Birkhoff Theorem and Radiation

Affine connection for diagonal $g_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}}\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right) \quad \text{NO SUM OVER } \lambda$$

Ricci Tensor:

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\kappa\eta} - \frac{\Gamma^{\eta}_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^{\eta}} \quad \text{FULL SUMMATION}$$

N.B.: In this problem set, we will set $c = 1$.

1a.) The Birkhoff theorem states that outside of a spherical distribution of matter, the metric tensor must be independent of time and equal to the Schwarzschild metric. A corollary is that in the hollow of a spherical distribution of matter, the metric tensor is Minkowski space-time. These are the Newtonian analogues of a point mass $1/r$ potential anywhere outside a spherical distribution of matter, and the vanishing of the gravitational field inside a spherical cavity in a spherical system. Birkhoff's theorem is critical to formulating cosmology.

To prove the theorem is straightforward but painful, because we need to calculate the Ricci tensor $R_{\mu\kappa}$, and that is always a nuisance. But Birkhoff's theorem is very important, so here we will go through its main step. (You can then fill in the rest at your leisure using Weinberg as your guide, if you so choose.)

Consider the line element for a general time-dependent spherical system,

$$-d\tau^2 = -B(r,t)dt^2 + A(r,t)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \equiv g_{\mu\nu}dx^{\mu}dx^{\nu}$$

Evaluating the nonvanishing affine connection components Γ^{a}_{bc} from the metric tensor $g_{\mu\nu}$, it is not difficult to show that we recover precisely the set we found for Schwarzschild in equation (252)—do this later if you'd like convincing—plus three others that were zero before. Do this bit now. Show that, in particular,

$$\Gamma^{r}_{rt} = \frac{\dot{A}}{2A}, \qquad \Gamma^{t}_{rr} = \frac{\dot{A}}{2B} \qquad \Gamma^{t}_{tt} = \frac{\dot{B}}{2B}$$

where we will use notation $\dot{A}$ for a time derivative and A' for an r derivative. These three are the new members of our affine connection set.

1b.) We will now show that $R_{tr} = -\dot{A}/rA$, which sure looks simple but in fact involves a large cancellation. The point now is that since all the $R_{\mu\kappa}$ terms must vanish in a vacuum, $\dot{A} = 0$, and A cannot depend on time. The other components of the Ricci tensor then all revert back to their Schwarzschild forms. (We won't show this explicitly, only because it is a long and dull exercise, but it is not particularly difficult). Thus, B doesn't depend on time either. This is Birkoff's theorem.

Using the Ricci tensor above, show that the first two groupings

$$\frac{1}{2}\frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda}$$

cancel one another out precisely. This is progress. (g is the determinant of $g_{\mu\nu}$.)

1c.) Show next that for $R_{\mu\kappa} = R_{tr}$,

$$\Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2}\frac{\partial \ln|g|}{\partial x^\eta} = -\frac{\dot{A}}{rA}$$

(Use [252] from the notes for any Γ's you need, or Weinberg [8.1.11].) You will find that everything cancels once again, except for one final term in the $\ln|g|$ derivative, shown on the right. With $\dot{A} = 0$, Birkhoff's theorem follows relatively easily, as the remaining $R_{\mu\kappa} = 0$ equations reduce to the Schwarzschild problem.

2.) *Desert island GR.* Here we will construct a linear, weak field theory gravity from scratch. Then we will construct GR from scratch! (Well, practically.)

Imagine that it is 1912. Minkowski has formulated the concept of his spacetime geometry (1908). Einstein has had his happy (1907) Equivalence Principle thought, and has just understood that gravity is a Riemannian geometric theory of a distorted Minkowski spacetime, and that the name of the game is to relate the coordinate derivatives of $g_{\mu\nu}$ to $T_{\mu\nu}$. But he knows nothing more. Let's help him out.

2a.) Our weak gravity field equation will need, on the left (curvature) side, a sum of second derivatives of $g_{\mu\nu}$. More conveniently, we use derivatives of the small quantity $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Not only is the background spacetime geometry flat Minkowski, our coordinates are very close to Cartesian. So, with $h \equiv h^\rho{}_\rho$, there are but five combinations that could possibly appear:

$$\Box h_{\mu\nu},\ \partial_\mu\partial_\nu h,\ (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu),\ \eta_{\mu\nu}\Box h,\ \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda}$$

($\Box \equiv \partial^\rho\partial_\rho$. We use the handy notation $\partial^\mu = \partial/\partial x_\mu$, $\partial_\mu = \partial/\partial x^\mu$, and raise and lower indices on $h_{\mu\nu}$ with $\eta^{\rho\mu}$.) Justify this statement and explain fully.

2b.) We accordingly search for an equation of the form:

$$\Box h_{\mu\nu} + \alpha(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h + \eta_{\mu\nu}(\gamma\Box h + \delta\partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

where $\alpha, \beta, \gamma, \delta$ and C are constants to be determined. You remember, of course, the stress tensor $T_{\mu\nu}$, now in Newtonian guise. We demand that $\partial^\mu T_{\mu\nu} = 0$ as an identity. What is the reason for this? Show that $\alpha = -1, \delta = 1, \gamma = -\beta$ follow:

$$\Box h_{\mu\nu} - (\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu) + \beta\partial_\mu\partial_\nu h - \eta_{\mu\nu}(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}) = CT_{\mu\nu}$$

2c.) By taking the trace of this last equation and using $T_{00} \gg T_{ii}$ (valid in the Newtonian limit — why?), show that

$$\partial_\rho\partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2}\Box h - \frac{CT_{00}}{2}$$

Be careful with signs and up-down indices.

2d.) Taking the static Newtonian limit of the (2b) final equation, show that

$$\nabla^2 h_{00} + \frac{1-\beta}{2}\nabla^2 h = \frac{C}{2} T_{00}$$

where ∇^2 is the usual Laplacian operator. Explain why this implies $\beta = 1$ and $C = -16\pi G$:

$$\Box h_{\mu\nu} - (\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu) + \partial_\mu \partial_\nu h - \eta_{\mu\nu}(\Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}) = -16\pi G T_{\mu\nu}$$

Compare this with equations (339), (340) and (348) in the notes and comment.

2e.) Given that the Ricci tensor $R_{\mu\nu}$ and $g_{\mu\nu} R^\rho{}_\rho$ are the only second rank tensors that are linear in the second derivatives of the metric tensor $g_{\mu\nu}$ when the curvature is weak, explain why the general field equations must take the form

$$R_{\mu\nu} - \frac{g_{\mu\nu} R}{2} = -8\pi G T_{\mu\nu}$$

where $R \equiv R^\rho_\rho$. Notice: not a Bianchi identity in sight. If Einstein could only have seen this in 1912.

3a.) *Coordinate sinuosities, the speed of gravitational radiation, and the harmonic gauge.* Recall the linear fully covariant curvature tensor:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right).$$

For a plane wave of the form $h_{\mu\nu} = A_{\mu\nu} \exp(ik_\rho x^\rho)$ travelling in vacuum, show that

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\mu k_\nu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

and that the linear vacuum field equation is

$$k_\kappa k^\rho \bar{h}_{\rho\mu} + k_\mu k^\rho \bar{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h/2$ and $k^2 = k^\rho k_\rho$. We do *not* yet assume that $k^2 = 0$, but shall try to deduce this.

3b.) Show that if $k^2 \neq 0$ then $R_{\lambda\mu\nu\kappa} = 0$. Yikes! No curvature. *A mere coordinate sinuosity propagating at the speed of thought.*

3c.) Finally, show that if we consider only disturbances propagating at the speed of light, then we must have $k^\rho \bar{h}_{\rho\sigma} = 0$. In other words, the harmonic gauge condition *must* be satisfied. You want gravitational radiation to travel at the speed of light and to actually produce curvature? No choice: use a harmonic gauge.

4.) *Radiation from a parabolic fly by.* The Peters—Mathews formula for the time-averaged gravitational wave luminosity of a binary system in an elliptical orbit (with semi-major axis

a, masses m_1 and m_2, $M \equiv m_1 + m_2$, eccentricity e) is given by (c is now back in the equation):

$$\langle L_{GW} \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5} \left[\frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{(1-\epsilon^2)^{7/2}} \right]$$

It's derivation is outlined in the notes (§7.7), or you may take it on perfect good faith from your humble instructor, however startling it may seem. Using this result, show that the total gravitational wave energy emitted by a single parabolic encounter between two bodies is

$$E_{GW} = \frac{85\pi\sqrt{2}}{24} \frac{G^{7/2} M^{1/2} m_1^2 m_2^2}{c^5 b^{7/2}}$$

where b is radius of closest approach. Recall that for a parabolic orbit, the radius r and aximuth ϕ are related by $r(1 + \cos\phi) = L$, where $L = a(1-\epsilon^2)$ is the "semi-latus rectum," a constant. A parabola corresponds to the $\epsilon \to 1$ limit, with $a(1-\epsilon^2) = L$ finite. You may find the material in §6.7.1 useful.