THE SIGN OF THE FOUR: COORDINATES, 4-VECTORS and TENSORS

"That's its business," said Lambert. "If Balbus says it's the same bulk, why, it's the same bulk, you know."

"Well, I don't believe it," said Hugh.

"You needn't," said Lambert. "Besides, it's dinner-time. Come along." They found Balbus waiting dinner for them, and to him Hugh at once propounded his difficulty.

"Let's get you helped first," said Balbus, briskly cutting away at the joint. "You know the old proverb, 'Mutton first, mechanics afterwards'?"

The boys did not know the proverb, but they accepted it in perfect good faith, as they did every piece of information, however startling, that came from so infallible an authority as their tutor. They ate on steadily in silence, and, when dinner was over, Hugh set out the usual array of pens, ink, and paper, while Balbus repeated to them the problem he had prepared for their afternoon's task.

— Excerpt from *A Tangled Tale,* by Lewis Carroll

1.) Consider the following thought:

"Special relativity holds for frames moving at constant relative velocity, but of course acceleration requires general relativity because the frames are noninertial."

Ineffable twaddle. Special relativity certainly doesn't cower before simple kinematical accleration. On the other hand, acceleration, even just uniform accleration in one dimension, is not without its connections with general relativity. We shall explore some of them here. *For ease of notation, we set $c = 1$.* In part (d) we'll put c back.

1a.) Let us first ask what we mean by "uniform acceleration." After all, a rocket approaching the speed of light c can't change its velocity at a uniform rate forever without exceeding c at some point. Go into the frame moving instantaneously at velocity v *with* the rocket relative to the "lab." The instantaneous rocket velocity, v', vanishes in this frame. Wait a time dt' later, as measured in this frame. The rocket now has velocity dv' in this same frame. What we mean by constant acceleration is $dv'/dt' \equiv a'$ is constant. The acceleration measured in the lab is certainly not constant! The question is, how is the lab acceleration $a = dv/dt$ related to the constant a'?

To answer this, let $V = v/\sqrt{1-v^2}$, the spatial part of the 4-vector V^α associated with the ordinary velocity $\boldsymbol{v}$. The same relation holds for V' and v'. Assume for the moment that the primed and unprimed frames differ by some arbitrary velocity w. The 4-velocity differentials are given by:

$$dV' = (dV - w\,dV^0)/\sqrt{1-w^2}$$

where $V^0 = 1/\sqrt{1-v^2}$. Explain.

1b.) Now, set $w = v$. We thereby go into the frame in which $v' = 0$; the rocket is instantaneously at rest. Prove that $dv = dv'(1-v^2)$. (Remember, v and v' are ordinary velocities.) From here, prove that

$$\frac{dv}{dt} = a'(1-v^2)^{3/2}.$$

1c.) Show that, starting from rest at $t = t' = 0$,

$$v = \frac{a't}{\sqrt{1 + a'^2 t^2}}, \qquad a't = \sinh(a't'),$$

and hence show that (for $x = 0$ at $t = t' = 0$):

$$v = \tanh(a't'), \qquad x = \frac{1}{a'}[\cosh(a't') - 1]$$

Do your own integrals!

1d.) Let's use these results to construct a full coordinate transformation from the lab frame x, t to the accelerating x', t' frame. We guess a transform of the form

$$t = A\sinh(a't') + B, \qquad x = A\cosh(a't') + C$$

where A, B, and C depend only upon x'. Then on $x' =$ constant surfaces, $dx/dt = \tanh(a't') = v$, which is indeed what we need. Prove that if i) surfaces of constant t' are surfaces of constant time in a frame moving instantaneously at v, and ii) t matches with t' at early times *and small* x', while x self-consistently agrees with x' at early times, then this uniquely determines A, B, and C. Put the speed of light c back into the equations, and show that

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh(a't'/c), \qquad x = \left(\frac{c^2}{a'} + x'\right)\cosh(a't'/c) - \frac{c^2}{a'}$$

1e.) Show that for the invariant Minkowski line element

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 = \left(1 + \frac{a'x'}{c^2}\right)^2 c^2 dt'^2 - dx'^2$$

Provide a physical interpretation of your result in terms of gravitational redshift.

2.) *Recognising tensors.* One way to prove that something is a vector or tensor is to show explicitly that it satisfies the coordinate transformation laws. This can be a long and arduous procedure if the tensor is complicated. There is another way.

Show that if V_ν is an arbitrary covariant vector and the combination $T^{\mu\nu}V_\nu$ is known to be a contravariant vector (note the free index μ), then

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}\right)V'_\nu = 0$$

Why does this prove that $T^{\mu\nu}$ is a tensor? Does your proof actually depend on the rank of the tensors involved?

3.) *What about $d^2x_\mu/d\tau^2$?* The geodesic equation in standard form gives us an expresssion for $d^2x^\mu/d\tau^2$ in terms of the affine connection, $\Gamma^\mu_{\nu\lambda}$. For the covariant coordinate x_μ, show that

$$\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^\mu}$$

(Hint: start with the standard geodesic equation for $d^2x^\mu/d\tau^2$, call it $dV^\mu/d\tau$, multiply by $V_\mu \equiv dx_\mu/d\tau$, and take it from there.) Under what conditions is $V_0 \equiv V_t$ a constant of the motion?

4.) *Hydrostatic Equilibrium in GR.* Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -(1 - 2GM/rc^2)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = R_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See §4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$.

5.) *Bondi Accretion: go with the flow.* To get some more practise working with the equations of GR, consider the problem of Bondi Accretion, the (exactly) spherical flow of gas into a black hole. Here we take the diagonal metric $g_{\mu\nu}$ as known. Later in the course we will derive $g_{\mu\nu}$ for a simple black hole, and build on the results we find here.

5a.) First, let us assume that particles are neither created or destroyed, just to keep things simple. So particle number is conserved. We use the usual r, θ, ϕ spherical coordinates. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu = nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that if particle number conservation implies:

$$J^\mu{}_{;\mu} = 0.$$

If nothing depends upon time, show that this integrates to

$$nU^r|g'|^{1/2} = \text{constant},$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$, and U^r is...well, you tell me what U^r is.

5b.) We move on to energy conservation, $T^{t\nu}_{\ \ ;\nu} = 0$. Refer to §4.6 in the notes, as in problem (4) above. Show that the only nonvanishing affine connection that we need to use is

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln|g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$(P + \rho c^2)U^r U_t |g'|^{1/2} = \text{constant}$$

where $U_t = g_{t\mu}U^\mu$, and ρ is the total energy density of the fluid in its rest frame, including any thermal energy. These two surprisingly simple equations will be solved in the next problem set.