Date & Time: 26.10.2016, 14:00 | Location: Univ – 91A | Please return by: 24.10.2016

Spacetime and Lorentz transformation

1. The Lorentz transformation Λ is defined such that $\Lambda^T g \Lambda = g$ where g is the Minkowski metric, taken as $g = \text{diag}(-1, 1, 1, 1)$. Show that for any pair of 4-vectors A, B, the scalar product $\mathsf{A} \cdot \mathsf{B} \equiv \mathsf{A}^T g \mathsf{B}$ is Lorentz-invariant.

2. Using a spacetime diagram, or otherwise, prove that
   a) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval,
   b) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval.

3. Define *proper time*. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by (ct, x, y, z) in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

4. Two particles have velocities $\mathbf{u}$, $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative velocity $\mathbf{w}$ is given by

$$\gamma(w) = \gamma(u)\gamma(v)\left(1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right) \tag{1}$$

   Prove this twice, by using each of the following two methods:
   a) In the given frame, the worldline of the first particle is $\mathsf{X} = (ct, \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

$$t' = \gamma_v t\left(1 - \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}\right) \tag{2}$$

   Obtain dt'/dt and apply the result of the previous question.
   b) Use the invariant $\mathsf{U} \cdot \mathsf{V}$, first showing that it is equal to $-c^2\gamma(w)$.

Doppler effect

1. The emission spectrum from a source in the sky is observed to have a periodic fluctuation, as shown in the data displayed in figure 1. It is proposed that the source is a binary star system. Explain how this could give rise to the data. Extract an estimate for the component of orbital velocity in the line of sight and, assuming the stars have equal mass, estimate the distance between them and their mass.

2. **Moving mirror.** A plane mirror moves uniformly with velocity $\mathbf{v}$ in the direction of its normal in a frame S. An incident light ray has angular frequency ω_i and is reflected with angular frequency ω_r.

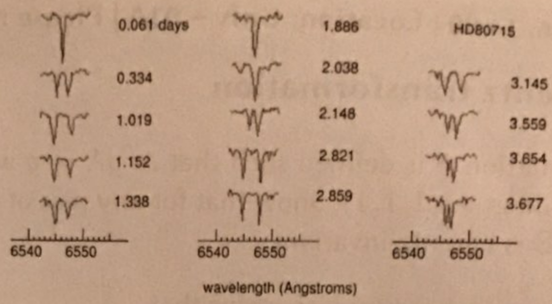

Figure 1: Spectra of light received from a astronomical object at specific times during an observation period of a few days.

a) Show that

$$\omega_i \sin\theta_i = \omega_r \sin\theta_r \tag{3}$$

where θ_i, θ_r are the angles of incidence and reflection.

b) Also show that

$$\frac{\tan(\theta_i/2)}{\tan(\theta_r/2)} = \frac{1+v/c}{1-v/c}. \tag{4}$$

[Hint: First establish by trigonometric manipulation that $\cos\theta = (1-t^2)/(1+t^2)$ where $t = \tan(\theta/2)$, then employ this in the Doppler formula relating $\cos\theta$ to $\cos\theta_0$ in order to obtain a relation between t and t_0. Then apply this relation to the two rays.]

Motion under a given force

1. **Twin paradox.**

   a) Evaluate the acceleration due to gravity at the Earth's surface in units of light years.

   b) In the twin paradox, the travelling twin leaves Earth on board a spaceship undergoing motion at constant proper acceleration of $9.8\ \mathrm{m/s^2}$. After 5 years of proper time for the spaceship, the direction of the rockets are reversed so that the spaceship accelerates towards Earth for 10 proper years. The rockets are then reversed again to allow the spaceship to slow and come to rest on Earth after a further 5 years of spaceship proper time. How much does the traveling twin age? How much does the stay-at-home twin age?

2. **Constant force.** Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows.

   a) Write down the solution for $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$.

   b) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\alpha = (\mathbf{p}_0 + \mathbf{f}t)/mc$.

   c) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.

d) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

$$x = \frac{c}{f}\left(m^2c^2 + p_0^2 + f^2t^2\right)^{1/2} + \text{const} \tag{5}$$

$$y = \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const} \tag{6}$$

where you may quote that $\int (a^2 + t^2)^{-1/2}\, dt = \log\left(t + \sqrt{a^2 + t^2}\right)$

e) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation.

[N.B. The calculation as a function of proper time is best done another way, see later problems].

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B2 Problem Set 1 Ziyan Li

Spacetime / Lorentz Transformation:

(1) Results of transformation:

1/1

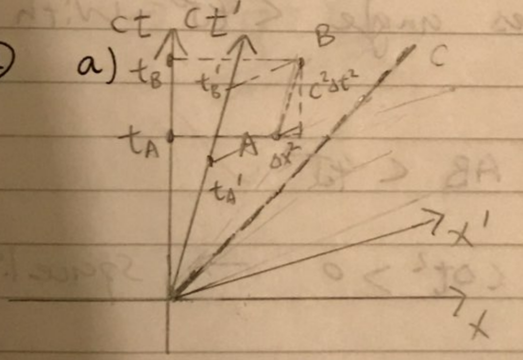
$$A' = \Lambda A \quad , \quad B' = \Lambda B$$

$$A' \cdot B' = (\Lambda A)^T g (\Lambda B) = A^T \underbrace{\Lambda^T g \Lambda}_{=g} B$$

$$= A^T g B = A \cdot B \rightarrow \text{as expected}$$

$\therefore$ $A \cdot B$ invariant

(2) a)



2/2

Total

A1	1/2
A2	2/2
A3	2/2
A4	5/5
B1	4/4
B2	5/6
C1	4/4
C2	6/6

→ 29/30

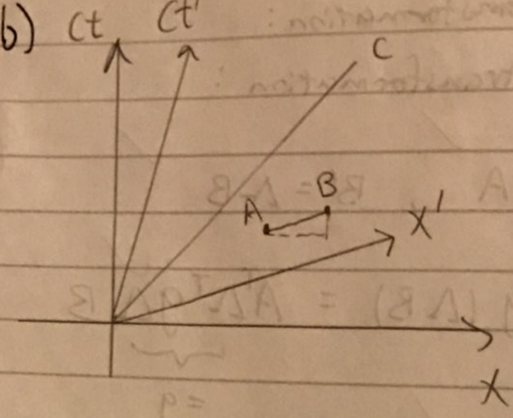
For time like intervals, $-c^2\Delta t^2 + \Delta x^2 < 0$

$\therefore$ the line joining events A, B has angle $> 45°$ with respect to the x-axis ~~(assume~~

$\because$ x'-axis can only approach $45°$ (when $v \sim c$) and never go beyond

$\therefore$ $t_A' < t_B'$ always iff $t_A < t_B$

$\Rightarrow$ temporal order unchanged $\Leftrightarrow$ A, B separated by time-like interval

b) 

Simultaneous events A and B has the line joining them parallel to x'-axis

$\because$ x'-axis makes angle < 45° with respect to x-axis

$\therefore$ Angle of AB < 45° ✓

$\therefore$ $\Delta x^2 - c\Delta t^2 > 0$ → Spacelike interval

Conversely if interval is spacelike, one can always find x'-axis parallel to AB

→ A,B are simultaneous / QED ✓

~~Good, but in both~~ case you ~~have only~~ proven "~~if~~" ~~not "and only if"~~

~~-1~~

I gave this some more thought. You ds² < 0 or ds² > 0 a condition, hence this proves "only if".

2/2

③ proper time is the time registered by a clock in its rest frame.

$$\cancel{d\tau =}\ d\tau^2 = \frac{1}{c^2}(c^2dt^2 - dx^2 - dy^2 - dz^2)$$

$$\rightarrow d\tau = dt\left(1 - \frac{\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2}{c^2}\right)^{\frac{1}{2}}$$

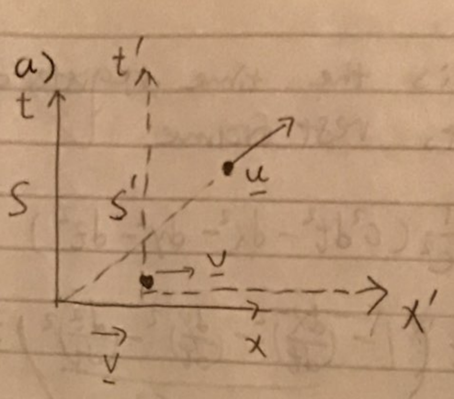
$$= dt\left(1 - \frac{u^2}{c^2}\right)^{1/2} = \frac{dt}{\gamma}$$

$$\rightarrow \frac{dt}{d\tau} = \gamma$$

5/15

④ a)

3P



let $\underline{v} = v_x \hat{x}$ $\Rightarrow$ particle $\underline{v}$ moves ~~with~~ relative to the lab frame (S) in the x-direction. Frame (S') is the rest frame of particle $\underline{v}$

~~Lorentz trans~~

Lorentz transformation $t \to t'$ from frames S to S':

$$\cancel{t' = \gamma t} \quad t' = \gamma\left(t - \frac{vx}{c^2}\right) = \gamma_v\left(t - \frac{v_x x_u}{c^2}\right)$$

world line for particle $\underline{u}$ in S:

$$X = \begin{pmatrix} ct \\ \underline{u}t \end{pmatrix} \Rightarrow x_u = u_x t \quad , \quad u_x = \underline{u}\cdot\hat{\underline{x}}$$

$$\therefore \; t' = \gamma_v\left(t - \frac{v_x x_u}{c^2}\right) = \gamma_v\left(t - \frac{(\underline{u}\cdot\hat{\underline{x}})v_x t}{c^2}\right)$$

$$= \gamma_v t\left(1 - \frac{\underline{u}\cdot(v_x\hat{\underline{x}})}{c^2}\right) = \boxed{\gamma_v t\left(1 - \frac{\underline{u}\cdot\underline{v}}{c^2}\right)}$$

where $\gamma_v = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$$\therefore \; \frac{dt'}{dt} = \cancel{\gamma_v t} \; \gamma_v\left(1 - \frac{\underline{u}\cdot\underline{v}}{c^2}\right)$$

let τ be the proper time of particle $\underline{u}$
(time measured in its rest frame)

then $\frac{dt}{d\tau} = \gamma_u \qquad \left(\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \right)$

γ_w, by definition, is given by $\gamma_w = \frac{dt'}{d\tau}$

because t' is particle $\underline{u}$'s time in frame S' (rest frame of particle $\underline{v}$) and $\underline{w}$ is the relative velocity between $\underline{u}$ and $\underline{v}$

$$\Rightarrow \quad \gamma_w = \frac{dt'}{d\tau} = \frac{dt'}{dt}\frac{dt}{d\tau} = \boxed{\gamma_u \gamma_v \left(1 - \frac{\underline{u}\cdot\underline{v}}{c^2}\right)}$$

QED.

b)

2P

$U = \begin{pmatrix} \gamma_u c \\ \gamma_u \underline{u} \end{pmatrix}$ in S

$V = \begin{pmatrix} \gamma_v c \\ \gamma_v \underline{v} \end{pmatrix}$ in S

$U' = \begin{pmatrix} \gamma_w c \\ \gamma_w \underline{w} \end{pmatrix}$ is S'

$V' = \begin{pmatrix} c \\ 0 \end{pmatrix}$ in S' because S' is the rest frame of $\underline{v}$

$\therefore \quad U' \cdot V' = -\gamma_w c^2$

$\because \quad U \cdot V = U' \cdot V' \quad \therefore \boxed{U \cdot V = -\gamma_w c^2}$

Okay, but would have been nice to see explicitly
$U \cdot V = (-c^2 + \vec{u}\cdot\vec{v})\gamma_u \gamma_v$

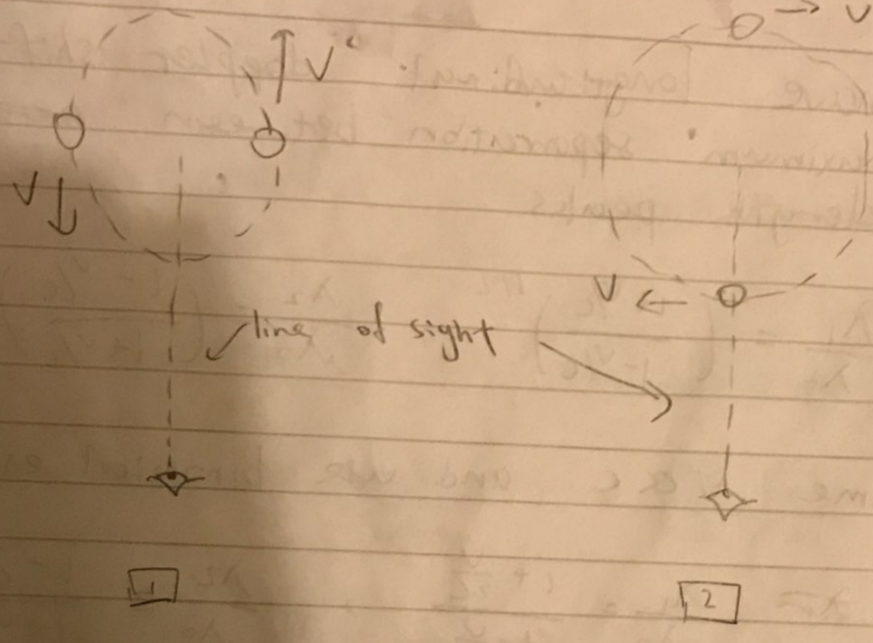
$$\therefore \quad -c^2\gamma_w = -\gamma_u\gamma_v c^2 + \gamma_u\gamma_v \underline{u}\cdot\underline{v}$$

$$\Rightarrow \quad \boxed{\gamma_w = \gamma_u\gamma_v\left(1 - \frac{\underline{u}\cdot\underline{v}}{c^2}\right)}$$

QED

⑤ Doppler effect (4/4)

①



$\boxed{1}$ shows the case when one star is approaching and the other star is receding from the line of sight. In this case the doppler shifted frequencies from ~~shows~~ two stars are different, so we get two peaks

$\boxed{2}$ shows the case when two stars are both moving vertically to the line of sight. In this case there is no doppler shift and we simply get ~~one~~ one peak.

At maximum separation between peaks
$\lambda_1 = 654.8$ nm $\lambda_2 = 654.5$ nm ~~(from~~
(reading from graphs)

If λ_0 is the ~~freq~~ frequency in the rest frame of the source ($\lambda_0 \approx 654.65$)

→ We have longitudinal doppler shift at maximum separation between ~~peaks~~ wavelength peaks.

$$\therefore \quad \frac{\lambda_1}{\lambda_0} = \left(\frac{1+v/c}{1-v/c}\right)^{1/2} \qquad \frac{\lambda_2}{\lambda_0} = \left(\frac{1-v/c}{1+v/c}\right)^{1/2}$$

Assume $v \ll c$ and use binomial expansion

$$\rightarrow \quad \cancel{\lambda_1 =} \ \frac{\lambda_1}{\lambda_0} = \frac{1+\frac{v}{2c}}{1-\frac{v}{2c}}, \qquad \frac{\lambda_2}{\lambda_0} = \frac{1-v/2c}{1+v/2c}$$

$$\therefore \quad \frac{\lambda_1}{\lambda_2} = \left(\frac{1+v/2c}{1-v/2c}\right)^2 \rightarrow \frac{1+v/2c}{1-v/2c} = \sqrt{\frac{\lambda_1}{\lambda_2}}$$

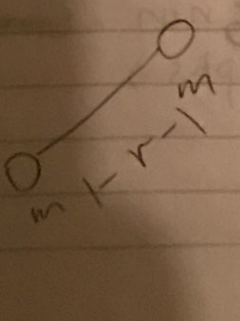
$$\sqrt{\frac{\lambda_1}{\lambda_2}} - \frac{v}{2c}\sqrt{\frac{\lambda_1}{\lambda_2}} = 1 + \frac{v}{2c}$$

$$\sqrt{\frac{\lambda_1}{\lambda_2}} - 1 = \frac{v}{2c}\left(1 + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)$$

$$\therefore \quad v = 2c\left(\frac{\sqrt{\lambda_1/\lambda_2} - 1}{\sqrt{\lambda_1/\lambda_2} + 1}\right)$$

$$= 2 \times 3 \times 10^8 \times (0.000115) = \boxed{6.87 \times 10^4 \text{ m/s}}$$

(agrees with $v \ll c$)



Binary star system (equal mass)

~~reduced mass~~ $\cancel{\mu = \frac{m \cdot m}{m+m} = \frac{m}{2}}$

r is the separation

Gravity is the centripetal force

$$\Rightarrow \quad \frac{Gm^2}{r^2} = \frac{mv^2}{(r/2)} \Rightarrow v = \sqrt{\frac{Gm}{2r}} \qquad \boxed{3}$$

orbital period T and angular velocity ω are related by $\omega = \frac{2\pi}{T}$

$$\omega = \frac{v}{(r/2)} = \frac{2v}{r} \Rightarrow \frac{2\pi}{T} = \frac{2}{r}\sqrt{\frac{Gm}{2r}}$$

$$\Rightarrow \quad T = \frac{\sqrt{2}\,\pi}{\sqrt{Gm}}\, r^{3/2} = \sqrt{\frac{2\pi^2}{G}} \cdot \sqrt{\frac{r^3}{m}} \qquad \boxed{4}$$

$$\boxed{3} \Rightarrow \sqrt{\frac{2r}{Gm}} = \frac{1}{v} \Rightarrow \frac{r}{m} = \frac{G}{2v^2} = \frac{\cancel{2}\, 6.67\times10^{-11}}{2\times(6.87\times10^{4})^2}$$

$$= 7.066\times10^{-21}\ \mathrm{m\,kg^{-1}} \qquad \boxed{5}$$

$\boxed{4} \Rightarrow$ From graphs we observe that

$$T = 2\times(1.886-0.061)\ \text{days} = 3.1536\times10^{5}\,\mathrm{s}$$

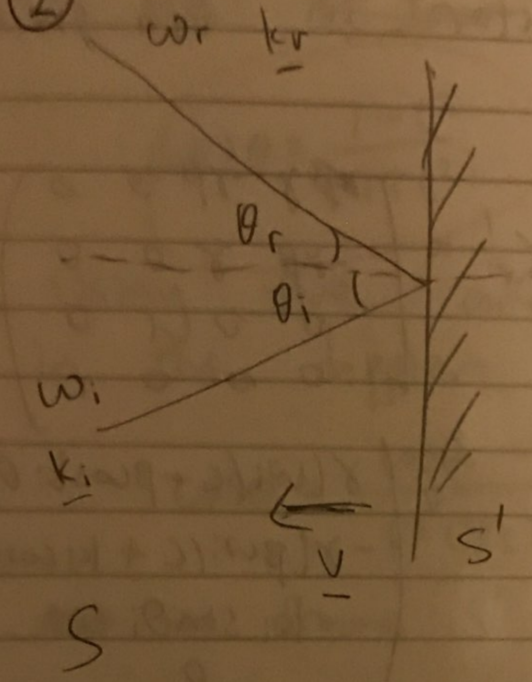
$$\frac{r^3}{m} = \frac{T^2 G}{2\pi^2} = 0.3361\ \mathrm{m^3\cdot kg^{-1}} \qquad \boxed{6}$$

$$\frac{\boxed{6}}{\boxed{5}} \Rightarrow \quad \cancel{r^2} \quad \boxed{r = 6.90\times10^{9}\,\mathrm{m}}$$

then $\boxed{5} \Rightarrow \boxed{m = 9.76\times10^{29}\ \mathrm{kg}}$

5/6

②



→ S is lab frame

→ S' is rest frame of mirror

→ initial incident wave 4-vector in S is

$$\cancel{K_i' = \begin{pmatrix} \omega_0/c \\ k_0\cos\theta_0 \\ k_0\sin\theta_0 \\ 0 \end{pmatrix}} \qquad K_i = \begin{pmatrix} \omega_0/c \\ k_i\cos\theta_i \\ k_i\sin\theta_i \\ 0 \end{pmatrix}$$

incident wave in S' is, by Lorentz transformation:

$$\cancel{K_i = \Lambda^{-1} K_i}$$

$$K_i' = \Lambda^{-1} K_i$$

$$\therefore K_i' = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_0/c \\ k_i\cos\theta_i \\ \cancel{k_i\cos\theta}\ k_i\sin\theta_i \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma(\omega_i/c + \beta k_i\cos\theta_i) \\ \gamma(\beta\omega_i/c + k_i\cos\theta_i) \\ k_i\sin\theta_i \\ 0 \end{pmatrix}$$

reflected wave 4-vector in S' is

$$K_r' = \begin{pmatrix} \gamma(\omega_i/c + \beta k_i\cos\theta_i) \\ -\gamma(\beta\omega_i/c + k_i\cos\theta_i) \\ k_i\cos\theta_i \\ 0 \end{pmatrix}$$

simply reverse the Y-component of k

reflected wave 4-vector in lab frame S is

$$\begin{pmatrix} \omega_r/c \\ k_r\cos\theta_r \\ k_r\sin\theta_r \\ 0 \end{pmatrix} = K_r = \Lambda K_r' = \begin{pmatrix} \cancel{\gamma\beta}\,\gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \gamma(\omega_i/c+\beta\cos k_i\theta_i) \\ -\gamma(\beta\omega_i/c + k_i\cos\theta_i) \\ k_i\sin\theta_i \\ 0 \end{pmatrix}$$

the third equation from above matrix equality yields

$$k_r\sin\theta_r = k_i\sin\theta_i$$

$\because$ wave is light $\therefore$ $\dfrac{\omega_i}{k_i} = \dfrac{\omega_r}{k_r} = c$

$\therefore \Rightarrow \boxed{\omega_i\sin\theta_i = \omega_r\sin\theta_r}$ QED

2P

This result simply follows from the invariant of the direction perpendicular to the relative motion between frames ~~under~~ under Lorentz transformation.

-1

b) let ~~t = tan~~ $t = \tan\frac{\theta}{2}$, then

$$\cos\theta = \frac{1-t^2}{1+t^2}, \quad \sin\theta = \frac{2t}{1+t^2}$$

According to the problem you were suppose to establish this identity.

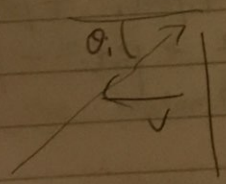
Using the angle transformation formula in the doppler effect:

$$\cos\theta = \frac{\cos\theta_0 + v/c}{1 + (v/c)\cos\theta_0} \qquad (\beta = \frac{v}{c})$$

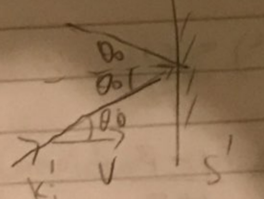
in frame S' incident angle = reflected angle

$$\theta_i' = \theta_r' = \theta_0 \rightarrow \cos\theta_i' = \cos\theta_r' = \cos\theta_0$$

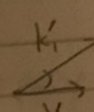
$$\therefore \cos\theta_0 = \frac{\cos\theta_i - \beta}{1 - \beta\cos\theta_i} = \frac{\cos\theta_r + \beta}{1 + \beta\cos\theta_r}$$



$-\beta$ for θ_i and $+\beta$ for θ_r is because lab frame is the source ~~frame~~ frame for incident light



From point of view S', $\underline{k_i'}$ and $\underline{v}$



have angle θ_0 between them

$$\therefore \cos\theta_i = \frac{\cos\theta_0 + v/c}{1 + \frac{v}{c}\cos\theta_0} \Rightarrow \cos\theta_0 = \frac{\cos\theta_i - \beta}{1 - \beta\cos\theta_i}$$

Similar argument works for ~~the~~ θ_0 and θ_r deriving the relationship between let $t = \tan\frac{\theta}{2}$, we have

$$\frac{\frac{1-t_i^2}{1+t_i^2}-\beta}{1-\beta\frac{1-t_i^2}{1+t_i^2}} = \frac{\frac{1-t_r^2}{1+t_r^2}+\beta}{1+\beta\frac{1-t_r^2}{1+t_r^2}}$$

$$\frac{1-t_i^2-\beta-\beta t_i^2}{1+t_i^2-\beta+\beta t_i^2} = \frac{1-t_r^2+\beta+\beta t_r^2}{1+t_r^2+\beta-\beta t_r^2}$$

$$\Rightarrow \cancel{1}+t_r^2+\cancel{\beta}-\beta t_r^2-t_i^2-\cancel{t_i^2 t_r^2}-\beta t_i^2+\cancel{\beta t_r^2 t_i^2}$$
$$-\cancel{\beta}-\cancel{\beta t_r^2}-\cancel{\beta^2}+\beta^2 t_r^2-\beta t_i^2-\cancel{\beta t_i^2 t_r^2}-\beta^2 t_i^2$$
$$+\cancel{\beta^2 t_i^2 t_r^2}$$

$$= \cancel{1}-t_r^2+\cancel{\beta}+\beta t_r^2+t_i^2-\cancel{t_i^2 t_r^2}+\beta t_i^2+\cancel{\beta t_i^2 t_r^2}$$
$$-\cancel{\beta}+\cancel{\beta t_r^2}-\cancel{\beta^2}-\cancel{\beta^2 t_r^2}+\beta t_i^2-\cancel{\beta t_i^2 t_r^2}$$
$$+\beta^2 t_i^2+\cancel{\beta^2 t_i^2 t_r^2}-\beta^2 t_r^2$$

$$\Rightarrow t_r^2-\beta t_r^2-t_i^2-\beta t_i^2-\beta t_r^2+\beta^2 t_r^2-\beta t_i^2-\beta^2 t_i^2$$
$$= -t_r^2+\beta t_r^2+t_i^2+\beta t_i^2+\beta t_r^2-\beta^2 t_r^2+\beta t_i^2+\beta^2 t_i^2$$

$$\Rightarrow t_r^2(1-2\beta+\beta^2)-t_i^2(1+2\beta+\beta^2)=0$$

$$\Rightarrow \frac{t_i^2}{t_r^2}=\left(\frac{1-\beta}{1+\beta}\right)^2 \Rightarrow \frac{t_i}{t_r}=\frac{1-\beta}{1+\beta}$$

3P

* I defined positive v to be the opposite to positive k in the above calculation.
If I reverse the positive direction of v so that it is the same as positive k, then

$$\frac{t_i}{t_r}=\frac{1+\beta}{1-\beta} \Rightarrow \boxed{\frac{\tan(\theta_i/2)}{\tan(\theta_r/2)}=\frac{1+v/c}{1-v/c}}$$

4/4

Motion under a given force

① a) 1 year = ~~3~~ $3.156 \times 10^7\,s$

1 light year $= 9.461 \times 10^{15}\,m$

$$9.8\,m/s = \frac{1/(9.461\times10^{15})}{1/(3.156\times10^{7})^2} \times 9.8$$

$$\Rightarrow \quad a_0 = \boxed{1.03 \text{ light year / year}^2}$$

TP

b)

Consider rapidity $\rho = \tanh^{-1}(\frac{v}{c})$

$$\tanh\rho = \frac{v}{c} \quad \Rightarrow \quad \operatorname{sech}^2\rho \,\frac{d\rho}{d\tau} = \frac{1}{c}\frac{dv}{d\tau}$$

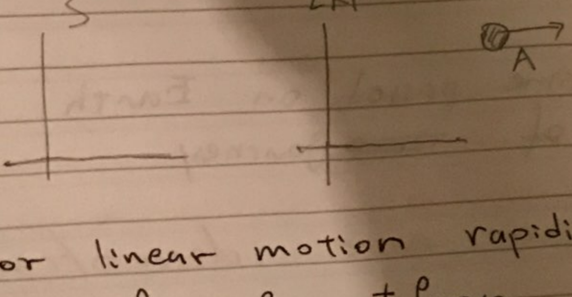
In the instanteneous rest frame (IRF):

$$v=0 \rightarrow \rho=0 \rightarrow \operatorname{sech}^2\rho = 1$$

$$\frac{dv}{d\tau} = a_0 = \text{proper acceleration}$$

$$\therefore \quad \frac{d\rho_{IRF}}{d\tau} = \frac{a_0}{c}$$

Consider an arbitrary inertial frame S



particle A is accelerating

For linear motion rapidity adds:

$$\rho_{A/S} = \rho_{A/IRF} + \rho_{IRF/S}$$

$$\therefore \quad \frac{d\rho_{A/S}}{d\tau} = \frac{d\rho_{A/IRF}}{d\tau} + 0 = \frac{a_0}{c}$$

$\rho_{IRF/S} = \text{const}$

$\therefore$ In frame S we also have $\frac{d\rho}{d\tau} = \frac{a_0}{c}$

$\Rightarrow$ $\underline{\frac{d\rho}{d\tau} = \frac{a_0}{c}}$ for all inertial frames moving along particles velocity / acceleration ~~same~~

Integrate gives $\rho = \frac{a_0 \tau}{c} + \text{const}$

Choose S so that at $\tau = 0$, $v = 0$

$\Rightarrow$ (the very initial IRF, in fact)

(this ensures $t=0$, $\tau=0$)

$\rightarrow \quad \rho = \frac{a_0 \tau}{c}$

$\because \quad \frac{dt}{d\tau} = \gamma = \cosh \rho \quad \therefore \Rightarrow t = \frac{c}{a_0} \sinh \rho$

$$\therefore \boxed{t = \frac{c}{a_0} \sinh\left(\frac{a_0 \tau}{c}\right)}$$ ✓

where τ is the time elapsed registered by rocket's clock

t is the time passed on Earth at the same stage of ~~travel~~ journey.

This was a nice derivation.

the traveling twin aged $T_1 = 5$ years during
the first ~~quarter of~~ quarter of ~~the~~ journey

According to ~~the~~ the Earth twin, the time passed
is

$$t_1 = \frac{c}{a_0} \sinh\left(\frac{a_0 T_1}{c}\right)$$

$c = 1$ lightyear / year

$a_0 = 1.03$ light year / year^2

$$\therefore t_1 = \frac{1}{1.03} \sinh\left(\frac{1.03 \times 5}{1}\right) = 83.7 \text{ years.}$$

the process is symmetric. i.e. each quarter of journey takes the same time ~~&~~ for both the rocket clock and the Earth clock

$\therefore$ the traveling twin aged ~~t =~~ $\boxed{T = 20 \text{ years}}$

the stay-at-home twin aged $\boxed{t = 335 \text{ years}}$

6/6

② a) $\cancel{t}$ $\frac{d\underline{P}}{dt} = \underline{f}$ if $\underline{f}$ = const

then $\boxed{\underline{P} = \underline{P}_0 + \underline{f}t}$

1p

b) $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\underline{v}\cdot\underline{v}}{c^2}}}$

$\underline{\alpha} = \frac{1}{mc}(\underline{P}_0 + \underline{f}t) = \frac{\underline{P}}{mc}$ $\because$ ~~$\underline{P} = \gamma mc$~~

$\because$ $\underline{P} = \gamma m \underline{v}$ ~~$\therefore \alpha^2 = \underline{\alpha}\cdot\underline{\alpha}$~~

$\therefore \alpha^2 = \underline{\alpha}\cdot\underline{\alpha} = \frac{1}{m^2c^2}\gamma^2 m^2 v^2 = \gamma^2 \frac{v^2}{c^2}$

$\therefore 1+\alpha^2 = \left(\gamma^2\frac{v^2}{c^2} + 1\right) = \frac{1}{1-\frac{v^2}{c^2}}\left(\frac{v^2}{c^2}\right) + \frac{1-\frac{v^2}{c^2}}{1-\frac{v^2}{c^2}}$

$= \frac{1}{1-\frac{v^2}{c^2}} = \gamma^2$

1p

$\Rightarrow \gamma^2 = 1+\alpha^2$ QED

c) magnitude :

$\frac{1}{1-\frac{v^2}{c^2}} = 1+\alpha^2 \Rightarrow \frac{1}{1+\alpha^2} = 1-\frac{v^2}{c^2}$

~~$\frac{v^2}{c^2}$~~ $\Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{1+\alpha^2} = \frac{\alpha^2}{1+\alpha^2}$

$\Rightarrow v^2 = \frac{\alpha^2 c^2}{1+\alpha^2} \Rightarrow v = \frac{|\alpha c|}{\sqrt{1+\alpha^2}}$

direction:

$\underline{v}$ is along $\underline{p} = \underline{p_0} + \underline{f}t$ which is along $\underline{\alpha}$

$$\Rightarrow \quad \underline{v} = \frac{\underline{\alpha} c}{\sqrt{1+\alpha^2}} = \frac{(\underline{p_0} + \underline{f}t)/m}{\sqrt{1 + \frac{(\underline{p_0} + \underline{f}t)^2}{m^2c^2}}}$$

$$= \boxed{\frac{\underline{p_0} + \underline{f}t}{\sqrt{m^2 + (\underline{p_0} + \underline{f}t)^2/c^2}}}$$

1p

~~ᵽ~~

d) $\underline{f} = f\hat{\underline{x}}$ $\quad \cancel{\underline{p_0}}$. $\underline{p_0} = p_0 \hat{\underline{y}}$

$$\Rightarrow \underline{f} \cdot \underline{p_0} = 0$$

$$\Rightarrow (\underline{p_0} + \underline{f}t)^2 = (\underline{p_0} + \underline{f}t) \cdot (\underline{p_0} + \underline{f}t)$$

$$= p_0^2 + f^2t^2$$

$$\Rightarrow \cancel{\underline{v} = \frac{p_0 \hat{x} + \hat{p}}{}} \quad \underline{v} = \frac{p_0 \hat{\underline{y}} + ft\hat{\underline{x}}}{\sqrt{m^2 + (p_0^2 + f^2t^2)/c^2}}$$

$$\therefore \frac{dx}{dt} = \frac{ft}{\sqrt{m^2 + (p_0^2 + f^2t^2)/c^2}}, \quad \frac{dy}{dt} = \frac{p_0}{\sqrt{m^2 + (p_0^2 + f^2t^2)/c^2}}$$

$\Downarrow$ ① $\qquad \Downarrow$ ②

$$\frac{dx}{dt} = \cancel{fc^2t} \frac{fct}{\sqrt{m^2c^2 + p_0^2 + f^2t^2}} \quad ①$$

$$\frac{dy}{dt} = \frac{P_0 c}{\sqrt{m^2c^2 + P_0^2 + f^2t^2}} \quad (2)$$

integrate (1) gives

$$\boxed{X(t) = \frac{c}{f}(m^2c^2 + P_0^2 + f^2t^2)^{1/2} + \text{const}}$$

integrate (2) gives

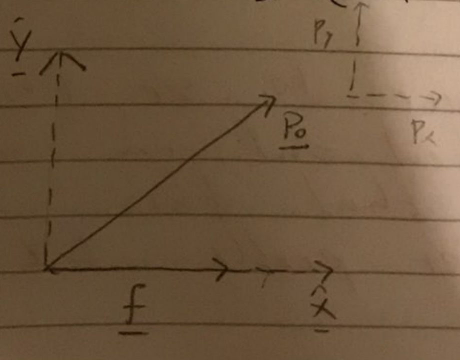
$$\boxed{Y(t) = \frac{cP_0}{f} \log\left(ft + \sqrt{m^2c^2 + P_0^2 + f^2t^2}\right) + \text{const}}$$

e) ~~initial~~ momentum ~~(t=0)~~ (at t=0) is ~~$\underline{P_0} = P_0\hat{x}$~~ ~~$\underline{P_0} = P_0\hat{y}$ and the force is~~

~~$\underline{f} = f\hat{x}$~~

Consider $\underline{P_0}$ and $\underline{f}$ arranged in arbitrary relative directions (the general case)

let $\underline{f} = f\hat{\underline{x}}$ (but $\underline{P_0} \neq P_0\hat{\underline{y}}$, $\underline{P_0} = P_x\hat{\underline{x}} + P_y\hat{\underline{y}}$)

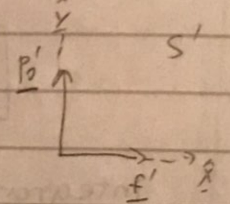


in frame S

then we transform to frame S', where the x-component of $\underline{P_0}$, P_x, is zero

S' has velocity $\underline{u}$ relative to S and $\underline{u} = u\hat{\underline{x}}$
$(\underline{u} \parallel \underline{f})$

$\because$ in that case $\underline{f}'$ is still along x-direction



$$P_x' = \gamma_u (P_x - \frac{uE}{c^2})$$

So if we let $P_x = \frac{uE}{c^2}$

i.e. $\boxed{\underline{u} = \frac{c^2}{E}(\underline{P_0} \cdot \hat{\underline{x}})\hat{\underline{x}}}$, then in frame $(E = \sqrt{P_0^2c^2 + m^2c^4})$

S', $P_x' = 0$. $\therefore \underline{P_0'} = P_y' \hat{\underline{y}}$

We can then use the method in d) to calculate the trejectory of particle in frame S' because in S', $\underline{f}' \perp \underline{P_0'}$

then we transform back to frame S to get the trajectory in S.

It would be nice to see the integrals worked out in more detail. Besides that this solution was perfect for problem (2.)

Date & Time: 09.11.2016, 14:00 | Location: Univ – 91A | Please return by: 07.11.2016

A Energy-momentum conservation

1. Show that if a 4-vector has a component which is zero in all frames, then the entire vector is zero. What insight does this offer into energy and momentum?

2. A particle of rest mass m and kinetic energy $3m$ strikes a stationary particle of rest mass $2m$ and sticks to it. Find the rest mass and speed of the composite particle.

   From this question on we will use $c = 1$ throughout – and I would encourage students to do the same.

   Then one has

$$E^2 - p^2 = m^2 \tag{1}$$

   and

$$E = p \quad \text{for} \quad m = 0. \tag{2}$$

B Particle formation

1. a) Pion formation can be achieved by the process $p + p \to p + p + \pi_0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.

   b) A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge.

   c) A particle formation experiment creates reactions of the form $b+t \to b+t+n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the 'efficiency' of the experiment as the ratio of the rest energy of the new particle to the supplied kinetic energy of the incident particle. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2} \tag{3}$$

2. Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m. High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

C Decay

1. A particle with known rest mass M and energy E decays into two products with known rest masses m_1 and m_2. Find the the energies E_1 , E_2 (in the lab frame) of the products, by the following steps:

a) Find the energies E_1', E_2' of the products in the CM frame.

b) Show that the momentum of either decay product in the CM frame is

$$p = (c/2M)\left[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2\right]^{1/2}$$ (4)

c) Find the Lorentz factor and the speed v of the CM frame relative to the lab.

d) Write down, in terms of v, γ, p, E_1' and E_1', expressions for E_1, E_2 when the products are emitted (a) along the line of flight and (b) at right angles to the line of flight in the CM frame.

2. **Compton scattering**. Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $\mathsf{P} + \mathsf{P}_e = \mathsf{P}' + \mathsf{P}_e'$.

[Hint: we would like to eliminate the final electron 4-momentum P_e', so make this the subject of the equation and square.]

A collimated beam of X rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of $90°$, including a quantitative indication of the scale.

D Four-gradient

1. Describe the way density and flux transform under the Lorentz transformation. Write down the continuity equation in 4-vector notation.

2. A wave motion has a phase ϕ given by $\phi(x, y, z, t) = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$ where $\boldsymbol{k}$ is a constant vector and ω is a constant frequency. Evaluate $\Box\phi$ and comment.

Julian Merten

(4)

B2 Problem Set 2 Ziyan Li

A1 2/2

→ Consider some 4-vector Q. pick a component such as the x-component, and supposed this component vanishes in all frames. If there is a frame in which the y or z component is non-zero, then we can rotate axes to make the x-component non-zero, contrary to the claim that it is zero in all reference frames. Therefore the y and z components are zero also. If there is a reference frame in which the time component Q^0 is non-zero, ~~contrary to the claim~~ then we can apply a Lorentz transformation to make Q' non-zero, contrary to the Claim. Therefore Q^0 is zero.

→ Similar arguments can be made ~~for~~ from y or z components.

1P

→ If the time component vanishes in all frames, and in some frame x-component is non-zero, then we can apply Lorentz transformation to make Q^0 non zero in some other frame. So x component has to be zero. Similarly for y and z components.

→ We conclude that the entire 4-vector has to be zero.

Consider 4-vector

$$Q = P_{after} - P_{before} = P_a - P_b = \begin{pmatrix} E_a/c \\ \underline{p_a} \end{pmatrix} - \begin{pmatrix} E_b/c \\ \underline{p_b} \end{pmatrix}$$

Total:

A1	2/2
A2	3/3
B1	4/5
B2	3/3
C1	5/7
C2	7/7
D1	2/2
D2	1/1

→ 27/30 ≈ 90%

excellent

$$\rightarrow \quad Q = \begin{pmatrix} \frac{E_a - E_b}{c} \\ \underline{p}_a - \underline{p}_b \end{pmatrix}.$$

If In one frame we know energy and momentum are conserved so $Q = 0$, then by the zero component lemma, $Q = 0$ in all ~~for~~ inertial ~~frames~~ frames

$\rightarrow \quad E_a = E_b$, $\underline{p}_a = \underline{p}_b$ in all frames.

$\rightarrow$ energy, momentum, and energy-momentum 4-vector are conserved in all inertial frames.

Also, by zero-component lemma, we can ~~postu~~ postulate the conservation of ~~4-momentum~~ one component of momentum ~~, and~~ ~~and~~ ~~that automatically ensures that~~ in all frames, and that ensures the conservation of momentum and energy, i.e. the 4-momentum in all frames.

1P

A2. let $c=1$

3/3

$\rightarrow P_1$ | | $\rightarrow P_3$

$K_1 = 3m$ | $K_2 = 0$ | $\Rightarrow$ | $E_3 = E_1 + E_2$

$m_1 = m$ | $M_2 = 2m$ | | M_3

$E_1 = 3m + m = 4m$ | $E_2 = 2m$ | | $P_3 = P_1 + P_2$

P_1 | $P_2 = 0$ | | $= P_1$

"~" for 4-vector.

4-vector conservation $\Rightarrow \tilde{P}_1 + \tilde{P}_2 = \tilde{P}_3$

$$\Rightarrow (\tilde{P}_1 + \tilde{P}_2)^2 = \tilde{P}_3^{\,2}$$

$$\Rightarrow \cancel{\tilde{P}_1 + \tilde{P}_2} \quad \tilde{P}_1^{\,2} + \tilde{P}_2^{\,2} + 2\tilde{P}_1 \cdot \tilde{P}_2 = \tilde{P}_3^{\,2}$$

$$\tilde{P}_1^{\,2} = -m^2 \qquad \tilde{P}_2^{\,2} = -4m^2$$

$$\tilde{P}_1 \cdot \tilde{P}_2 = \cancel{E_1 E_2} \; -E_1 \cdot E_2 + \underbrace{\underline{P_1} \cdot \underline{P_2}}_{0} = -8m^2$$

$$\Rightarrow \cancel{-13m^2 = \tilde{P}_3^{\,2} = -M_3^2 \Rightarrow M}$$

$$-21m^2 = \tilde{P}_3^{\,2} = -M_3^2 \Rightarrow M_3 = \boxed{\sqrt{21}\,m}$$ 1P

is the rest mass of composite particle.

$P_1 = \sqrt{E_1^2 - m_1^2} = \sqrt{16m^2 - m^2} = \sqrt{15}m$

$P_2 = 0$

$\rightarrow \quad P_3 = P_1 + P_2 = \sqrt{15}m$ 1P

$$E_3 = E_1 + E_2 = 4m + 2m = 6m$$

$$\beta_{cm} = \frac{P_3}{E_3} = \frac{\sqrt{15}m}{6m} = \frac{\sqrt{15}}{6}$$

$\Rightarrow$ speed of composite particle is

$$\boxed{V_{cm} = \frac{\sqrt{15}}{6} c}$$

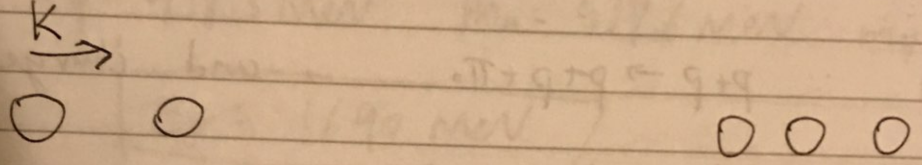
1p

B1. Proton has mass m_p and pion has mass m_π

4/15

a) $p + p \rightarrow p + p + \pi_0$ let $c = 1$

the threshold energy (minimum kinetic energy) is required when the two protons and the formed pion are all stationary in the centre of mass frame



$E_1 = K + m_p$ $E_2 = m_p$ $E_1' = m_p$ $E_2' = m_p$ $E_\pi = m_\pi$

$P_1 = P_1$ $P_2 = 0$ $P_1' = 0$ $P_2' = 0$ $P_\pi = 0$

Lab frame / before CM frame / after

4-vector ~~invariant~~ dot product is invariant

$$(\tilde{P_1} + \tilde{P_2})^2 = (\tilde{P_1'} + \tilde{P_2'} + \tilde{P_\pi})^2$$

$\cancel{\rightarrow K + m_p}$

$$\rightarrow \tilde{P_1}^2 + \tilde{P_2}^2 + 2\tilde{P_1}\cdot\tilde{P_2} = -(2m_p + m_\pi)^2$$

$$\rightarrow -m_p^2 - m_p^2 - 2(K + m_p)(m_p) = -(2m_p + m_\pi)^2$$

$$\rightarrow -\cancel{2m_p^2} - 2Km_p - \cancel{2m_p^2} = -\cancel{4m_p^2} - 4m_p m_\pi - m_\pi^2$$

$$\Rightarrow \boxed{K = \frac{4m_p m_\pi - m_\pi^2}{2m_p}}$$

1P

$$K = \frac{4m_p m_\pi - m_\pi^2}{2m_p} = \frac{(4m_p - m_\pi)m_\pi}{2m_p}$$

$$= \boxed{260.3\ MeV}$$

$(m_p = 938.3\ MeV/c^2,\quad m_\pi = \cancel{134}\ 135.0\ MeV/c^2.)$

This is about twice the rest energy of pion.

b) $\because$ $p + p \to p + p + \pi_0$, and charge conservation

$\therefore$ π_0 has charge 0

$\to$ $\gamma + p \to p + n + \pi_0$

photon

proton stays to conserve charge.

I am afraid that this violates baryon number conservation

use: $\gamma + p \to n + \pi_+$

-1P

γ c

$E_\gamma = P_\gamma c$

P_γ

$m_\gamma = 0$

p

$E_p = m_p$

$P_p = 0$

m_p

$\Longrightarrow$

p n π_0.

m_p m_n m_π

Lab before

cm after

$$\tilde{P_\gamma} = \begin{pmatrix} E_\gamma \\ P_\gamma \end{pmatrix} \qquad \tilde{P_p} = \begin{pmatrix} m_p \\ 0 \end{pmatrix}.$$

$$(\tilde{P_\gamma} + \tilde{P_p})^2 = (\tilde{P_p}' + \tilde{P_n}' + \tilde{P_\pi}')^2 = -(m_p + m_n + m_\pi)^2$$

$$\tilde{P_\gamma}^2 + 2\tilde{P_\gamma}\cdot\tilde{P_p} + \tilde{P_p}^2 = -(m_p + m_n + m_\pi)^2$$

$\rightarrow$ $\cancel{0 \mp 2E_\gamma m_p}$

$$0 - 2E_\gamma m_p - m_p^2 = -(m_p + m_n + m_\pi)^2.$$

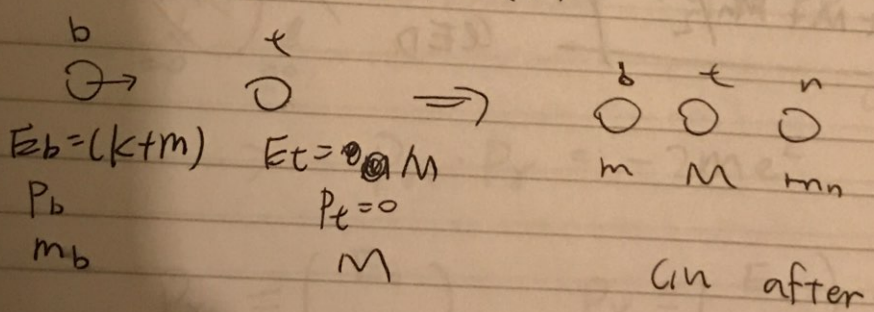
$$\rightarrow \boxed{E_\gamma = \frac{(m_p + m_n + m_\pi)^2 - m_p^2}{2m_p}}$$

$m_p = 938.3$ MeV $\quad m_n = 939.6$ MeV $\quad m_\pi = 135.0$ MeV

$$\boxed{E_\gamma = 1690 \text{ MeV}}$$

c)

$$b + t \rightarrow b + t + n.$$



At threshold the 3 formed particles are at rest in CM frame.

$$\Rightarrow -(m + M + m_n)^2 = (\tilde{P}_b + \tilde{P}_t)^2 = \tilde{P}_b^2 + \tilde{P}_t^2 + 2\tilde{P}_b \cdot \tilde{P}_t$$

$$= -m^2 - M^2 - 2(K+m)M$$

$$\Rightarrow K + m = \frac{(m + M + m_n)^2 - m^2 - M^2}{2M}$$

✓ 1P

1P

$$\text{efficiency} = \frac{m_n}{K} = \frac{m_n}{\frac{(m+M+m_n)^2 - m^2 - M^2}{2M} - m}$$

$$= \frac{2M m_n}{(m+M+m_n)^2 - m^2 - M^2 - 2Mm}$$

$$= \frac{2M m_n}{\cancel{m^2} + \cancel{M^2} + m_n^2 + \cancel{2mM} + 2m_n M + 2m m_n - \cancel{m^2} - \cancel{M^2} - \cancel{2Mm}}$$

$$= \frac{2M m_n}{m_n^2 + 2m_n M + 2m_n m}$$

$$= \boxed{\frac{M}{m + M + m_n/2}}$$

QED.

1P

B2.

3/3

B2.

$\gamma_0 \xrightarrow{c}$	$\xleftarrow{c} \gamma$	$\Rightarrow$	e^+	e^-
o	o		O	O
E_0	E		m_e	m_e.

3/3

lab before   cm after

~~4-vectors:~~

Threshold → products at rest in CM frame

$\cancel{P_{\gamma_0} + P_\gamma = P_{me} + P}$

Using 4-vectors:

$$(P_{\gamma_0} + P_\gamma)^2 = (P_{e^-} + P_{e^+})^2 = -(m_e + m_e)^2$$
$$= -4m_e^2$$

$$\underbrace{\cancel{P_{\gamma_0}^2}}_{=0} + \underbrace{\cancel{P_\gamma^2}}_{=0} \cancel{+2\tilde{P}_{\gamma_0} \cdot \tilde{P}_\gamma} + 2P_{\gamma_0} \cdot P_\gamma = -4m_e^2$$

✓ 1P

$$\rightarrow \quad P_{\gamma_0} \cdot P_\gamma = -2m_e^2$$

$$P_{\gamma_0} = \begin{pmatrix} E_0 \\ E_0 \end{pmatrix} \qquad P_\gamma = \begin{pmatrix} E \\ -E \end{pmatrix}$$

$$\therefore \quad P_{\gamma_0} \cdot P_\gamma = -EE_0 - EE_0 = -2EE_0$$

$$\Rightarrow \quad -2EE_0 = -2m_e^2$$

$$\rightarrow \quad E = \frac{m_e^2}{E_0} = \boxed{\frac{m^2}{E_0}}$$

✓ 1P

$E_0 = 2.3\times10^{-4}\,\text{eV}$ $\quad$ ~~me~~ $m = 0.511\,\text{MeV}$

$$E = \frac{m^2}{E_0} = \frac{(0.511\times10^6)^2}{2.3\times10^{-4}} = \boxed{1.135\times10^{15}\,\text{eV}}$$

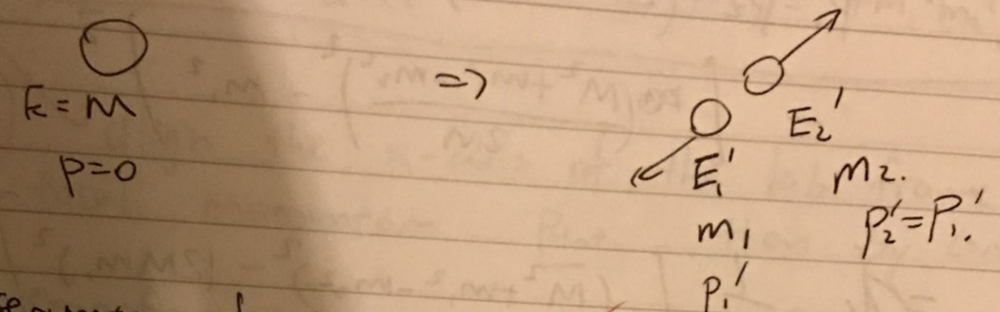
7P

C1. let $c=1$

5/7

a) in CM frame

before



conservation of energy:

$$E_1' + E_2' = E = M$$

conservation of momentum:

no, $p_1'^2 = p_2'^2$

$$p_1' = p_2' \Leftrightarrow E_1'^2 - m_1^2 = E_2'^2 - m_2^2$$

-1P

$$\rightarrow E_1'^2 - E_2'^2 = m_1^2 - m_2^2$$

$$\rightarrow \underbrace{(E_1' + E_2')}_{M}(E_1' - E_2') = m_1^2 - m_2^2.$$

$$\rightarrow \begin{cases} E_1' - E_2' = \dfrac{m_1^2 - m_2^2}{M} \\ E_1' + E_2' = M \end{cases}$$

$$\rightarrow \boxed{\begin{aligned} E_1' &= \frac{M^2 + m_1^2 - m_2^2}{2M} \\ E_2' &= \frac{M^2 - m_1^2 + m_2^2}{2M} \end{aligned}}$$

1P

For both p_1, p_2

$$p_1 = p_2 = \left(E_1'^2 - m_1^2\right)^{\frac{1}{2}}$$

$$= \left[\left(\frac{M^2+m_1^2-m_2^2}{2M}\right)^2 - m_1^2\right]^{\frac{1}{2}}$$

1P

$$= \frac{1}{2M}\left[(M^2+m_1^2-m_2^2)^2 - (2Mm_1)^2\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[(M^2+m_1^2+2Mm_1-m_2^2)(M^2+m_1^2-2Mm_1-m_2^2)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[\left[(M+m_1)^2-m_2^2\right]\left[(M-m_1)^2-m_2^2\right]\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[(M+m_1-m_2)(M+m_1+m_2)(M-m_1+m_2)(M-m_1-m_2)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[(m_1+m_2+M)(m_1+m_2-M)(m_1-m_2+M)(m_1-m_2-M)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[\left((m_1+m_2)^2-M^2\right)\left((m_1-m_2)^2-M^2\right)\right]$$

$$= \frac{1}{2M}\left[\left(m_1^2+m_2^2-M^2+(2m_1m_2)\right)\times\left(m_1^2+m_2^2-M^2-(2m_1m_2)\right)\right]^{\frac{1}{2}}$$

$$= \frac{1}{2M}\left[(m_1^2+m_2^2-M^2)^2 - 4m_1^2m_2^2\right]^{\frac{1}{2}}$$

1P

If you want, but there is really no need.

~~if~~ we put c back gives $\Rightarrow$

$$P = \boxed{P_1' = P_2' = \frac{c}{2M}\left[(m_1^2+m_2^2-M)^2 - 4m_1^2m_2^2\right]^{\frac{1}{2}}}$$

c) align the x-axis of the lab frame with total momentom $\underline{P}_{tot}$, then by Lorentz transformation

$$P_{tot,x}' = \gamma(-E_{tot}V/c^2 + P_{tot,x})$$

$$P_{tot,y}' = P_{tot,y} = 0$$

$$P_{tot,z}' = P_{tot,z} = 0$$

For $\underline{P_{tot,x}} = \underline{0}$, need $\underline{P_{tot,x}'} = 0$.

$$\Rightarrow -\frac{E_{tot}V}{c^2} + P_{tot,x} = 0$$

$$\rightarrow V = \frac{P_{tot,x}c^2}{E_{tot}}.$$

$$\Rightarrow \boxed{\underline{v} = \frac{\underline{P}_{tot}c^2}{E_{tot}}}.$$

For lab frame before decay:

$$V = \frac{P_{tot}c^2}{E_{tot}} = \frac{\sqrt{E^2-M^2}}{E} = \boxed{\frac{\sqrt{E^2-m^2c^4}}{E}c}$$

is the CM velocity

7P

Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\cancel{\frac{v}{c} = \frac{\sqrt{E^2-(Mc^2)^2}}{E} = \sqrt{1-m}}$

$\frac{v}{c} = \frac{\sqrt{E^2-M^2}}{E} = \sqrt{1-\left(\frac{M}{E}\right)^2} \rightarrow \frac{v^2}{c^2} = 1-\left(\frac{M}{E}\right)^2$

$\Rightarrow \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{\cancel{1+}\left(\frac{M}{E}\right)^2}} = \frac{E}{M}$ ✓

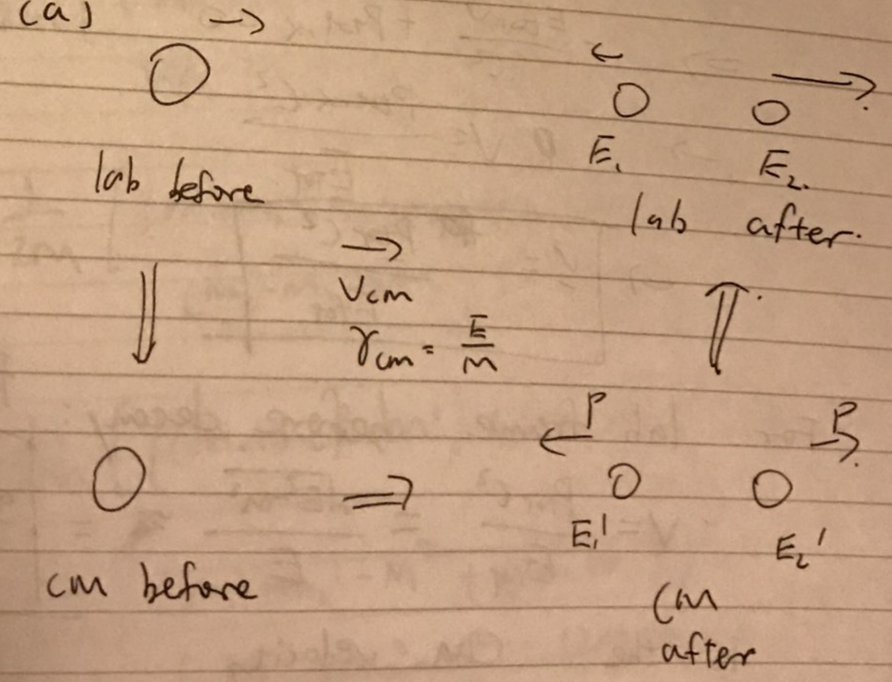
$= \boxed{\frac{E}{Mc^2}}$

(consistent with $E = \gamma Mc^2$)

That would have been the faster route.

d)

(a)

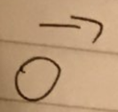


$$\boxed{\begin{aligned} E_1 &= \gamma(E_1' - vp) \\ E_2 &= \gamma(E_2' + vp) \end{aligned}}$$

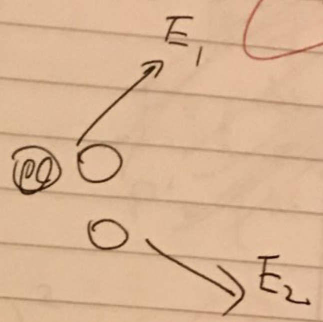
Okay, but what exactly are you doing here?

-1P

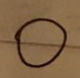
(b)



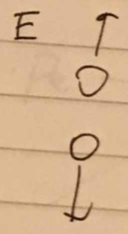
lab before



$\Downarrow$



$\Rightarrow$



cm before

$$E = \gamma(E + vP_x)$$

~~E_1~~

in this case $P_x = 0$

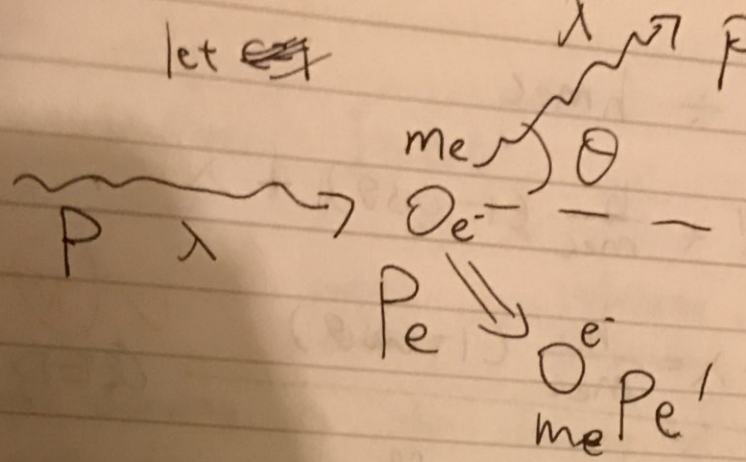
1P

$$\Rightarrow \boxed{\begin{aligned} E_1 &= \gamma E_1' \\ E_2 &= \gamma E_2' \end{aligned}}$$

C2

7/7

let ~~e~~



$\cancel{P_e = \begin{pmatrix} m_e c^2 \\ 0 \end{pmatrix}}$

$\cancel{P_e} \quad \cancel{P_e = \begin{pmatrix} m_e c^2 \\ 0 \end{pmatrix}}$

$\cancel{P = \begin{pmatrix} hc/\lambda \\ h/\lambda \end{pmatrix}}$

$\cancel{P' = (hc}$

$$P + P_e = P' + P_e'$$

$$\rightarrow \quad P_e'^2 = (P + P_e - P')^2$$ ✓ 1p

✓ 1p

$$\rightarrow \cancel{P_e'^2} = \cancel{P^2} + \cancel{P_e^2} + \cancel{P'^2} + 2P\cdot P_e - 2P\cdot P' - 2P_e\cdot P'$$

For photons $\quad P^2 = P'^2 = 0$ ✓ 1p

✓

$\cancel{P_e^2 = -m_e^2c^4}$ $\quad P_e^2 = -m_e^2c^2 = P_e'^2$

$$\Rightarrow \quad P\cdot P_e - P\cdot P' - P_e\cdot P' = 0$$ ✓

$$P_e = \begin{pmatrix} m_e c \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} \cancel{hc} \; h/\lambda \\ h/\lambda \\ 0 \\ 0 \end{pmatrix}$$

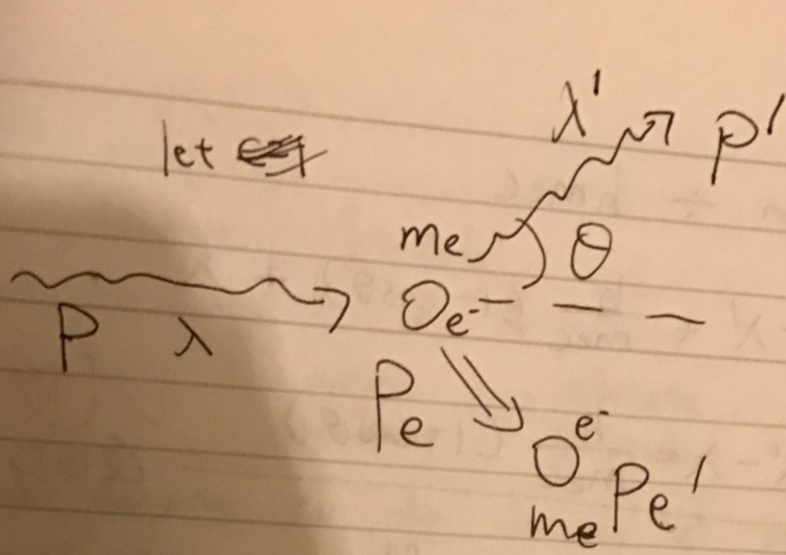
$$P' = \begin{pmatrix} \frac{h}{\lambda'} \\ \frac{h}{\lambda'}\cos\theta \\ \frac{h}{\lambda'}\sin\theta \\ 0 \end{pmatrix}.$$ ✓ 1p

$$\Rightarrow 0 = -\frac{h}{\lambda} m_e c + \frac{h^2}{\lambda\lambda'}(1 - \cos\theta) + \frac{h}{\lambda'} m_e c$$

C2

7/7

let ~~e~~



$\cancel{P_e = \begin{pmatrix} m_e c^2 \\ 0 \end{pmatrix}}$

$\cancel{P_e = \begin{pmatrix} m_e c^2 \\ 0 \end{pmatrix}}$ $\cancel{P = \begin{pmatrix} \frac{hc}{\lambda} \\ h/\lambda \end{pmatrix}}$ $\cancel{P' = \begin{pmatrix} hc \end{pmatrix}}$

$$P + P_e = P' + P_e'$$

$$\rightarrow \quad P_e'^2 = (P + P_e - P')^2$$ ✓ 1P

$$\rightarrow \cancel{P_e'^2} = \cancel{P^2} + \cancel{P_e^2} + \cancel{P'^2} + 2P\cdot P_e - 2P\cdot P' - 2P_e \cdot P'$$ ✓ 1P

For photons $P^2 = P'^2 = 0$ ✓ 1P ✓

$\cancel{P_e^2 = -m_e^2 c^4}$ $P_e^2 = -m_e^2 c^2 = P_e'^2$

$$\Rightarrow \quad P\cdot P_e - P\cdot P' - P_e \cdot P' = 0$$ ✓

$$P_e = \begin{pmatrix} m_e c \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad P = \begin{pmatrix} \cancel{hc/\lambda}\ h/\lambda \\ h/\lambda \\ 0 \\ 0 \end{pmatrix}$$

$$P' = \begin{pmatrix} \frac{h}{\lambda'} \\ \frac{h}{\lambda'}\cos\theta \\ \frac{h}{\lambda'}\sin\theta \\ 0 \end{pmatrix}.$$ ✓ 1P

$$\Rightarrow 0 = -\frac{h}{\lambda} m_e c + \frac{h^2}{\lambda\lambda'}(1-\cos\theta) + \frac{h}{\lambda'} m_e c$$

5

$\times \lambda\lambda'$, then $\div\ h m_e c$

$$\Rightarrow \quad 0 = -\lambda' + \frac{h}{m_e c}(1-\cos\theta) + \lambda$$

$$\Rightarrow \quad \underline{\lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)}$$ ✓ 10

QED

This is Compton effect.

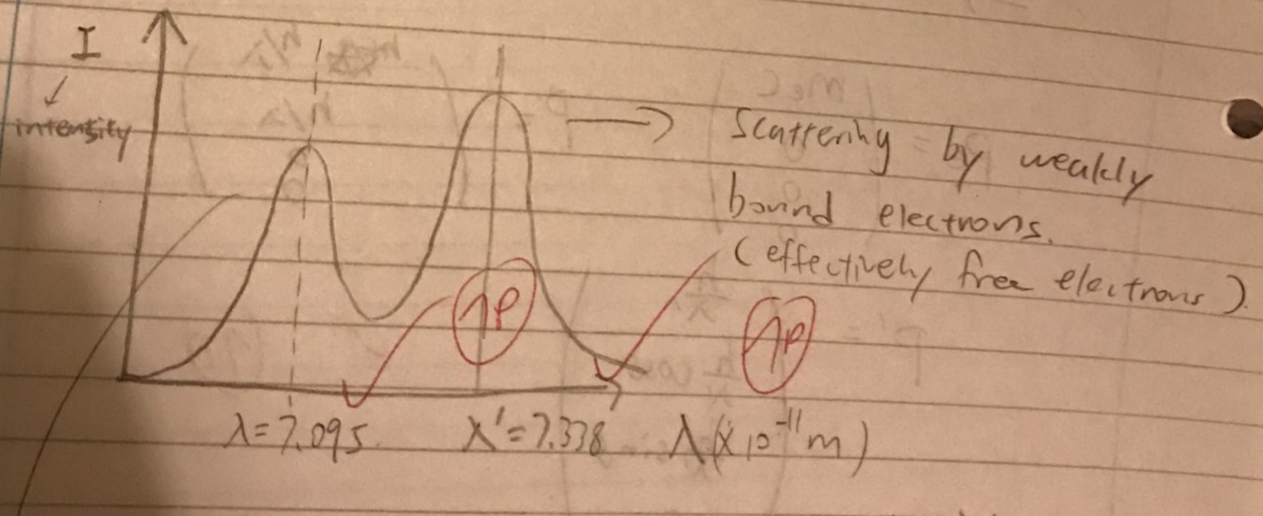
If $E = \frac{hc}{\lambda} = 17.52\,\text{keV}$

$$\rightarrow \lambda = \frac{6.63\times10^{-34} \times 3\times10^{8}}{17.52\times10^{3}\times1.6\times10^{-19}} = 7.095\times10^{-11}\,\text{m}$$

If $\theta = 90^\circ \rightarrow \cos\theta = 0$

$$\rightarrow \lambda' = \lambda + \frac{h}{m_e c} = 7.338\times10^{-11}\,\text{m}.$$

The wavelength spectrum is then:



Scattering by tightly bound electrons and nuclei
(m large, $\Delta\lambda$ small ~ 0)

Nice!

D1. 2/2

Define $J \equiv \rho_0 U$

ρ_0 ↓ proper density

U ↓ velocity 4-vector

Proper density is the density of a fluid particle in its rest frame, so by definition it is Lorentz ~~invar~~ invariant.

→ J is a 4-vector.

In the local rest frame $J = \begin{pmatrix} \rho_0 c \\ 0 \end{pmatrix}$

Transform to another frame

$$J' = \Lambda J = \begin{pmatrix} \gamma \rho_0 c \\ \gamma \rho_0 \underline{v} \end{pmatrix}$$

In that frame $\rho = \gamma \rho_0$, because any region in the local rest frame will be Lorentz contracted in this new frame by a facter of γ. The number of particles is invariant. Hence the density increases by a factor of γ

→ $\rho = \gamma \rho_0$. ⇒ ~~$J = \begin{pmatrix} \rho c \\ \rho \underline{v} \end{pmatrix}$~~

$J = \begin{pmatrix} \rho c \\ \rho \underline{v} \end{pmatrix}$ and $\rho \underline{v} = \underline{j}$ is ~~be~~ the flux

→ $J = \begin{pmatrix} \rho c \\ \underline{j} \end{pmatrix}$ ✓ 2P

3

D2.

If particles are conserved, then the rate of change of number of particles in a region is the net flow of particles in or out of the ~~region~~ region per time.

$$\frac{\partial}{\partial t}\int_R \rho \, dV = -\int_{\partial(R)} \rho \underline{u}\cdot d\underline{s} = \int_{\partial(R)} -\underline{j}\cdot d\underline{s}$$

$$= \int_R -\underline{\nabla}\cdot\underline{j}\, dV$$

divergence theorem

this is true $\forall$ dV $\Rightarrow$ $\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot\underline{j} = 0$

2r

$\Rightarrow$ $\boxed{\Box\cdot J = 0}$ is the continuity equation, where $\Box = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\left(\underline{\Box}\cdot J = \Box^T(gJ)\right)$.

7/7

D2.

$$\phi(x,y,z,t) = \underline{k}\cdot\underline{r} - \omega t. = k_x x + k_y y + k_z z - \omega t$$

$$\rightarrow \Box\phi = \begin{pmatrix} -\frac{1}{c}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \phi = \begin{pmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{pmatrix} = \boxed{\begin{pmatrix} \omega/c \\ \underline{k} \end{pmatrix}}$$

As wave propagates, all frames will agree on those events where the displacement is maximal. So the wave crest locations are Lorentz invariant. And because Lorentz transformation is ~~linear~~ linear, all frames agree on how far through the cycle the oscillation is between wave crests.

→ the phase ϕ is Lorentz invariant.

→ $\Box\phi$ is a 4-vector, let $K = \Box\phi$

→ K is the wave 4-vector. ✓

1P

nice

To: Julian Merten

B2 Problem Set 3

Ziyan Li

$$\boxed{\Box^2 A = -\frac{J}{\varepsilon_0 c^2}}$$

$$\Box \cdot A = 0$$

$\Rightarrow$

total:
A1 3/3
A2 ~~6/7~~ 7/7
B1 4/4
B2 4/4
B3 4/4
B4 8/8
C1 7/7
C2 ~~7/8~~ 8/8

45/45

$\Rightarrow$ ~~(43/45)~~

~~…~~ %

good work

= 100%

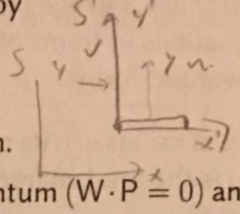
excellent.

Date & Time: 22.11.2016, 17:30 | Location: Univ – 91A | Please return by: 21.11.2016

A Angles in relativistic kinematics

1. In S' a rod parallel to the x'- axis moves in the y'-direction with velocity u. Show that in S the rod is inclined to the x-axis at an angle $-\tan^{-1}(\gamma uv/c^2)$.

2. The Pauli-Lubanski spin vector W is a 4-vector related to angular momentum. For a particle of energy E and momentum $\boldsymbol{p}$ its components are given by



$$\mathsf{W} = (\boldsymbol{s}\cdot\boldsymbol{p},\ (E/c)\,\boldsymbol{s}) \tag{1}$$

where $\boldsymbol{s}$ is the 3-spin, i.e. the intrinsic angular momentum.

   a) Show that this 4-vector is orthogonal to the 4-momentum ($\mathsf{W}\cdot\mathsf{P} = 0$) and that in the limit $v \to c$, W is proportional to P [hint: start in the rest frame and apply a boost].

   b) For a massive particle, we may define a spin 4-vector $s^a = W^a/mc$. In the absence of an applied torque, the spin 4-vector of an accelerating particle evolves as

$$\frac{ds^a}{d\tau} = \frac{s_\lambda \dot{u}^\lambda}{c^2} u^a \tag{2}$$

   where u^a is the 4-velocity and the dot signifies $d/d\tau$. Show that the 3-spin evolves as

$$\frac{d\boldsymbol{s}}{d\tau} = \frac{\gamma^2}{c^2}\left[(\boldsymbol{s}\cdot\dot{\boldsymbol{v}})\,\boldsymbol{v} - (\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\,\boldsymbol{s}\right] \tag{3}$$

   and find $s(\tau)$ for a particle accelerated along a straight line with speed $v(\tau) = c\,[1 - \exp(-2\Gamma\tau)]^{1/2}$, where Γ is a constant.

B Electromagnetism

$S_x\dot{v}\cancel{v\hat{x}} - \dot{v}v\cancel{S_x}\hat{x} - \dot{v}vS_y\hat{y} - \dot{v}vS_z\hat{z}$

1. How does a 2nd rank tensor change under a Lorentz transformation? By transforming the field tensor, and interpreting the result, prove that the electromagnetic field transforms as:

$$\begin{aligned} \boldsymbol{E}'_\parallel &= \boldsymbol{E}_\parallel & \boldsymbol{E}'_\perp &= \gamma\,(\boldsymbol{E}_\perp + \boldsymbol{v}\wedge\boldsymbol{B})\,, \\ \boldsymbol{B}'_\parallel &= \boldsymbol{B}_\parallel & \boldsymbol{B}'_\perp &= \gamma\,(\boldsymbol{B}_\perp - \boldsymbol{v}\wedge\boldsymbol{E}/c^2) \end{aligned} \tag{4}$$

   [Hint: you may find the algebra easier if you treat $\boldsymbol{E}$ and $\boldsymbol{B}$ separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]

2. Obtain the electric field of a uniformly moving charge, as follows. Place the charge at the origin of the primed frame S' and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\boldsymbol{B} = \boldsymbol{v}\wedge\boldsymbol{E}/c^2$.

3. a) Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\boldsymbol{A}$, ϕ such that

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \qquad \boldsymbol{E} = -\left(\frac{\partial \boldsymbol{A}}{\partial t}\right) - \nabla\phi \tag{5}$$

b) Express the other two of Maxwell's equations in terms of $\boldsymbol{A}$ and ϕ.

c) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators and Lorentz scalars (a manifestly covariant form).

4. A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ where q is the total charge. Find the fields due a moving charged sphere by two methods, as follows.
N.B. it will be useful to let the rest frame of the sphere be S' (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S' be in the standard configuration.

a) Field method. Write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ and $r' \geq a$ where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.

b) Potential method. In the rest frame of the sphere the 3-vector potential is zero, and the scalar potential is

$$\begin{aligned}
\phi' &= \frac{q}{8\pi\epsilon_0 a}\left(3 - r'^2/a^2\right) && \text{for } r' < a \\
\phi' &= \frac{q}{4\pi\epsilon_0 r'} && \text{for } r' \geq a.
\end{aligned} \tag{6}$$

Form the 4-vector potential, transform it, and thus show that both ϕ and $\boldsymbol{A}$ are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle $\partial/\partial x$ and $\partial/\partial x'$, etc.]

C Retarded potentials and radiative emission

1. a) Write down the solution to Poisson's equation for the case of a point charge q.

b) In electrostatics, how is the electric potential at a point in space obtained if the charge distribution is known?

c) Now consider the wave equation

$$\Box^2\phi = -\frac{\rho}{\epsilon_0} \tag{7}$$

Show that the spherical wave form $\phi = \kappa g(t - r/c)/r$ (where $\kappa \equiv 1/4\pi\epsilon_0$) is a solution of the wave equation for $r \neq 0$ if ρ is zero everywhere except at the origin.

d) We would like to show that this is a solution also as $r \to 0$, if the charge density ρ is concentrated at a point at the origin. Using your knowledge of Poisson's equation, or otherwise, show that this is true as long as $g(t) = \int \rho(t)dV$.

e) Hence write down the solution to the wave equation for a given arbitrary time-dependent distribution of charge.

f) Why is this called a retarded solution?

2. The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\begin{aligned} \boldsymbol{E} &= \frac{q}{4\pi\epsilon_0\kappa^3}\left(\frac{\boldsymbol{n}-\boldsymbol{v}/c}{\gamma^2 r^2} + \frac{\boldsymbol{n}\wedge[(\boldsymbol{n}-\boldsymbol{v}/c)\wedge\boldsymbol{a}]}{c^2 r}\right) \\ \text{where} \quad \boldsymbol{n} &= \boldsymbol{r}/r, \quad \kappa = 1 - v_r/c = 1 - \boldsymbol{n}\cdot\boldsymbol{v}/c \\ \boldsymbol{B} &= \boldsymbol{n}\wedge\boldsymbol{E}/c \end{aligned} \tag{8}$$

where $\boldsymbol{r}$ is the vector from the source point to the field point, and $\boldsymbol{v}$, $\boldsymbol{a}$ are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified? A charged particle moves along the x axis with constant proper acceleration ('hyperbolic motion'), its worldline being given by

$$x^2 - t^2 = \alpha^2 \tag{9}$$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows.

$t' = t - \frac{\sqrt{\alpha^2+y^2}}{c}$

a) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$t_s = \frac{x_{st}}{c}$

$$x_s = \alpha + \frac{y^2}{2\alpha} \tag{10}$$

b) Show that the velocity and acceleration at the source event are

$x = \sqrt{\alpha^2+t^2}$

$$v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}, \qquad a_s = \frac{\alpha^2}{x_s^3}. \tag{11}$$

c) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of x_s, v_s, a_s. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\boldsymbol{n}$, $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{n}\wedge(\boldsymbol{n}\wedge\boldsymbol{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.

d) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis so that their separation is constant, comment on the forces they exert on one another.

d) We would like to show that this is a solution also as $r \to 0$, if the charge density ρ is concentrated at a point at the origin. Using your knowledge of Poisson's equation, or otherwise, show that this is true as long as $g(t) = \int \rho(t)dV$.

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$$x^2 - t^2 = \alpha^2 \tag{9}$$

$x = \sqrt{\alpha^2 + t^2}$

in units where $c = 1$. Find the electric field at $t = 0$ at points in the plane $x = \alpha$, as follows.

$t' = t - \frac{\sqrt{\alpha^2+y^2}}{c}$

a) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$. Show that the source event is at

$$x_s = \alpha + \frac{y^2}{2\alpha} \tag{10}$$

b) Show that the velocity and acceleration at the source event are

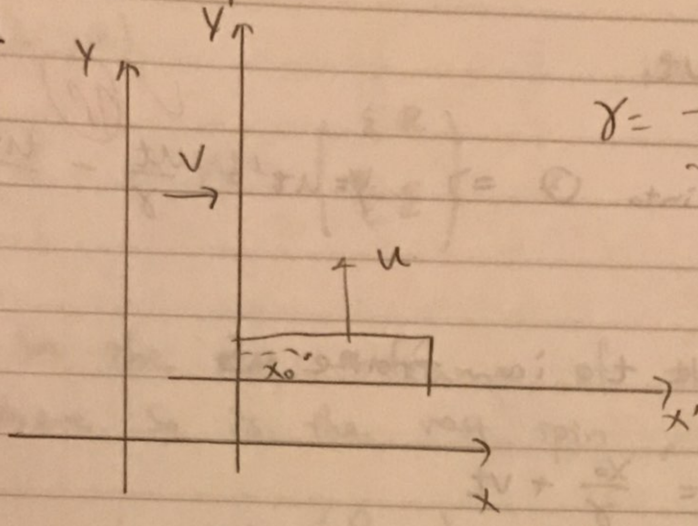
$$v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}, \qquad a_s = \frac{\alpha^2}{x_s^3}. \tag{11}$$

c) Consider the case $\alpha = 1$, and the field point $y = 2$. Write down the values of x_s, v_s, a_s. Draw on a diagram the field point, the source point, and the location of the charge at $t = 0$. Mark at the field point on the diagram the directions of the vectors $\boldsymbol{n}$, $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{n}\wedge(\boldsymbol{n}\wedge\boldsymbol{a})$. Hence, by applying the formula above, establish the direction of the electric field at $(t, x, y, z) = (0, 1, 2, 0)$.

d) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x axis so that their separation is constant, comment on the forces they exert on one another.

A1

3/3



$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Consider a point with x coordinate ~~x_0~~ x_0 on the rod in frame S'

then the event

$$x' = x_0$$
$$y' = ut'$$
$$z' = 0$$
$$t' = t'$$

when transformed to frame S, gives

$$x = \gamma(x' + vt') = \gamma(x_0 + vt') \quad ①$$

~~$y' = \ldots$~~

$$y = y' = ut' \quad ②$$

✓ 1p

$$z = z' = 0$$
$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) = \gamma\left(t' + \frac{vx_0}{c^2}\right) \quad ③$$

③ $\Rightarrow$ $t' = \frac{t}{\gamma} - \frac{vx_0}{c^2}$

Substitute into ① $\Rightarrow$ $x = \gamma x_0 + \gamma v\left(\frac{t}{\gamma}\right) - \frac{\gamma v^2}{c^2} x_0$

$$= \underbrace{\gamma\left(1 - \frac{v^2}{c^2}\right)}_{\frac{1}{\gamma}} x_0 + \cancel{\gamma} vt$$

$\rightarrow x = \frac{x_0}{\gamma} + vt$

✓ 1P

Substitute ③ into ② $\Rightarrow$ $y = ut' = \frac{ut}{\gamma} - \frac{uvx_0}{c^2}$

$\therefore$ At time t in frame S

$$x = \frac{x_0}{\gamma} + vt$$

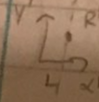
$$y = -\frac{uvx_0}{c^2} + \frac{ut}{\gamma}$$

At given time t in S, the inclination angle of the rod is given by

$\cancel{\tan\theta = \frac{dy/dx_0}{dx/dx_0}} \Rightarrow$ ✓ 1P

$$\tan\theta = \frac{dy}{dx} = \frac{dy/dx_0}{dx/dx_0} = \frac{-\frac{uv}{c^2}}{\frac{1}{\gamma}} = -\frac{uv\gamma}{c^2}$$

$$\therefore \boxed{\theta = -\tan^{-1}\left(\frac{\gamma uv}{c^2}\right)}$$



$R' = \begin{pmatrix} ct' \\ L \\ ut' \\ 0 \end{pmatrix}$ $\quad R = \Lambda^{-1} R' = \begin{pmatrix} \gamma(t' + \beta x') \\ \gamma(x' + \beta t') \\ y' \\ z' \end{pmatrix} \Rightarrow \begin{aligned} t &= \gamma t' + \gamma\beta L \\ x &= \gamma L + \gamma\beta t' \\ y &= ut' \end{aligned}$

$\beta = v$

$x' = L$
$y' = ut'$
$z' = 0$

for $t = 0$ $\quad t' = \frac{\beta L}{c}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad \frac{y}{x} = -\frac{u\beta L}{\gamma L \underbrace{(1-\beta^2)}_{\gamma^{-2}}} = -\frac{u\beta}{c}\gamma = -\frac{uv\gamma}{c^2}.$

Go into the rest frame

$$W_0 = \begin{pmatrix} 0 \\ m\underline{s} \end{pmatrix} \quad P_0 = \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$W_0 \cdot P_0 = 0 = \underline{\underline{W \cdot P}}$$

A2 a)

6/7

$$W = \begin{pmatrix} \underline{S} \cdot \underline{p} \\ \frac{E}{c} \underline{S} \end{pmatrix}$$

in the rest ~~a~~ frame of the particle, $\underline{p} = 0$, $\underline{S} = \underline{S_0}$, where $\underline{S_0}$ is the rest spin, $E = mc^2$

$$\therefore \quad W_0 = \begin{pmatrix} 0 \\ mc\underline{S_0} \end{pmatrix}, \quad \text{4-velocity} \quad U_0 = \begin{pmatrix} c \\ \underline{0} \end{pmatrix}$$

$\therefore \quad W_0 \cdot U_0 = 0 \quad \because$ 4-vector product is invariant

$\therefore \quad \underline{\underline{W \cdot U = 0}}$ in all ~~in~~ inertial frames.

$\because P = m_0 U$

$\therefore W \cdot P = 0$

X

1P

Start from rest frame and Lorentz transform

$$W = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ m\underline{s} \end{pmatrix} = \begin{pmatrix} m\gamma\beta S_{\parallel} \\ m\gamma \\ mS_{\perp} \end{pmatrix}$$

$$\begin{pmatrix} \underline{S} \cdot \underline{p} \\ \frac{E}{c} \underline{S} \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} 0 \\ mc\underline{S_0} \end{pmatrix}$$

$$P = \begin{pmatrix} \gamma m \\ \gamma m \underline{v} \end{pmatrix} \quad \text{for } v \to c = 1$$

$$W = P\{S_{\parallel}\}$$

$$\begin{pmatrix} \underline{S} \cdot \underline{p} \\ \frac{E}{c} S_x \\ \frac{E}{c} S_y \\ \frac{E}{c} S_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ mcS_{0x} \\ mcS_{0y} \\ mcS_{0z} \end{pmatrix} = \begin{pmatrix} \gamma\beta mcS_{0x} \\ \gamma mcS_{0x} \\ mcS_{0y} \\ mcS_{0z} \end{pmatrix}$$

$$\therefore \quad \frac{E}{c} S_x = \frac{E}{c} S_{0x} \Rightarrow \underline{S_{\parallel}} = \underline{S_{0\parallel}}$$

b)

$\frac{E}{c} S_y = \frac{E}{\gamma c} S_{0y}$, $\frac{E}{c} S_z = \frac{E}{\gamma c} S_{0z}$ $\Rightarrow$ $\underline{S}_\perp = \frac{S_{0\perp}}{\gamma}$

$\therefore$ ~~As~~ As $v \to c$ $\gamma \to \infty$ $\Rightarrow$ $\underline{S}_\perp \to \underline{0}$

$\therefore$

I don't understand this part. You need to look at rest frame p^α $\Rightarrow \Lambda p^\alpha = \begin{pmatrix} m \\ \underline{0} \end{pmatrix}$ ~~γmv~~

Spin directed along velocity (momentum)
~~and ...~~

$\therefore W = \begin{pmatrix} S_{0x} p \\ S_{0x} \frac{E}{c} \\ 0 \\ 0 \end{pmatrix}$ as $v \to c$

Λp

As $v \to c$ ie $\therefore$ ~~$E = \sqrt{p^2c^2 + m_0^2c^4}$ and $p = \gamma m_0 v \to \gamma m_0 c$~~

$p = \gamma m_0 v \to \gamma m_0 c$

~~$\therefore E = \sqrt{(\gamma^2+1)} m_0 c^2 \approx \gamma m_0 c \approx p$~~

$E = \sqrt{p^2c^2 + m_0^2c^4} = \sqrt{\gamma^2 m_0^2 c^4 + m_0^2 c^4} \approx \gamma m_0 c^2 \approx pc$

$\therefore E \to pc$ as $v \to c$, ~~Also ...~~

$\therefore W = \begin{pmatrix} S_{0x} p \\ S_{0x} p \\ 0 \\ 0 \end{pmatrix} = S_{0x} P = |\underline{S}_{\parallel}| P$

~~$\Rightarrow S_{0x}$~~ $|\underline{S}_{\parallel}| = S_{0x} = \underline{S} \cdot \hat{\underline{x}} = \underline{S} \cdot \frac{\underline{p}}{|\underline{p}|} = \frac{\underline{S} \cdot \underline{p}}{p}$

$\therefore$ $\boxed{W = \frac{\underline{S} \cdot \underline{p}}{p} P}$ $\Rightarrow$ W proportional to P

ΛP

$S^\alpha = \frac{1}{m} W^\alpha = \frac{1}{m} \begin{pmatrix} \underline{S} \cdot \underline{p} \\ E \underline{S} \end{pmatrix} = \begin{pmatrix} \frac{1}{m} \underline{S} \cdot \underline{p} \\ \gamma \underline{S} \end{pmatrix}$

$\dot{S} = \begin{pmatrix} \frac{1}{m} (\dot{\underline{S} \cdot \underline{p}}) \\ (\dot{\gamma \underline{S}}) \end{pmatrix} = \begin{pmatrix} \frac{1}{m} (\dot{\underline{S}} \cdot \underline{p} + \underline{S} \cdot \dot{\underline{p}}) \\ \gamma \dot{\underline{S}} + \dot{\gamma} \underline{S} \end{pmatrix}$

$\dot{S}^\alpha = S_\lambda \dot{u}^\lambda u^\alpha = \frac{1}{m} \begin{pmatrix} -\underline{S} \cdot \underline{p} \\ E \underline{S} \end{pmatrix} \begin{pmatrix} \dot{\gamma} \\ \dot{\gamma} \underline{u} + \gamma \dot{\underline{u}} \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma \underline{u} \end{pmatrix}$

$\frac{1}{m}(\underbrace{-\gamma \underline{s}\cdot\underline{f} + \gamma E \underline{s}\cdot\underline{u}}_{0} + \gamma E \underline{s}\cdot\underline{u}) \begin{pmatrix} \gamma \\ \gamma \underline{u} \end{pmatrix}$

$= \frac{1}{m}(\gamma E \underline{s}\cdot\underline{u}) \begin{pmatrix} \gamma \\ \gamma \underline{u} \end{pmatrix}$

b)

$$\frac{dS^a}{d\tau} = \frac{S_\lambda \dot{U}^\lambda}{c^2} U^a \qquad ①$$

$S =$ spin 4-vector $= \frac{W}{mc}$

$\therefore$ spatial part of $S = \frac{(\frac{E}{c})\underline{s}}{mc} = \frac{E}{mc^2}\underline{s} = \gamma \underline{s}$

~~Sub in equation ① ⇒~~

$\cancel{\frac{d}{d\tau}(\gamma \underline{s}) =}$ time part of S

$= \frac{\underline{s}\cdot\underline{p}}{mc} = \frac{\underline{s}\cdot\gamma m \underline{v}}{mc} = \frac{\gamma \underline{s}\cdot\underline{v}}{c}$

$\therefore S = \begin{pmatrix} \frac{\gamma \underline{s}\cdot\underline{v}}{c} \\ \gamma \underline{s} \end{pmatrix}$

Apply equation ① to the spatial part of S

$\frac{d}{d\tau}(\gamma \underline{s}) = \left(-\frac{\gamma \underline{s}\cdot\underline{v}}{c}\frac{\dot{E}}{c} + \gamma \underline{s}\cdot\dot{\underline{p}} \right) \frac{\gamma \underline{v}}{mc^2}$ ✓ 1p

$\dot{U} = m_0 \dot{P}$

Assume $\frac{dm_0}{d\tau} = 0$

$(\frac{\dot{E}}{c} = (\gamma m c)^{\cdot} = \dot{\gamma} m c = \dot{U}_0 m = \dot{P}_0)$

$= (-\cancel{\gamma \underline{s}\cdot\underline{v}\dot{\gamma}} + \gamma \underline{s}\cdot(\cancel{\dot{\gamma}\underline{v}} + \gamma \dot{\underline{v}})) \frac{\gamma \underline{v}}{c^2}$

$= \gamma^3 (\underline{s}\cdot\dot{\underline{v}}) \frac{\underline{v}}{c^2}$

$\Rightarrow \gamma \dot{\underline{s}} + \dot{\gamma}\underline{s} = \frac{1}{m}\gamma^2 E(\underline{s}\cdot\dot{\underline{u}})\cdot\underline{u}$

$\Rightarrow \frac{d\underline{s}}{d\tau} = \dot{\underline{s}} = \frac{1}{m\gamma}\gamma^2 E(\underline{s}\cdot\dot{\underline{u}})\underline{u} - \frac{1}{\gamma}\dot{\gamma}\underline{s}$

$\dot{\gamma} = \gamma^3 \underline{v}\cdot\dot{\underline{v}}$

$\Rightarrow \dot{\underline{s}} = \gamma^2((\underline{s}\cdot\dot{\underline{u}})\underline{u} - \frac{1}{\gamma}\gamma^3(\underline{u}\cdot\dot{\underline{u}})\underline{s})$

$\Rightarrow \dot{\underline{s}} = \gamma^2[(\underline{s}\cdot\dot{\underline{u}})\underline{u} - (\underline{u}\cdot\dot{\underline{u}})\underline{s}]$. qed □

$$\frac{d}{d\tau}(\gamma S) = \dot{\gamma}\underline{S} + \gamma\dot{\underline{S}}$$

TP

$$\dot{\gamma} = \frac{d}{d\tau}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \frac{d}{d\tau}\left(\frac{1}{\sqrt{1-\frac{\underline{v}\cdot\underline{v}}{c^2}}}\right)$$

$$= \frac{d}{d\tau}\left(1-\frac{\underline{v}\cdot\underline{v}}{c^2}\right)^{-\frac{1}{2}} = -\frac{1}{2}\left(1-\frac{\underline{v}\cdot\underline{v}}{c^2}\right)^{-\frac{3}{2}}\frac{d}{d\tau}\left(-\frac{\underline{v}\cdot\underline{v}}{c^2}\right).$$

$$= -\frac{1}{\cancel{2}}\underbrace{\left(\left(1-\frac{\underline{v}\cdot\underline{v}}{c^2}\right)^{-\frac{1}{2}}\right)^3}_{\gamma^3}(-\cancel{2}\,\underline{v}\cdot\dot{\underline{v}})\left(\frac{1}{c^2}\right)$$

$$= \frac{\gamma^3}{c^2}\,\underline{v}\cdot\dot{\underline{v}}$$

$$\therefore \quad \gamma\dot{\underline{S}} = -\frac{\gamma^3}{c^2}(\underline{v}\cdot\dot{\underline{v}})\underline{S} + \frac{\gamma^3}{c^2}(\underline{S}\cdot\dot{\underline{v}})\underline{v}$$

$$\Rightarrow \quad \underline{\dot{\underline{S}} = \frac{\gamma^2}{c^2}\left[(\underline{S}\cdot\dot{\underline{v}})\underline{v} - (\underline{v}\cdot\dot{\underline{v}})\underline{S}\right].}$$

TP

QED.

Assume $\underline{v}$ along x $\quad \underline{v} = v\hat{\underline{x}} \quad \dot{\underline{v}} = \dot{v}\hat{\underline{x}}$

$$v = c\left(1-e^{-2\Gamma\tau}\right)^{\frac{1}{2}}$$

$$\dot{\underline{S}} = \frac{\gamma^2}{c^2}\left[(\underline{S}\cdot\dot{v}\hat{\underline{x}})v\hat{\underline{x}} - (\underline{v}\cdot\dot{\underline{v}})\,\underline{S}\right]$$

$$= \frac{\gamma^2}{c^2}\left[\cancel{S_x\dot{v}v\hat{\underline{x}} - v\dot{v}S_x\hat{\underline{x}}} - v\dot{v}S_y\hat{\underline{y}} - v\dot{v}S_z\hat{\underline{z}}\right]$$

$$= -\frac{\gamma^2 v\dot{v}}{c^2}\left(S_y\hat{\underline{y}} + S_z\hat{\underline{z}}\right)$$

$$\Rightarrow \quad \dot{S}_x = 0 \quad \Rightarrow \boxed{S_x = S_x(0)} \quad 1P$$

$$\dot{S}_y = -\frac{\gamma^2 v \dot{v}}{c^2} S_y$$

$$\Rightarrow \frac{dS_y}{S_y} = -\frac{\gamma v \dot{v}}{c^2} d\tau$$

$$\because v = c(1 - e^{-2\Gamma\tau})^{1/2}$$

$$\therefore \left(\frac{v}{c}\right)^2 = 1 - e^{-2\Gamma\tau}$$

$$\gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = \frac{1}{e^{-2\Gamma\tau}} = e^{2\Gamma\tau}$$

$$\dot{v} = \frac{c}{\cancel{2}}(1 - e^{-2\Gamma\tau})^{-\frac{1}{2}}\left[-(-\cancel{2}\Gamma)e^{-2\Gamma\tau}\right]$$

$$= \frac{c\Gamma e^{-2\Gamma\tau}}{(1-e^{-2\Gamma\tau})^{\frac{1}{2}}} = \frac{c^2\Gamma e^{-2\Gamma\tau}}{v} = \frac{c^2\Gamma}{\gamma^2 v}$$

$$\therefore \frac{dS_y}{S_y} = -\frac{\gamma^2 v \dot{v}}{c^2} d\tau = -\frac{1}{\cancel{c^2}} \cancel{\gamma^2 v} \frac{\cancel{c^2}\Gamma}{\cancel{\gamma^2 v}} d\tau$$

$$= -\Gamma d\tau$$

$$\therefore \boxed{S_y = S_y(0) e^{-\Gamma\tau}} \quad 1P$$

Similarly $\boxed{S_z = S_z(0) e^{-\Gamma\tau}}$

→ Consistent with

$\underline{S}_{\parallel}$ stay the same

$\underline{S}_{\perp}$ goes to $\underline{0}$ as $v \to c$

$\cancel{S(\tau)}$

$$\dot{\underline{S}} = \gamma^2 \begin{pmatrix} \cancel{S_{\parallel} u_x' u_x - u_x u_x' S_x} \\ S_y \cdot 0\,0 - u_y u_y' S_y \\ S_z\, 0\,0 - u_z u_z' S_z \end{pmatrix}$$

$S_y = -u_y u_y' S_y \gamma^2 = S_2.$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1-(1-\exp(\,))}}$$

$$= \exp(\Gamma\tau)$$

$$\dot{v} = \frac{1}{2\sqrt{1-\exp(\,)}} \cdot (-\exp(-2\Gamma\tau))(-2\Gamma)$$

$$v\dot{v} = \Gamma \exp(-2\Gamma\tau) = \Gamma\gamma^2$$

B1

4/4

Transformation of 2nd Rank tensor $\mathbb{F}$

is $\underline{\underline{\mathbb{F}' = \Lambda \mathbb{F} \Lambda^T}}$ 1P (Λ is lorentz transformation)

The field tensor $\mathbb{F}$ is given by

$$\mathbb{F} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

transform $\mathbb{F} \Rightarrow \mathbb{F}' = \Lambda \mathbb{F} \Lambda^T$

$$\Rightarrow \mathbb{F}^{a'b'} = \Lambda^{a'}{}_{\mu} \mathbb{F}^{\mu\nu} (\Lambda^T)_{\nu}{}^{b'}$$

$$= \Lambda^{a'}{}_{\mu} \Lambda^{b'}{}_{\nu} \mathbb{F}^{\mu\nu}$$ 1P

$\because \mathbb{F}^{ab}$ is antisymmetric

and Λ is symmetric

$\therefore \mathbb{F}^{a'b'}$ is antisymmetric

Now $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\mathbb{F}^{0'1'} = \Lambda^{0'}{}_{\mu} \Lambda^{1'}{}_{\nu} \mathbb{F}^{\mu\nu}$$

$\Lambda^{0'}{}_{\mu} \neq 0$ only for $\mu = 0, 1$

$\Lambda^{1'}{}_{\nu} \neq 0$ only for $\nu = 0, 1$

$$\therefore \mathbb{F}^{0'1'} = \Lambda^{0'}{}_{0} \Lambda^{1'}{}_{1} \mathbb{F}^{01} + \Lambda^{0'}{}_{1} \Lambda^{1'}{}_{0} \mathbb{F}^{10}$$

$$= (\gamma^2 - \gamma^2\beta^2) \mathbb{F}^{01} = \mathbb{F}^{01}$$

1P

$$\Rightarrow \frac{E_x'}{c} = \frac{E_x}{c} \Rightarrow E_x' = E_x \Rightarrow \boxed{E_{//}' = E_{//}}$$

$$\mathbb{F}^{0'2'} = \Lambda^{0'}{}_{0} \Lambda^{2'}{}_{2} \mathbb{F}^{02} + \Lambda^{0'}{}_{1} \Lambda^{2'}{}_{2} \mathbb{F}^{12}$$

$$= \gamma \mathbb{F}^{02} - \gamma\beta \mathbb{F}^{12}$$

$$\Rightarrow \frac{E_y'}{c} = \gamma \frac{E_y}{c} - \gamma\beta B_z \Rightarrow E_y' = \gamma(E_y - v_x B_z).$$

$$\mathbb{F}^{0'3'} = \Lambda^{0'}{}_{0} \Lambda^{3'}{}_{3} \mathbb{F}^{03} + \Lambda^{0'}{}_{1} \Lambda^{3'}{}_{3} \mathbb{F}^{13}$$

$$\Rightarrow \frac{E_z'}{c} = \gamma \frac{E_z}{c} + \gamma\beta B_y \Rightarrow E_z' = \gamma(E_z + v_x B_y)$$

1P

$$\Rightarrow \boxed{\underline{E}_\perp = \gamma(\underline{E}_\perp + \underline{v} \times \underline{B})}$$

$$F^{2'3'} = \underset{1}{\Lambda^{2'}{}_2}\,\underset{1}{\Lambda^{3'}{}_3}\,F^{23} = F^{23}$$

$$\rightarrow \quad B_x' = B_x \quad \Rightarrow \quad \boxed{\underline{B}_{\parallel}' = \underline{B}_{\parallel}}$$

$$F^{1'2'} = \underset{-\gamma\beta}{\Lambda^{1'}{}_0}\,\underset{1}{\Lambda^{2'}{}_2}\,F^{02} + \underset{\gamma}{\Lambda^{1'}{}_1}\,\underset{1}{\Lambda^{2'}{}_2}\,F^{12}$$

$$B_z' = -\gamma\beta\left(+\frac{E_y}{c}\right) + \gamma \cdot B_z.$$

$$= \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$F^{1'3'} = \underset{-\gamma\beta}{\Lambda^{1'}{}_0}\,\underset{1}{\Lambda^{3'}{}_3}\,F^{03} + \underset{\gamma}{\Lambda^{1'}{}_1}\,\underset{1}{\Lambda^{3'}{}_3}\,F^{13}$$

$$-B_y' = -\gamma\beta\left(\frac{E_z}{c}\right) + \gamma(-B_y)$$

$$\Rightarrow \quad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$\rightarrow \quad \boxed{\underline{B}_{\perp}' = \gamma\left(\underline{B}_{\perp} - \frac{\underline{v}\times\underline{E}}{c^2}\right)}$$

B2.

4/4

In rest frame of charge S' ($\because$ charge at origin)

$$\underline{E}' = \frac{Q}{4\pi\varepsilon_0 r'^3}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad \underline{B}' = \underline{0}$$

Consider the event $[ct', x', y', z']$. The fields at this same event, but evaluated in frame S, is

$$E_x = E_x'$$

$$E_y = \gamma E_y'$$

$$E_z = \gamma E_z'$$

$\because \underline{B} = 0$

$$\therefore \quad E_x = \frac{Q}{4\pi\varepsilon_0}\frac{x'}{r'^3}$$

$$E_y = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma y'}{r'^3}$$

1P

$$E_z = \frac{Q}{4\pi\varepsilon_0}\frac{\gamma z'}{r'^3}$$

Lorentz transformation

$$x' = \gamma(x - vt)$$ 1P

$$y' = \cancel{\gamma t} y$$

$$z' = z$$

$$\therefore \quad E_x = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r'^3}\gamma(x - vt)$$

$$E_y = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r'^3}\gamma y$$

$E_z = \frac{q}{4\pi\epsilon_0} \frac{1}{r'^3} \gamma z$

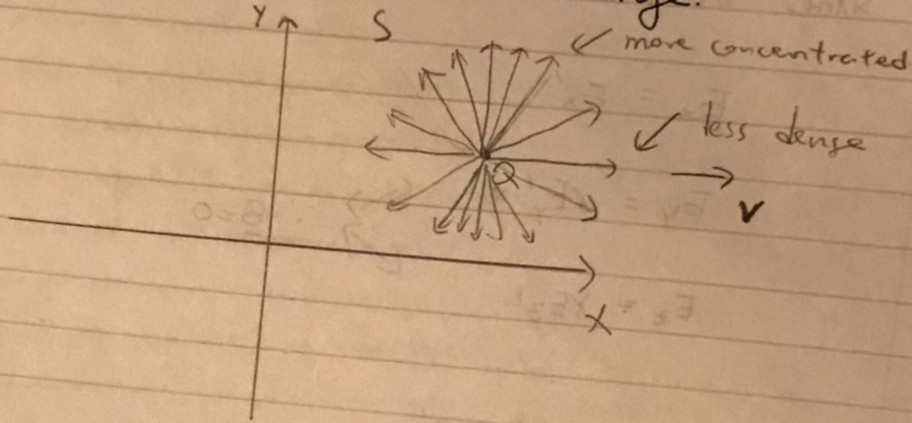
$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$$\therefore \boxed{\underline{E} = \frac{\gamma Q}{4\pi\epsilon_0(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \underbrace{\begin{pmatrix} x-vt \\ y \\ z \end{pmatrix}}_{\text{position of charge}}}$$

1P

assuming charge at (0,0,0) at t=0

→ The field lines directed radially from the charge.



Magnetic field

$\cancel{B_{\parallel}' = B}$ $\quad \underline{B}_{\parallel} = \underline{B}_{\parallel}' = 0$

$\rightarrow \quad B_x = 0 \quad \Rightarrow \quad \underline{B} = \underline{B}_{\perp}$

$$\underline{B}_{\perp} = \gamma(\underbrace{\underline{B}_{\perp}'}_{=0} + \frac{\underline{v}\times\underline{E}'}{c^2}) = \gamma\frac{\underline{v}\times\underline{E}'}{c^2}$$

$$\underline{v}\times\underline{E}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v & 0 & 0 \\ E_x' & E_y' & E_z' \end{vmatrix} = \begin{pmatrix} 0 \\ -vE_z' \\ vE_y' \end{pmatrix}$$

$$\therefore \underline{B}_{\perp} = \gamma\frac{\underline{v}\times\underline{E}'}{c^2} = \frac{\gamma}{c^2}\begin{pmatrix} 0 \\ -vE_z' \\ vE_y' \end{pmatrix} = \frac{1}{c^2}\begin{pmatrix} 0 \\ -v\gamma E_z' \\ v\gamma E_y' \end{pmatrix}$$

$$= \frac{1}{c^2}\begin{pmatrix} 0 \\ -vE_z \\ vE_y \end{pmatrix} = \frac{1}{c^2}\underline{v}\times\underline{E}$$

1P

QED

B3 a)

4/4

If ~~$\nabla \times B =$~~ $\underline{B} = \underline{\nabla} \times \underline{A}$, $\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$ ✓ 1P

then $\underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) = 0$ satisfied.

$$\underline{\nabla} \times \underline{E} = -\frac{\partial}{\partial t}\underbrace{(\underline{\nabla} \times \underline{A})}_{\underline{B}} - \underbrace{\underline{\nabla} \times \underline{\nabla}\phi}_{0} = -\frac{\partial \underline{B}}{\partial t} \quad \text{satisfied}$$ 1P

b) other two equations:

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$$

$$\rightarrow \frac{\rho}{\varepsilon_0} = \underline{\nabla} \cdot \left(-\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi\right) = -\nabla^2\phi - \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{A})$$

$$\Rightarrow \boxed{-\nabla^2\phi - \frac{\partial}{\partial t}(\underline{\nabla} \cdot \underline{A}) = \frac{\rho}{\varepsilon_0}}$$ ✓

$$\underline{\nabla} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}. \Rightarrow c^2 \underline{\nabla} \times \underline{B} = \frac{\underline{j}}{\varepsilon_0} + \frac{\partial \underline{E}}{\partial t}.$$

$$\rightarrow c^2 \underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \frac{\underline{j}}{\varepsilon_0} - \frac{\partial}{\partial t}\underline{\nabla}\phi - \frac{\partial^2 \underline{A}}{\partial t^2}.$$ ✓

$$\Rightarrow \boxed{c^2 \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) + \frac{\partial}{\partial t}\underline{\nabla}\phi + \frac{\partial^2 \underline{A}}{\partial t^2} - c^2 \nabla^2 \underline{A} = \frac{\underline{j}}{\varepsilon_0}}$$ 1P

(using $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}$)

c) introduce lorenz gauge

$$\underline{\nabla} \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$ ✓

then $\underline{\nabla}\cdot\underline{A} = \cancel{-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t}} - \frac{1}{c^2}\frac{\partial\phi}{\partial t}.$

$\therefore \; -\nabla^2\phi - \frac{\partial}{\partial t}\left(-\frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \frac{\rho}{\varepsilon_0} \Rightarrow -\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = -\frac{\rho}{\varepsilon_0}$

$\Rightarrow \boxed{\Box^2\phi = -\frac{\rho}{\varepsilon_0}} \qquad \Rightarrow \Box^2\left(\frac{\phi}{c}\right) = -\frac{(\rho c)}{c^2\varepsilon_0}$

$-c^2\underline{\nabla}(\underbrace{\underline{\nabla}\cdot\underline{A} + \frac{\partial\phi}{\partial t}}_{0}) + c^2\left(-\frac{1}{c^2}\frac{\partial^2\underline{A}}{\partial t^2} + \nabla^2\underline{A}\right) = -\frac{\underline{j}}{\varepsilon_0}$

$\Rightarrow \boxed{\Box^2\underline{A} = -\frac{\underline{j}}{\varepsilon_0}}$

let $A = \begin{pmatrix}\phi/c \\ \underline{A}\end{pmatrix}$ $\quad J = \begin{pmatrix}\rho c \\ \underline{j}\end{pmatrix}$

$\Rightarrow \boxed{\Box^2 A = -\frac{1}{\varepsilon_0 c^2} J}$

B4. a)

8/8

$r' \geq a \rightarrow \underline{E}' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'^3}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \underline{B}' = \underline{0}$

$r' < a \rightarrow \underline{E}' = \frac{q}{4\pi\varepsilon_0 a^3}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ 1p $\underline{B}' = \underline{0}$

$\cancel{E_x = E_x'} \quad \cancel{B_x = B_x'}$

$\cancel{E_y = \gamma E_y'}$

$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\because \underline{B}' = 0 \quad \therefore \begin{cases} E_x = E_x' \\ E_y = \gamma E_y' \\ E_z = \gamma E_z' \end{cases}$

$\cancel{x' = \gamma(x'}$

$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases}$

$\rightarrow$ in frame S : $\cancel{E_x}$

$\underline{\underline{r' \geq a}}$

$E_x = E_x' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'^3}x' = \cancel{\frac{q}{4\pi\varepsilon_0}}$

$E_y = \gamma E_y' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'^3}\gamma y'$

$E_z = \gamma E_z' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r'^3}\gamma z'$ 1p

see B2

$\Rightarrow \boxed{\underline{E} = \frac{q}{4\pi\varepsilon_0}\frac{\gamma}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}\begin{pmatrix} x-vt \\ y \\ z \end{pmatrix}} \quad (r \geq a)$

$B_x = B_x' = 0 \quad B_y = \gamma(B_y' - \frac{1}{c^2}vE_z') = -v\gamma E_z'/c^2$

$B_z = \gamma(B_z' + \frac{1}{c^2}vE_y') = v\gamma E_y'/c^2$

$$\boxed{\underline{B} = \frac{\mu_0 q}{4\pi \cancel{\varepsilon_0 c^2}} \frac{v\gamma Q}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}} \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}}$$ $(r \geq a)$

$\underline{\underline{r<a}}$

$$E_x = E_x' = \frac{q}{4\pi\varepsilon_0 a^3} x'$$

$$E_y = \gamma E_y' = \frac{q}{4\pi\varepsilon_0 a^3} \gamma y'$$

$$E_z = \gamma E_z' = \frac{q}{4\pi\varepsilon_0 a^3} \gamma z'$$

$$\rightarrow \boxed{\underline{E} = \frac{q\gamma}{4\pi\varepsilon_0 a^3} \begin{pmatrix} x-vt \\ y \\ z \end{pmatrix}}$$ (1P) $(r<a)$

$B_x = 0$. $B_y = -v\gamma E_z' / c^2$

$B_z = v\gamma E_y' / c^2$

$$\cancel{\underline{B} = \frac{q\gamma}{4\pi\varepsilon_0 a^3 c^2}}$$

$$\boxed{\underline{B} = \frac{\mu_0 q \gamma v}{4\pi a^3} \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}}$$ $(r<a)$

b)

The four potential $A' = \begin{pmatrix} \phi'/c \\ \underline{A}' \end{pmatrix}$

in frame S'.

$$A' = \begin{pmatrix} \phi'/c \\ \underline{0} \end{pmatrix} = \begin{pmatrix} \phi'/c \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in frame S.

$$A = \Lambda^{-1} A'$$

$$\Rightarrow \begin{pmatrix} \phi/c \\ A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi'/c \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \phi/c = \gamma\frac{\phi'}{c} \rightarrow \underline{\phi = \gamma\phi'}$$

1p

$$\underline{A_x = \gamma\beta\frac{\phi'}{c}}, \quad A_y = 0, \quad A_z = 0 \Rightarrow \underline{A} = \begin{pmatrix} \gamma\beta\frac{\phi'}{c} \\ 0 \\ 0 \end{pmatrix}$$

For $r \geq a$

$$\rightarrow \phi = \gamma\phi' = \frac{\gamma q}{4\pi\varepsilon_0 r'} = \frac{\gamma q}{4\pi\varepsilon_0(\gamma^2(x-vt)^2 + y^2 + z^2)^{1/2}}$$

1p

$$\rightarrow \underline{A} = \gamma\frac{v}{c^2}\phi'\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{v\gamma q\,\hat{x}}{4\pi\varepsilon_0 c^2(\gamma^2(x-vt)^2 + y^2 + z^2)^{1/2}}$$

$$\therefore\ \underline{B} = \underline{\nabla} \times \underline{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ \partial_z A_x \\ -\partial_y A_x \end{pmatrix}$$

$$\partial_z A_x = \frac{\mu_0}{4\pi}\gamma q v \left(-\frac{1}{\cancel{2}}\right) \frac{1}{(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} (\cancel{2}z)$$

$$\partial_y A_x = \frac{\mu_0}{4\pi}\gamma q v \left(-\frac{1}{2}\right) \frac{1}{(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} (2y)$$

$$\rightarrow \boxed{\underline{B} = \frac{\mu_0}{4\pi}\gamma q v \frac{1}{(\gamma^2(x-vt)^2 + y^2 + z^2)} \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} \quad (r \geq a)}$$

7p

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi = \cancel{\partial_t A_x} \begin{pmatrix} -\partial_t A_x - \partial_x \phi \\ -\partial_y \phi \\ -\partial_z \phi \end{pmatrix}$$

$$-\partial_t A_x = \cancel{\phi} \frac{v\gamma q}{4\pi\varepsilon_0 c^2} \frac{1}{(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} \left(\cancel{-}\frac{1}{\cancel{2}}\right) \cancel{2}\gamma^2 (x-vt)(-v)$$

$$= \left(-\frac{v^2}{c^2}\gamma^2\right) \frac{\gamma q}{4\pi\varepsilon_0 (\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} (x-vt).$$

$$-\partial_x \phi = \cancel{-} \frac{\gamma q}{4\pi\varepsilon_0} \left(-\frac{1}{\cancel{2}}\right) \frac{1}{(\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} (\cancel{2}\gamma^2(x-vt)).$$

$$= (\gamma^2) \frac{\gamma q}{4\pi\varepsilon_0 (\gamma^2(x-vt)^2 + y^2 + z^2)^{3/2}} (x-vt)$$

$$\because\quad \gamma^2 - \frac{v^2}{c^2}\gamma^2 = \left(\frac{1}{\gamma^2}\right)\cdot(\gamma^2) = 1$$

$\rightarrow \therefore \quad E_x = \dfrac{\gamma q}{4\pi\varepsilon_0(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}(x-vt)$

$-\partial_y\phi = \dfrac{\gamma q y}{4\pi\varepsilon_0(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}$

$-\partial_z\phi = \dfrac{\gamma q z}{4\pi\varepsilon_0(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}$

$$\therefore \boxed{\underline{E} = \frac{\gamma q}{4\pi\varepsilon_0}\frac{1}{(\gamma^2(x-vt)^2+y^2+z^2)^{3/2}}\begin{pmatrix} x-vt \\ y \\ z \end{pmatrix} \quad r \geq a}$$

7P

$\underline{\underline{r<a}}$

$\cancel{B}$ $\quad \phi = \gamma\phi' = \dfrac{q\gamma}{8\pi\varepsilon_0 a}\left(3-\dfrac{r'^2}{a^2}\right)$

$\underline{A} = \dfrac{\gamma v}{c^2}\phi'\begin{pmatrix}1\\0\\0\end{pmatrix} = \dfrac{\gamma v q}{8\pi\varepsilon_0 c^2 a}\left(3-\dfrac{r'^2}{a^2}\right)\hat{\underline{x}}$

$\underline{B} = \begin{pmatrix} 0 \\ \partial_z A_x \\ -\partial_y A_x \end{pmatrix} \qquad r'^2 = \gamma^2(x-vt)^2+y^2+z^2$

$\partial_z A_x = -\dfrac{\gamma v q}{8\pi\varepsilon_0 c^2 a^3}(2z) = -\dfrac{\mu_0\gamma v q}{4\pi a^3}z$

$-\partial_y A_x = -(-)\dfrac{\gamma v q}{8\pi\varepsilon_0 c^2 a^3}(2y) = \dfrac{\mu_0\gamma v q}{4\pi\varepsilon_0 a^3}y$

$$\therefore \boxed{\underline{B} = \frac{\mu_0 q\gamma v}{4\pi a^3}\begin{pmatrix} 0 \\ -z \\ y \end{pmatrix} \quad r<a}$$

$$\underline{E} = \begin{pmatrix} -\partial_t A_x - \partial_x \phi \\ -\partial_y \phi \\ -\partial_z \phi \end{pmatrix}$$

$$-\partial_t A_x = + \frac{\gamma v q}{8\pi\varepsilon_0 c^2 a^3}(-v) 2\gamma^2 (x - vt)$$

$$-\partial_x \phi = \frac{q\gamma}{8\pi\varepsilon_0 a^3} 2\gamma^2 (x - vt)$$

$\because$ $$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

$\therefore$ $$E_x = \frac{q\gamma}{4\pi\varepsilon_0 a^3}(x - vt)$$

$$-\partial_y \phi = \frac{q\gamma}{4\pi\varepsilon_0 a^3} y \qquad -\partial_z \phi = \frac{q\gamma}{4\pi\varepsilon_0 a^3} z$$

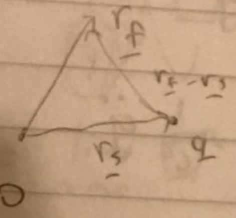
$\therefore$ $$\boxed{\underline{E} = \frac{q\gamma}{4\pi\varepsilon_0 a^3}\begin{pmatrix} x - vt \\ y \\ z \end{pmatrix} \qquad r < a}$$

C1

7/7

a) Solution $\cancel{\phi}$ to $\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$

$\cancel{\phi = \frac{q}{4\pi\varepsilon_0 r}}$ for point charge q



$$\boxed{\phi = \frac{q}{4\pi\varepsilon_0 |\underline{r}_f - \underline{r}_s|}}$$ 1P

$\underline{r}_f$ = position we are finding the field (potential)

$\underline{r}_s$ = position of charge

b) If distribution of charge density $\rho(\underline{r}_s)$ is known

then

$$\boxed{\phi = \int \frac{\rho(\underline{r}_s)}{4\pi\varepsilon_0 |\underline{r}_f - \underline{r}_s|} \, d^3\underline{r}_s}$$ 1P

c) $\Box^2\phi = -\frac{\rho}{\varepsilon_0}$

At $r \neq 0$ $\rho = 0$ $\Rightarrow$ $-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = 0$

try $\phi = k\,\frac{g(t - \frac{r}{c})}{r}$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi)$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(k\,g\left(t - \frac{r}{c}\right)\right)$$

let $t_0 = t - \frac{r}{c}$ and $\frac{dg}{dt_0} = \dot{g}$ $\frac{d^2g}{dt_0^2} = \ddot{g}$

then $\frac{\partial g}{\partial t} = \frac{dg}{dt_0}\frac{\partial t_0}{\partial t} = \frac{dg}{dt_0} = \dot{g}$

$\Rightarrow \frac{\partial^2 g}{\partial t^2} = \frac{d\dot{g}}{dt_0}\frac{\partial t_0}{\partial t} = \frac{d^2 g}{dt_0^2} = \ddot{g}$

$\Rightarrow \frac{\partial g}{\partial r} = -\frac{1}{c}\dot{g}$ $\frac{\partial^2 g}{\partial r^2} = \frac{1}{c^2}\ddot{g}$ ✓ 1P

$\therefore \nabla^2\phi = \frac{\kappa}{rc^2}\ddot{g}$ ✓ 2P

$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\kappa}{rc^2}\ddot{g} \Rightarrow \underline{\Box^2\phi = 0}$ ~~st~~ satisfied

d)
If $r \to 0$. then we cannot do

$r\phi = (\kappa)\frac{\kappa}{\cancel{\kappa}} g(t-\frac{r}{c}) = \kappa g(t-\frac{r}{c})$ because the cancellation is not well defined.

$\therefore$ look at first derivatives

$\frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\kappa}{r}g(t-\frac{r}{c})\right) = -\frac{\kappa g(t-\frac{r}{c})}{r^2} - \frac{1}{c}\frac{\dot{g}(t-r/c)}{r}$ ① ②

$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\kappa}{r}g(t-\frac{r}{c})\right) = \frac{\kappa}{r}\dot{g}(t-\frac{r}{c})$ ③

As $r \to 0$ ① >> ②, ③

$\rightarrow \frac{\partial\phi}{\partial r} \rightarrow -\frac{\kappa g(t-\frac{r}{c})}{r^2}$

$\frac{\partial\phi/\partial t}{\partial\phi/\partial r} \rightarrow 0$

→ We treat spatial ~~deri~~ derivatives without differentiating g, and ignore time derivatives.

$$\therefore \quad \lim_{r\to 0} \Box^2 \phi = \lim_{r\to 0} \Box^2 \left(\frac{k}{r} g(t - r/c) \right).$$

$$= \lim_{r\to 0} \Box^2 \left(\frac{k}{r} g(t) \right) = k g(t) \lim_{r\to 0} \left(\nabla^2 \left(\frac{1}{r} \right) \right).$$

$$= -4\pi g(t) \delta^3(\underline{r}) k = -4\pi \frac{1}{4\pi\varepsilon_0} g(t) \delta^3(\underline{r})$$

$$= -\frac{1}{\varepsilon_0} g(t) \delta^3(\underline{r})$$

$\left(\Delta + \frac{k^2}{\omega^2} \right)$

$\int G(\underline{r}, \underline{r}') = \frac{e^{i\omega t}}{|\underline{r} - \underline{r}'|}$

At $r = 0 \Rightarrow \Box^2 \phi = -\frac{\rho(t)}{\varepsilon_0}$

$$\Rightarrow -\frac{1}{\varepsilon_0} g(t) \delta^3(\underline{r}) = -\frac{1}{\varepsilon_0} \rho(t)$$

$$\Rightarrow g(t) \delta^3(\underline{r}) = \rho(t)$$

integrate over whole space.

$$g(t) \underbrace{\int \delta^3(\underline{r})\, d^3\underline{r}}_{1} = \int \rho(t)\, dV$$

$$\Rightarrow \underline{\underline{g(t) = \int \rho(t)\, dV}}$$

e) the solution

$$\boxed{\phi(\underline{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\underline{r_s}, t-\frac{r_{sf}}{c})}{r_{sf}}\, d^3\underline{r_s} \quad \text{where } r_{sf} = |\underline{r_f} - \underline{r_s}|}$$

1P

f) this is call "retarded" because the potential at time $\underline{t}$ depends on the charge density at a previous time $t-\frac{r_{sf}}{c}$

1P

Green's formalism

~~$D\,\phi\ \rho$~~

$D\,G(\underline{r}-\underline{r_0}) = \delta(\underline{r}-\underline{r_0})$

~~$D\phi(\underline{r}) = \rho(\underline{r})$~~

operator

$\phi(\underline{r}) = \int \rho(\underline{r}') \, G(\underline{r},\underline{r}')\, dV$

$G(\underline{r},\underline{r}') = \frac{1}{|\underline{r}-\underline{r}'|}$

Green's function:

$\frac{\partial^2\phi}{\partial t^2} = \frac{g'}{r}$

C2.

7/8

The 4-vector potential of a charge in arbitrary motion is

$$A = \frac{q}{4\pi\epsilon_0} \frac{U/c}{(-R\cdot U)} = \begin{pmatrix} \phi/c \\ \underline{A} \end{pmatrix}$$

U = 4 velocity $= \begin{pmatrix} \gamma c \\ \gamma \underline{v} \end{pmatrix}$

R = 4 displacement from source event to field event. $= \begin{pmatrix} r_{sf} \\ \underline{r_{sf}} \end{pmatrix}$ $\left(\because \cancel{r_{sf}}\ \tau_{sf} = \frac{r_{sf}}{c}\right)$

After substituting, calculating $\underline{A}$ and ϕ, we can use $\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \underline{\nabla}\phi$ and $\underline{B} = \underline{\nabla}\times\underline{A}$ to get the desired equation.

a) ~~Source~~ Field event $(0, \alpha, y, 0)$

Source event $(t_s, x_s, 0, 0)$

($\because$ particle moves along x $\therefore$ $y_s = z_s = 0$)

Also $t_s = -\frac{r_{sf}}{c} = \frac{-\sqrt{(x_s-\alpha)^2+y^2}}{c} \underbrace{=}_{c=1} -\sqrt{(x_s-\alpha)^2+y^2}$ ✓ 1P

~~we~~ substitute this into worldline of source

$x_s^2 - t_s^2 = \alpha^2$ $\quad (\Rightarrow$ ~~α t_s~~ $\left(t_s = -\sqrt{x_s^2-\alpha^2}\right)$

$\therefore$ $x_s^2 - (x_s-\alpha)^2 - y^2 = \alpha^2$ $\Rightarrow$ $x_s^2 - x_s^2 + 2\alpha x_s - \alpha^2 - y^2 = \alpha^2$

→ source event needs to follow worldline.

$\rightarrow \quad 2\alpha x_s = 2\alpha^2 + y^2$

$\therefore \boxed{x_s = \alpha + \frac{y^2}{2\alpha}}$ ✓ (1P)

b) $\because \quad x^2 - t^2 = \alpha^2$

$\therefore \quad x\,dx - t\,dt = 0 \qquad \therefore \frac{dx}{dt} = \frac{t}{x}$

$\therefore \quad v_s = \left.\frac{dx}{dt}\right|_{x=x_s} = \frac{t_s}{x_s} = \boxed{-\frac{\sqrt{x_s^2-\alpha^2}}{x_s}}$ ✓ (1P)

$a_s = \left.\frac{d^2x}{dt^2}\right|_{x=x_s} = \left.\frac{d}{dt}\left(\frac{t}{x}\right)\right|_{x=x_s} = \left(-\frac{t}{x^2}\frac{dx}{dt} + \frac{1}{x}\right)_{x=x_s}$

$= -\frac{t_s}{x_s^2}\left(\frac{t_s}{x_s}\right) + \frac{1}{x_s}$

$= \frac{1}{x_s^3}\left(\underbrace{-t_s^2 + x_s^2}_{\alpha^2}\right) = \boxed{\frac{\alpha^2}{x_s^3}}$ ✓ (1P)

c) If $\alpha = 1 \quad y = 2$

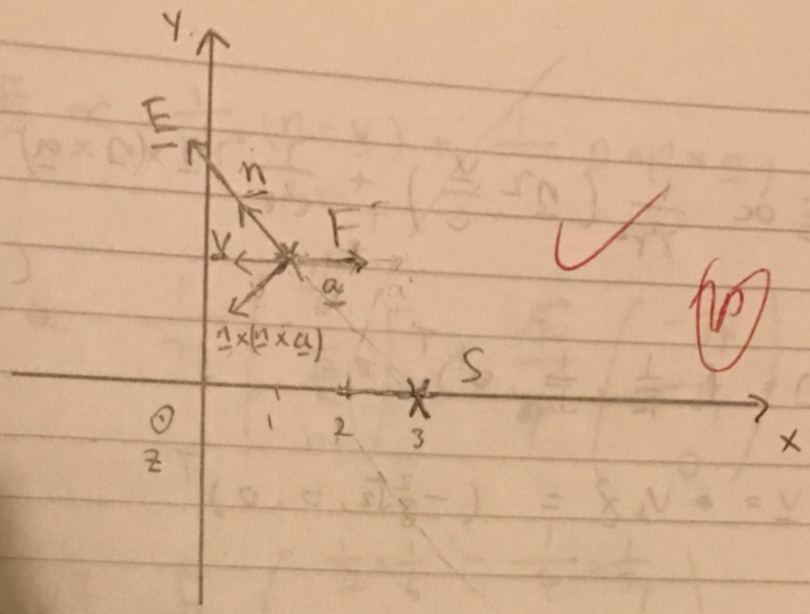
$\rightarrow \quad x_s = \alpha + \frac{y^2}{2\alpha} = 1 + \frac{2^2}{2\times1} = 1+2 = \underline{\underline{3}}$ ✓

$\rightarrow \quad v_s = -\frac{\sqrt{x_s^2-\alpha^2}}{x_s} = -\frac{\sqrt{3^2-1^2}}{3} = -\frac{2\sqrt{2}}{3} = \underline{\underline{-0.943}}$ ✓

(1P)

$\rightarrow \quad a_s = \frac{\alpha^2}{x_s^3} = \frac{1^2}{3^3} = \frac{1}{27} = \underline{\underline{0.037}}$ ✓

$\rightarrow \quad t_s = -\sqrt{3^2-1^2} = -2\sqrt{2} = \underline{\underline{-2.828}}$



Field event (0, 1, 2, 0)

Sourse event (−2.828, 3, 0, 0)

→ ∵ $\underline{n} = \frac{\underline{r}}{r}$ ∴ $\underline{n}$ is in the direction of S→F

→ $\underline{v}$ and $\underline{a}$ are both in the $\underline{x}$ direction, with $\underline{v}$ negative, $\underline{a}$ positive.

→ $\underline{n} \times \underline{a}$ is in the $-\hat{z}$ direction

∴ $\underline{n} \times (\underline{n} \times \underline{a})$ is in the xy plane and perpendicular to $\underline{n}$

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 k^3}\left(\frac{\underline{n} - \underline{v}/c}{\gamma^2 r^2} + \frac{\underline{n} \times ((\underline{n} - \frac{\underline{v}}{c}) \times \underline{a})}{c^2 r}\right)$$

∵ $\underline{v}$ and $\underline{a}$ are both along $\hat{\underline{x}}$

∴ $\underline{v} \times \underline{a} = 0$

$$\therefore \underline{E} \propto \frac{1}{\gamma^2 r^2}\left(\underline{n} - \frac{\underline{v}}{c}\right) + \frac{1}{c^2 r}\, \underline{n} \times (\underline{n} \times \underline{a})$$

$(c=1)$

$$\underline{n} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$$

$$\underline{v} = v_s \hat{\underline{x}} = \left(-\frac{2}{3}\sqrt{2}, 0, 0\right)^T$$

$$\underline{a} = \underline{n} - \frac{\underline{v}}{c} = \underline{n} - \underline{v} = \left(\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{2}, 0\right)^T$$

$$\gamma^2 = \frac{1}{1-v_s^2} = \frac{1}{1-\frac{8}{9}} = 9$$

$$\underline{r} = (-2, 2, 0) \qquad \therefore r = 2\sqrt{2} \rightarrow r^2 = 8$$

$$\underline{a} = \left(\frac{1}{27}, 0, 0\right)^T$$

$$\underline{n} \times \underline{a} = \begin{vmatrix} x & y & z \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{27} & 0 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{2}}{54} \end{pmatrix}$$

$$\underline{n} \times (\underline{n} \times \underline{a}) = \begin{vmatrix} x & y & z \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{54} \end{vmatrix} = \begin{pmatrix} -\frac{1}{54} \\ -\frac{1}{54} \\ 0 \end{pmatrix}$$

$$\therefore \underline{E} \propto \frac{1}{\gamma^2 r^2}(\underline{n}-\underline{v}) + \frac{1}{r}\,\underline{n}\times(\underline{n}\times\underline{a})$$

$$\propto \frac{1}{72}\begin{pmatrix} \frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} + \frac{\sqrt{2}}{4}\begin{pmatrix} -\frac{1}{\sqrt{4}} \\ -\frac{1}{\sqrt{4}} \\ 0 \end{pmatrix}$$

$$= \sqrt{2}\begin{pmatrix} \frac{1}{72}\times\frac{1}{6} - \frac{1}{4}\times\frac{1}{\sqrt{4}} \\ \frac{1}{72}\times\frac{1}{2} - \frac{1}{4}\times\frac{1}{\sqrt{4}} \\ 0 \end{pmatrix}$$

Just some wrong numbers. $-\frac{1}{2}$

$$\propto \sqrt{2}\begin{pmatrix} -\frac{1}{432} \\ \frac{1}{432} \\ 0 \end{pmatrix} \propto \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $\underline{E}$ is along $\underline{n}$ at field event $(0, 1.2, 0)$

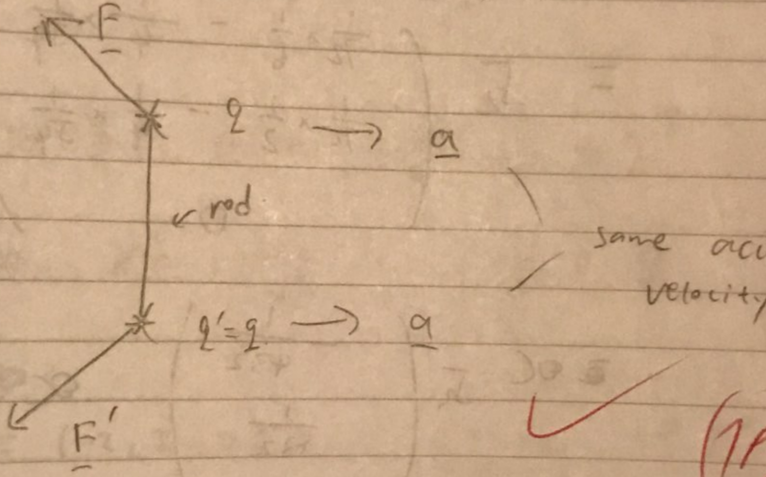
nice

d)

In this example, ~~...~~ At field event $(0,1,2,0)$, the charge would ~~appear~~ be at $(0,1,0,0)$

→ Same position in x.

(so appropriate to model charges fixed to a rod perpendicular to x.)



1P

This force is a <u>self force</u> or <u>radiation reaction</u>

→ It arises from the motion of particles themselves

→ Any external force on the rod will cause acceleration and thus induce this self force.