Problem Set I: Scattering Theory

1.1 Given the Lorentz transforms for the momentum p and the total energy E of a single particle of rest mass m moving at speed u relative to the laboratory, show that

$$E^2 - c^2p^2 = m^2c^4$$

where c is the speed of light. For a system of particles with total energy E_t and total momentum p_t the quantity $E_t^2 - c^2p_t^2$ is an invariant. Justify and explain the importance of this result and explain the significance of the invariant quantity.

A particle of rest mass m travelling at speed u in the laboratory collides with a stationary particle of equal rest mass and they combine to form a new particle. Calculate the speed v in the laboratory and the rest mass M of the new particle expressing your result in terms of m, u, and $\gamma = (1 - u^2/c^2)^{-1/2}$. The new particle subsequently breaks up into two particles of equal rest mass αm. Show that the momentum of each particle in the zero momentum (center-of-mass) frame is

$$p = mc\sqrt{\frac{1+\gamma-2\alpha^2}{2}}$$

and hence deduce a greatest value for α. What is the threshold energy for $e^+e^- \longrightarrow \mu^+\mu^-$ using a beam of positrons on stationary electrons? ($m_ec^2 = 0.511$ MeV, $m_\mu c^2 = 105.66$ MeV)

1.2 The scattering formalism we have discussed in the lecture must be consistent with the quantum mechanics we have learned so far. We can verify this in one dimension:

a)* Derive the Lippmann-Schwinger equation in position representation in one dimension:

$$\Psi(x) = \Phi(x) - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} e^{ik|x-x'|}V(x')\Psi(x')dx',$$

with $k = \sqrt{2mE/\hbar^2}$.

b) Use this equation to develop the Born-approximation for one-dimensional scattering. That is, choose $\Phi(x) = Ae^{ikx}$, and assume $\Psi(x') \simeq \Phi(x')$. Evaluate the integral for $x \ll 0$ and hence show that the reflection coefficient takes the form

$$R \simeq \left(\frac{m}{\hbar^2 k}\right)^2 \left|\int_{-\infty}^{\infty} e^{2ikx}V(x)dx\right|^2.$$

c) Use this to compute the transmission coefficient ($T = 1 - R$) for scattering from a delta function ($V(x) = -\alpha\delta(x)$) and from a finite square well ($V(x) = -V_0$ for $-a < x < a$, $V(x) = 0$ otherwise). Compare your results with the exact solutions

$$T_\delta = \left(1 + \frac{m\alpha^2}{2\hbar^2 E}\right)^{-1} \quad \text{and} \quad T_{sq} = \left(1 + \frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)\right)^{-1}$$

in the case of high energy ($E \gg m\alpha^2/2\hbar^2$ and $E \gg V_0$).

1.3 The Klein-Gordon equation is a quantum mechanical wave equation compatible with special relativity. This equation can be written as

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\Phi(\vec{x},t) = 0.$$

a) Let $\Phi(\vec{x},t) = \exp(-iEt/\hbar + i\vec{p}\vec{x}/\hbar)$ define a quantum mechanical plane wave solution describing a relativistic particle of mass m. Show that this solution satisfies the Klein-Gordon equation.

b) Define the 4-vector $k^\mu = (E/c\hbar, \vec{p}/\hbar)$. Show that the 4-scalar product $k \cdot x = k^\mu x_\mu$, where $x_\mu = (ct, -\vec{x})$, is dimensionless.

c) The Green's function for the Klein-Gordon equation is defined by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)G(x'-x) = \delta^4(x'-x) = \delta(\vec{x}'-\vec{x})\delta(t'-t).$$

Show that $G(x'-x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x'-x)} G(k)$ with

$$G(k) = -\frac{1}{k^{02} - \vec{k}\cdot\vec{k} - m^2c^2/\hbar^2}$$

satisfies this definition. (Hint: use the following definition for the Dirac function in 4d: $\delta^4(x'-x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x'-x)}$).

1.4 Consider the case of low-energy scattering from a spherical delta-function shell $V(r) = \alpha\delta(r-a)$, where α and a are constants. Calculate the scattering amplitude $f(\Theta)$, the differential cross-section $d\sigma/d\Omega$, and the total cross-section σ. Assume $ka \ll 1$, so that only the $l = 0$ term contributes significantly. (To simplify matters, throw out all $l \neq 0$ terms right from the start.) Express your answer in terms of the dimensionless quantity $\Phi \equiv 2ma\alpha/\hbar^2$. (Answer: $\sigma = 4\pi a^2\Phi^2/(1+\Phi)^2$.)

1.5 Draw all the lowest order electromagnetic Feynman diagram(s) for the following processes:

a) $e^- + e^+ \longrightarrow e^- + e^+$

b) $e^- + e^- \longrightarrow e^- + e^-$

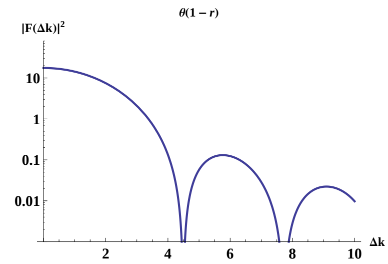
c) $e^- + e^- \longrightarrow e^- + e^- + \mu^+ + \mu^-$

d) $\gamma \longrightarrow e^+e^-$ in the presence of matter

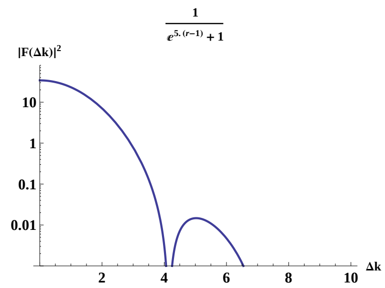
e) $\gamma + \gamma \longrightarrow \gamma + \gamma$

Draw some 2$^{\text{nd}}$ and 3$^{\text{rd}}$ order diagrams for case a).

1.6 Graphs (i) and (ii) below are examples of nuclear form factors for a uniform unit sphere and the Saxon-Woods potential shown in iii) ($\rho(r) \propto [1 + \exp((r - R)/a)]^{-1}$, $R = 1$, $a = 0.2$). How would these form factors scale along the Δk-axis if the radius r of the corresponding sphere of charge was doubled? By relating the momentum transfer to the scattering angle, use the data from p-Ag scattering to estimate the size of the silver nucleus. Compare to the expectation for an incompressible nucleus, $r = r_0 A^{1/3}$ with $r_0 = 1.25$ fm.



(i)



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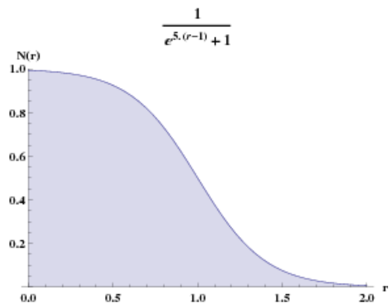
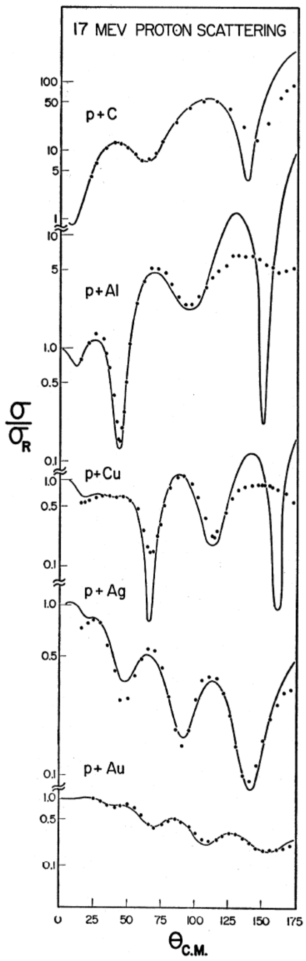


(iii)



Problem Set II: Nuclear Physics

2.1 What assumptions underlie the radioactivity law

$$\frac{dN}{dt} = -\Gamma N \ ?$$

A sample consists originally of nucleus A only, but subsequently decays according to

$$A \xrightarrow{\Gamma_A} B \xrightarrow{\Gamma_B} C.$$

Write down differential expressions for $\frac{dA}{dt}$, $\frac{dB}{dt}$ and $\frac{dC}{dt}$. Solve for, and then sketch, the fractions of, $A(t)$, $B(t)$ and $C(t)$. At what time is the decay rate of B maximum?

2.2 The radius r of a nucleus with mass number A is given by $r = r_0 A^{1/3}$ with $r_0 = 1.2$ fm. What does this tell us about the nuclear force?

a) Use the Fermi gas model (assuming $N \approx Z$) to show that the energy ϵ_F of the Fermi level is given by

$$\epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}.$$

b) Estimate the total kinetic energy of the nucleons in an ^{16}O nucleus.

c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{a_A(N-Z)^2}{A}.$$

Assuming that $(N - Z) \ll A$ use the Fermi gas model to justify this form and to estimate the value of a_A. Comment on the value obtained.

2.3 $^9_4$Be is the only stable isotope of beryllium. Some values of $B(A, Z)$ computed from the SEMF in MeV are

	$A = 6$	$A = 7$	$A = 8$	$A = 9$	$A = 10$	$A = 11$
$Z = 2$	24.3	14.5	11.8	−0.5	−5.7	−19.0
$Z = 3$	28.1	39.3	36.5	39.2	31.7	30.2
$Z = 4$	19.7	36.7	54.2	57.5	64.9	61.9
$Z = 5$	−40.1	6.8	31.0	54.5	63.7	76.0
$Z = 6$	−112.2	−50.4	0.7	30.1	58.5	72.5
$Z = 7$	−235.6	−135.0	−70.5	−15.6	18.7	51.5

a) Would you expect the SEMF to work well in this range of A and Z?

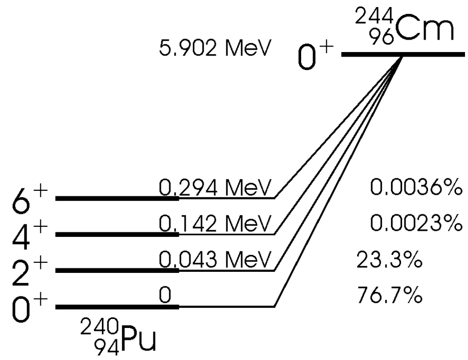
b) Illustrate how the binding energies predicted by the SEMF can be used to understand why $^9_4$Be is stable and $^{11}_4$Be is unstable.

c) The results of the SEMF given above above predict that $^{10}_4$Be is stable. In reality, it decays slowly to $^{10}_5$B. Indicate the mechanism by which the decay occurs and calculate by how much the binding energy of $^{10}_4$Be would need to be adjusted from the SEMF value of 64.9 MeV to make the decay energetically possible.

d) $^7_4$Be is found to decay only by electron capture. In what way are the binding energies predicted by the SEMF for $^7_4$Be and $^7_3$Li inconsistent with this fact?

e) Why is ${}^{8}_{4}$Be unstable even though in low-mass elements the $A = 2Z$ isotope is often the most stable? (The measured binding energy of ${}^{4}_{2}$He is 28.3 MeV.)

2.4 The figure below shows the α-decay scheme of ${}^{244}_{96}$Cm and ${}^{240}_{94}$Pu.



Justify that we would expect the rates to satisfy an equation of the form

$$\log R = A - \frac{BZ}{\sqrt{Q}}$$

with the parameters A and B. The Q value for the ground state to ground state transition is 5.902 MeV and for this transition $A = 132.8$ and $B = 3.97$ (MeV)$^{1/2}$ when R is in s^{-1}. The branching ratio for this transition is given in the figure. Calculate the mean life of ^{244}Cm.

Estimate the transition rate from the ground state of ^{244}Cm to the 6^+ level of ^{240}Pu using the same values for A and B and compare to the branching ratio given in the figure. Suggest a reason for any discrepancy.

[Hint: what form does the Schrödinger equation take for angular momentum quantum number $l \neq 0$?]

2.5 Which terms in the SEMF are responsible for the existence of a viable chain reaction of thermal-neutron-induced uranium fission? What distinguishes the isotopes of uranium that support such a reaction?

The fission of ^{235}U by thermal neutrons is asymmetric, the most probable mass numbers of fission fragments being 93 and 140. Where are the daughter nuclei in relation to the valley of stability and what happens to them subsequently? Use the semi-empirical mass formula to find the most probable value for Z for thse mass numbers and estimate the energy released in fission of ${}^{235}_{92}$U and hence the mass of ${}^{235}_{92}$U consumed each second in typical commercial nuclear reactor with a power of 1 GW.

In the construction of a nuclear fission reactor an important role is often played by water, heavy water or graphite. Describe this role and explain why are these materials are suitable.

Why is the fissile material not completely mixed up with the moderator?

2.6 The Fermi theory of neutron β-decay predicts that the rate of electrons emitted with momentum between p and $p + dp$ is given by

$$\Gamma(p_e)dp_e = \frac{G_F^2}{2\pi^3\hbar^7c^3}p_e^2(Q - T_e)^2dp_e,$$

where T_e is the kinetic energy of the electron, and Q is the energy released in the reaction.

a) Sketch this spectrum and justify the form of this result. What are the assumptions used to derive this result?

b) Show that for $Q \gg m_e c^2$ the total rate is proportional to Q^5.

c) What spin states are allowed for the combined system of the electron and the neutrino?

d) What modification to the above equation will be required for nuclear β-decay? Why are transitions between initial and final nuclei with angular momenta differing by more than $\hbar$ suppressed?

Problem Set III: Particle Physics I

3.1 The cross-section for the reaction $\pi^- p \to \pi^0 n$ shows a prominent peak when measured as a function of the π^- energy. The peak corresponds to the Δ resonance which has a mass of 1232 MeV, with $\Gamma = 120\,\text{MeV}$. The partial widths for the incoming and outgoing states are $\Gamma_i = 40\,\text{MeV}$, and $\Gamma_f = 80\,\text{MeV}$ respectively for this reaction.

a) The pions can be represented in an isospin triplet ($I = 1$) and the nucleons form an isospin doublet ($I = \frac{1}{2}$), while the Δ series of resonances have $I = \frac{3}{2}$.

   By assuming that the isospin operators I, I_3, $I_\pm$ obey the same algebra as the quantum mechanical angular momentum operators J, J_z, $J_\pm$, explain that the ratio $\Gamma_i/\Gamma_f = \frac{1}{2}$. (Remember that $|j_1 j_2 JM\rangle = \sum \langle j_1 j_2 m_1 m_2 | j_1 j_2 JM\rangle |j_1 j_2 m_1 m_2\rangle$ with the Clebsch-Gordan coefficients $c_{J,M,m_1,m_2} = \langle j_1 j_2 m_1 m_2 | j_1 j_2 JM\rangle$.)

b) At what pion beam energy will the cross-section be maximal for a stationary proton?

c) On the same plot draw the cross-sections as a function of the center-of-mass energy for the processes $\pi^- p \to \pi^0 n$ and the one for $\pi^- p \to \pi^- p$ around the resonance, giving values for the variables in the Breit-Wigner formula where possible.

d) By considering the quark content of the intermediate states, discuss whether you would expect similar peaks in the cross-sections for the reactions (i) $K^- p \to$ products and (ii) $K^+ p \to$ products.

3.2 ^{60}Co nuclei ($J^P = 5^+$) are polarised by immersing them at low temperature in a magnetic field. When these nuclei β-decay to ^{60}Ni (4^+) more electrons are emitted opposite to the aligning B field than along it (Reported in C.S.Wu et al., Phys. Rev. 105, 1413 (1957)). Explain why this demonstrates parity violation in the weak interaction.

3.3 Write down the valence quark content for each of the different particles in the reactions below and check that the conservation laws of electric charge, flavour, strangeness and baryon number are satisfied throught.

$$\begin{array}{llll} (1) & \pi^- + p & \to & K^0 + \Lambda \\ (2) & K^- + p & \to & K^0 + \Xi^0 \\ (3) & \Xi^- + p & \to & \Lambda + \Lambda \\ (4) & K^- + p & \to & K^+ + K^0 + \Omega^- \end{array}$$

Draw a quark flow diagram for the last reaction.

3.4 Consider the decay of the ρ^0 meson ($J^P = 1^-$) in the following decay modes:

a) $\rho^0 \to \pi^0 + \gamma$,

b) $\rho^0 \to \pi^+ + \pi^-$,

c) $\rho^0 \to \pi^0 + \pi^0$.

Draw diagrams in each case to show the quark flow.

Consider the symmetry of the wave-function required for $\pi^0 + \pi^0$ and explain why this decay mode is forbidden.

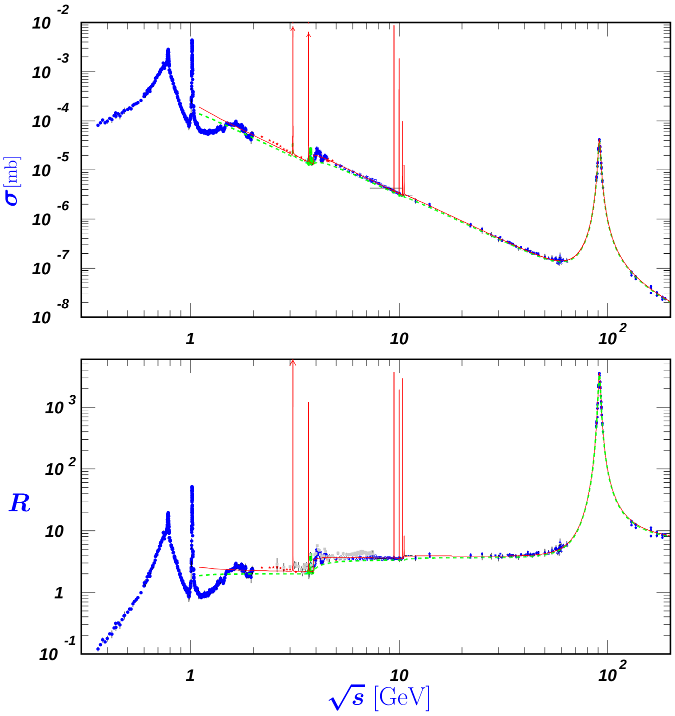
From consideration of the relative strength of the different fundamental forces, determine which of the other two decay modes will dominate.

3.5 Draw leading order electromagnetic Feynman diagrams for the processes

$$e^+ + e^- \to \mu^+ + \mu^- \quad \text{and} \quad e^+ + e^- \to q + \bar{q}.$$

How do the vertex and propagator factors compare?

The figure below shows the cross-section $\sigma(e^+e^- \to \text{hadrons})$ and the ratio of cross-sections $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ as a function of the center-of-mass energy.



Considering the number of quarks that can be created at particular centre-of-mass energy, what values of R would you expect for centre-of-mass energy in the range $2\,\text{GeV} < \sqrt{s} < 20\,\text{GeV}$?. How do your predictions match the data? How do these measurements support the existence of quark colour? What can you conclude on the structure of the quarks from these plots?

What is causing the sharp peaks in R at a centre-of-mass energy of about 3 GeV and 10 GeV, and the broader peak at about 100 GeV?

3.6 To a first approximation the potential in a $q\bar{q}$ system is

$$V(r) = -\frac{4}{3}\frac{\hbar c \alpha_s}{r} + kr$$

a) Sketch $V(r)$ against r. Show that for $r \ll r_0 = \sqrt{\hbar c \alpha_s / k}$ the $1/r$ term dominates.

b) Now let's assume we can ignore the linear term. Calculate values for α_s for two cases:

   i. The splitting between the $n = 2$ and $n = 1$ states in the Ψ system ($c\bar{c}$) is 588 MeV, and $m_c = 1870\,\text{MeV}/c^2$.

   ii. The splitting between the $n = 2$ and $n = 1$ states in the Υ system ($b\bar{b}$) is 563 MeV, and $m_b = 5280\,\text{MeV}/c^2$.

Why do these results differ?

c) Still ignoring the linear term, compute the Bohr radius a_0 for the J/Ψ and the Υ. The value of $\sqrt{k}$ is about $400\,\mathrm{MeV}/\sqrt{\hbar c}$; is it a reasonable approximation to ignore the linear term in $V(r)$ to compute the energies of the lowest bound states?

d) Calculate the expectation value of the kinetic energy in the $n = 1$ states of the J/Ψ and the Υ. Comment on whether we are justified in using a non-relativistic approximation for these states; are relativistic corrections more significant in positronium or in these $q\bar{q}$ states?

Problem Set IV: Particle Physics II

4.1 Draw Feynman diagrams showing a significant decay mode of each of the following particles:
a) π^0 meson
b) π^+ meson
c) μ^-
d) τ^- to a final state containing hadrons
e) K^0
f) top quark
What type of interaction is responsible for each of these decays? What is the reason why the decay you listed is more significant than others?

4.2 At the HERA collider 27 GeV positrons collided with 920 GeV protons.

a) By considering the de Broglie wavelength of the positron can you justify that these collisions can be considered to be due to positrons scattering off the quarks in the protons?

b) For these collisions draw **one** example of a Feynman diagram for **each** of the cases of weak charged-current, weak neutral-current and electromagnetic interaction.

c) Calculate the center-of-mass energy of the quark–positron system assuming that the 4-momentum of the quark P_q can be represented as a fixed fraction f of the proton 4-momentum P_p, in the approximation where both particles are massless.

d) What is the highest-mass particle that can be produced in such a collision in the approximation that a quark carries about 1/3 of the proton momentum?

e) How does the propagator for the weak charged current and electromagnetic interactions vary with 4-momentum transfer P^2? Hence explain the fact that at low values of the momentum transfer it is found that the ratio of weak interactions to electromagnetic interactions is very small whereas at very high values it is found that the ratio is of the order of unity.

4.3 Draw the lowest-order Feynman diagrams for the process $e^+ + e^- \to \mu^+ + \mu^-$. Well above threshold, the expression for the total cross-section is

$$\sigma(e^+ + e^- \to \mu^+ + \mu^-) = \frac{4\pi}{3}\frac{(\alpha\hbar c)^2}{E^2} \quad \text{for} \quad E \ll M_Z c^2$$

where α is the electromagnetic fine-structure constant, M_Z is the mass of the Z^0 boson and E is the center-of-mass energy.

a) Would you expect the same expression to be valid for the processes $e^+ + e^- \to e^+ + e^-$ and $e^+ + e^- \to \tau^+ + \tau^-$?

b) Starting from this equation, derive an expression for $\sigma(e^+ + e^- \to \text{hadrons})$ for $E > 10\,\text{GeV}$.

c) An e^+e^- operates at a center-of-mass energy of 30 GeV with a luminosity of $3\times10^{35}\,\text{m}^{-2}\text{s}^{-1}$. What is the rate of production of hadronic events at such a collider?

4.4 Draw Feynman diagrams for the decays of the muon and the tau lepton. Are hadronic decays possible? Making use of Sargent's rule explain why you would expect for the ratio

$$\frac{\Gamma(\tau^- \to e^- + \nu + \bar{\nu})}{\Gamma(\mu^- \to e^- + \nu + \bar{\nu})} = \left(\frac{m_\tau}{m_\mu}\right)^5 .$$

Test this prediction using the following data:

$$m_\tau = 1777.0\,\mathrm{MeV}/c^2 \qquad \tau_\tau = 2.91\times10^{-13}\,\mathrm{s}$$
$$m_\mu = 105.66\,\mathrm{MeV}/c^2 \qquad \tau_\mu = 2.197\times10^{-6}\,\mathrm{s}$$
$$BR(\tau^- \to e^- + \nu + \bar{\nu}) = 17.8\%$$

4.5 a) Draw Feynman diagrams for the production of $W^\pm$ bosons being produced at a $p\bar{p}$ collider. If the W^+ boson is close to its Breit-Wigner peak, what possible decays may it have? (Which final states are kinematically accessible?)

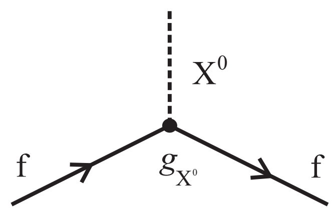
b) What fraction of W^+ decays would you expect to produce positrons?

c) Suggest why the W was discovered in the leptonic rather than hadronic decay channels.

d) How could the outgoing (anti-)electron momentum be determined? How might the components of the neutrino momentum perpendicular to the beam be determined?

e) Draw diagrams for the decays $D^0 \to K^- + \pi^+$ and $D^0 \to K^- + e^+ + \nu_e$. Disregarding the differences in the 2-body and 3-body density of states factors, what do you expect for the relative rates of these decays?

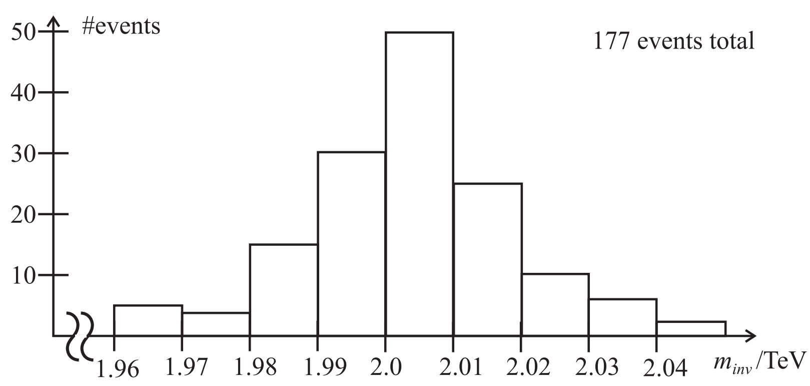
4.6 Let us assume that scientists at the LHC would discover a new particle X^0 of mass M_{X^0}. The only interactions it has are described by the Feynman diagram below, where g_{X^0} is a universal coupling constant valid for all fermions f.



Draw a parton level Feynman diagram of lowest order in g_{X^0} for the production of the X^0 at the LHC. Make sure to explain how any partons in your diagram connect to the valence quarks of the protons.

If the two constituents of the proton which ultimately produce an X^0 have momentum fraction x_1 and x_2 respectively and if the proton-proton centre of mass energy is $\sqrt{s}$ find an approximate relationship between m_{X^0}, $\sqrt{s}$, x_1 and x_2 that is independent of the proton mass. Clearly state any approximations you make.

The X^0 has been observed in its jet-jet, $\mu^+\mu^-$ and other final states. For an integrated luminosity of 2 fb^{-1} the graph below shows the number of events $n_{jj}(m_{inv})$ per invariant mass interval in which the X^0 decays into a two-jet final state (background was subtracted) as a function of the invariant mass of its decay products m_{inv}.



Derive an expression for the ratio $N_{\mu^+\mu^-}/N_{jj}$ where $N_{\mu^+\mu^-}$ (N_{jj}) are the total number of events you expect to observe in the $\mu^+\mu^-$ (jet-jet) final states. Compare the shape and normalisation of the corresponding graph for $\mu^+\mu^-$ final states to the one shown above.

Write down the functional form of n_{jj} as a function of m_{inv} and the particle spins. Under which assumptions is this form accurate? You may ignore any normalisation constants. Explain the meaning of all terms in your functional form. Deduce the production cross-section $\sigma(pp \rightarrow \mathrm{X}^0)$, the mass and the lifetime of the X^0 from the diagram. You may assume that that the detector which obtained this result is fully efficient and covers the full solid angle around the collision point.

Deduce the Baryon number of the X^0.