Problem Set 4. **General Relativity, HT17**

Cosmology

RECALL:

Dynamical evolution equation:

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E \text{ (Energy Form)} = -\frac{c^2}{a^2} \text{ (Curvature Form)} = -kc^2 \text{ (FRW Form)}$$

FRW metric, R dimensions of length, $k = 0, \pm 1$:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - kr^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Curvature form, with $R_0 = 1$ and R dimensionless, a^2 positive or negative:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1 - r^2/a^2} + R^2 r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

1a.) *A big bang in empty space??* Show that the dynamical field equation for the scale factor $R(t)$ for an *empty space* $\rho = 0$ leads to an FRW metric of the form

$$-d\tau^2 = -dt^2 + \frac{t^2 dr^2}{1 + r^2} + r^2 t^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Use $c = 1$ for this problem!

1b.) But surely empty space must be Minkowski spacetime! Though this metric does not look static, there *must* be a coordinate transformation that turns this metric into a static Minkowski form. In other words, we ought to be able to find two functions, s and T,

$$s = s(r, t), \quad T = T(r, t) \qquad \text{or equivalently} \qquad r = r(s, T), \quad t = t(s, T)$$

that transform the metric of part (1a) into an old friend:

$$-d\tau^2 = -dT^2 + ds^2 + s^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

By inspection, we must have

$$s(r, t) = rt.$$

Why "by inspection?" Explain convincingly why it is as simple as this, in just one to two sentences.

1c.) Using $s = rt$,and by then demanding that the coefficient of dT^2 be -1 after the coordinate change, show that $T = \sqrt{s^2 + t^2}$ (up to an additive function of s which you may safely discard), and thereby derive the second coordinate transformation:

$$T = t\sqrt{1 + r^2}.$$

Give the explicit functional forms for $r(s,T)$ and $t(s,T)$.

1d.) Complete the full coordinate transformation for $d\tau^2$ and verify in detail that the Minkowski metric emerges. You may find it to your advantage to express $\partial t/\partial s$ and $\partial r/\partial s$ in terms of r and t, and $\partial r/\partial T$ in terms of $\partial t/\partial T$, before you begin.

2.) *A radiation/matter universe.* Repent now, or face a calculation for an Eternal, Infinite Universe of Fire and Brimstone! Well...radiation and matter, actually. Much the same. Anyway, it's too late to repent, the calculation begins. Solve the dynamical cosmological equation (Energy form) for $R(t)$ for the case of an arbitrary mixture of radiation and non-relativistic matter in a spatially flat universe ($E = 0$). Assume a current energy density of $\rho_{\gamma 0}c^2$, and a matter density ρ_{m0}. In terms of the "inferno ratio" $I = \rho_{\gamma 0}/\rho_{m0}$, you should find

$$(R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3\Omega_{m0}^{1/2} H_0 t}{2}$$

3.) *Drifting in an E-dS universe.* On its maiden voyage to explore an Einstein-de Sitter universe, the starship *Titanic* strikes an iceberg comet within moments of this universe's birth and the ship must then drift at a constant velocity for all eternity. (Don't ask where the early universe iceberg comes from. It's the *Titanic,* OK?) The question is, how far does the ship get? More precisely, what is the largest comoving coordinate distance r the ship attains if it starts at $r = 0$ at $R = 0$? The metric is standard E-dS:

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2 dr^2 + R^2 r^2 d\Omega^2$$

$R(t)$ is the usual scale factor. We will use $d\varpi = R dr$ for the proper physical distance. Other standard notation and results for reference: t_0 is the current age of the universe, $R = (t/t_0)^{2/3}$ for E-dS, $H_0 \equiv \dot{R}_0$.

3a.) A constant *Titanic* velocity is *not* just $d\varpi/dt = V_0$ (some constant), because $d\varpi/dt$ measures the velocity relative to expanding comoving embedded observers who are all trying to run away from the *Titanic.* Show that if the *Titanic* has a measured velocity V at some instant when it passes one such observer, then when the ship overtakes another observer, a distance $d\varpi$ farther away, the velocity V' this observer measures is

$$V' = V - \frac{\dot{R}\, d\varpi}{R}\left(1 - \frac{V^2}{c^2}\right)$$

to first order in $d\varpi$. (You will need the special relativity velocity addition formula in addition to Hubble's law.) From this equation, show that the rate at which the measured $V = d\varpi/dt$ is changing with cosmic time is given by the differential equation

$$\frac{\dot{V}}{V(1 - V^2/c^2)} = -\frac{\dot{R}}{R}$$

where as usual $\dot{V} = dV/dt$. Solve this equation and show that with $V = V_0$ at $t = t_0$, the solution is

$$\frac{V}{\sqrt{1 - V^2/c^2}} = \frac{U_0}{R}$$

where U_0 is the spatial component of the *Titanic* 4-velocity at time t_0. (N.B.: In this problem, subscript 0 will always denote "current time," not the 4-vector time-like component.)

3b.) The result of 3a.) says that the product $\mathcal{P}R$ is constant, where $\mathcal{P}$ is the spatial 4-momentum. Show that, in this form, this is equivalent to an adiabatic expansion, either of photons (extreme relativistic particles), or classical particles (classical nonrelavistic gas). [Cosmic adiabatic expansion for photons correponds to the temperature T obeying $TR \sim constant$, while for a classical gas adiabatic behaviour is $T\rho^{-2/3} \sim constant$, where ρ is the mass density.]

3c.) Solve the equation $d\varpi/dt = V(R)$ for the comoving coordinate r in an E-dS universe to obtain for our problem:

$$r(R) = \frac{c}{H_0}\int_0^R \frac{dx}{[x + c^2x^3/U_0^2]^{1/2}}$$

and show therefore that as $R \to \infty$, the comoving coordinate $r \to r_{max}$, where

$$r_{max} = \frac{3.708\sqrt{U_0 c}}{H_0}$$

The numerical factor is

$$3.708 = \int_0^\infty \frac{dy}{(y + y^3)^{1/2}}$$

Even after an infinite amount of time, and even though this universe is decelerating, a constant velocity *Titanic* only reaches a finite value of comoving coordinate r for any finite U_0. But the ship can reach arbitrarily large r, if V_0 approaches the speed of light.

4.) *Starlight energy density.* In a simple Euclidian universe, with n galaxies per unit volume each with Luminosity L, the net flux of starlight received at Earth from sources a distance r away is

$$dF = \frac{L}{4\pi r^2} \times n \times 4\pi r^2\, dr = Ln\, dr$$

Integrating this from 0 to D and dividing by c (to convert to a radiation energy density) gives LnD/c. If $D \to \infty$ this is infinite! This is known as Olber's Paradox. It is not a problem for the Big Bang Theory because the universe began at some finite time in the past. Let's see how it works.

Consider an E-dS Universe, $R(t) = (t/t_0)^{2/3} = (3H_0t/2)^{2/3}$. We assume that the current number density of galaxies is n_0, that galaxy number is conserved, and that all galaxies have the same luminosity L at all times. Prove that the current energy density is given by the expression

$$Ln_0 \int_0^{t_0} R(t)\, dt$$

What is the value of this integral in terms of L, n_0, and H_0?

5a.) *Once around the Universe?* For a closed, matter-dominated universe with current mass density ρ_0, show that

$$H_0^2(\Omega_{M0} - 1) = c^2/a^2$$

where

$$\Omega_{M0} = \frac{8\pi G\rho_0}{3H_0^2}$$

5b.) Consider the path of a photon (null geodesic) through this universe. With η defined from §8.5 in the notes:

$$R = \frac{1 - \cos\eta}{2(1 - \Omega_{M0}^{-1})}$$

show that

$$\eta = \sin^{-1}(r/a)$$

where r follows the proper coordinate of the photon. In other words, r goes from zero to a and back again to zero (and R goes from zero to a maximum to zero again), as η advances by 2π! How many times could a photon travel around such a universe? Could you see the back of your head with a powerful telescope, looking straight ahead?