To: James Sadler

B3 Problem Set 4

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hc $\nu = \frac{1}{\lambda}$ $\omega = \frac{2\pi}{}$ $\frac{\omega}{c}$ k

$2\pi c\nu = \omega$

$c = \lambda f$ $\nu c = f$

$c = \lambda f = \frac{f}{\nu}$

Problem Set 4 *Light propagation through matter and lasers*

8. (a) Explain what is meant by the terms **homogeneous broadening** and **inhomogeneous broadening**. Give two examples of each class of broadening.

(b) Describe in outline how the natural linewidth of a transition is consistent with the Uncertainty Principle. What is the natural linewidth of a transition between two levels with radiative lifetimes of τ_1 and τ_2?

(c) Show that the full-width at half-maximum linewidth of a Doppler-broadened transition is given by

$$\Delta\nu_D = \nu_0\sqrt{8\ln 2}\sqrt{\frac{k_BT}{Mc^2}},$$

where T is the temperature of the atoms, M their mass, and ν_0 the frequency emitted on the transition by a stationary atom.

9. Figure 1 shows data from measurements of the homogeneous linewidth of the D_1 ($6p\,^2P_{1/2} \to 6s\,^2S_{1/2}$) and D_2 ($6p\,^2P_{3/2} \to 6s\,^2S_{1/2}$) transitions in Cs at 894 and 852 nm respectively.

(a) Calculate the Doppler width in MHz of these transitions assuming that the temperature of the Cs vapour is 21 °C, and comment on the relative magnitudes of the inhomogeneous and homogeneous linewidths.

(b) The radiative lifetimes of the D_1 ($6p\,^2P_{1/2}$ and D_2 ($6p\,^2P_{3/2}$ levels are 34.75 and 30.41 ns respectively. What is the natural linewidth of the D_1 and D_2 transitions? Is your calculated value consistent with Fig. 1?

(c) Use the data presented to deduce the rate of increase in the homogeneous linewidth in units of MHz Torr^{-1} for each of the two transitions. Explain briefly the cause of this increase in homogenous linewidth with pressure.

(d) What is the mean collision time at a He pressure of 100 Torr for He-Cs collisions?

[The molar mass of Cs is 132.9 g.]

10. (a) The He-Ne laser operates on several $s \to p$ transitions in neon, including the $5s \to 3p$ transition at 632.8 nm. Under the operating conditions of the laser, the fluorescence lifetimes of the upper and lower levels are aproximately 100 ns and 10 ns respectively for this transition, and the Einstein-A coefficient is 10^7 s^{-1}. Taking the upper and lower laser levels to have equal degeneracies, determine whether or not it is possible, in principle, for continuous-wave laser oscillation to be observed on this transition.

(b) Repeat the calculation for the $3d^{10}4p\,^2P_{3/2} \to 3d^9 4s^2\,^2D_{5/2}$ transition at 510 nm in the copper-vapour laser, given that the Einstein A coefficient is 2×10^6 s^{-1} and the fluorescence lifetime of the lower laser level is approximately 10 μs.

11. (a) Show that the optical gain cross-section of a homogeneously broadened laser transition may be written as,

$$\sigma_{21}(\omega - \omega_0) = \frac{\pi^2 c^2}{\omega_0^2}A_{21}g_H(\omega - \omega_0),$$

where $g_H(\omega - \omega_0)$ is the lineshape function of the transition.

$\frac{2\pi^2 c^2}{\omega_0^2} \times \cancel{\frac{\lambda}{\lambda^2}}$

$\omega_0 = 2\pi\nu_0 = \frac{2\pi c}{\lambda_0}$

$= \frac{2c^2}{\omega_0^2}$

$\therefore \sigma_{21}(0) = \frac{2c^2\pi\,\lambda_0^2}{(2\pi)^2 c^2} = \frac{2\lambda_0^2\pi}{4\pi^2} = \frac{\lambda_0^2}{2\pi}$

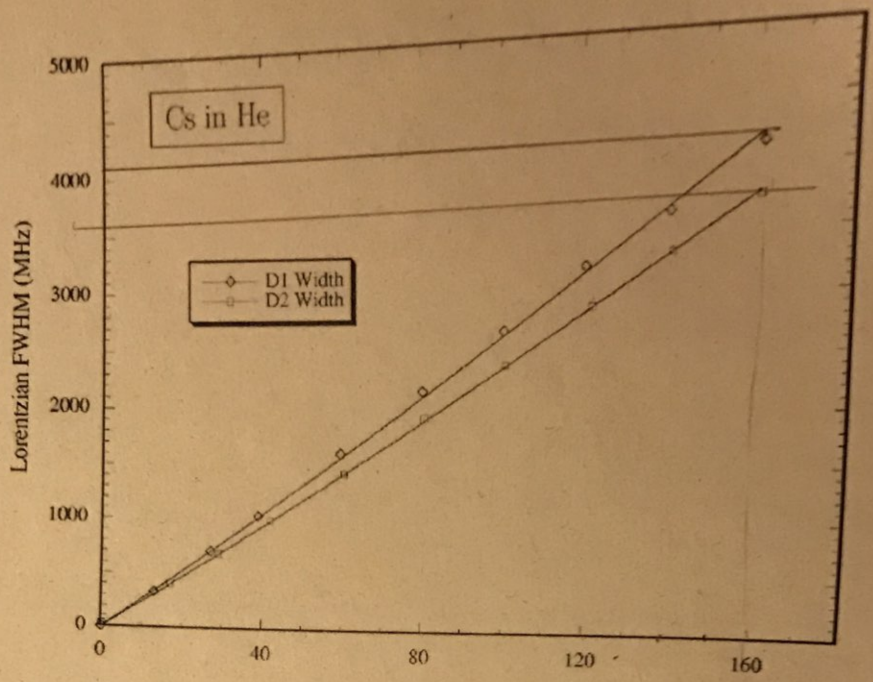


Figure 1: Measured full width at half maximum of the homogeneously broadened component of the D_1 and D_2 lines of Cs as a function of the pressure of helium (data from A. Andalkar and R. B. Warrington *Phys. Rev. A* **65** 032708 (2002)).

(b) Show that for the special case of a purely lifetime broadened transition from an upper level 2 which decays only radiatively to a long-lived lower level 1 the peak optical gain cross-section is given by:

$$\sigma_{21}(0) = \frac{\lambda_0^2}{2\pi},$$

where λ_0 is the vacuum wavelength of the transition.

(c) A laser operates on a transition from an excited electronic energy level of a diatomic molecule in which the two constituent atoms form a molecular bond with each other. This level has a lifetime of 10 ns against radiative decay which is entirely on the laser transition at 250 nm to the unstable ground electronic level, which has a lifetime of 3×10^{-14} s against dissociation into its constituent atoms.

- Calculate the peak optical gain cross-section of the laser transition, assuming that it is purely lifetime broadened.
- What upper level population density would be needed to provide a small signal gain of 0.1cm^{-1}?
- Assuming that 10% of the power input leads to formation of molecules in the upper laser level, calculate the minimum power input per unit volume required to sustain the population of the upper level at the value calculated above. Comment briefly on the implications of this result.

12*. (a) Use a rate equation analysis to show that the gain coefficient of a homogeneously broadened laser transition is modified by the presence of narrow-band radiation of total intensity I to,

$$\alpha_I(\omega - \omega_0) = \frac{\alpha_0(\omega - \omega_0)}{1 + I/I_s(\omega_L - \omega_0)},$$

where ω_L is the laser frequency. Give an expression for the saturation intensity I_s.

(b) Explain in physical terms why the saturation intensity depends on the detuning of the intense beam from the centre frequency of the transition.

(c) On the same graph plot the gain coefficient as a function of frequency ω:

- as measured by a weak probe beam in the absence of any other radiation;
- as measured by a weak probe beam in the presence of a narrow-band beam of intensity $I_s(\omega_L - \omega_0)$;
- as measured by an intense, narrow-band beam of constant intensity $I_s(0)$.

13. A saturated amplifier: A steady-state laser amplifier operates on the homogeneously broadened transition between two levels of equal degeneracy. Population is pumped exclusively into the upper level at a rate of $1.0 \times 10^{18}\ \mathrm{s^{-1}\,cm^{-3}}$. The lifetimes of the upper and lower levels are 5 ns and 0.1 ns respectively. A collimated beam of radiation enters the 2 m long amplifier with an initial intensity $I(0)$. The gain cross section of the medium is $4 \times 10^{-12}\ \mathrm{cm^2}$ at the 400 nm wavelength of the monochromatic beam. Calculate the intensity of the beam at the exit of the amplifier when:

- $I(0) = 0.1\ \mathrm{W\,cm^{-2}}$
- $I(0) = 500\ \mathrm{W\,cm^{-2}}$
- $I(0) = 50\ \mathrm{W\,cm^{-2}}$

$\frac{dN_2}{dt} = R - \frac{N_2}{\tau_2}$

$\frac{dN_1}{dt} = \frac{N_2}{\tau_2} - \frac{N_1}{\tau_1}$

[*Hint:* In the latter case guess a solution and proceed by iteration]

14. (a) Explain briefly what is meant by the terms **three-level** and **four-level laser**. Discuss why there is a large difference in the threshold power which is required to achieve laser oscillation in these two classes of laser.

(b) A laser cavity is formed by two mirrors of reflectivity 100% and 95%. Calculate the energy absorbed by the active ions which is necessary to achieve pulsed laser oscillation for rods of ruby and Nd:YAG each 50 mm long, 5 mm diameter, and each with an active ion concentration of $4 \times 10^{19}\ \mathrm{cm^{-3}}$. Take the pump bands to be at $20\,000\ \mathrm{cm^{-1}}$ and $12\,000\ \mathrm{cm^{-1}}$ for the ruby and Nd:YAG laser respectively. For the Nd:YAG laser, you may assume that the peak optical gain cross-section is $\sigma_{21}(0) = 6 \times 10^{-19}\ \mathrm{cm^2}$.

(c) How will these values compare with the electrical energy which must be supplied to the laser?

8. (a)

→ homogeneous broadening: All individual atoms behave in the same way while the spectral frequency distri-bution is ~~bro~~ broadened. This generally produces a Lorentzian line shape

e.g ~~Natural~~ → Natural ~~Br~~ Broadening
→ Pressure Broadening ✓

→ inhomogeneous broadening: Individual atoms behave differently and contribute to different parts of the spectrum. This tends to produce a Gaussian spectral line.

e.g → Doppler Broadening
→ Phonon Broadening in ~~An~~ amorphous solids. ✓

(b)

~~Natural~~ Natural linewidth of spontaneous emission is $\Delta\omega$

Consider the lineshape

$$\phi(\omega) = \frac{\gamma/2\pi}{(\omega-\omega_0)^2 + \frac{\gamma^2}{4}}$$

FWHM $\quad \Delta\omega = 2 \times \sqrt{\frac{\gamma^2}{4}} = \frac{\gamma}{2} \times 2 = \gamma = A_{21} = \frac{1}{\tau}$

$\Rightarrow \quad \Delta\omega = \gamma = \frac{1}{\tau}$ → natural linewidth

From uncertainty principle

$\Delta t \cdot \Delta E \sim \hbar$

$\therefore \quad \Delta\omega \sim \frac{\Delta E}{\hbar} \sim \frac{1}{\Delta t} \sim \frac{1}{\tau} \sim \gamma$

$\rightarrow$ consistent.

For a 2 level system

the ~~natu~~ natural linewidth is the sum of width of each level

$\therefore \quad \boxed{\Delta\omega_N = \frac{1}{\tau_1} + \frac{1}{\tau_2}}$.

(c)

Consider z to be the direction of observation.

Velocity distribution in v_z is

$$P(v_z)\,dv_z = \sqrt{\frac{M}{2\pi k_B T}} \exp\left(-\frac{M v_z^2}{2 k_B T}\right) dv_z$$

Doppler effect gives shift in frequency

$$\omega - \omega_0 = \frac{v_z}{c}\,\omega_0$$

$$\therefore \quad V_z = c\,\frac{\omega-\omega_0}{\omega_0}$$

$$dV_z = \frac{c}{\omega_0}\,d\omega$$

substitute gives frequency distribution

$$\phi(\omega) = \frac{c}{\omega_0}\sqrt{\frac{M}{2\pi k_B T}}\,\exp\left(-\frac{Mc^2}{2k_BT}\left(\frac{\omega-\omega_0}{\omega_0}\right)^2\right) d\omega$$

✓

FWHM

$\Delta\omega_D = 2\times$ [$\Delta\omega_{1/2}$ at which $\phi(\omega)$ gets $\frac{1}{2}$ of maximum]

$$\cancel{\left(\frac{\Delta\omega_D/2}{\omega_0}\right)^2} \qquad \exp\left(-\frac{Mc^2}{2k_BT}\left(\frac{\Delta\omega_{1/2}}{\omega_0}\right)^2\right) = \frac{1}{2}$$

$$\therefore \quad \ln 2 = \frac{Mc^2}{2k_BT}\left(\frac{\Delta\omega_{1/2}}{\omega_0}\right)^2$$

$$\rightarrow \quad \Delta\omega_{1/2} = \sqrt{2\ln 2}\,\frac{\omega_0}{c}\sqrt{\frac{k_BT}{M}}$$

$$\therefore \quad \Delta\omega_D = \sqrt{8\ln 2}\,\omega_0\sqrt{\frac{k_BT}{Mc^2}} \qquad \left(\Delta\nu_D = \frac{\Delta\omega_D}{2\pi}, \quad \nu_0 = \frac{\omega_0}{2\pi}\right)$$

$$\rightarrow \quad \Delta\nu_D = \sqrt{8\ln 2}\,\nu_0\sqrt{\frac{k_BT}{Mc^2}}$$

✓

9. (a)

For D_1. $\nu_1 = \frac{c}{\lambda_1} = 1.12 \times 10^6 \, m^{-1} \times 3 \times 10^8 \, m/s$

$= 3.354 \times 10^{14} \, Hz.$

Doppler width

$$\Delta\nu_{D_1} = \nu_1 \sqrt{8 \ln 2} \times \sqrt{\frac{k_B T}{M c^2}}$$

$T = 21°C = 294 K$ $M = 2.21 \times 10^{-25} kg$

$\rightarrow \boxed{\Delta\nu_{D_1} = 357 \, MHz}$

~~$\therefore \Delta\nu_D = 1.19 \, m^{-1}$~~

For D_2 $\nu_2 = \frac{c}{\lambda_2} = 3.521 \times 10^{14} \, Hz$

$\Delta\nu_{D_2} = \boxed{375 \, MHz}$

→ The Doppler width is smaller than the ~~natural~~ homogeneous linewidth at high pressure, and roughly equal at low pressure.

(b) the natural linewidth of

$\Delta\omega = \frac{1}{\tau_{grad}} + \frac{1}{\tau_{...}}$

→ D_1 $\Delta\nu_{N_1} = \frac{1}{2\pi\tau_1} = \boxed{4.58 \, MHz}$

→ D_2 $\Delta\nu_{N_2} = \frac{1}{2\pi\tau_2} = \boxed{5.23 \, MHz}$

Not consistent with Fig 1

3 orders of magnitude off

→ ~~effective~~ effect of pressure very significant

(c) Rate of increase:

For D_1

$$\text{Rate} = \frac{4100\ \text{MHz}}{160\ \text{torr}} = \underline{\underline{25.625\ \text{MHz/torr}}}$$

For D_2

$$\text{Rate} = \frac{3600\ \text{MHz}}{160\ \text{torr}} = \underline{\underline{22.5\ \text{MHz/torr}}}$$

This is pressure Broadening, in which rapid collision interrupts the transition and shortens the lifetime. This increases the energy uncertainty and thus the line width. So the line is further broadened.

(d) At 100 torr, the homogeneous broadening

For D_1:

$$\Delta\nu_{P_1} = \left(\gamma + \frac{2}{\tau_c}\right)\frac{1}{2\pi} = 25.625 \times 100 = 2562.5\ \text{MHz}$$

$$\because \gamma \ll \frac{1}{\tau_c} \therefore \frac{1}{\pi\tau_c} = 2562.5\ \text{MHz}$$

$$\rightarrow \tau_c^1 = 1.24 \times 10^{-10}\ \text{s}$$

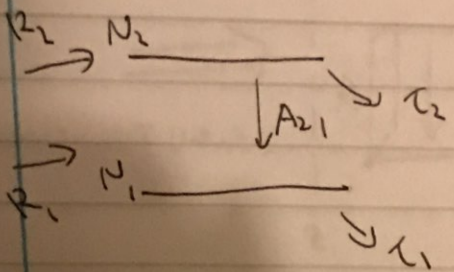
For D_2: $\Delta\nu_{P_2} = \left(\gamma + \frac{2}{\tau_c}\right)\frac{1}{2\pi} \simeq \frac{1}{\pi\tau_c} = 22.5 \times 100 = 2250\ \text{MHz}$

$$\rightarrow \tau_c^2 = 1.42 \times 10^{-10}\ \text{s}$$

$$\rightarrow \text{average } \underline{\underline{\tau_c = 1.33 \times 10^{-10}\ \text{s}}}$$

10.

The ~~condit~~ rate equations for 2 levels



$$\frac{g_2}{g_1} A_{21} \tau_1 < 1$$

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}$$

$$\frac{dN_1}{dt} = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1}$$

At steady state $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

$$\therefore \quad N_2 = R_2 \tau_2$$

$$N_1 = R_1 \tau_1 + R_2 \tau_2 A_{21} \tau_1$$

* Condition for steady state population inversion

$$\frac{N_2}{g_2} > \frac{N_1}{g_1}$$

steady state laser "continuous laser"

$$\therefore \quad \frac{R_2 \tau_2 g_1}{R_1 \tau_1 g_2} \left(1 - \frac{g_2}{g_1} A_{21} \tau_1 \right) > 1$$

if not satisfied, then a "pulse laser"

The ~~ps~~ pumping rate R_1, R_2 are parameters that we can control

$$\therefore \quad \text{we need} \quad 1 - \frac{g_2}{g_1} A_{21} \tau_1 > 0$$

~~both~~ · For ~~both~~ $g_1 = g_2$, we need

$$1 - A_{21}\tau_1 > 0$$

$$\therefore \quad \underline{A_{21} < \frac{1}{\tau_1}}$$

(a)

$$A_{21} = 10^7 s^{-1}$$

$$\frac{1}{\tau_1} = \frac{1}{10ns} = \cancel{10^{-8}} \quad 10^8 s^{-1}$$

$$\Rightarrow A_{21} < \frac{1}{\tau_1} \Rightarrow \underline{\underline{\text{possible}}}$$

✓

(b) $A_{21} = 2 \times 10^6 s^{-1}$

$$\frac{1}{\tau_1} = \frac{1}{10 \times 10^{-6} s} = 0.2 \times 10^6 s^{-1}$$

$$A_{21} > \frac{1}{\tau_1} \Rightarrow \underline{\underline{\text{impossible}}}$$

✓

use $g_2 = 4$ $g_1 = 6$

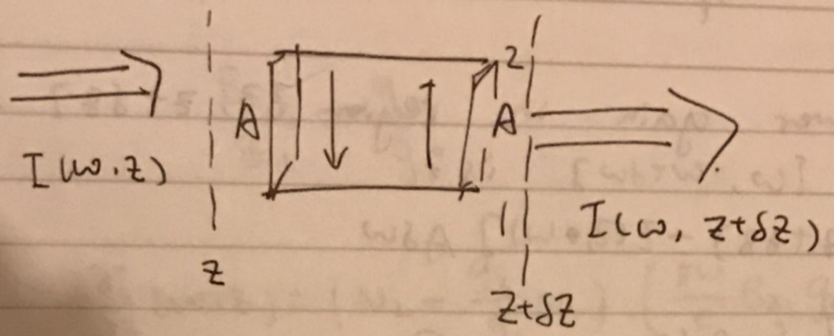
Degeneracy $= 2J + 1$

$\frac{4}{6} 2 \times 10^6 \times 10^{-5} = 13$

$g_2 = 2\left(\frac{3}{2}\right) + 1 = 4$ $\qquad > 1$

$g_1 = 2\left(\frac{5}{2}\right) + 1 = 6$ $\qquad$ Not possible

11.



→ ignore spontaneous emission as it is small in laser systems.

→ the rate ~~of~~ atoms transfer from 2 to 1 is

$$[N_2 B_{21} g_H(\omega-\omega_0)\rho(\omega,z)\delta\omega - N_1 B_{12} g_H(\omega-\omega_0)\rho(\omega,z)\delta\omega]A\delta z$$

Note that since total numbers of atoms N_1 and N_2 appear in this expression, homogeneous broadening is implied.

power gained by the beam = rate of transfer

× energy of emitted photon ($\hbar\omega$)

∴ Power at frequency range $[\omega, \omega+d\omega]$ is

$$\delta P = [N_2 B_{21} g_H(\omega-\omega_0)\delta\omega\,\rho(\omega,z) - N_1 B_{12} g_H(\omega-\omega_0)\rho(\omega,z)\delta\omega]\hbar\omega A\delta z \quad (1)$$

$$= (N_2 B_{21} - N_1 B_{12}] g_H(\omega-\omega_0)\rho(\omega,z)\hbar\omega\,\delta\omega A\delta z$$

$\Rightarrow$ $I(\omega, z) A$ = power per frequency interval carried by the beam across plane $z = z$

the power gain in region $[z, z+\delta z]$ ~~is~~ for frequency $[\omega, \omega+\delta\omega]$ is:

$$\delta P = [I(z+\delta z) - I(z, \omega)] A \delta\omega$$

$$= \frac{\partial I}{\partial z} \delta\omega A \delta z \quad (2)$$

equating (1) and (2) gives.

$$\frac{\partial I}{\partial z} = [N_2 B_{21} - N_1 B_{12}] g_H(\omega - \omega_0) \frac{\hbar\omega}{c} [\rho(\omega, c) c]$$

~~$\rho(\omega, c) \cdot c$ = (energy per volume) x (rate of change~~ of distance ~~along z) per frequency range~~

~~= rate of change of~~

$\rightarrow$ Also

$I(\omega, z) A$ = rate of energy transfer per frequency

$$= \frac{\rho \delta z A}{\delta t} = \rho c A$$

$\Rightarrow$ ~~$I(\omega, c$~~ $I(\omega, z) = \rho(\omega, z) c$

$$\therefore \frac{\partial I}{\partial z} = [N_2 B_{21} - N_1 B_{12}] g_H(\omega - \omega_0) \frac{\hbar\omega}{c} I$$

(I is specific intensity)

rewrite this equation as

$$\frac{\partial I}{\partial z}(\omega, z) = \left(N_2 - \frac{B_{12}}{B_{21}} N_1\right)\left(\frac{\hbar\omega}{c} B_{21}\, g_H\right) I$$

$$\because \quad \frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}$$

$$\therefore \quad \frac{\partial I}{\partial z}(\omega, z) = \underbrace{\left(N_2 - \frac{g_2}{g_1} N_1\right)}_{N^*} \underbrace{\left(\frac{\hbar\omega}{c} B_{21}\, g_H(\omega-\omega_0)\right)}_{\sigma_{21}(\omega-\omega_0)} I(\omega, z)$$

define $\quad N^* = N_2 - \frac{g_2}{g_1} N_1 \quad$ = population inversion density

$\sigma_{21}(\omega-\omega_0) = \frac{\hbar\omega}{c} B_{21}\, g_H(\omega-\omega_0) =$ optical gain cross-section

Using $\quad A_{21} = \frac{\hbar\omega_{21}^3}{\pi^2 c^3} B_{21} \quad \rightarrow B_{21} = \frac{\pi^2 c^3}{\hbar\omega_0^3} A_{21}$

$$\cancel{\rightarrow \sigma_{21}(\omega-\omega_0) = \frac{\pi^2 c^2}{}}$$

Assume emission occurs near $\omega \approx \omega_0$

$$\therefore \quad \sigma_{21}(\omega-\omega_0) = \frac{\hbar\omega}{c} B_{21}\, g_H(\omega-\omega_0)$$

$$\approx \frac{\hbar\omega_0}{c} B_{21}\, g_H(\omega-\omega_0)$$

$$= \frac{\hbar\omega_0}{c} \times \frac{\pi^2 c^3}{\hbar\omega_0^3} A_{21}\, g_H(\omega-\omega_0)$$

$$\rightarrow \boxed{\sigma_{21} \approx \frac{\pi^2 c^2}{\omega_0^2} A_{21}\, g_H(\omega-\omega_0)}$$

(b) → pure ~~radiative~~ lifetime Broadened transition

→ upper level decays radiatively

→ lower level long lived (doesn't decay)

~~$\frac{dN_2}{dt} = -N_2A_{21} \rightarrow N_2(t) = N_2(0)\exp(-A_{21}t)$~~

~~$\frac{1}{2}m_e\omega_0^2r^2 + \frac{1}{2}m_e\dot{r}^2 \rightarrow$ energy decays $\sim \exp(-\gamma t)$~~

⟹ we have $A_{21} = \gamma$ ← damping factor.

$$g_H(\omega-\omega_0) = \frac{1}{\pi}\frac{\gamma/2}{(\omega-\omega_0)^2+(\gamma/2)^2}$$

at peak $g_H(0) = \frac{1}{\pi}\frac{1}{\gamma/2} = \frac{2}{\pi\gamma}$

$$\sigma_{21}(0) = \frac{\pi^2c^2}{\omega_0^2}A_{21}g_H(0) = \frac{\pi^2c^2}{\omega_0^2}\gamma\frac{2}{\pi\gamma} = \frac{2\pi c^2}{\omega_0^2}$$

$\because \omega_0 = 2\pi f_0 = \frac{2\pi c}{\lambda_0}$

$$\therefore \sigma_{21}(0) = \frac{2\pi c\cdot c}{2\pi c\cdot 2\pi c}\lambda_0^2 = \boxed{\frac{\lambda_0^2}{2\pi}}$$

✓

(c) • $A_{21} = \frac{1}{\tau_2}$ = ~~…~~

$\gamma = \frac{1}{\tau_1} + \frac{1}{\tau_2}$

At peak ~~$\sigma_{21}(0) = \frac{2}{\pi\gamma}$~~ $g_H(0) = \frac{2}{\pi\gamma}$

$\therefore \quad \sigma_{21}(0) = \frac{\pi^2 c^2}{\omega_0^2} A_{21} \frac{2}{\pi\gamma} = \frac{2\pi c^2}{\omega_0^2} \frac{A_{21}}{\gamma}$

$= \frac{\lambda_0^2}{2\pi} \frac{A_{21}}{\gamma}$

$= \frac{\lambda_0^2}{2\pi} \frac{\tau_1}{\tau_1 + \tau_2}$

$= \frac{(250 \times 10^{-9})^2}{2\pi} \frac{(3 \times 10^{-14})}{(3 \times 10^{-14}) + (10 \times 10^{-9})} = \underline{\underline{2.98 \times 10^{-20}\ m^2}}$

✓

• $N^* \sigma_{21} = (0.1\ cm^{-1})$

$\rightarrow (N_2 - \frac{g_2}{g_1} N_1) \sigma_{21} = (0.1\ cm^{-1}) = (10\ m^{-1})$

Assume population inversion is large that

$N^* \simeq N_2$

$\therefore \quad N_2 \sigma_{21} = 10\ m^{-1}$

$N_2 = \frac{10}{2.98 \times 10^{-20}} = \underline{\underline{3.36 \times 10^{20} / m^3}}$

✓

• $\frac{dN_2}{dt} = -A_{21} N_2$

So the ~~rate~~ rate of formation of molecules should be $\frac{+A_{21} N_2}{N_2}$ to keep a constant population N_2

power is input to ~~excited~~ excite lower state (1) back to higher state (2)

$\therefore (10\%) P_{in} = \hbar\omega_0 A_{21} N_2 = \frac{hc}{\lambda_0} \frac{1}{\tau_2} N_2.$

$$P_{in} = \frac{10 hc N_2}{\lambda_0 \tau_2} = \underline{\underline{2.67 \times 10^{11} \ W/m^3}}$$

→ A very high power need to be input to the system to keep the population inversion

12. let R be the pumping rate

ω_L be the laser frequency ~~(assume I...)~~

(assume $I(\omega, z) = I(z)\,\delta(\omega - \omega_L)$)

~~the rate equations~~

then the rate equations $\quad (\frac{I}{c} = \rho)$

$$\frac{dN_2}{dt} = R_2 - \cancel{}(N_2 B_{21} - N_1 B_{12}) \int_0^\infty g_H(\omega - \omega_0)\, \underset{\substack{\downarrow \\ I(\omega) = I\delta(\omega - \omega_L)}}{I(\omega)}\, \frac{1}{c}\, d\omega - \frac{N_2}{\tau_2}$$

$$= R_2 - N^* \sigma_{12}(\omega_L - \omega_0)\frac{I}{\hbar\omega_L} - \frac{N_2}{\tau_2}$$

$$\therefore \frac{dN_1}{dt} = R_1 + N^*\sigma_{12}(\omega_L - \omega_0)\frac{I}{\hbar\omega_L} - \frac{N_1}{\tau_1} + N_2 A_{21}.$$

Steady state : $\frac{dN_1}{dt} = \frac{dN_2}{dt} = 0$

$$\Rightarrow \quad N_2 = R_2\tau_2 - N^*\sigma_{21}\frac{I}{\hbar\omega_L}\tau_2$$

$$N_1 = R_1\tau_1 + N^*\sigma_{21}\frac{I}{\hbar\omega_L}I_1 + N_1 A_{21}\tau_1$$

show working

$$\therefore \quad N^* = \frac{R_2\tau_2[1 - (g_2/g_1)A_{21}\tau_1] - (g_2/g_1)R\tau_1}{1 + \sigma_{21}\frac{I}{\hbar\omega_L}[\tau_2 + (\frac{g_2}{g_1})\tau_1 - (\frac{g_2}{g_1})A_{21}\tau_1\tau_2]}$$

let $N^*(0) = R_2\tau_2[1 - (\frac{g_2}{g_1})A_{21}\tau_1] - \frac{g_2}{g_1}R\tau_1$

$$\boxed{I_s = \frac{\hbar\omega_L}{\sigma_{21}}\Big(\underbrace{\tau_2 + \frac{g_2}{g_1}\tau_1 - \frac{g_2}{g_1}A_{21}\tau_1\tau_2}_{\frac{1}{\tau_R}}\Big)^{-1}} = \frac{\hbar\omega_L}{\sigma_{21}\tau_R}$$

asymmetrical

then ~~N^*_I~~ $N^*(I) = \frac{N^*(0)}{1+ I/I_s}$

the gain coefficient

$\alpha_I(\omega-\omega_0) = N^*(I)\,\sigma_{21}(\omega-\omega_0) = \frac{N^*(0)\,\sigma_{21}(\omega-\omega_0)}{1+ I/I_s}$

for $I=0$, $\alpha_0(\omega-\omega_0) = N^*(0)\,\sigma_{21}(\omega-\omega_0)$

$\therefore$ $\boxed{\alpha_I(\omega-\omega_0) = \frac{\alpha_0(\omega-\omega_0)}{1+ I/I_s(\omega-\omega_0)}}$

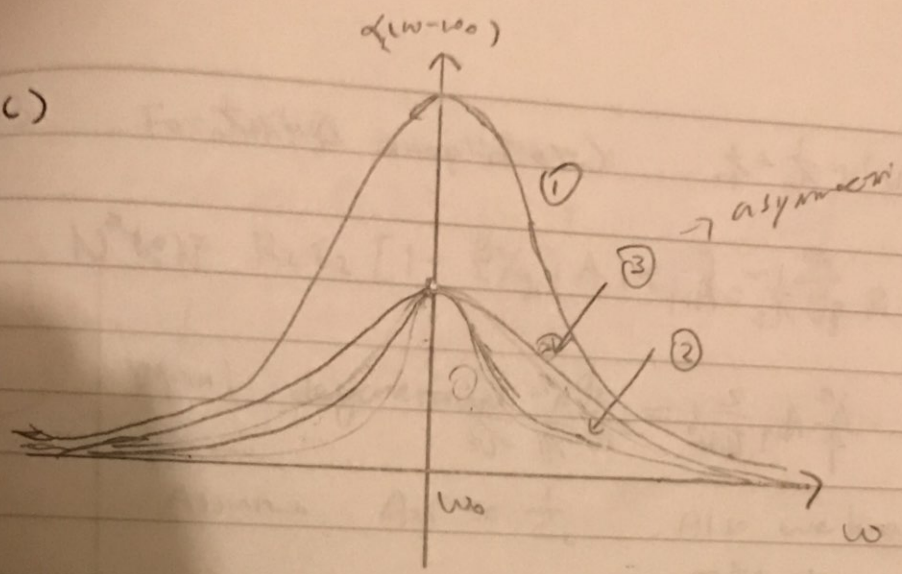
(b)

If detuning is large ($|\omega_L - \omega_0|$ is large), then stimulated emission is weak, and ~~We assume that~~ we need higher intensity of beam to achieve the same gain ~~coef~~ coefficient. So ~~I_s depend~~ at larger $\omega_L-\omega_0$, larger I will achieve same value of I/I_s.

$\Rightarrow$ I_s depends on $\omega_L-\omega_0$

✓

(c)



① weak probe, no other radiation

$$\alpha_I(\omega-\omega_0) = N^*(0)\,\sigma_{21}(\omega-\omega_0) = \alpha_0(\omega-\omega_0)$$

② weak probe , $I = I_s$

$$\therefore \alpha_I = \frac{\alpha_0}{1+I/I_s} = \frac{\alpha_0}{2}$$

③ strong probe $\alpha_I(\omega-\omega_0) = \dfrac{\alpha_0(\omega-\omega_0)}{1+I/I_s(\omega-\omega_0)}$

If $I = I_s(0)$

then $1+\dfrac{I}{I_s} = 1+\dfrac{I_s(0)}{I_s(\omega-\omega_0)} < 2$

$(\because I_s(\omega-\omega_0) > I_s(0))$

$\therefore$ in ③ the gain is reduced less than in ②

for $\omega \neq \omega_0$

13. For this amplifier

$$N^*(0) = R_2\tau_2\left[1 - (g_2/g_1)A_{21}\tau_1\right] - \frac{g_2}{g_1}R_1\tau_1$$

equal degeneracy $\frac{g_2}{g_1} = 1$

Assume $A_{21} \approx \frac{1}{\tau_2}$, Also we know $R_1 = 0$

$$\therefore \quad N^*(0) = R_2\tau_2\left[1 - \frac{\tau_1}{\tau_2}\right] = \underline{R_2(\tau_2 - \tau_1)}$$

$$I_S \approx \frac{\hbar\omega_L}{\sigma_{21}\tau_2} \qquad (A_{21} \approx \tfrac{1}{\tau_2} \rightarrow \tau_R \approx \tau_2)$$

$\cancel{I_S = \hbar\omega_L}$ $\quad I_S = \underline{\frac{hc}{\sigma_{21}\tau_2\lambda_L}}$ $\qquad \alpha_0 = N^*(0)\,\sigma_{21}$

✓

$$\frac{dI}{dz} = \alpha_\tau(\omega-\omega_0)\,I(\omega,z)$$

$$\rightarrow \quad \frac{dI}{dz} = \frac{\alpha_0}{1+I/I_S}\,dI.$$

$$\therefore \quad \int_{I(0)}^{I(z)} \frac{1 + I/I_S}{I}\,dI = \int_0^z \alpha_0\,dz$$

$$\rightarrow \quad \underline{\ln\left[\frac{I(z)}{I(0)}\right] + \frac{I(z)-I(0)}{I_S} = \alpha_0 z}$$

$\cancel{I_S = R_2(\tau_2-\tau_1) =}$

$$I_s = \cancel{} \frac{hc}{\sigma_{21}\tau_2\lambda_L} \simeq 24.9 \text{ W cm}^{-2}$$ ✓

~~$I(0) = 0.1$~~ $\quad \alpha_0 = N^*(0)\,\sigma_{21}$

$$= R_2\sigma_{21}(\tau_2 - \tau_1)$$

$$= 0.0196$$

$$z = 2\text{m} = 200 \text{ cm}$$

- $I(0) = 0.1 \text{ W cm}^{-2} \rightarrow I \ll I_s$

$$\therefore I(z) \simeq I(0)\exp(\alpha_0 z)$$

$$= (0.1)\, e^{0.0196 \cdot 200}$$

$$= \underline{\underline{5.04 \text{ W cm}^{-2}}}$$ ✓ 5.46 w/cm^2

- $I(0) = 500 \text{ W/cm}^2 \rightarrow I \gg I_s$

$$I(z) \simeq I(0) + \alpha_0 I_s z$$

$$= 500 + 0.0196 \times 24.9 \times 200$$

$$\simeq \underline{\underline{598 \text{ W cm}^{-2}}}$$ ✓

• $I(0) = 50 \text{ W cm}^{-2} \rightarrow I \sim I_s$

$$\ln\left[\frac{I(z)}{I(0)}\right] + \frac{I(z) - I(0)}{I_s} = \alpha_0 z$$

By writing

$$z = \frac{1}{0.0196}\left[\log\left(\frac{I_z}{50}\right) + \frac{I_z - 50}{24.9}\right].$$

and do trial and error

we find at $\underline{\underline{I_z = 125 \text{ W cm}^{-2}}}$, we recover

$z \simeq 200$

14. (a)

→ A 3 level laser system is a system in which the upper laser transition level is ~~the~~ intermediate between the three levels, and transition ~~to~~ is to the ground level

→ A 4 level laser is a system that the laser transition is between two intermediate levels of the 4 levels.

✓

→ Both of them are solid state lasers, The ground state is pumped ~~so population~~ to the highest level (the pump band), it then rapidly decays to the second highest level which is called upper level, it then decays radiatively to the lower level

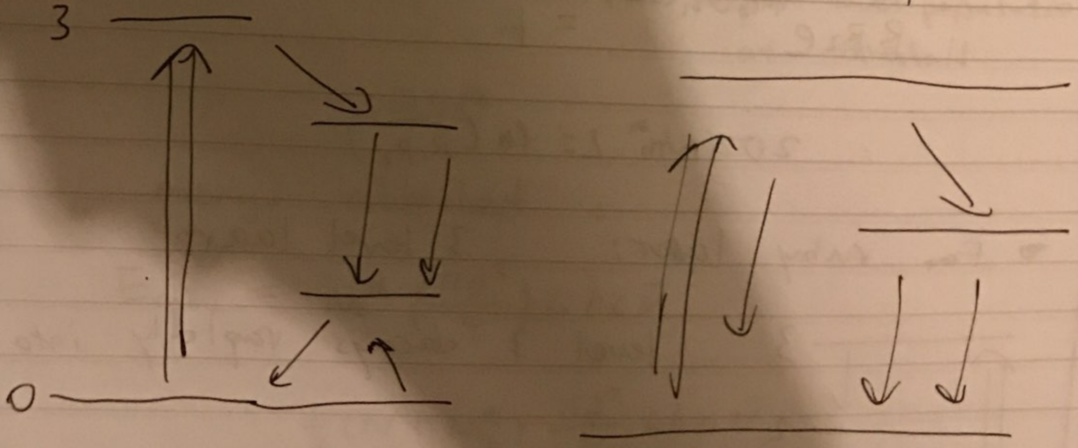
✓

→ If lower level is the ground state, then system is a 3-level laser, If not, then system is a 4-level laser.

✓

4 level

3level



→ The pump power required to achieve population inversion is much higher for 3-level laser than for 4-level laser

This is because in a 3-level laser a large fraction of the total population of active ions resides in the lower level prior to pumping. Essentially half of this population must be excited to the proper laser level for population inversion > 0. In contrast, a 4-level laser system the population in lower level prior to pumping is essentially zero. So it is only necessary to transfer sufficient ions into the upper laser level for the threshold population inversion density to be reached.

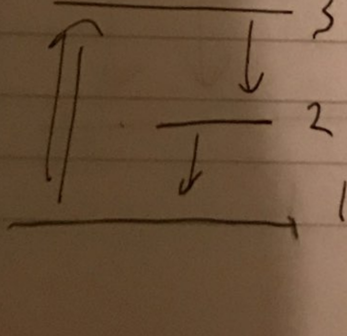
(b) At threshold, let N^*_{th} = threshold population inversion

net gain per round ~~trp~~ trip need

$$R_1R_2\, e^{N^*_{th}\sigma_{21}(2L)} = 1$$

$$\therefore\ 2\sigma_{21}N^*_{th}L = \ln\left(\frac{1}{R_1R_2}\right)$$

- For ruby laser: 3 level laser



level 3 decays rapidly into level 2

so $N_3 \simeq 0$

$\rightarrow \cancel{N = N_2 + N_1}$

Rate equations:

$$\frac{dN_3}{dt} = \Gamma_{31} N_1 - A_{31} N_3 - N_3/\tau_{coll}$$

$$\frac{dN_2}{dt} = -N^* \sigma_{21} \frac{I}{\hbar\omega_2} - A_{21} N_2 + N_3/\tau_{coll}$$

$$\frac{dN_1}{dt} = -\Gamma_{31} N_1 + N^* \sigma_{21} \frac{I}{\hbar\omega_2} + A_{21} N_2 + A_{31} N_3$$

Assuming $g_1 \simeq g_2 \simeq 1$, $N_3 \simeq 0$

$\rightarrow$ at steady state

$\cancel{N = N_3 +}$ $\quad N = N_2 + N_1$ = total density of active ion

$N^*_{th} = N_2 - N_1$ = population inversion at threshold.

$$\rightarrow N_2 = \frac{N + N^*_{th}}{2} = \text{upper level population at threshold.}$$

$\equiv N_2^{th}$

$\therefore$ energy absorbed is

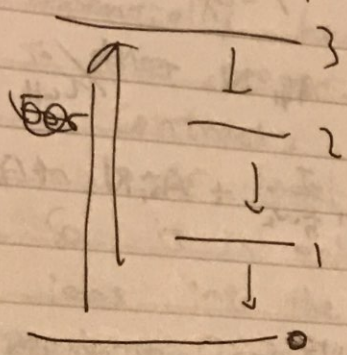
$$E_{ruby} = N_2^{th} \left(\frac{\pi d^2}{4}\right) L h c \bar{\nu}$$

$$= \frac{1}{2}(N + N^*_{th}) \frac{\pi d^2}{4} L h c \bar{\nu}$$

$$E_{ruby} = \frac{1}{2}\left(N + \frac{1}{2\sigma_{21} L}\ln\left(\frac{1}{R_1 R_2}\right)\right)\left(\frac{\pi d^2}{4}\right) L h c \bar{\nu}$$

$$= \underline{\underline{7.81\ \text{J}}}$$

For Nd:YAG laser, population of lower level prior to pumping is essentially 0



Assume lower level (level 1) has short lifetime, then

we set $N_1 \simeq 0$

$$\therefore N^{*}_{th} = N_2^{th} - N_1 = N_2^{th}.$$

$$\therefore E_{Nd} = N_2^{th}\left(\frac{\pi d^2}{4}\right) L h c \bar{\nu}$$

$$= \frac{1}{2\sigma_{21} \cancel{L}}\ln\left(\frac{1}{R_1 R_2}\right)\frac{\pi d^2}{4}\cancel{L} h c \bar{\nu}$$

$$= \frac{1}{2\sigma_{21}}\ln\left(\frac{1}{R_1 R_2}\right)\frac{\pi d^2}{4} h c \bar{\nu}$$

$$= \underline{\underline{2 \times 10^{-3}\ \text{J}}}$$

(c) In order to overcome the losses in the cavity, mirrors, geometry, electrical components, etc... the actual pump energy should be $\underline{\underline{\sim 60}}$ times higher