Problem Set 2. **General Relativity, HT17**

Schwarzschild Astrophysics

Reference:

Schwarzschild metric:

$$-c^2 d\tau^2 = -c^2 dt^2 \left(1 - \frac{2GM}{rc^2}\right) + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Affine connection for diagonal g_{ab}:

$$\Gamma^a_{ba} = \Gamma^a_{ab} = \frac{1}{2g_{aa}} \frac{\partial g_{aa}}{\partial x^b} \quad (a = b \text{ permitted, NO SUM})$$

$$\Gamma^a_{bb} = -\frac{1}{2g_{aa}} \frac{\partial g_{bb}}{\partial x^a} \quad (a \neq b, \quad \text{NO SUM})$$

$$\Gamma^a_{bc} = 0, \quad (a, b, c \quad \text{distinct})$$

This is a long Problem Set, designed to give you lots of practise working with the GR equations! Leave time, or pick and choose problems with your tutor.

1.) *What is wrong with the following argument?* Let V^ρ be a contravariant vector. Consider

$$V^\rho_{;\mu;\nu} - V^\rho_{;\nu;\mu}$$

Go into local freely falling inertial coordinates. Then, the semi-colon covariant derivatives just become ordinary derivatives. Mixed derivatives commute, so this is just zero. But if a tensor is zero in one frame, it is zero in any frame! Hence, the above is identically zero.

This is in clear contradiction to the finding of section 5.1 in the notes. What has gone wrong?

2a.) *Black hole orbits.* In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathcal{E}$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathcal{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S$$

where r is the standard radial coordinate, l_S and $\mathcal{E}_S$ are constants, $\Phi_S(r)$ is an effective potential function, and $v_S = dr/d\tau$. Determine l_S and $\mathcal{E}_S$ in terms of the fundamental angular momentum and energy constants J and E from lecture (or the notes). Express $\Phi_S(r)$ in terms of l_S, $\mathcal{E}_S$, the speed of light c, GM and r. The form of l_S, $\mathcal{E}_S$, and Φ_S should be chosen to go over to their Newtonian counterparts in the limit $E \to c^2$, $c \to \infty$, $E - c^2 \to$ finite.

2b.) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for Φ_S, and what does it mean physically if it does not exist?

2c.) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3},$$

exactly the Newtonian form. Here $\Omega \equiv d\phi/dt$ at the coordinate location r, where dt is the proper time interval at infinity. Derive expressions for E and J in terms of GM, c^2 and r.

2d.) Below what value of r does Φ_S not have any local extrema? (Answer: $6GM/c^2$.)

3.) *Bondi accretion onto a Schwarzschild black hole.* In Problem Set 1, we found that the equations for spherically symmetric inflow of a perfect gas on to a black hole were described by the equations:

$$nU^r|g'|^{1/2} = \text{constant} \;=\; \mathrm{C}_1,$$

$$(P+\rho c^2)U^r U_t|g'|^{1/2} = \text{constant} \;=\; \mathrm{C}_2,$$

where U_μ is a 4-velocity, n a particle number density, g' the determinant of $g_{\mu\nu}$ without the factor of $\sin^2\theta$, P the pressure, and ρ the total energy density. We now examine these equations for the Schwarzschild metric. We define

$$\varpi = \mu n,$$

where μ is the rest mass per particle and ϖ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2. P and ϖ are assumed to be related by a simple power law relationship,

$$P = K\varpi^\gamma$$

where K is a constant, and γ is called the adiabatic index. The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it.) Thus

$$\rho = \varpi + \frac{P}{c^2(\gamma-1)}.$$

This is not an artificial problem: it is directly valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The speed of sound a in a nonrelativistic gas is given by

$$a^2 = \gamma P/\varpi,$$

and we shall use variable throughout.

3a.) Verify that

$$|g'| = r^4$$

and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2\right]^{1/2}$$

(Take care to distinguish U^t and U_t.)

3b.) Combine our C_1 and C_2 equations to show that

$$\left(c^2 + \frac{a^2}{\gamma - 1}\right)^2 \left(c^2 + U^2 - \frac{2GM}{r}\right) = \text{constant}$$

where we have dropped the superscript r on U^r for greater clarity. The other equation we use is just the C_1 equation by itself, which is mass conservation. Show that it may be written as

$$4\pi\varpi r^2 U = \dot{m},$$

which defines the constnat mass accretion rate $\dot{m}$. How is a^2 related to ϖ?

3c.) A full analysis of these equations is complicated enough for a problem set on its own. Here we will do three simple tasks:

i) Show that the "constant" on the right of the first equation of problem (3b) is

$$c^2\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2$$

where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

ii) Show that the Newtonian limit of our equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy is conserved, and is also the original Bondi 1951 solution for accretion onto a star.

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c^2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c(1 - R_S/r).$$

4a.) *Kinematic and gravitational redshifts.* One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R)V^\mu(R)}{p_\mu(E)V^\mu(E)}$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

4b.) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits

$\odot$———$\longrightarrow$ observer $\gtrdot$

Show that in t, r, θ, ϕ coordinates for the $0, 1, 2, 3$ components,

$$V^\mu(R) = (1, 0, 0, 0), \quad V^\mu(E) = V_E^0(1, 0, 0, d\phi/cdt), \text{ with } V_E^0 = dt/d\tau$$

Then, using $g_{\mu\rho}V^\mu V^\rho = -1$ and problem (2c), conclude that

$$V_E^0 = (1 - 3GM/rc^2)^{-1/2}$$

4c.) Finally, using the results of problem (3) from Problem Set 1, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2}\left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1}, \quad \Omega^2 = GM/r^3.$$

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$(1 - 3GM/rc^2)^{1/2}.$$

Why? From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - 3GM/rc^2\right)^{1/2} / \left(1 \pm (rc^2/GM - 2)^{-1/2}\right)$$

(Hint: $g^{\nu\rho}p_\nu p_\rho = 0$.) Interpret the $\pm$ sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

5a.) *The perihelion advance of Mercury.* In the notes we found that the differential equation for $u = 1/r$ for Mercury's orbit could be written as follows. $u = u_N + \delta u$ with the Newtonian solution u_N given by

$$u_N = (GM/J^2)(1 + \epsilon\cos\phi)$$

and the differential equation for δu is

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi).$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2)$$

where $a = 3(GM)^3/(c^2J^4)$ and $b = 1 + \epsilon^2/2$.

To do this, try a solution of the form

$$\delta u = A_0 + A_1 \phi e^{i\phi} + A_2 e^{2i\phi}$$

where the A's are constants. Why do we need an additional factor of ϕ in the A_1 term?

5b.) Show that the solution for u is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon \cos\phi + \epsilon a \phi \sin\phi$$

Since a is very small, show that this equivalent to

$$u = ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}[1 + \epsilon(\cos\phi(1-\alpha))]$$

where

$$\alpha = aJ^2/GM = 3(GM/Jc)^2$$

5c.) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\Delta\phi = 6\pi\left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.