B2 Problem Set 4

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To Julian Merten

A1 4/4
A2 3/3
A3 1/1
A4 4/4
A5 3/3
B1 6/6
C1 9/10
D1 3/3
E1 5/5
E2 5/5

43/44 ≈ 98%

very good

Date & Time: TBD (suggestion: 18.1.2017, 2:00p) | Location: Univ – Garden 1 (full group)
Please return by: TBD - 3 days

A Revision

1. Using algebra, or otherwise, show that
   a) for any time-like vector there exists a frame in which its spatial part is zero,
   b) with one exception, any vector orthogonal to a null vector is spacelike, and describe the exception.

2. For motion under a pure (rest mass preserving) inverse square law force $\boldsymbol{f} = -\alpha \boldsymbol{r}/r^3$, where α is a constant, derive the energy equation $\gamma m c^2 - \alpha/r = \text{const.}$

3. For an isolated system of particles, let
$$s^2 = \left(\sum E_i\right)^2 - \left(\sum \boldsymbol{p}_i\right)^2 \tag{1}$$
where the sums are taken over the particles in the system at some given time. What is s for a single particle of mass m?

4. Particle tracks are recorded in a bubble chamber subject to a uniform magnetic field of 2 tesla. A vertex consisting of no incoming and two outgoing tracks is observed. The tracks lie in the plane perpendicular to the magnetic field, with radii of curvature 1.67 m and 0.417 m, and separation angle 21°. It is believed that they belong to a proton and a pion respectively. Assuming this, and that the process at the vertex is decay of a neutral particle into two products, find the rest mass of the neutral particle.

5. The far field due to an elementary wire segment dz carrying oscillating current I is given by
$$dE = \frac{I \sin\theta \, dz}{2\epsilon_0 c r \, \lambda} \cos(kr - \omega t) \tag{2}$$
Compare and contrast the case of a short antenna and the *half-wave dipole antenna.* Roughly estimate E in the the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennas?

B Reflection symmetry and angular momentum

1. The 4-angular momentum of a single particle about the origin is defined
$$L^{ab} \equiv \mathsf{X}^a \mathsf{P}^b - \mathsf{X}^b \mathsf{P}^a \tag{3}$$
   a) Prove that, in the absence of forces, $dL^{ab}/d\tau = 0$.
   b) Exhibit the relationship between the space-space part and the 3-angular momentum vector $\boldsymbol{L} = \boldsymbol{x} \wedge \boldsymbol{p}$.

c) The total angular momentum of a collection of particles about the pivot R is defined

$$L^{ab}_{\text{tot}} = \sum_i (X^a_i - R^a)\, P^b_i - (X^b_i - R^b)\, P^a_i \tag{4}$$

where the sum runs over the particles (that is, X and P are 4-vectors not 2nd rank tensors, i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

C Lagrangian mechanics

1. a) How is a canonical momentum related to a Lagrangian?

   b) Show that the Lagrangian

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{v}, t) = -\frac{mc^2}{\gamma} + q(-\phi + \boldsymbol{v} \cdot \boldsymbol{A}) \tag{5}$$

leads to the canonical momentum $(\gamma m\boldsymbol{v} + q\boldsymbol{A})$ and to the equation of motion

$$\frac{d}{dt}(\gamma m\boldsymbol{v}) = q\,(\boldsymbol{E} + \boldsymbol{v} \wedge \boldsymbol{B})\,. \tag{6}$$

In this formalism, write down the Hamiltonian function $\mathcal{H}$ for a particle of charge q moving in a magnetic field $\boldsymbol{B} = \nabla \wedge \boldsymbol{A}$. Make sure you express $\mathcal{H}$ in terms of the appropriate variables.

D Electromagnetism

1. Assuming the relation of fields $\boldsymbol{E}$, $\boldsymbol{B}$ to potentials ϕ, $\boldsymbol{A}$, show that the field tensor can be written

$$\mathbb{F}^{ab} = \partial^a \mathsf{A}^b - \partial^b \mathsf{A}^a. \tag{7}$$

(Note, the right hand side here is the 4-vector equivalent of a curl operation). [Hint: use cyclic permutation to avoid unnecessary repetition]. Now write down $\partial^c \mathbb{F}^{ab}$ in terms of ∂ operators and $\boldsymbol{A}$. By keeping track of the sequence of indices, show that

$$\partial^c \mathbb{F}^{ab} + \partial^a \mathbb{F}^{bc} + \partial^b \mathbb{F}^{ca} = 0. \tag{8}$$

(In an axiomatic approach, one could argue in the opposite direction, asserting the above as an axiom and then deriving the relation of fields to potentials).

E Field energy and momentum

1. a) The electric field in a linear accelerator is $10^6\ V/m$. Find the power emitted by an electron traveling down the accelerator. Express your result in eV per metre as- suming the electrons travel at close to the speed of light. *You may quote Larmor's formula for emitted power.*

b) A magnetic field of 1 tesla is used to maintain electrons in their orbits around a synchrotron of radius 10 m. Show that the electron energy is approximately 3 GeV. Find the radiative energy loss per revolution.

c) What is the main reason why the loss rate is so much higher in part a) than in part b)?

2. Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction. Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u and frequency ν in the two frames satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2} \tag{9}$$

(Optional: can you prove this for any relative motion of the frames? [Hint: write $\mathbb{T}^{ab}$ in terms of K^a]). A student proposes that these quantities should transform like ν'/ν not ν'^2/ν^2, on the grounds that energy-momentum $\mathsf{N} = (uc, \boldsymbol{N})$ is a 4-vector and so should transform in the same way as the wave-vector. What is wrong with this argument?

A1 a) time like vector $A = \begin{pmatrix} A_0 \\ \underline{A} \end{pmatrix} \Rightarrow -A_0^2 + \underline{A} \cdot \underline{A} < 0$

To find the frame such that the spatial part of A is $\underline{0}$:

→ First we rotate the axes so that the spatial part of A points towards the x-axis. Now A becomes $A = \begin{pmatrix} A_0 \\ A_x \\ 0 \\ 0 \end{pmatrix}$

→ Then we apply a Lorentz boost : $A' = \Lambda A$

$$A' = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma A_0 - \gamma\beta A_x \\ -\gamma\beta A_0 + \gamma A_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(A_0 - \beta A_x) \\ \gamma(A_x - \beta A_0) \\ 0 \\ 0 \end{pmatrix}$$

We want A' to have zero spatial part, so need $A_x = 0$

$\therefore A_x - \beta A_0 = 0 \Rightarrow \beta = \frac{A_x}{A_0}$

$\because$ A is a time like vector $\therefore -A_0^2 + \underline{A} \cdot \underline{A} = -A_0^2 + A_x^2 < 0$

$\therefore \left(\frac{A_x}{A_0}\right)^2 < 1 \quad \therefore \left|\frac{A_x}{A_0}\right|^2 < 1 \Rightarrow -1 < \frac{A_x}{A_0} < 1$ ✓ 1P

Hence to make the spatial part of a time like vector vanish,

we need $-1 < \beta < 1$. If we only consider the magnitude of velocity

then $\beta < 1$.

→ This is always possible because $\beta = \frac{v}{c}$, and $\beta < 1 \Leftrightarrow v < c$

So QED #

b) null vector $A = \begin{pmatrix} A_0 \\ \underline{A} \end{pmatrix} \Rightarrow -A_0^2 + \underline{A}\cdot\underline{A} = 0 \Rightarrow A_0^2 = \underline{A}\cdot\underline{A} = |\underline{A}|^2$

A vector orthogonal to null vector A : $B \cdot A = 0$ ✓ 1P

$\rightarrow \begin{pmatrix} B_0 \\ \underline{B} \end{pmatrix} \cdot \begin{pmatrix} A_0 \\ \underline{A} \end{pmatrix} = 0 \quad \therefore -A_0 B_0 + \underline{A}\cdot\underline{B} = 0 \Rightarrow A_0 B_0 = \underline{A}\cdot\underline{B}$

$\rightarrow A_0 B_0 = |\underline{A}||\underline{B}|\cos\theta$ (θ is the angle between $\underline{A}$ and $\underline{B}$)

Square the above equation $\Rightarrow A_0^2 B_0^2 = |\underline{A}|^2 |\underline{B}|^2 \cos^2\theta$

$\because |\underline{A}|^2 = A_0^2 \quad \therefore B_0^2 = |\underline{B}|^2 \cos^2\theta$ ✓ 1P

So $-B_0^2 + \underline{B}\cdot\underline{B} = -B_0^2 + |\underline{B}|^2 < 0$ i.e. $\underline{B}$ is <u>space-like</u>

<u>except</u> if $\cos\theta = 1$, in which case $B_0^2 = \underline{B}\cdot\underline{B}$, and $\underline{B}$ is another null vector ✓ 1P

QED #

$$F = \frac{dP}{d\tau} = \left(\frac{1}{c}\frac{dE}{d\tau}, \frac{d\vec{P}}{d\tau} \right) = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\vec{f} \right)$$

$$\left(\vec{f} = \frac{d\vec{P}}{dt} \right)$$

$$U \cdot F = \gamma^2 \left(-\frac{dE}{dt} + \underline{u} \cdot \underline{f} \right) \Rightarrow \underline{u} = 0$$

pure force : $\frac{dm}{d\tau} = 0$

A2

$$U \cdot F = -c^2 \frac{dm}{d\tau}$$

rest mass persevering

$\because \vec{f} = -\frac{\alpha \vec{r}}{r^3}$ is a pure force $\therefore \frac{dE}{dt} = \underline{f} \cdot \underline{u}$ 1P

$\underline{u}$ = velocity $= \dot{\vec{r}}$ $\therefore \frac{dE}{dt} = -\frac{\alpha \vec{r} \cdot \dot{\vec{r}}}{r^3} = -\frac{\alpha}{2} \frac{1}{r^3} \frac{d}{dt}(r^2)$ ①

Integrate ① with respect to time $\Rightarrow$

$$E = \text{const} - \frac{\alpha}{2} \int \frac{1}{r^3} \frac{d}{dt}(r^2)\, dt = \text{const} - \frac{\alpha}{2} \int \frac{1}{r^3} d(r^2)$$ 1P

$$= \text{const} - \frac{\alpha}{\cancel{2}} \int \frac{1}{r^{\cancel{3}_2}} \cancel{2r}\, dr = \text{const} - \alpha \int \frac{1}{r^2} dr$$

$$= \text{const} + \frac{\alpha}{r}$$

~~strong force~~

~~other~~

$\because E = \gamma m c^2 \quad \left(\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$

1P

$\therefore \underline{\gamma m c^2 - \frac{\alpha}{r} = \text{const}}$ QED #

pure force $\Rightarrow$ any force you can treat the object as a point like particle

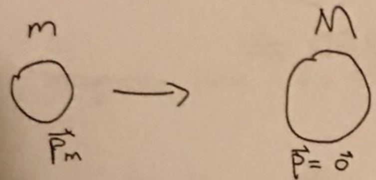
non pure force $\Rightarrow$ object has some structure

A3

→For a single particle of mass m :

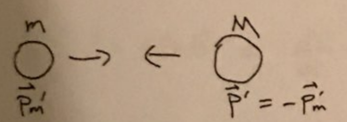
$$S^2 = E_1^2 - P_1^2 = E^2 - p^2 = \gamma^2 m^2 - \beta^2\gamma^2 m^2 = \gamma^2(1-\beta^2)m^2$$

$\because \gamma^2 = \frac{1}{1-\beta^2}$ $\therefore S^2 = m^2 \Rightarrow \underline{\underline{S = m}}$ ✓ 7P



Lab Before

S^2 is invariant because it is a dot product of 4-vector P_{tot} with itself ($S^2 = -P_{tot} \cdot P_{tot}$)



CM before

Lab frame : $S^2 = (\Sigma E_i)^2 - (\Sigma \vec{P}_i)^2 =$

$$(\sqrt{P_m^2 + m^2} + M)^2 - P_m^2$$

$$= \cancel{P_m^2} + m^2 + M^2 + 2M\sqrt{P_m^2 + m^2} - \cancel{P_m^2}$$

$$= m^2 + M^2 + 2M\sqrt{P_m^2 + m^2}$$

CM frame : $S^2 = m^2 + M^2 + 2M\sqrt{P_m^2 + m^2}$, and total momentum

$$\Sigma \vec{P}_i = \vec{P}_m' + \vec{P}' = \vec{0}$$

$\therefore$ Available energy $(\Sigma E_i')^2 = S^2$

$$\therefore \Sigma E_i' = (m^2 + M^2 + 2M\sqrt{P_m^2 + m^2})^{\frac{1}{2}} = \underline{\underline{S}}$$

the velocity of CM is $\beta = \frac{P_{tot}}{E_{tot}} = \frac{\Sigma P_i}{\Sigma E_i} = \frac{P_m}{\sqrt{P_m^2 + S^2}}$

Lorentz transform the momentum of M :

$P = 0 \quad P' = -P_m' \qquad P' = \gamma(P - \beta E)$, with $P = 0$, $E = M$

$\therefore P' = -\gamma\beta M \Rightarrow P_m' = \gamma\beta M$

$$P_m' = \frac{\beta}{\sqrt{1-\beta^2}} M = \frac{\frac{P_m}{\sqrt{P_m^2+S^2}} M}{\sqrt{1-\frac{P_m^2}{P_m^2+S^2}}} = \frac{P_m M}{(\cancel{P_m^2}+S^2-\cancel{P_m^2})^{1/2}} = \frac{P_m M}{S}$$

QED #

$P = 0 \quad P' = -P_m' \qquad P' = \gamma(P - \beta E)$, with $P = 0$, $E = M$

$\therefore P' = -\gamma\beta M \implies P_m' = \gamma\beta M$

$$P_m' = \frac{\beta}{\sqrt{1-\beta^2}} M = \frac{\frac{P_m}{\sqrt{P_m^2+S^2}} M}{\sqrt{1-\frac{P_m^2}{P_m^2+S^2}}} = \frac{P_m M}{(\cancel{P_m^2}+S^2-\cancel{P_m^2})^{1/2}} = \frac{P_m M}{S}$$

QED #

A4

$\vec{f} = \frac{d\vec{p}}{dt}$, $\vec{p} = \gamma m_0 \vec{v}$ $\therefore$ $\vec{f} = \frac{d}{dt}(\gamma m_0 \vec{v}) = \frac{d\gamma}{dt} m_0 \vec{v} + \gamma \frac{dm_0}{dt}\vec{v} + \gamma m_0 \frac{d\vec{v}}{dt}$

$\Rightarrow$ for a pure force $\frac{dm_0}{dt} = 0$ (magnetic force is pure)

$\therefore$ $\vec{f} = \frac{d\gamma}{dt} m_0 \vec{v} + \gamma m_0 \frac{d\vec{v}}{dt} = \frac{d\gamma}{dt} m_0 \vec{v} + \gamma m_0 \vec{a}$

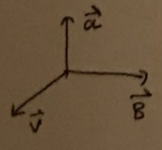
magnetic force $\vec{f} = q\vec{v} \times \vec{B}$, for a pure force $\frac{dE}{dt} = \vec{f}\cdot\vec{v}$

$\because$ $E = \gamma m_0 c^2$ $\therefore$ $\frac{dE}{dt} = m_0 c^2 \frac{d\gamma}{dt}$ $\Rightarrow$ $\frac{d\gamma}{dt} = \frac{1}{m_0 c^2}\frac{dE}{dt} = \frac{\vec{f}\cdot\vec{v}}{m_0 c^2}$

for $\vec{f} = q\vec{v}\times\vec{B}$ $\Rightarrow$ $\frac{d\gamma}{dt} = \frac{1}{m_0 c^2}\underbrace{(q\vec{v}\times\vec{B})\cdot\vec{v}}_{0} = 0$

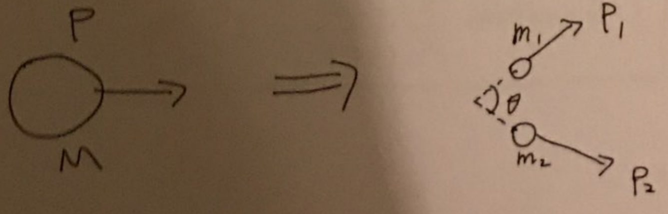
$\Rightarrow$ $\vec{f} = \gamma m_0 \vec{a}$ $\therefore$ $q\vec{v}\times\vec{B} = \gamma m_0 \vec{a}$ If $\vec{v} \perp \vec{B}$ (as in the question),

$qvB = \gamma m_0 a$, and $\vec{v} \perp \vec{a}$ $\Rightarrow$ circular motion $a = \frac{v^2}{r}$ ← radius



$\therefore$ $\frac{v^2}{r} m_0 \gamma = qvB$ $\Rightarrow$ $r = \frac{\gamma m_0 v}{qB} = \frac{p}{qB}$ ← momentum

$\therefore$ $p = qBr$ is the relation between magnitude of momentum and radius of curvature for a particle in magnetic field



(let $c = 1$ $\therefore$ $P = \binom{E}{\vec{p}}$)

Consider two-body decay $\Rightarrow$

Conservation of 4-momentum : $P = P_1 + P_2$

Taking scalar product of each side : $P \cdot P = P^2 = P_1^2 + P_2^2 + 2P_1 \cdot P_2$

$$\Rightarrow \quad -M^2 = -m_1^2 - m_2^2 + 2\begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} \cdot \begin{pmatrix} E_2 \\ \vec{p}_2 \end{pmatrix}$$

$$\therefore \quad M^2 = m_1^2 + m_2^2 + 2(E_1E_2 - \vec{p}_1 \cdot \vec{p}_2)$$

$$\therefore \quad M^2 = m_1^2 + m_2^2 + 2(E_1E_2 - p_1p_2\cos\theta)$$ 7P $\quad (E_1^2 = p_1^2 + m_1^2 ; \; E_2^2 = p_2^2 + m_2^2)$

particle 1 : proton p^+
particle 2 : pion π^-

neutral particle : $M \rightarrow p^+ + \pi^-$

$B = 2T \quad r_1 = 1.67m$

$q_1 = q_2 = e \quad r_2 = 0.417\,m$

$p_1 = eBr_1 = (2)(1.67)(3\times10^8) = 1002 \frac{MeV}{c}$

$p_2 = eBr_2 = (2)(0.417)(3\times10^8) = 250.2 \frac{MeV}{c}$ 7P

$m_1 = 938.28 \frac{MeV}{c^2}$

$m_2 = 139.57 \frac{MeV}{c^2}$

$\theta = 21^\circ$

$$\Rightarrow M = \left[(m_1^2 + m_2^2 + 2(E_1E_2 - p_1p_2\cos\theta))\right]^{\frac{1}{2}}$$

$$= \underline{1103.8 \; MeV/c^2}$$ 7P

A5

The far field due to wire segment dz is

$$dE = \frac{I \sin\theta}{2\varepsilon_0 c r}\frac{dz}{\lambda}\cos(kr - \omega t)$$

→ Short Antenna :

The current oscillations are maximal at the centre and zero at the ends.

If Antenna is short we can approximate the current distribution as roughly linear

If length of Antenna $= L$, let $I(z) = I_0\left(1 - 2\frac{|z|}{L}\right)$

So $$E = \frac{\sin\theta}{2\varepsilon_0 c r \lambda}\cos(kr-\omega t)\, I_0 \int_{-L/2}^{L/2} \left(1 - 2\frac{|z|}{L}\right) dz$$

where $$\int_{-L/2}^{L/2}\left(1 - 2\frac{|z|}{L}\right) dz = 2\int_0^{L/2} 1 - \frac{2z}{L}\, dz = 2\left[z - \frac{z^2}{L}\right]_0^{L/2}$$

$$= 2\left(L/2 - L/4\right) = \frac{L}{2}$$

⇒ $$\boxed{E = \frac{I_0 L \sin\theta}{4\varepsilon_0 c r \lambda}\cos(kr - \omega t)}$$

✓ 7P

→ half-wave dipole antenna :

This is a centre fed antenna of length $\frac{\lambda}{2}$

The current distribution is a standing wave $I(z) = I_0 \cos(kz)$

So $E = \frac{\sin\theta}{2\varepsilon_0 c r \lambda} \cos(kr-\omega t)\, I_0 \int_{-\lambda/4}^{\lambda/4} \cos(kz)\, dz$

Where $\int_{-\lambda/4}^{\lambda/4} \cos(kz)\, dz = \frac{1}{k} \sin(kz) \Big|_{-\lambda/4}^{\lambda/4} = \frac{\sin(\frac{\pi}{2}) + \sin(\frac{\pi}{2})}{2\pi/\lambda} = \frac{\lambda}{\pi}$

$\underbrace{}_{k=\frac{2\pi}{\lambda}}$

$\therefore \boxed{E = \frac{I_0 \sin\theta}{2\pi \varepsilon_0 c r} \cos(kr-\omega t)}$ ✓ 1p

→ For antenna longer than $\frac{\lambda}{2}$, futher increase in length alter the directional distribution of the radiation field significantly, rather than the total emitted power. ✓ 1p

Also, as the antenna gets longer, diffraction effect becomes more significant.

B1

a)

4-Angular Momentum: $L^{ab} \equiv x^a p^b - x^b p^a$

$$\frac{dL^{ab}}{d\tau} = p^b \frac{dx^a}{d\tau} + x^a \frac{dp^b}{d\tau} - p^a \frac{dx^b}{d\tau} - x^b \frac{dp^a}{d\tau}$$; In the absence of forces,

1P

$$F = \frac{dP}{d\tau} = 0 \rightarrow \frac{dL^{ab}}{d\tau} = p^b \frac{dx^a}{d\tau} - p^a \frac{dx^b}{d\tau} = p^b U^a - p^a U^b = m_0 U^b U^a - m_0 U^a U^b$$
$$= 0$$ QED

1P

b)

→ The space-space part of L^{ab} is $\begin{pmatrix} L^{11} & L^{12} & L^{13} \\ L^{21} & L^{22} & L^{23} \\ L^{31} & L^{32} & L^{33} \end{pmatrix}$

$\because L^{ab}$ is antisymmetric by definition $\therefore L^{11} = L^{22} = L^{33} = 0$

$$X = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad P = \begin{pmatrix} E/c \\ P_x \\ P_y \\ P_z \end{pmatrix} \Rightarrow L^{12} = -L^{21} = x^1 p^2 - x^2 p^1 = xP_y - yP_x$$

$$L^{13} = -L^{31} = xP_z - zP_x$$

1P

$$L^{23} = -L^{32} = yP_z - zP_y$$

→ the 3-angular momentum vector $\vec{L} = \vec{x} \times \vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$

$$\Rightarrow \vec{L} = \begin{vmatrix} i & j & k \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = \begin{pmatrix} yP_z - zP_y \\ zP_x - xP_z \\ xP_y - yP_x \end{pmatrix} = \begin{pmatrix} L^{23} \\ L^{31} \\ L^{12} \end{pmatrix}$$

1P

⇒ space-space part of L^{ab} is related to $\vec{L}$ by $\vec{L} = \begin{pmatrix} L^{23} \\ L^{31} \\ L^{12} \end{pmatrix}$

$\because L^{ab}$ is constant along worldline

$\therefore$ from the space-space part: $\vec{L}$ is conserved

c) $L^{ab}_{tot} = \sum_i (x_i^a - R^a) P_i^b - (x_i^b - R^b) P_i^a = L^{ab}_{tot}(R)$

$$= \underbrace{\sum (x^a p^b - x^b p^a)}_{\equiv L^{ab}_{tot}(0)} + (R^a \sum P^b - R^b \sum P^a)$$

✓ 7p

$$= L^{ab}_{tot}(0) + R^a \sum p^b - R^b \sum p^a$$

In CM frame, the space part of P is 0

So $L^{ab}_{tot}(R) = L^{ab}_{tot}(0)$ if $a, b \neq 0$ ✓ 7p

→ The space-space part of $L^{ab}_{tot}(R)$ is independent of R in CM frame

→ The 3-angular momentum $\vec{L}$ is independent of pivot in CM frame. // QED #

C1

a) The canonical momentum of the generalised coordinate q_i is $\tilde{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, where $\mathcal{L}$ is the Lagrangian and $\dot{q}_i = \frac{dq_i}{dt}$ 1P

b) The Lagrangian $\mathcal{L}(\vec{x}, \vec{v}, t) = -\frac{mc^2}{\gamma} + q(-\phi + \vec{v}\cdot\vec{A})$ $\quad (\vec{v} = \dot{\vec{x}})$

Canonical momentum $\vec{\tilde{p}}_{\vec{x}} = \vec{\nabla}_{\vec{v}}(\mathcal{L}) = \begin{pmatrix} \frac{\partial}{\partial v_x} \\ \frac{\partial}{\partial v_y} \\ \frac{\partial}{\partial v_z} \end{pmatrix} \mathcal{L} = \frac{\partial}{\partial \vec{v}} \mathcal{L}$

$$\therefore \vec{\tilde{p}}_{\vec{x}} = \frac{\partial}{\partial \vec{v}} \mathcal{L} = \frac{\partial}{\partial \vec{v}} \left(-mc^2 \left(1 - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{\frac{1}{2}} + q(-\phi + \vec{v}\cdot\vec{A})\right)$$ 1P

$$= \cancel{+} m\cancel{c^2}\left(\frac{1}{\cancel{2}}\right)\underbrace{\left(1 - \frac{\vec{v}\cdot\vec{v}}{c^2}\right)^{-\frac{1}{2}}}_{\gamma}\left(+\frac{1}{\cancel{c^2}}\right)\underbrace{(2\vec{v})}_{\vec{\nabla}_{\vec{v}}(\vec{v}\cdot\vec{v}) = 2\vec{v}} + \underbrace{q\vec{A}}_{\vec{\nabla}_{\vec{v}}(\vec{v}\cdot\vec{A}) = \vec{A}}$$ 1A

$$= \gamma m \vec{v} + q\vec{A}$$ 1A

QED #

The Euler-Lagrange equation : $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$ 1P

$$\therefore \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}}\right) = \frac{\partial \mathcal{L}}{\partial \vec{x}} \qquad \therefore \frac{d}{dt}\left(\gamma m \vec{v} + q\vec{A}\right) = q(-\vec{\nabla}\phi + \vec{\nabla}(\vec{v}\cdot\vec{A}))$$

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{A}$$ 1P

$$\Rightarrow \frac{d}{dt}(\gamma m \vec{v}) = -q\underbrace{\left(\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}\right)}_{= -\vec{E}} + q\underbrace{\left(\vec{\nabla}(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\vec{\nabla})\vec{A}\right)}_{= \vec{v}\times(\vec{\nabla}\times\vec{A}) = \vec{v}\times\vec{B}}$$ 1P 1P

$$\Rightarrow \frac{d}{dt}(\gamma m \vec{v}) = q(\vec{E} + \vec{v}\times\vec{B})$$

QED #

The Hamiltonian $H(q_i, \vec{p}_i, t) = \sum_i \vec{p}_i \dot{q}_i - \mathcal{L}(q_i, \vec{p}_i, t)$

$$= \vec{P}_{\dot{x}} \cdot \vec{v} - \mathcal{L}$$

✓ 1P

$$= (\gamma m \vec{v} + q\vec{A}) \cdot \vec{v} + \frac{mc^2}{\gamma} + q(\phi - \vec{v} \cdot \vec{A}) = \gamma mc^2 + q\phi$$

$$= (m_0^2 c^4 + p^2 c^2)^{\frac{1}{2}} + q\phi$$

$$\Rightarrow \quad \underline{H = \left((\vec{p} - q\vec{A})^2 c^2 + m_0^2 c^4 \right)^{1/2} + q\phi}$$

why are these two lines equal?

For magnetic field only: $\underline{H = ((\vec{p} - q\vec{A})^2 c^2 + m_0^2 c^4)^{1/2}}$

+1P

D1

$$A^a = \begin{pmatrix} \phi/c \\ \vec{A} \end{pmatrix} \quad \mathbb{F}^{ab} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \quad \partial^a = \begin{pmatrix} -\frac{1}{c}\partial_t \\ \vec{\nabla} \end{pmatrix}$$

Consider $\partial^a A^b - \partial^b A^a$:

All the diagonal elements $(a=b)$ vanish because the tensor is antisymmetric

$$\text{For } \begin{matrix} a=0 \\ b=1,2,3 \end{matrix} \Rightarrow \partial^0 A^b - \partial^b A^0 = -\frac{1}{c}\partial_t A^b - \partial^b \frac{\phi}{c}$$

$$= \frac{1}{c}\left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)^b$$

$$= \left(\frac{\vec{E}}{c}\right)^b$$

By antisymmetry : $\text{For } \begin{matrix} a=1,2,3 \\ b=0 \end{matrix} \Rightarrow \partial^a A^0 - \partial^0 A^a = \left(-\frac{\vec{E}}{c}\right)^a$

For $a, b \neq 0 \Rightarrow$

If $a=1,\ b=2$ $\quad \partial^1 A^2 - \partial^2 A^1 = \partial_x A_y - \partial_y A_x = (\vec{\nabla}\times\vec{A})_z = B_z = -(\partial^2 A^1 - \partial^1 A^2)$

By cyclic permutation : $\partial^1 A^3 - \partial^3 A^1 = -B_y = -(\partial^3 A^1 - \partial^1 A^3)$

$\partial^2 A^3 - \partial^3 A^2 = B_x = -(\partial^3 A^2 - \partial^2 A^3)$

$$\therefore \partial^a A^b - \partial^b A^a = \left(\begin{array}{c|ccc} 0 & & \vec{E}/c & \\ \hline & 0 & B_z & -B_y \\ -\frac{\vec{E}}{c} & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{array}\right) = \mathbb{F}^{ab}$$

QED#

$$\rightarrow \partial^c \mathbb{F}^{ab} = \partial^c(\partial^a A^b - \partial^b A^a) = \underline{\underline{\partial^c\partial^a A^b - \partial^c\partial^b A^a}}$$

$$\partial^c F^{ab} + \partial^a F^{bc} + \partial^b F^{ca} = \cancel{\partial^c \partial^a A^b} - \cancel{\partial^c \partial^b A^a} + \cancel{\partial^a \partial^b A^c} - \cancel{\partial^a \partial^c A^b}$$

$$+ \cancel{\partial^b \partial^c A^a} - \cancel{\partial^b \partial^a A^c}$$

$$\underset{\partial^a \partial^b = \partial^b \partial^a}{=} \underline{\underline{0}} \quad \text{QED}$$

second order derivatives commute

7p

E1

a) $\because$ Velocity parallel to acceleration

$\therefore$ force is the same in lab frame and instantaneous rest frame $\Rightarrow$ $f = eE = m_0 a_0$ $\Rightarrow$ $a_0 = \frac{eE}{m_0}$ (1p)

$\Rightarrow$ proper acceleration $a_0 = \frac{(1.6\times10^{-19})(10^6)}{(9.11\times10^{-31})} = 1.76\times10^{17}\ m/s^2$

Larmor's formula for arbitrary velocity :

$$P_L = \frac{2}{3}\frac{q^2 a_0^2}{4\pi\varepsilon_0 c^3} = \frac{e^2 a_0^2}{6\pi\varepsilon_0 c^3} = 1.76\times10^{-19}\ W$$

$$= \underline{1.1\ eV/s}$$

$\because$ Electron speed $\sim c$

$\therefore$ Power in eV per metre $= \frac{P_L dt}{dx} \approx \frac{P_L}{c} = \underline{3.67\times10^{-8}\ eV/m}$ (1p)

b) In general $P_L = \frac{2}{3}\frac{q^2}{4\pi\varepsilon_0 c^3}\gamma^6\left(a^2 - \frac{(\vec{v}\times\vec{a})^2}{c^2}\right)$

For circular motion : $|\vec{v}\times\vec{a}| = va$, $a = \frac{v^2}{r}$

$\rightarrow P_L = \frac{e^2}{4\pi\varepsilon_0 c^2}\gamma^6 \underbrace{(1-\frac{v^2}{c^2})}_{\frac{1}{\gamma^2}}\left(\frac{v^2}{r}\right)^2 = \frac{e^2}{4\pi\varepsilon_0 c^2}\frac{\gamma^4 v^4}{r^2}$

$\rightarrow$ Radiative loss per unit revolution is

$$\Delta E = P_L \cdot \overset{\text{period}}{T} = P_L\left(\frac{2\pi r}{v}\right) = \frac{e^2}{3\varepsilon_0 r}\gamma^4\left(\frac{v}{c}\right)$$

Electron energy $E = \gamma m c^2 = (p^2c^2 + m_0^2c^4)^{1/2}$

Electron momentum in circular motion $p = eBr = 1.6\times10^{-18}\ kg\,m/s$

$\therefore E = 4.8\times10^{-10}\,J = \underline{3\,GeV}$ ✓ 7P

$\therefore \gamma = \frac{E}{m_0c^2} = 5854$ This means $v \rightarrow c$

$\therefore \Delta E = \frac{e^2\gamma^4}{3\varepsilon_0 r} = 1.132\times10^{-13}\,J = \underline{7\times10^5 eV}$ ✓ 7P

c) loss rate is higher in b) than in a) because

$\vec{f_\perp} = \gamma m \vec{a_\perp}$ and $\vec{f_\parallel} = \gamma^3 m \vec{a_\parallel}$ $\rightarrow$ for same magnitude

✓ 7P

of force, longtitudinal force provides larger acceleration than transverse one does by a factor of γ^2. And large acceleration means large radiative power.

E2

For Electromagnetic plane wave in X-direction,

Stress-energy tensor : $\mathbb{T}^{ab} = \varepsilon_0 E_0^2 \cos^2(kx-\omega t) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ✓ 1P

4 - wave vector : $K = \begin{pmatrix} k \\ k \\ 0 \\ 0 \end{pmatrix}$

Between any relative motion of the frames:

In 4-vector notation : $\mathbb{T}^{ab} = \varepsilon_0 c^2 \frac{E_0^2}{\omega^2} \cos(X_\nu K^\nu) K^a K^b$ ✓ 1P

$\because \mathbb{T}^{ab}$ and $K^a K^b$ are second rank tensors

$\therefore$ By the quotient rule, $\frac{E_0^2}{\omega^2}$ is a Lorentz invariant no matter what the transformation of 2nd rank tensor is (Λ may not be the same as in standard configuration) 1P

$\rightarrow \frac{E_0'^2}{\omega'^2} = \frac{E_0^2}{\omega^2} \Rightarrow$ frequency $\frac{\nu'^2}{\nu^2} = \frac{(\omega'/2\pi)^2}{(\omega/2\pi)^2} = \frac{\omega'^2}{\omega^2} = \frac{E_0'^2}{E_0^2}$ 1P

$\mathbb{T}^{ab} = \left(\begin{array}{c|c} u & \vec{g}c \\ \hline \vec{g}c & \sigma_{ij} \end{array}\right) = \left(\begin{array}{c|ccc} u & gc & 0 & 0 \\ \hline gc & p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad K = \begin{pmatrix} k \\ k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} w/c \\ w/c \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \quad \frac{u'}{u} = \frac{K^{0'}K^{0'}}{K^0 K^0} = \frac{\omega'^2}{\omega^2}$$

$$\frac{g'}{g} = \frac{K^{0'}K^{1'}}{K^0 K^1} = \frac{\omega'^2/c^2}{\omega^2/c^2} = \frac{\omega'^2}{\omega^2}$$

$$\frac{P'}{P} = \frac{K^{1'}K^{1'}}{K^1 K^1} = \frac{\omega'^2/c^2}{\omega^2/c^2} = \frac{\omega'^2}{\omega^2}$$

Hence $\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2}$

$\rightarrow N = \begin{pmatrix} uc \\ \vec{N} \end{pmatrix}$ is <u>not</u> a 4-vector, So the student is wrong

✓ 7/8

$$\frac{u'}{u} = \frac{K^{0'}K^{0'}}{K^0 K^0} =$$

$K^0 K$