infty} l\alpha_l r_0^{l-1} P_l(\cos\theta) = -\varepsilon_r^{-1} E_0 \cos\theta + \sum_{l=0}^{\infty} (-l-1)\frac{\varepsilon_r^{-1}\beta_l}{r_0^{l+2}} P_l(\cos\theta)$$

(Griffith pp 9:)

The potential is continuous at the Boundary $r = r_0$

$$② \quad \therefore \sum_{l=0}^{\infty} \alpha_l r_0^l P_l(\cos\theta) = \left(\sum_{l=0}^{\infty} \frac{\beta_l}{r_0^{l+1}} P_l(\cos\theta)\right) - E_0 r_0 \cos\theta$$

① gives :

$l=1 \rightarrow \quad \alpha_1 = -\varepsilon_r^{-1} E_0 - \frac{2\varepsilon_r^{-1}\beta_1}{r_0^3}$

$l \neq 1 \rightarrow \quad l\alpha_l r_0^{l-1} = -(l+1)\frac{\varepsilon_r^{-1}\beta_l}{r^{l+2}}$

$\rightarrow \quad \alpha_l r_0^{2l+1} = -\frac{(l+1)\varepsilon_r^{-1}}{l}\beta_l.$ ③

② gives :

$l=1 \rightarrow \quad \alpha_1 r_0 = \frac{\beta_1}{r_0^2} - E_0 r_0$

$l \neq 1 \rightarrow \quad \alpha_l r_0^l = \frac{\beta_l}{r_0^{l+1}} \rightarrow \alpha_l r_0^{2l+1} = \beta_l.$ ④

③ and ④ gives that if $l \neq 1$, ~~then~~
and $\alpha_l, \beta_l \neq 0$

$-\frac{(l+1)\varepsilon_r^{-1}}{l} = 1 \quad \therefore \quad -\varepsilon_r^{-1}l - \varepsilon_r = l.$

$\therefore \quad l = -\frac{\varepsilon_r^{-1}}{\varepsilon_r^{-1}+1}$ but $l \geq 0$ and is a integer $\therefore$ for $l \neq 1$, $\alpha_l = 0$, $\beta_l = 0$

For $l=1$: $\alpha_1 r_0 = \frac{\beta_1}{r_0^2} - E_0 r_0$

$\alpha_1 = -\varepsilon_r^{-1} E_0 - \frac{2\varepsilon_r^{-1}\beta_1}{r_0^3}$

$\alpha_1 r_0 = \frac{\beta_1}{r_0^2} - E_0 r_0 = -\varepsilon_r^{-1} E_0 r_0 - \frac{2\varepsilon_r^{-1}\beta_1}{r_0^2}$

$\frac{(2\varepsilon_r^{-1}+1)\beta_1}{r_0^2} = (1-\varepsilon_r^{-1}) E_0 r_0$

$\rightarrow \quad \beta_1 = \frac{1-\varepsilon_r^{-1}}{2\varepsilon_r^{-1}+1} E_0 r_0^2$

$$\frac{\beta_1}{r_0^2} = \alpha_1 \cancel{r_0} + E_0 \cancel{r_0} = -\frac{\alpha_1 \cancel{r_0} + \varepsilon_r^{-1} E_0 \cancel{r_0}}{2\varepsilon_r^{-1}}$$

$$\therefore \quad 2\varepsilon_r^{-1}\alpha_1 + 2\varepsilon_r^{-1}E_0 = -\alpha_1 - \varepsilon_r^{-1}E_0$$

$$\rightarrow \quad 3\varepsilon_r^{-1}E_0 = \cancel{2} -(2\varepsilon_r^{-1}+1)\alpha_1$$

$$\rightarrow \quad \alpha_1 = \cancel{-\frac{2\varepsilon_r+1}{3\varepsilon_r}} \quad -\frac{3\varepsilon_r^{-1}}{2\varepsilon_r^{-1}+1}E_0$$

Call $\alpha_1 = -A_1$, $-E_0 = -A_2$, $\beta_1 = A_3$

$\phi_{in} = \phi_1$, $\phi_{out} = \phi_2$

We have

$$\phi_1 = -\frac{3\varepsilon_r^{-1}}{2\varepsilon_r^{-1}+1}E_0 r\cos\theta \quad (r \le r_0)$$

$$\phi_2 = -E_0 r\cos\theta + \frac{1-\varepsilon_r^{-1}}{2\varepsilon_r^{-1}+1}E_0\cos\theta\frac{r_0^3}{r^2} \quad (r > r_0)$$

$$\rightarrow \boxed{\begin{aligned}\phi_1 &= -\frac{3}{\varepsilon_r+2}E_0 r\cos\theta \quad (r \le r_0)\\ \phi_2 &= -E_0 r\cos\theta + \frac{\varepsilon_r-1}{\varepsilon_r+2}E_0\cos\theta\frac{r_0^3}{r^2} \quad (r > r_0)\end{aligned}}$$

$$\therefore \quad A_1 = \frac{3E_0}{\varepsilon_r+2}, \quad A_2 = E_0, \quad A_3 = \frac{\varepsilon_r-1}{\varepsilon_r+2}E_0 r_0^3$$

✓ ✓ ✓

Electric field inside sphere is

$$\cancel{E_2} \quad \underline{E}_1 = -\nabla\phi_1$$

$\therefore$ let $\underline{E}_1 = (E_r, E_\theta, E_\phi)$

by symmetry $\rightarrow E_\phi = 0$

$$E_r = -\frac{\partial \phi_1}{\partial r} = \frac{3E_0 \cos\theta}{\varepsilon_r + 2}$$

$$E_\theta = -\frac{1}{r}\frac{\partial \phi_1}{\partial \theta} = \frac{3}{\varepsilon_r+2} E_0 (-\sin\theta)$$

$$\cancel{\therefore \underline{E}_1 = \frac{3E_0}{\varepsilon_r+2}(\cos\theta, -\sin\theta, 0)}$$

$$\therefore \underline{E}_1 = \frac{3E_0}{\varepsilon_r+2}[\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}] = \frac{3E_0}{\varepsilon_r+2}\hat{\underline{z}}$$

$$\cancel{= \frac{3}{\varepsilon_r+2}E_0}$$

Polarisation : $\underline{P} = \frac{P_1}{\cancel{\varepsilon_0}} = E_1\varepsilon_0$

$$= \frac{\varepsilon_r \varepsilon_0 \underline{E}_1}{\cancel{\varepsilon_0}} - \underline{E}_1\varepsilon_0 = (\varepsilon_r - 1)\underline{E}_1\varepsilon_0$$

$$= \frac{3(\varepsilon_r - 1)\varepsilon_0}{\varepsilon_r + 2}\underline{E}_0$$

Electric field outside the sphere

$$\underline{E}_2 = -\underline{\nabla}\phi_2 = \cancel{E_0\hat{z}} \; \underline{E}_0 - A_3 \underline{\nabla}\left(\frac{\cos\theta}{r^2}\right)$$

$$\underline{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right) = \frac{-2\cos\theta}{r^3}$$

$$\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right) = \frac{-\sin\theta}{r^2}$$

$$\therefore \underline{E}_2 = \underline{E}_0 + \frac{2A_3\cos\theta}{r^3}\hat{\underline{r}} + \frac{A_3\sin\theta}{r^3}\hat{\underline{\theta}}$$

$$\cancel{= \underline{E}_0 + \frac{A_3\cos\theta}{r^3}(2\hat{\underline{r}} + \hat{\underline{\theta}})}$$

$$= \underline{E}_0 + \frac{A_3}{r^3}(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}})$$

$\therefore$ the change in ambient field is

$$\Delta\underline{E} = \underline{E}_2 - \underline{E}_0 = \frac{A_3}{r^3}(2\cos\theta\,\hat{\underline{r}} + \sin\theta\,\hat{\underline{\theta}})$$

at $r \sim r_0$, the local enhancement of $\underline{E}_0$ in the z direction is

$$\cancel{\langle \Delta E \rangle = \frac{A_3}{r_0^3}\left\langle (2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})\cdot(\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{\hat{z}})\right\rangle}$$

$$\cancel{= \frac{A_3}{r_0^3}\langle 2\cos^2\theta - \sin^2\theta\rangle = \frac{A_3}{r_0^3}\left[\underbrace{\langle 2\cos^2\theta\rangle}_{1} - \underbrace{\langle\sin^2\theta\rangle}_{\frac{1}{2}}\right]}$$

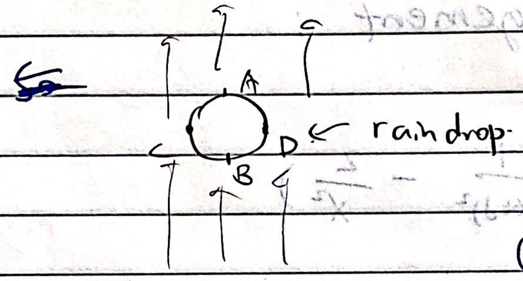
$$= \frac{A_3}{2r_0^3}$$

$$\Delta E_z = \Delta E \cdot \hat{z} = \frac{A_3}{r^3}\left[(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \cdot (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\right]$$

$$\underbrace{\phantom{(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})}}_{\hat{z}}$$

$$= \underline{\frac{A_3}{r^3}\left(2\cos^2\theta - \sin^2\theta\right)}$$

Where $A_3 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} E_0 r_0^3$



So at the poles $\underbrace{}_{(A,B)}$ of the drop the ~~enh~~ ~~enh~~ enhancement in electric field is large, and at the sides (C, D) of the raindrop the ambient field is reduced by the raindrop (enhancement is negative)

$\because \varepsilon_r = 80 \gg 1 \quad \therefore A_3 \simeq E_0 r_0^3$

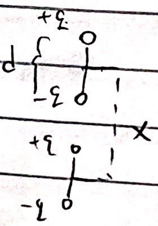
at $r = r_0$, average of ΔE_z is

$$|\Delta E_z| = \frac{A_3}{r_0^3}\langle 2\cos^2\theta - \sin^2\theta \rangle = \frac{A_3}{2r_0^3} = \frac{E_0}{2}$$

$\therefore$ Average enhancement is about $\boxed{50\%}$

The raindrops can be modeled as ~~electrop~~ electric dipoles. and the maximum force happens when the poles of dipoles are facing each other → because this causes the maximum enhancement.

← $F = F_{max}$



Force between two dipoles in this arrangement :

$$F = \frac{q^2}{4\pi\varepsilon_0}\left(\frac{1}{(x-d)^2} + \frac{1}{(x+d)^2} - \frac{2}{x^2}\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0 x^2}\left(\left(1-\frac{d}{x}\right)^{-2} + \left(1+\frac{d}{x}\right)^{-2} - 2\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0 x^2}\left(\cancel{1} + \cancel{2\frac{d}{x}} + 3\frac{d^2}{x^2} + \cancel{1} - \cancel{2\frac{d}{x}} + 3\frac{d^2}{x^2} - \cancel{2}\right)$$

$$= \frac{3(qd)^2}{2\pi\varepsilon_0 x^4} = \frac{3p^2}{2\pi\varepsilon_0 x^4}$$ (p is the dipole moment)

dipole moment $p = PV$

polarisation density

Volume $= \frac{4}{3}\pi r_0^3$

$\cancel{P = \frac{4}{3}\pi r_0^3 \frac{3\varepsilon_0(\varepsilon_r-1)}{\varepsilon_r+2} E_0 = 4\pi\varepsilon_0\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right) r_0^3 E_0}$

$$p = PV = \frac{4}{3}\pi r_0^3 \frac{3\varepsilon_0(\varepsilon_r-1)}{\varepsilon_r+2} E_0 = 4\pi\varepsilon_0 \frac{\varepsilon_r-1}{\varepsilon_r+2} r_0^3 E_0$$

$E_0 = 1000\,V/m$

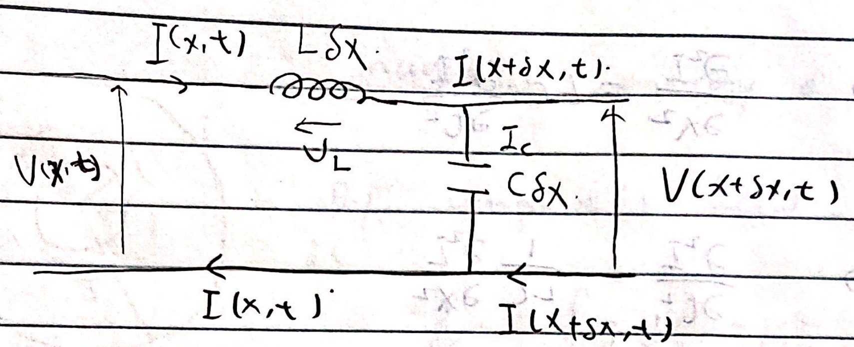
$\varepsilon_r \approx 1$ $\quad r_0 = 1\,mm$

$= 1.11 \times 10^{-16}\ C/m^2$

$x = 5 \times 10^{-3}\, m$

$$\rightarrow F = \frac{3(1.11 \times 10^{-16})^2}{2\pi(8.854 \times 10^{-12})(5 \times 10^{-3})^4} = \boxed{1.07 \times 10^{-12}\, N}$$

8.



loss-less transmission line → current in return is the same as current in active wire.

$$V(x+\delta x, t) = V(x,t) - V_L = V(x,t) - L\delta x \frac{\partial i(x,t)}{\partial t}.$$

$$I(x+\delta x, t) = I(x,t) - I_c = I(x,t) - C\delta x \frac{\partial V(x+\delta x, t)}{\partial t}$$

$$\simeq I(x,t) - C\delta x \frac{\partial V(x,t)}{\partial t}$$

$$\therefore \quad \frac{V(x+\delta x, t) - V(x,t)}{\delta x} = -L\frac{\partial I}{\partial t}(x,t).$$

$$\frac{I(x+\delta x, t) - I(x,t)}{\delta x} = -C\frac{\partial V}{\partial t}(x,t)$$

$$\rightarrow \quad \frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \quad \frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}.$$

(the telegrapher's equation).

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) = -L\frac{\partial^2 I}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{C}\frac{\partial I}{\partial x}\right) = -\frac{1}{C}\frac{\partial^2 I}{\partial x^2}$$

$$\rightarrow \frac{\partial^2 I}{\partial x^2} = LC \frac{\partial^2 I}{\partial t^2}$$

$$\rightarrow \frac{\partial^2 I}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 I}{\partial x^2}$$

similarly $\frac{\partial^2 V}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 V}{\partial x^2}$

→ ~~Vel~~ phase velocity of wave

$$\boxed{V = \frac{1}{\sqrt{LC}}}$$

~~Forw~~ Forward wave solution

$V_+(x,t) = V_0 \cos(\omega t - kx)$

$I_+(x,t) = I_0 \cos(\omega t - kx)$

Substitute into the wave equations we get.

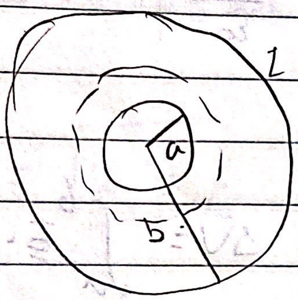
$$k^2 = LC\omega^2 \rightarrow \frac{\omega}{k} = \frac{1}{\sqrt{LC}}$$

Substitute into the telegrapher equation we get

$$\cancel{-k} = \quad -kV_0 = -L\omega I_0$$

∴ characteristic impedance

$$Z = \frac{V_0}{I_0} = \frac{L\omega}{k} = \frac{L}{\sqrt{LC}} = \boxed{\sqrt{\frac{L}{C}}}$$



magnetic energy $E = \frac{1}{2}LI^2 l$

By Ampere's Law, the B field is

$$\oint \underline{B} \cdot d\underline{l} = \mu_0 I$$

$$\rightarrow B(2\pi r) = \mu_0 I.$$

$$\rightarrow B = \frac{\mu_0 I}{2\pi r}$$

energy density $u = \frac{1}{2}\frac{B^2}{\mu_0} = \frac{\mu_0^{\cancel{2}} I^2}{2\cancel{\mu_0} 4\pi^2 r^2} = \frac{\mu_0 I^2}{8\pi^2 r^2}$

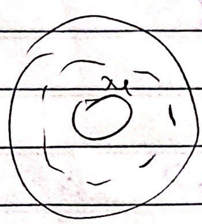
energy $E = \int_V u \, d\tau = \int_{r=a}^{r=b} \frac{\mu_0 I^2}{8\pi^2} \frac{1}{r^2} (2\pi r l) \, dr$

$$= \frac{\mu_0 I^2}{8\pi^2} \cdot 2\pi l \int_a^b \frac{1}{r} \, dr$$

$$= \frac{\mu_0 I^2}{4\pi} l \ln\left(\frac{b}{a}\right)$$

$\therefore \frac{1}{2}LI^2 l = \frac{\mu_0 I^2}{4\pi} l \ln\left(\frac{b}{a}\right)$

$$\rightarrow \underline{L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)}$$



λ = charge per unit length

l = length

Gauss's Law $\oiint \underline{E} \cdot d\underline{s} = \frac{Q}{\varepsilon_0}$

$$\rightarrow \quad E(2\pi r l) = \frac{Q}{\varepsilon_0} = \frac{\lambda l}{\varepsilon_0}$$

$$\rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

potential difference $\Delta V = \int_a^b \underline{E} \cdot \underline{dr}$

$$\Delta V = \int_a^b \frac{\lambda}{2\pi\varepsilon_0 r} dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

$\therefore$ capacitance per unit lengeh

$$C = \frac{Q}{\Delta V l} = \frac{\lambda l}{\frac{\lambda l}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_0}{\ln\left(\frac{b}{a}\right)}$$

$$\therefore Z_0 = \sqrt{\frac{L}{C}} = \left(\frac{\mu_0}{2\pi}\ln(b/a) \cdot \cancel{2\pi}\frac{1}{2\pi\varepsilon_0}\ln(b/a)\right)^{\frac{1}{2}}$$

$$= \left(\left(\frac{1}{2\pi}\ln\left(\frac{b}{a}\right)\right)^2 \frac{\mu_0}{\varepsilon_0}\right)^{\frac{1}{2}}$$

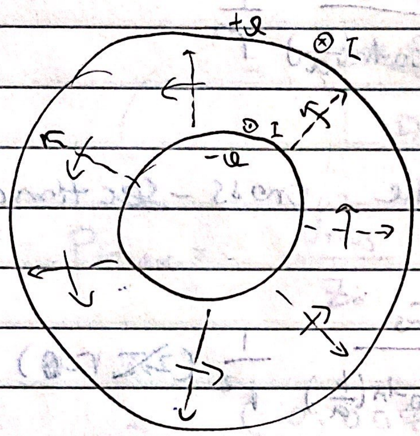
$$= \frac{1}{2\pi}\left(\frac{\mu_0}{\varepsilon_0}\right)^{\frac{1}{2}} \ln\left(\frac{b}{a}\right)$$

$$\rightarrow \quad 2\pi Z_0 \sqrt{\frac{\varepsilon_0}{\mu_0}} = \ln\left(\frac{b}{a}\right)$$

$$\therefore \quad \frac{b}{a} = \exp\left(2\pi Z_0 \sqrt{\frac{\varepsilon_0}{\mu_0}}\right)$$

$$= \exp\left(2\pi (50)\left(\frac{8.854\times10^{-12}}{4\pi\times10^{-7}}\right)^{\frac{1}{2}}\right)$$

$$= \boxed{2.3}$$



— $\underline{B}$

---- $\underline{E}$

⊙ propagation direction of EM wave.

$$\underline{S} = \cancel{\frac{\underline{E}\times\underline{B}}{\mu_0}} \quad \underline{E}\times\underline{H} = \frac{1}{\mu_0}\underline{E}\times\underline{B} = \frac{1}{\mu_0} EB\,\hat{\underline{z}}$$

↑ air-fill ↓ since $\underline{E} \perp \underline{B}$

$$S = \frac{1}{\mu_0}EB = \frac{1}{\mu_0}\left(\frac{\mu_0 I}{2\pi r}\right)\left(\frac{\lambda}{2\pi\varepsilon_0 r}\right)$$

$$= \frac{I\lambda}{4\pi^2\varepsilon_0 r^2}$$

$I = I_0\cos(\omega t - kx)$

$V = V_0\cos(\omega t - kx)$ $\because V \propto \lambda \therefore \lambda = \lambda_0\cos(\omega t - kx)$

$$\lambda = \frac{2\pi\varepsilon_0 V}{\ln\left(\frac{b}{a}\right)}$$

$$\therefore S = \frac{\cancel{2}\pi\varepsilon_0 V_0 I_0}{\cancel{4}_2\pi^2\varepsilon_0\ln\left(\frac{b}{a}\right)}\frac{1}{r^2}\underbrace{\cos^2(\omega t - kx)}_{\text{average to } \frac{1}{2}}$$

$$I_0 = \frac{V_0}{Z_0}$$

$$\rightarrow \langle S \rangle = \left[\frac{V_0^2}{4\pi Z_0 \ln(\frac{b}{a})} \right] \frac{1}{r^2}$$

$$\because \; V_{rms}^2 = \frac{V_0^2}{2}$$

$$\therefore \; \langle S \rangle = \frac{V_{rms}^2}{2\pi Z_0 \ln(\frac{b}{a})} \frac{1}{r^2}$$

integrate over the cross-sectional area

$$\int \langle S \rangle \, dA = \int_a^b \frac{V_{rms}^2}{\cancel{2\pi} Z_0 \ln(\frac{b}{a})} \frac{1}{r^2} (\cancel{2\pi} r) \, dr$$

$$= \frac{V_{rms}^2}{Z_0} \frac{1}{\ln(\frac{b}{a})} \int_a^b \frac{1}{r} \, dr$$

$$= \frac{V_{rms}^2}{Z_0} \left(\frac{\cancel{\ln(\frac{b}{a})}}{\cancel{\ln(\frac{b}{a})}} \right) = \frac{V_{rms}^2}{Z_0} = P$$

For $Z_0 = 50\Omega$, $b = 1cm = 0.01m$,

$$a = \frac{0.01m}{2.3} = 0.00435\,m$$

maximum E-field is

$$E_{max} = \frac{\lambda}{2\pi\varepsilon_0 a}$$, ~~where λ~~ is

$$\cancel{\lambda = \frac{2\pi\varepsilon_0 V_{max}}{\ln(\frac{b}{a})} = \frac{2\pi(8.854\times10^{-12})(3\times10^6\,V)}{\ln(2.3)}}$$

$$\cancel{= 2\times10^{-4}\,C/m}$$

$\to \quad \lambda_{max} = 2\pi\varepsilon_0 a E_{max} = 7.28 \times 10^{-7}$ C/m

$$V_{max} = \frac{\lambda_{max}}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$$

$$= \frac{\cancel{2\pi\varepsilon_0}\, a E_{max}}{\cancel{2\pi\varepsilon_0}}\ln\left(\frac{b}{a}\right) = a E_{max}\ln\left(\frac{b}{a}\right)$$

$$P_{max} = \frac{V_{rms}^2}{Z_0} = \frac{V_{max}^2}{2Z_0} = \frac{a^2 E_{max}^2 \ln^2\left(\frac{b}{a}\right)}{2Z_0}$$

$$= \frac{(0.00435)^2 (3\times10^6)^2 (2.\cancel{3})^2 \ln(2.3)^2}{2(50)}$$

$$= \boxed{9\times10^6\ \text{W}} \qquad = 1.18\times10^6\ \text{W}$$

12×10^6 W

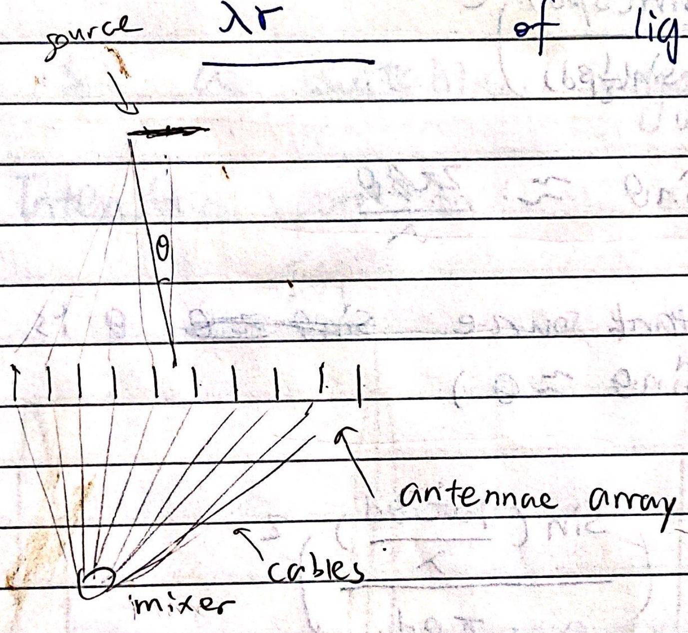
$$a E_0 = V_{rms}\left(\frac{\sqrt{2}}{\ln\left(\frac{b}{a}\right)}\right)$$

$$V_{rms}^2 = \frac{a^2 E_0^2 \ln^2\left(\frac{b}{a}\right)}{2}$$

9. Fraunhofer ~~diffraction~~ approximation is the approximation that the phase of the light at the observation point is a linear function of position for all points in the diffraction operature.

Fraunhofer condition is

$\frac{D^2}{\lambda r} < 1$ where λ is the wavelength of light.



Fraunhofer approximation: $\beta = \frac{2\pi}{\lambda} d \sin\theta$

$$U(\beta) = \int_{-\infty}^{\infty} u(x) e^{i\beta x} dx$$

$$= \int_{-\infty}^{\infty} \left(\sum_{m=1}^{10} \delta(x - md) \right) e^{i\beta x} dx$$

$$= \sum_{m=1}^{10} e^{i\beta md} = e^{i\beta d} + e^{2i\beta d} + \cdots + e^{10 i\beta d}$$

$$= \frac{e^{i\beta d}(1-e^{10i\beta d})}{1-e^{i\beta d}} = \frac{\cancel{e^{i\beta d}}\,\cancel{e^{5i\beta}}\, e^{i\beta d} e^{5i\beta d}(e^{5i\beta d}-e^{-5i\beta d})}{e^{\frac{i\beta d}{2}}(e^{\frac{i\beta d}{2}}-e^{-\frac{i\beta d}{2}})}$$

$$= e^{\frac{11 i}{2}\beta d}\,\frac{e^{5i\beta d}-e^{-5i\beta d}}{e^{\frac{i\beta d}{2}}-e^{-\frac{i\beta d}{2}}} = e^{\frac{11}{2}i\beta d}\left(\frac{2i\sin(5\beta d)}{2i\sin(\frac{1}{2}\beta d)}\right)$$

$\cancel{I(\theta)}$ $I(\beta) \propto |U(\beta)|^2$

$$= \left(\frac{\sin(5\beta d)}{\sin(\frac{1}{2}\beta d)}\right)^2.$$

$$\beta = \frac{2\pi}{\lambda}\cancel{d}\sin\theta \simeq \frac{2\pi\cancel{d}\theta}{\lambda}$$

(for distant source ~~$\sin\theta \simeq \theta$~~ θ is small, $\sin\theta \simeq \theta$)

$$\therefore I(\theta) \propto \left(\frac{\sin(\frac{10\pi\theta d}{\lambda})}{\sin(\frac{\pi\theta d}{\lambda})}\right)^2$$

first minima $\frac{10\pi\theta d}{\lambda} = \pm\pi$

$$\therefore \theta_{min} = \pm\frac{\lambda}{10d}$$

Angular width of resulting pattern is

$$\Delta\theta = \frac{\lambda}{10d} - \left(-\frac{\lambda}{10d}\right) = \boxed{\frac{\lambda}{5d}}$$

∴ ~~a~~ the larger the spacing d, the narrower the peak

the angular resolution needed for astronomical objects is 0.001

∴ let $\frac{\lambda}{5d} = 0.001$

$d = \frac{\lambda}{0.005} = \frac{0.21m}{0.005} = \boxed{42m}$

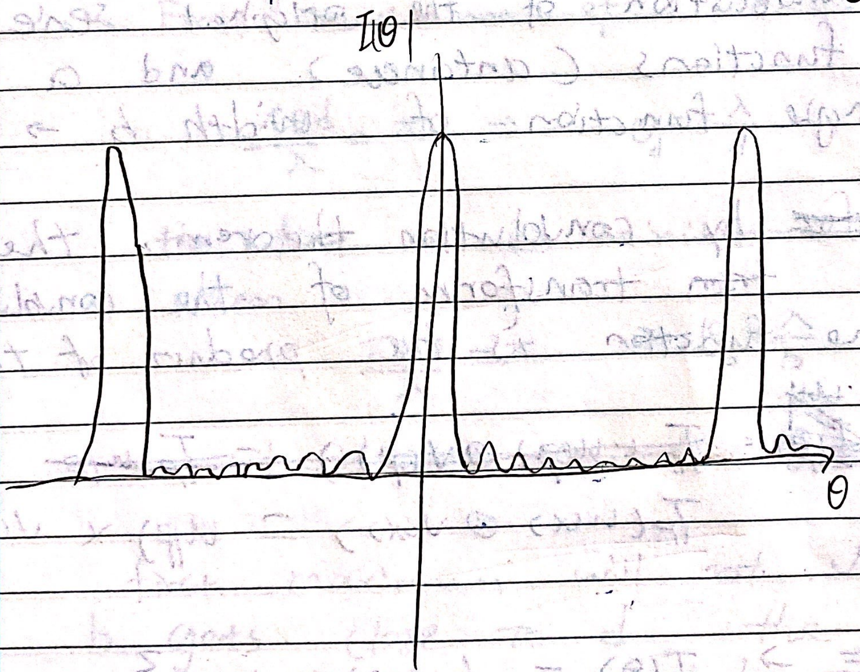
is a suitable suggestion.

$d = \lambda$

Intensity pattern:

$d \sin\theta = (1)\lambda$

$\sin\theta = 1 \rightarrow d = \lambda$



→ The principle direction of reception can be changed without moving the array by making the ~~ate anta~~ antennae pointing at an angle θ with respect to the normal.

With the reflectors, the reception pattern would be multiplied by an envelope functio that has the form of a sinc function determined by the value of b.

~~I~~ the ~~te~~ transmission function now is the convolution of the original series of delta functions (antanae) and a rectangle function of width b → rect(b)

So ~~the fur~~ by convolution theorem, the fourier ~~tran~~ transform of the convolution of two function is the product of them

$$\sqrt{I(\beta)} \cancel{I(\beta)} = \cancel{T_F(u(\beta) \otimes v(\beta)) = T_F(u(\beta}$$

$$T_F(u(x) \otimes v(x)) = u(\beta) \times v(\beta)$$

$$\cancel{u(\beta)} \Rightarrow I(\theta) = (u(\theta) \times v(\theta))^2$$

$$v(\theta) = \int_{-\infty}^{\infty} \text{rect}(b)\, e^{i\beta x}\, dx = \int_{-b/2}^{b/2} e^{i\beta x}\, dx$$

$$= \frac{1}{i\beta}\left(e^{\frac{i\beta b}{2}} - e^{-\frac{i\beta b}{2}}\right)$$

$$= \frac{2i\sin\left(\frac{\beta b}{2}\right)}{i\beta} = \cancel{d}\, b\,\frac{\sin\left(\frac{\beta b}{2}\right)}{\frac{\beta b}{2}} \propto \operatorname{sinc}\left(\frac{\beta b}{2}\right)$$

$$= \operatorname{sinc}\left(\frac{\pi \theta b}{\lambda}\right)$$

$$\therefore I(\theta) \propto \left(\frac{\sin\left(\frac{10\pi\theta d}{\lambda}\right)}{\sin\left(\frac{\pi\theta d}{\lambda}\right)}\right)^2 \left(\frac{\sin\left(\frac{\pi\theta b}{\lambda}\right)}{\left(\frac{\pi\theta b}{\lambda}\right)}\right)^2$$

First order ~~maximum of~~ $I(\theta)$ is when

$$\frac{\pi\theta d}{\lambda} = \pi \rightarrow \theta_d = \frac{\lambda}{d}$$

the first ~~minimum~~ minimum of envelope is when

$$\frac{\pi\theta b}{\lambda} = \pi \rightarrow \theta_b = \frac{\lambda}{b}$$

$\because b < d \quad \therefore \quad \cancel{\theta_d > \theta_b} \quad \theta_d < \theta_b.$

first maximum will not vanish, but if b gets close to d the magnitude of first minimum will be small

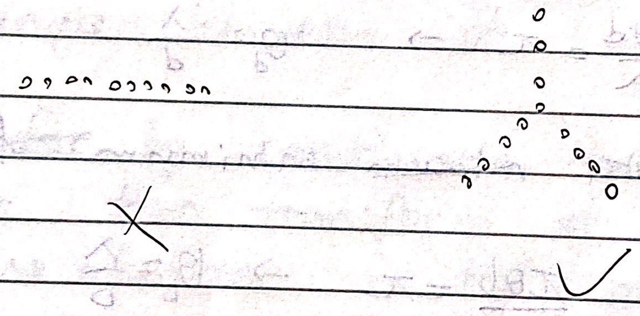
$\longrightarrow$ $\underline{b = 0.8m}$ is a good choice the first minimum of $I(\theta)$ is $\theta_{min} = \frac{\lambda}{10d}$.

$\because d > b \quad \therefore \frac{\lambda}{10d} < \frac{\lambda}{b}$

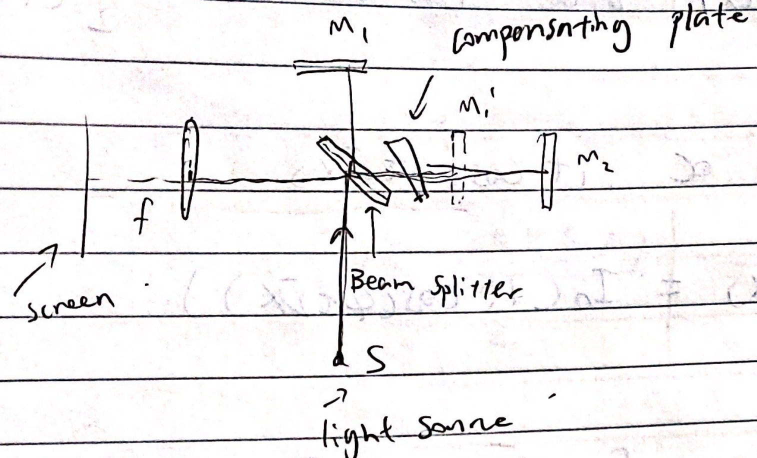
$\therefore$ The envelope does not affect the width of principle maximum

$\therefore$ we still have $\Delta\theta = \frac{\lambda}{5d}$

→ To improve the angular resolution we can design the array to be not linear but triangular



10.



→ The light source send light to the beam splitter, which split the ~~beam~~ incident beam in half.

→ The compensating plate makes sure that the ~~splitted~~ two light beams travel with the same phase inside the glass.

→ M_1' is the image of mirror M_1 with respect to the beam splitter, M_1 and M_2 form two images of the light source and the images give the interference pattern

amplitude $u(x) = u_0 + u_0 e^{i\delta}$

$\propto 1 + e^{i\delta}$

at the centre of the field of view the phase difference $\delta = \frac{2\pi}{\lambda} \cdot 2x = \frac{4\pi x}{\lambda} = 4\pi\bar{\nu}x \quad (\bar{\nu} = \frac{1}{\lambda})$

$\therefore u(x) \propto 1 + e^{i(4\pi\bar{\nu}x)} = e^{i2\pi\bar{\nu}x}\left(e^{i(2\pi\bar{\nu}x)} - e^{-i(2\pi\bar{\nu}x)}\right)$

$= 2e^{i2\pi\bar{\nu}x}\cos(2\pi\bar{\nu}x)$

$$I(x) \propto |u(x)|^2 \propto \cos^2(2\pi\bar{\nu}x) = \frac{1}{2}(1+\cos(4\pi\bar{\nu}x))$$

$$\rightarrow \quad I(x) \propto 1+\cos(4\pi\bar{\nu}x)$$

$$\rightarrow \quad I(x) = I_0(1+\cos(4\pi\bar{\nu}x))$$

Equal intensity for $\bar{\nu}_0$ and ~~$\bar{\nu}_0 + \delta\bar{\nu}$~~ $\bar{\nu}_0 + \delta\nu$

so total intensity

$$I(x) = I_0[1+\cos(4\pi\bar{\nu}_0 x) + 1 + \cos(4\pi(\bar{\nu}_0+\delta\bar{\nu})x)]$$

$$= I_0[2+\cos(4\pi\bar{\nu}_0 x) + \cos(4\pi(\bar{\nu}_0+\delta\nu)x)]$$

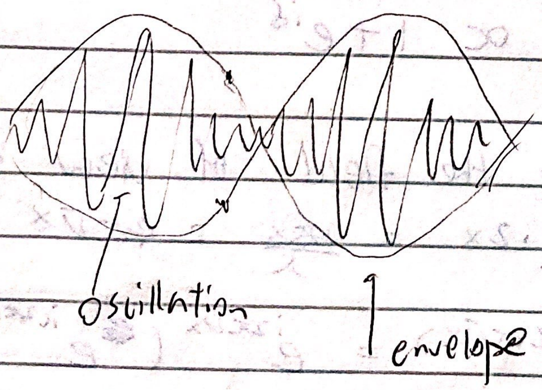
$$= I_0[2 + 2\cos(4\pi x \frac{2\bar{\nu}_0+\delta\nu}{2})\cos(4\pi x \frac{\delta\nu}{2})]$$

$$\simeq 2I_0[1+\cos(4\pi x\bar{\nu}_0)\cos(2\pi x\delta\nu)]$$

oscillation envelope

$\therefore$ at x, $I_{max} = I_0(2+2\cos(2\pi x\delta\nu))$

$I_{min} = I_0(2-2\cos(2\pi x\delta\nu))$



Visibility

$$V = \frac{I_{max}-I_{min}}{I_{max}+I_{min}}$$

$$V = \left| \frac{\cancel{2I_0}(\cancel{1} + \cos(2\pi x\,\delta\nu) - \cancel{1} + \cos(2\pi x\,\delta\nu))}{\cancel{2I_0}(1 + \cos(2\pi x\,\delta\nu) + 1 - \cos(2\pi x\,\delta\nu))} \right|$$

$$= \left| \frac{2\cos(2\pi x\,\delta\nu)}{2} \right| = \boxed{|\cos(2\pi x\,\delta\nu)|}$$

$\cos^2(2\pi x\,\delta\nu)$?

Consider the Gaussian doppler shift:

$\lambda = 656\,nm$, $\bar{\nu} = \frac{1}{\lambda}$

Full width Half maximum (FWHM) of doppler boardening is

$$\Delta\nu_{\frac{1}{2}} = \sqrt{\frac{8k_B T \ln 2}{Mc^2}}\,\bar{\nu}$$

The fourier cosine transform of the oscillation part of the intensity distribution is the power spectrum

the envelope function can be modeled as

$$f(x) = \exp(-a^2x^2)$$

$$S(\Delta\bar{\nu}) \propto \int_0^\infty \exp(-a^2x^2)\cos(4\pi\Delta\bar{\nu}x)\,d\Delta\bar{\nu})$$

$$\propto \exp\left(-\frac{(4\pi\Delta\nu)^2}{4a^2}\right)$$

→ The FWHM is

$$\frac{1}{2} = -\frac{4\pi^2 \Delta\nu_{\pm}^2}{a^2}$$

$$\to \Delta\nu_{\frac{1}{2}} = \Delta\nu_{+} - \Delta\nu_{-} = 2 \times \frac{a\sqrt{\ln 2}}{2\pi}$$

$$= \frac{a\sqrt{\ln 2}}{\pi}$$

From the visibility graph roughly at $x = 5\text{mm}$ the envelope function $f(x)$ goes to 0

$$\therefore \quad \exp(-a^2 (5\text{mm})^2) \approx 0.05$$

$$a = 346 \text{ m}^{-1}$$

$$\to \Delta\nu_{\frac{1}{2}} = \frac{a\sqrt{\ln 2}}{\pi} = 91.7 \text{ m}^{-1}$$

$$\Delta\nu_{\frac{1}{2}} = \sqrt{\frac{8k_BT\ln 2}{Mc^2}}\,\bar{\nu}$$

$$\to T = \frac{(\Delta\nu_{\frac{1}{2}}\lambda)^2 c^2}{8k_B\ln 2} M$$

$M \approx$ mass of hydrogen $\approx m_p = 1.67 \times 10^{-27}$ kg

$$\therefore \quad T = \frac{(91.7 \times 656 \times 10^{-9})^2 (3.0 \times 10^8)^2 (1.67 \times 10^{-27})}{8 \times 1.38 \times 10^{-23} \ln 2}$$

$$\approx \boxed{7100\text{K}}$$

A2 June 2015

Toturial Notes

1. ✓

2. ✓

3. $\vec{\nabla}\times\vec{B} = \mu_0\mu_r\vec{J} + \cancel{\varepsilon_0\varepsilon_r}\,\varepsilon_0\varepsilon_r\mu_0\mu_r\frac{\partial\vec{E}}{\partial t}$

$\vec{\nabla}\times\vec{B} = \mu_0\mu_r\vec{J}$, $\vec{J} = \sigma\vec{E}$

$\vec{\nabla}\times\vec{B} =$

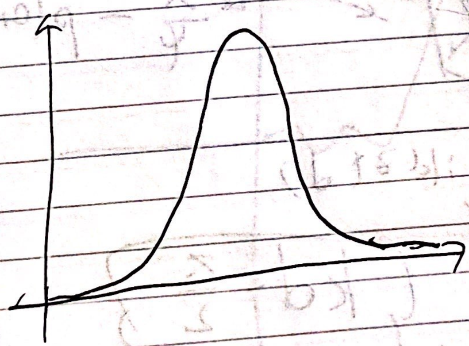
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

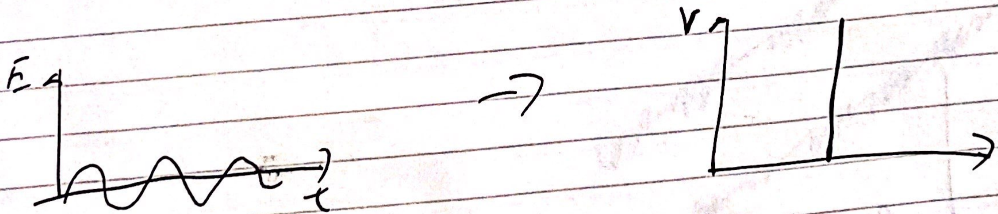
$\vec{\nabla}\times\vec{B} = \mu\vec{J}_f + \mu\varepsilon\frac{\partial\vec{E}}{\partial t}$

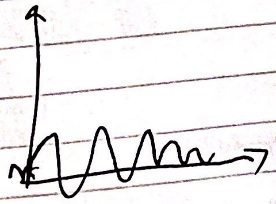
4. Hecht optic (polarisation & interferences)

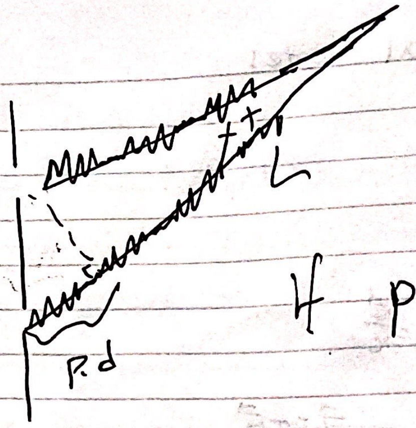
correlated → every time they have same frequency

same phase difference up any time





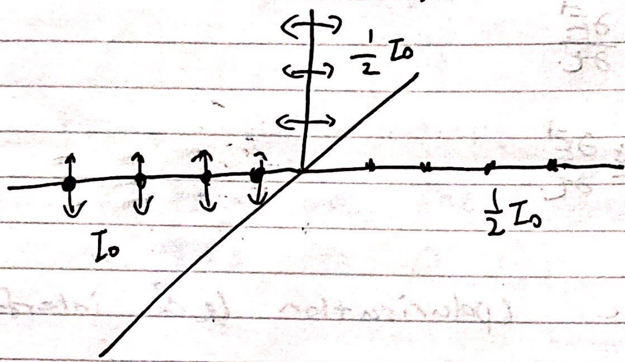




If $p.d > L$

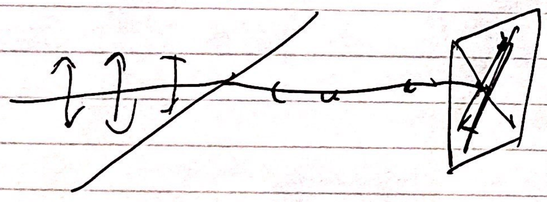
coherence length.

Polarising BS:



Circular:

Need $\frac{\pi}{2}$

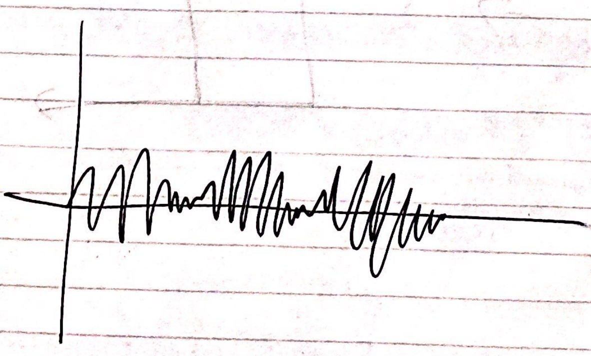


$\frac{\lambda}{4}$ - plate.

e^{ikz} , $e^{ik(z+d)}$

$\boxed{kd = \frac{\pi}{2}}$

e^{ikz}

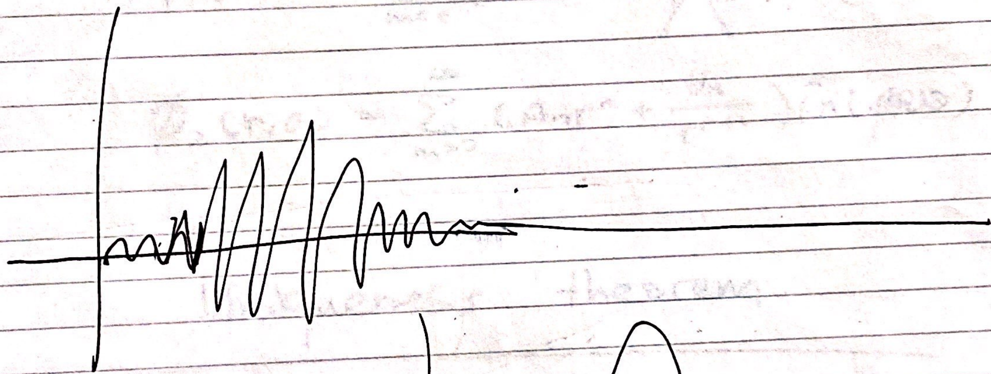


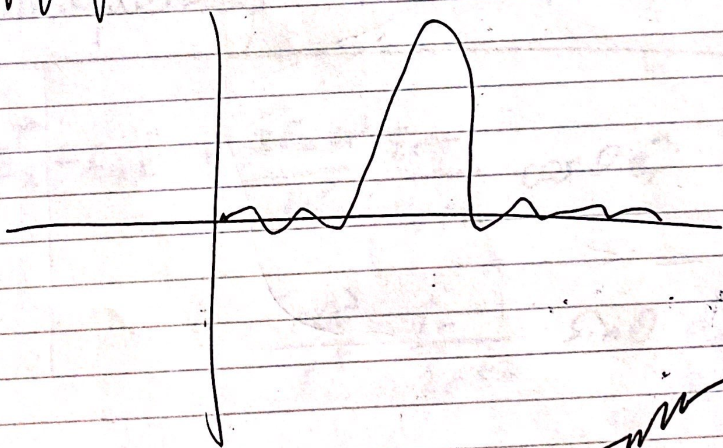
OPD >> l_c

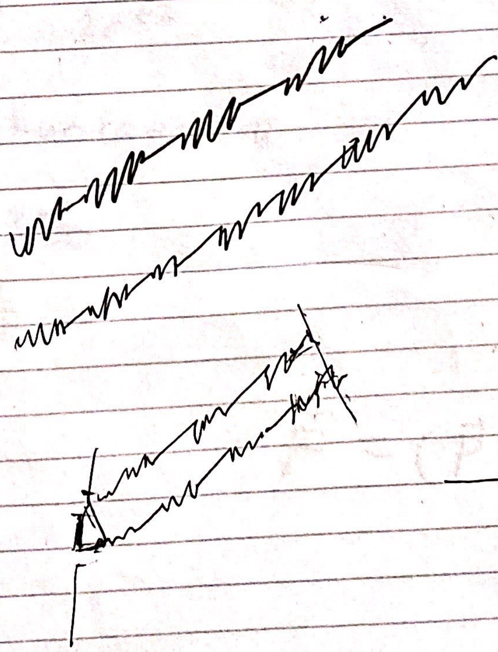
e^{ikz}

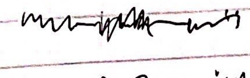
pulse

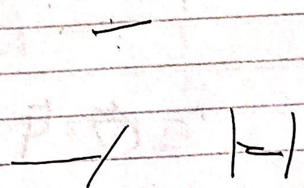
↳ Not necessarily regularly.





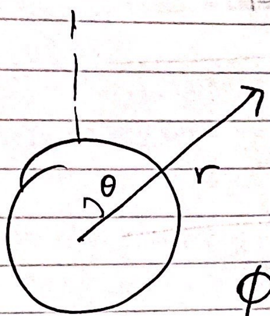






L > l_c

$\phi = -E_0 z \qquad r \to \infty$



$\phi_2 = -E_0 r\cos\theta$

$r \to 0$

ϕ_1 is finite

$r = r_0 \qquad \phi_1(r_0,\theta) = \phi_2(r_0,\theta)$

$P_0(\cos\theta) = 1$

$P_1(\cos\theta) = \cos\theta$

$\cancel{\phi_1 = (r,\theta) =}$

$$\phi_1(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \cancel{\frac{B_n}{r^{n+1}}}\right)P_n(\cos\theta)$$

$$\phi_2(r,\theta) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Uniqueness theorem.

$$\vec{E}_2 = E_0\hat{k} + \frac{2E_0 a^3}{r^3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\cos\theta\,\hat{e}_r + \frac{E_0 a^3}{r^3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\sin\theta\,\hat{e}_\theta$$

maximum enhancement

$|E_2|_{max} \sim 3E_0$

$\vec{p} = \frac{4}{3}\pi r_0^3 \vec{P}$

$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E}$

$$U = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p_1}\cdot\vec{p_2} - 3(\vec{p_1}\cdot\hat{r})(\vec{p_2}\cdot\hat{r})\right]$$

$2\times10^6 W$

$(1.8\times10^6 W)$

$$I(\theta) = \frac{\sin^2 5\delta}{\sin^2\frac{\delta}{2}}$$

$\frac{\lambda}{d} = 1$ $\rightarrow d \leq \lambda$.

$$I(\theta) = \left(\frac{\sin 5\delta}{\sin\frac{\delta}{2}}\right)^2 \frac{\sin^2\delta_1}{\delta_1^2}. \qquad \delta_1 = \frac{\pi b \sin\theta}{\lambda}.$$

$\theta \rightarrow \qquad \delta_1 = \pi.$

$$\sin\theta = \frac{\lambda}{b} = \frac{\lambda}{d}.$$

$\rightarrow b = d$?

put more detectors ?

$$\boxed{V = e^{-\frac{4\pi^2 x^2}{a^2}}}$$

$$\rightarrow f(v) = e^{-\frac{mv^2}{2k_BT}}.$$

$$\rightarrow \frac{\Delta v}{v} = \frac{v}{c}.$$