FLUIDS, FLOWS AND COMPLEXITY

PROBLEM SET 3 AND SOLUTIONS

Julia Yeomans

Comments and corrections to julia.yeomans@physics.ox.ac.uk please.

1. **Saddle node bifurcation: a biochemical switch**

   A gene G is activated by a biochemical signal substance S. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^2 + g^2}$$

   where the k's are positive constants.

   (a) Suggest physical motivations for each term.

   (b) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

   giving expressions for r and s.

   (c) Show that, if $s = 0$ there are two positive fixed points x^* if $r < r_c$ and find r_c. Sketch the flows as a function of r in an (r, x) plot.

   (d) For (a) $s = 0.05$ and (b) $s = 0.15$ give a rough sketch of the fixed points and flows in an (r, x) plot.

   (e) Find parametric equations for the bifurcation curves in (r, s) space and classify the bifurcations that occur. Plot the bifurcation curves (using eg Mathematica).

2. **Pitchfork bifurcation**

   Consider the dynamical system

$$\dot{x} = \epsilon + rx - x^3$$

   where ϵ is known as an imperfection parameter.

   (a) Derive the fixed points and their stability properties for $\epsilon = 0$. Thus confirm that a supercritical pitchfork bifurcation occurs at $r = 0$. Sketch the bifurcation diagram (r, x).

   (b) For $\epsilon = 0$, confirm that this system is symmetric under $x \Leftrightarrow -x$. Show that this symmetry is broken for $\epsilon \neq 0$.

   (c) By treating ϵ as a small parameter, determine how the fixed points and their stability are modified in the case of a small, non-zero value of ϵ. Note that this approach breaks down as $r \to 0$, and

   (d) obtain an exact expression for the position of any fixed points for $r = 0$.

   (e) Hence sketch the bifurcation diagram (r, x) for a small positive value of ϵ. This bifurcation is known as an *imperfect pitchfork bifurcation*. Using your diagram explain how abrupt jumps in x might occur as r is slowly varied.

   (f) Calculate the curves $\epsilon_c(r)$ where saddle node bifurcations occur, and plot them in the (ϵ, r) plane, indicating the number of fixed points in each region.

3. **Flows in phase space**

In each of the following dynamical systems, identify the fixed points, classify them, and sketch a phase portrait.

(i) $\dot{x} = y, \quad \dot{y} = -2x - 3y.$
Hint: The trajectories tend to the slowly decaying eigenvector near the fixed point and the fast eigenvector far from the fixed point. Why?
(ii) $\dot{x} = 5x + 2y, \quad \dot{y} = -17x - 5y.$
Hint: It helps to check where the trajectories cross the axes and/or to notice that this is a Hamiltonian system.
(iii) $\dot{x} = x - y, \quad \dot{y} = x^2 - 4.$

4. **Damped pendulum**

The motion of a damped pendulum is described by

$$\ddot{\theta} + b\dot{\theta} + \sin\theta = 0.$$

(a) Rewrite the equation as a two-dimensional linear system.

(b) Classify the fixed points for $b < 2$, (ii) $b > 2$. Sketch the phase portrait for small b and explain how it changes as b increases, relating your results to the motion of a under-damped and over-damped pendulum.

5. **Lorenz equations**

(a) Write down the Lorenz equations and solve for the fixed points, stating the values of r over which each exists. Comment on the physical significance of each of these fixed points in the context of Rayleigh-Bénard convection.

(b) Establish the stability criteria for the fixed point at the origin as a function of r. State the stability criteria for the remaining fixed points.

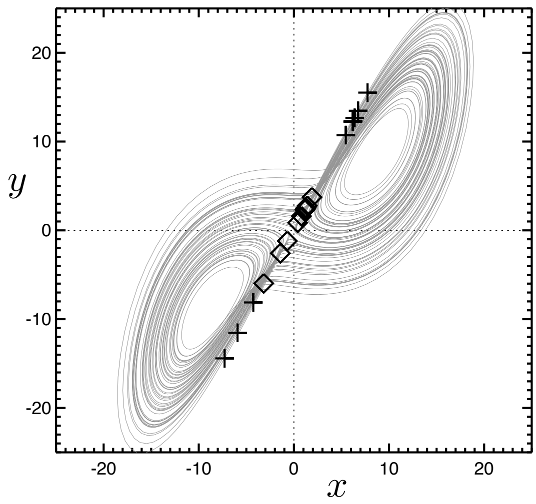
(c) The following figure shows a projection onto the $x - y$ plane of a trajectory of a particle governed by the Lorenz equations with $r = 28$, $\sigma = 10$, $b = 8/3$. The bold crosses and diamonds show points on a Poincaré section where the trajectory crosses the plane $z = r - 2\sigma > 0$. One symbol denotes trajectories passing downwards (decreasing z), the other upwards. Which is which?

(d) Show that a the length of a small displacement element, $|\delta\mathbf{x}|$, between two adjacent trajectories grows or decays exponentially if $\delta\mathbf{x}$ is aligned with one of the eigenvectors of $(\mathcal{J} + \mathcal{J}^T)/2$ where $\mathcal{J}$ is the Jacobian matrix.

(e) Write down the matrix $(\mathcal{J}+\mathcal{J}^T)/2$ at the point $(0, 0, r-2\sigma)$ and solve for its eigenvalues. Hence deduce that the dominant directions in which $\delta\mathbf{x}$ grows and decays are aligned parallel to the $x - y$ plane at this point, whereas displacements perpendicular to the $x - y$ plane decay slowly.

(f) Using the continuity equation for flows in phase space, show that volume elements contract exponentially in time at a rate

$$\delta\dot{V} = \delta V_0 \exp -(\sigma + 1 + b)t.$$



Since $(\mathcal{J} + \mathcal{J}^T)/2$ is a symmetric matrix, its eigenfunctions are orthogonal. Confirm that the volume of a rectangular volume element defined by three displacement elements aligned with each of the eigenvectors in your answer to part (e) decays at the same rate.

6. **A chaotic map**

Consider the decimal shift map on the unit interval given by

$$x_{n+1} = 10\; x_n \;(\text{mod}1)$$

where mod 1 means keep only the non-integer part of x eg 1.67 (mod 1) =0.67.

(i) Draw the graph of the map.
(ii) Find all the fixed points.
(iii) Show that the map has periodic points of all periods, but that all of them are unstable.
(iv) Show that the map has infinitely many aperiodic orbits.
(v) By considering the rate of separation between two nearby orbits, show that the map has sensitive dependence on initial conditions.