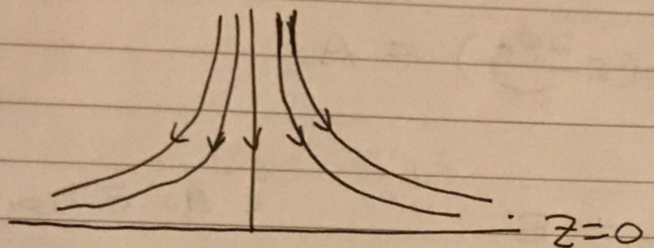
B1 2016

① irrotational flow $\nabla \times \underline{u} = 0$

$\Rightarrow \underline{u} = \nabla \phi$

incompressible flow $\nabla \cdot \underline{u} = 0$

$\therefore \nabla \cdot \nabla \phi = 0 \Rightarrow \underline{\underline{\nabla^2 \phi = 0}}$



Near the vicinity of the stagnation point

The surface is locally flat.

$\because$ object is impenetrable

$\therefore u_z = 0$ at $z = 0 \Rightarrow \left.\frac{\partial \phi}{\partial z}\right|_{z=0} = 0$ ①

(for all r)

The x, y directions are symmetric, so we switch to cylindrical polar coordinates.

$\underline{u} = (u_r, u_\theta, u_z)$ where $u_\theta = 0$

$\because \nabla^2 \phi = 0 \quad \therefore \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \phi}{\partial r}\right) + \frac{\partial^2 \phi}{\partial z^2} = 0$ ②

Also by symmetry $\left.\frac{\partial \phi}{\partial r}\right|_{r=0} = 0$ ③

(for all z)

~~② + ① + ③ →~~ $\cancel{\frac{\partial \phi}{\partial z}}$ ~~independent of r~~

$\cancel{\frac{\partial \phi}{\partial r}}$ ~~independent of z~~

The the trial solution

$$\phi(r,z) = \phi_r(r) + \phi_z(z)$$

$$\therefore \text{②} \Rightarrow \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_r}{\partial r}\right)}_{\text{a function only of } r} + \underbrace{\frac{\partial^2 \phi_z}{\partial z^2}}_{\text{a function only of } z} = 0$$

$$\therefore \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi_r}{\partial r}\right) = A$$

$$\frac{\partial^2 \phi_z}{\partial z} = B \qquad \text{(A, B constants)}$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial \phi_r}{\partial r}\right) = rA \Rightarrow r\frac{\partial \phi_r}{\partial r} = \frac{A}{2}r^2 + C$$

$$\Rightarrow \frac{\partial \phi_r}{\partial r} = \frac{A}{2}r + \frac{C}{r}$$

$\because$ at $r=0$, $\frac{\partial \phi_r}{\partial r}$ is not going to ∞

$\therefore C = 0 \qquad \therefore \frac{\partial \phi_r}{\partial r} = \cancel{A}\ \frac{A}{2}r$

$\therefore \phi_r = \frac{A}{4}r^2 + D$ (~~ar~~ additive constant can be set to 0 $\therefore D = 0$)

$\Rightarrow \phi_r = \cancel{A}\ A r^2$ (absorb $\frac{1}{4}$ into A)

$$\frac{\partial^2 \phi_z}{\partial z^2} = B \rightarrow \frac{\partial \phi_z}{\partial z} = Bz + E$$

$\because$ At $z=0$ $\frac{\partial \phi_z}{\partial z} = \frac{\partial \phi}{\partial z} = 0$ $\therefore$ $E=0$

$\therefore \frac{\partial \phi_z}{\partial z} = Bz$ $\therefore \phi_z = \frac{B}{2} z^2 + F$

absorb $\frac{1}{2}$ into B

can be set to 0

$\therefore \phi_z = Bz^2$

$\therefore$ we have $\phi(r, z) = Ar^2 + Bz^2$

Then $\nabla^2 \phi(r,z) = 0$

$\therefore$ $\cancel{\frac{1}{r}\frac{\partial}{\partial r}}$ $\frac{1}{r}\frac{\partial}{\partial r}(r \cdot 2Ar) + 2B = 0$

$\therefore$ $4A + 2B = 0$

~~let~~ $\because$ $u_z < 0$ $\therefore$ ~~$2B$~~ for $z > 0$

$\therefore$ $B < 0$ $\therefore$ let ~~$2B =$~~ $B = -2W$

then $A = W$, $B = -2W$

$\therefore \phi(r, z) = W(r^2 - 2z^2)$ $\because r^2 = x^2 + y^2$

$\therefore \phi(x, y, z) = W(x^2 + y^2 - 2z^2)$ $(W > 0)$

→ This solution is unique up to an additi[ve] constant due to the uniqueness theore[m] of first order derivative boundary conditions

The streamlines:

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

$$\underline{u} = \underline{\nabla}\phi = 2w(x, y, -2z)$$

$$\therefore \frac{dx}{x} = \frac{dy}{y} = -\frac{dz}{2z}$$

integrate $\ln x = \ln y + C_1 = -\frac{1}{2}\ln z + C_2$

$\therefore \ln\frac{y}{x} = \text{const}$ $\quad y = \text{const} \times x$

$\ln z = -2\ln x + \text{const}$

$\therefore \ln(zx^2) = \text{const}$ $\quad z = \frac{\text{const}}{x^2}$

$x = \frac{\text{const}}{\sqrt{z}}$

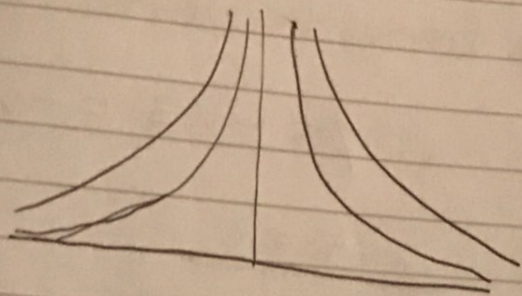
$\therefore$ stream-line is $\boxed{y = kx = \frac{m}{\sqrt{z}}}$

(k, m are constants)

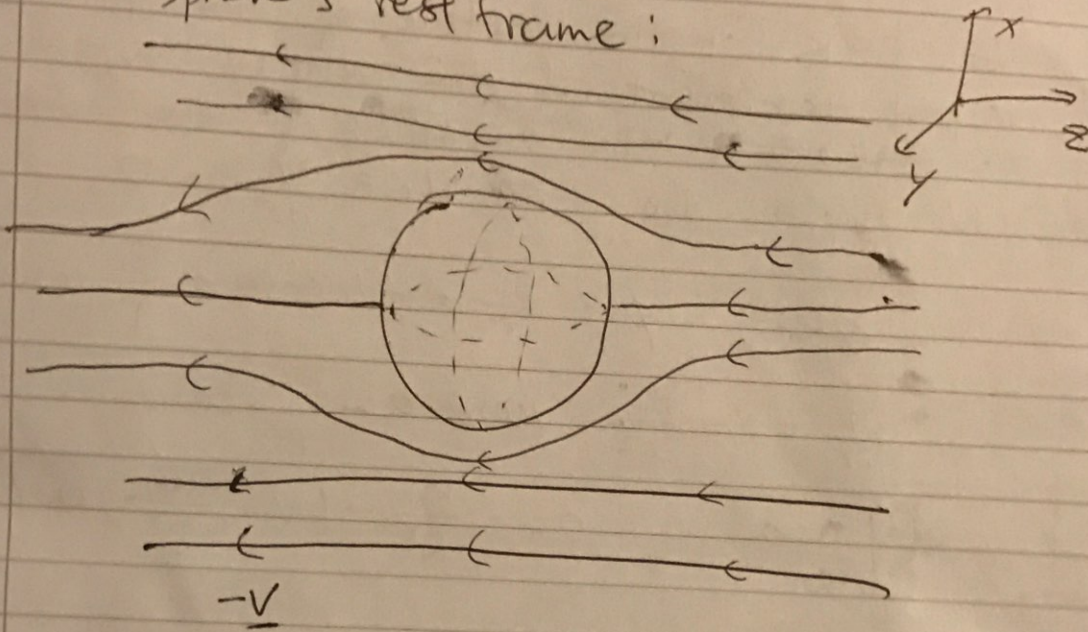
$\therefore z = \frac{c_1(y)}{x^2}$, $\quad z = \frac{c_2(x)}{y^2}$ $\cancel{\Rightarrow z = \frac{c}{x^2+y^2}}$

$\therefore x^2 z = \text{const}$
$y^2 z = \text{const}$
$\therefore (x^2+y^2)z = \text{const}$

$\therefore z = \frac{c}{x^2+y^2}$



→ Sphere's rest frame:



In the rest frame of the sphere, the fluid moves with a constant velocity $-\underline{V}$ (V in negative $\hat{\underline{z}}$ direction), while the sphere is at rest.

Assume $\phi(r,\theta) = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$ is the velocity potential. The system has azimulthal symmetry. so only r, θ are relevant.

• This form of ϕ automatically solves $\nabla^2\phi = 0$

We now only need to worry about the boundary conditions

→ At large r away from the sphere,

$\underline{u} = (0, 0, -V)$ $\quad \therefore \phi = -Vz = -Vr\cos\theta$

$\therefore \underline{A_1 = -V}$, $\underline{A_0 = 0}$

At $r = R$ on the surface of the sphere, the normal component of the velocity with respect to ~~$\underline{u}(\underline{r}) = u$~~ the surface of the sphere is 0 (No slip) i.e. $u_r = \dfrac{\partial \phi}{\partial r} = 0$

$\because \phi = \dfrac{B_0}{r} - Vr\cos\theta + \dfrac{B_1}{r^2}\cos\theta$

$\therefore \dfrac{\partial \phi}{\partial r} = -\dfrac{B_0}{r^2} - V\cos\theta - \dfrac{2B_1}{r^3}\cos\theta$

$\because \left.\dfrac{\partial \phi}{\partial r}\right|_{r=R} = 0 \quad \therefore 0 = -\dfrac{B_0}{R^2} - \left(V + \dfrac{2B_1}{R^3}\right)\cos\theta$

$\therefore$ We need $\underline{B_0 = 0}$ and $\underline{B_1 = -\dfrac{VR^3}{2}}$

$\therefore \underline{\phi(r,\theta) = -V\cos\theta\left(r + \dfrac{R^3}{2r^2}\right)}$

velocity $\underline{u}(\underline{r}) = \underline{\nabla}\phi = u_r\,\hat{\underline{r}} + u_\theta\,\hat{\underline{\theta}}$

$u_r = \dfrac{\partial \phi}{\partial r} = -V\cos\theta\left(1 - \dfrac{R^3}{r^3}\right)$

$u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = V\sin\theta\left(1 + \dfrac{R^3}{2r^3}\right)$

The stagnation points are $(u_r, u_\theta) = (0, 0)$

$\Rightarrow$ A $(r, \theta) = \underline{\underline{(R, 0)}}$

B $(r, \theta) = (R, \pi)$

velocity potential $\phi = -V\cos\theta\left(r + \frac{R^3}{2r^2}\right)$

$\rightarrow$ Near the stagnation point A. let $\rho^2 = x^2 + y^2$

so that $r^2 = \rho^2 + z^2$, $z = r\cos\theta$

$$\therefore \phi = -Vz\left(1 + \frac{1}{2}R^3(\rho^2 + z^2)^{-3/2}\right)$$

Near A, $z \simeq R$ $\rho \ll R$ $\therefore$ let $z = R + \delta$

$$\phi = -V(R+\delta)\left[1 + \frac{1}{2}R^3\left(\rho^2 + (R+\delta)^2\right)^{-3/2}\right]$$

$$= -V(R+\delta)\left[1 + \frac{1}{2}R^3\left((\rho^2+\delta^2) + 2\delta R + R^2\right)^{-3/2}\right]$$

$$= -V(R+\delta)\left[1 + \frac{1}{2}\left(1 + \frac{2\delta}{R} + \frac{\delta^2+\rho^2}{R^2}\right)^{-3/2}\right]$$

$\simeq -RV\left(\frac{3}{2} + \frac{3}{2}\frac{\delta^2}{R^2}\right)$

Taylor expansion for $(1+x)^{-3/2}$ is

At $\theta = 0$

$\cos\theta \simeq 1\left(1 - \frac{\theta^2}{2}\right)$

$$(1+x)^{-3/2} \simeq 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \cdots$$

$\therefore -RV\left(\frac{3}{2} + \frac{3}{2}\frac{\delta^2}{R^2} - \frac{3}{2}\frac{\theta^2}{2}\right)$

$\therefore$

$= -\frac{3}{2}RV + \frac{3V}{4R}\left((R\theta)^2 - 2\delta^2\right)$

$$\left[1 + \left(\frac{2\delta}{R} + \frac{\delta^2+\rho^2}{R^2}\right)\right]^{-3/2} \simeq 1 - \frac{3}{2}\left(\frac{2\delta}{R} + \frac{\delta^2+\rho^2}{R^2}\right)$$

$$+ \left(\frac{2\delta}{R} + \frac{\delta^2+\rho^2}{R^2}\right)^2 \times \frac{15}{8} = 1 - \frac{3\delta}{R} - \frac{3\delta^2}{2R^2} - \frac{3\rho^2}{2R^2}$$

$$+ \frac{15}{2}\frac{\delta^2}{R^2} + O\left(\frac{\delta}{R}\right)^3$$

$$\simeq 1 - \frac{3\delta}{R} + \frac{6\delta^2}{R^2} - \frac{3\rho^2}{2R^2}$$

$$\therefore \phi \simeq -V(\delta+R)\Big[\underbrace{1+\tfrac{1}{2}}_{\frac{3}{2}} - \frac{3\delta}{2R} + \frac{3\delta^2}{R^2} - \frac{3\rho^2}{4R^2}\Big]$$

$$= \cancel{-\tfrac{3}{2}V\delta} + \frac{3V\delta^2}{2R} - \frac{3VR}{2} + \cancel{\frac{3V\delta}{2}}$$

$$- \frac{3V\delta^2}{R} + \frac{3V\rho^2}{4R} + O(\delta^3)$$

$$\simeq -\frac{3VR}{2} + \frac{3V}{4R}(\rho^2 - 2\delta^2)$$

$\therefore \phi$ near A is (ignore constant $-\frac{3VR}{2}$)

$$\phi_A = \frac{3V}{4R}(\rho^2 - 2\delta^2)$$

$$= \frac{3V}{4R}(x^2+y^2-2(z-R)^2)$$

as expected.

→ Similar near point B

$\phi = -Vr\cos\theta\left(1+\frac{R^3}{2r^3}\right)$ let $\rho^2 = x^2+y^2$, $r^2 = \rho^2+z^2$

Near B $z<0$, $z \simeq -R$ ~~let $\delta = -R-z = -(z+R)$~~

let $\delta = -R - z$

$\therefore z = -R-\delta = -(R+\delta)$

$\therefore$ replace R by $-R$, δ by $-\delta$ in ϕ_A we get

$$\phi_B = -\frac{3V}{4R}(x^2+y^2-2(z+R)^2)$$ as expected

The Bernoulli equation $\frac{u^2}{2} + \frac{P}{\rho} = const$

$$u_r = -V\cos\theta\left(1 - \frac{R^3}{r^3}\right)$$

$$u_\theta = V\sin\theta\left(1 - \frac{R^3}{2r^3}\right)$$

At stagnation point $\underline{u} = 0$ $\therefore$ $P = P_0$

$\therefore$ $const = \frac{P_0}{\rho}$ $\therefore$ $\frac{u^2}{2} + \frac{P}{\rho} = \frac{P_0}{\rho}$

$$\therefore \quad P = P_0 - \frac{1}{2}\rho u^2$$

$\cancel{u^2 = u_r^2 + u_\theta^2 = V^2\cos^2\theta\left(1 - \frac{R^3}{r^3}\right.}$

on the surface of the sphere: $\cancel{u_\theta = 0}$ $u_r = 0$

$\cancel{u = 0}$ $\underline{u} = u_\theta \underline{\hat{\theta}}$ $\therefore u^2 = u_\theta^2(R)$

$\cancel{u_\theta = V\sin\theta}$ $u_\theta(R) = V\sin\theta\left(1 - \frac{R^3}{2R^3}\right) = \frac{1}{2}V\sin\theta$

$$\therefore \quad P = P_0 - \frac{1}{2}\rho\left(\frac{1}{2}V\sin\theta\right)^2$$

$$\rightarrow \quad \underline{\underline{P = P_0 - \frac{1}{8}\rho V^2\sin^2\theta}}$$

(2)

Naiver Stokes equation is

$$\underbrace{\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u}}_{\text{inertial term}} + \frac{1}{\rho}\underline{\nabla}p + g\underline{\hat{k}} = \underbrace{\nu\nabla^2\underline{u}}_{\text{viscous term}}$$

For slow viscous flow, Reynold number is small. The inertial term is negligible compare to the viscous term.

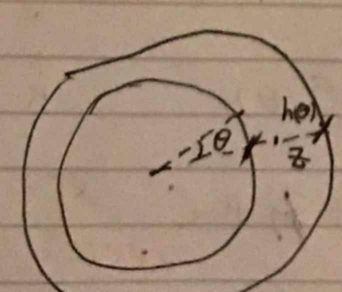
So ignore the inertial term

$$\underbrace{\frac{1}{\rho}\underline{\nabla}p}_{\text{pressure term}} + \underbrace{g\underline{\hat{k}}}_{\text{gravity term}} = \underbrace{\nu\nabla^2\underline{u}}_{\text{viscous term.}}$$

incompressibility requires

$$\underline{\nabla}\cdot\underline{u} = 0$$

~~keep creti crit~~ criteria: slow velocity
small length scale.
incompressible flow
high viscosity.



Under lubrication approximation,

$$\nabla^2 u \simeq \frac{\partial^2 u}{\partial z^2} \quad \text{for } u = u_\theta = u(\theta, z)$$

We ignore the pressure gradient ∇p since its effect is small compare to gravity

$\frac{\partial u}{\partial z} \sim \frac{U}{h}$ (The exact explanation to this, see

$\frac{\partial^2 u}{\partial z^2} \sim \frac{U}{h^2}$ H.K. Moffatt, "Behaviour of a viscous film on the outer surface of a rotating cylinder").

$\frac{1}{R}\frac{\partial u}{\partial \theta} \sim \frac{U}{L}$

So the Stokes's equation in θ direction is

$\sim \frac{U}{R}$

$\frac{\partial^2 u}{\partial \theta^2} \sim \frac{U}{R^2} \ll \frac{U}{h}$

$$\cancel{g\hat{k}\cdot\hat{\theta} = \nu\frac{\partial^2 u}{\ } \quad \nu\nabla^2 u =}$$

$u(\theta,z)|_{z=0}=0$

$\therefore \nabla^2 \sim \frac{\partial^2}{\partial z^2}$

$$\underbrace{g\hat{k}\cdot\hat{\theta}}_{\cos\theta} = \nu(\nabla^2 \underline{u})\cdot\hat{\theta} = \nu\frac{\partial^2 u}{\partial z^2}$$

$\frac{\partial u}{\partial z}\Big|_{z=h} = 0$

vertical u is small

$\therefore \nabla\cdot\underline{u} = 0$

$$\therefore \quad \nu\frac{\partial^2 u}{\partial z^2} = g\cos\theta$$

$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$ Boundary conditions for $u(\theta, z)$

$U \sim \frac{U_{x,y}}{L} + \frac{U_z}{h}$ No slip at $z=0 \Rightarrow$ $\underline{u(\theta,0) = U}$

No shear stress at $z=h$

$$\Rightarrow \quad \nu\frac{\partial u}{\partial z}\Big|_{z=h} = 0 \quad \therefore \frac{\partial u}{\partial z}\Big|_{z=h} = 0$$

$\therefore U_z \sim \frac{h}{L}U$

$\therefore U_z$ small

$$\therefore \quad \frac{\partial^2 u}{\partial z^2} = \frac{g}{\nu}\cos\theta$$

θ-direction $\frac{\partial p}{\partial s} \sim \rho\nu\frac{\partial^2 u_s}{\partial z^2}$

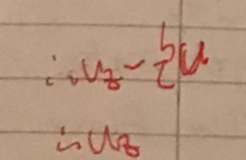
R $\quad p \sim \rho\nu\frac{U}{h^2}R$

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{g}{\nu}\cos\theta\, z + C(\theta)$$

At $z = h$ $\quad 0 = \frac{g}{\nu}\cos\theta\, h + C$

$\nabla p \quad \rho\nu\nabla^2 u$

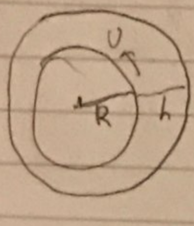
$\frac{\rho\nu U R}{h^2 (R,h)} \quad \rho\nu\frac{U}{h^2}$



$$\therefore C = -\frac{g}{\nu}\cos\theta\, h.$$

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{g}{\nu}\cos\theta\,(z - h)$$

Details of lubrication approximation



$(R \gg h)$

$$\frac{1}{\rho}\nabla p + g\hat{\underline{k}} = \nu \nabla^2 \underline{u}$$

z equation : $\frac{1}{\rho}\frac{\partial p}{\partial z} - g\sin\theta = \nu(\nabla^2 \underline{u})_z = \nu\Big[\underbrace{\nabla^2 u_z}_{\sim 0} - \underbrace{\frac{u_z}{(R+z)^2}}_{\sim 0} - \frac{2}{(R+z)^2}\underbrace{\frac{\partial u_\theta}{\partial\theta}}_{\sim \frac{U}{R^2} \sim 0}\Big]$

$\therefore \frac{\partial u_\theta}{\partial z} = \frac{U}{h} \quad \frac{\partial^2 u_\theta}{\partial z^2} = \frac{U}{h^2}$

$$\nabla^2 u_\theta = \frac{1}{R+z}\frac{\partial}{\partial z}\left((R+z)\frac{\partial u_\theta}{\partial z}\right) + \frac{1}{(R+z)^2}\frac{\partial^2 u_\theta}{\partial\theta^2}$$

$$= \cancel{\frac{1}{R+z}(R+z)}\underbrace{\frac{\partial^2 u_\theta}{\partial z^2}}_{\sim \frac{U}{h^2}\ ✓} + \underbrace{\frac{1}{R+z}\frac{\partial u_\theta}{\partial z}}_{\sim \frac{U}{hR}\ \times} + \underbrace{\frac{1}{(R+z)^2}\frac{\partial^2 u_\theta}{\partial\theta^2}}_{\sim \frac{U}{R^2}\ \times}$$

$$\simeq \frac{\partial^2 u_\theta}{\partial z^2}$$

$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{1}{R+z}\frac{\partial}{\partial z}((R+z)u_z) + \frac{1}{R+z}\frac{\partial u_\theta}{\partial\theta} = 0$

$\therefore \underbrace{\frac{\partial u_z}{\partial z}}_{\frac{u_z}{h}} + \underbrace{\frac{1}{R+z}u_z}_{\frac{u_z}{R}} + \underbrace{\frac{1}{R+z}\frac{\partial u_\theta}{\partial\theta}}_{\frac{u_\theta}{R}} = 0$

$\frac{u_z}{h} \gg \frac{u_z}{R} \quad + \quad \frac{u_\theta}{R} = 0$

$\Rightarrow u_z \sim \frac{h}{R}u_\theta \simeq 0 \quad (\because \ll u_\theta)$

$\therefore \nabla^2 u_z \sim 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial p}{\partial z} - g\sin\theta = 0$

$\therefore p \sim \rho g h$

θ equation

$\underbrace{\frac{1}{\rho}\frac{1}{R+z}\frac{\partial p}{\partial\theta}}_{\sim \frac{1}{\rho}\frac{p}{R} \sim \frac{\rho g h}{\rho R} \sim \frac{h}{R}g \ll g\ \times} + g\cos\theta = \nu\nabla^2 u_\theta$

$= \nu\frac{\partial^2 u_\theta}{\partial z^2} \Rightarrow g\cos\theta = \nu\frac{\partial^2 u_\theta}{\partial z^2}$

$$\cancel{u = \frac{g}{\nu} + \left(\frac{z^2}{}\right)}$$

$$u = \frac{g}{\nu}\cos\theta\left(\frac{z^2}{2} - zh\right) + D(\theta).$$

At $z = 0$, $u = U$

$\therefore\ U = D(\theta)$ $\quad\therefore\ \underline{\underline{u = U + \left(\frac{z^2}{2} - zh\right)\frac{g}{\nu}\cos\theta}}$

$$Q = \int_0^h u\,dz = \int_0^h \left[U + \left(\frac{z^2}{2} - zh\right)\frac{g}{\nu}\cos\theta\right] dz$$

$$= \left[Uz + \left(\frac{z^3}{6} - \frac{z^2 h}{2}\right)\frac{g}{\nu}\cos\theta\right]_0^h$$

$$= Uh + h^3\left(\frac{1}{6} - \frac{1}{2}\right)\frac{g}{\nu}\cos\theta$$

$$= \underline{\underline{Uh - \frac{h^3}{3}\frac{g}{\nu}\cos\theta}}$$

If $H = \frac{Uh}{Q}$ then $\frac{1}{H} = \frac{Q}{Uh}$. $h = \frac{QH}{U}$

$\therefore \frac{1}{H} = 1 - \frac{gh^2}{3U\nu}\cos\theta$ $\quad\left(\alpha = \frac{gQ^2}{3\nu U^3}\right)$

$$= 1 - \frac{gQ^2H^2}{3U^3\nu}\cos\theta = 1 - H^2\alpha\cos\theta$$

$\therefore \Rightarrow 1 - \frac{1}{H} = H^2\alpha\cos\theta$

$$\therefore\ \underline{\underline{\frac{1}{H^2} - \frac{1}{H^3} = \alpha\cos\theta}}$$

→ Q is constant because the conservation of mass ~~around one revolution of the fluid motion~~ and that the fluid is incompressible

~~Same amount of liquid is flux is~~
~~The flux of fluid is the same for every~~

So same amount of fluid has to flow through each cross-section of different θ per unit time.

$$\therefore \frac{1}{H^2} - \frac{1}{H^3} = \alpha\cos\theta \qquad \therefore \alpha\cos\theta H^3 - H + 1 = 0$$

incompressible ⇒ constant flux

$$\therefore H = 1 + \alpha\cos\theta H^3 \qquad \because \alpha \ll 1.$$

assume a H close to 1. then let $H = 1 + x$ $(x \ll 1)$

$$\therefore H = 1 + \alpha\cos\theta\,(1 + O(x))$$

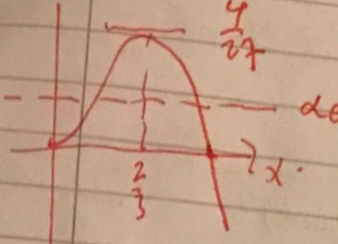
$$= \cancel{1 + \alpha\cos\theta + O(\alpha^2) + \ldots}$$

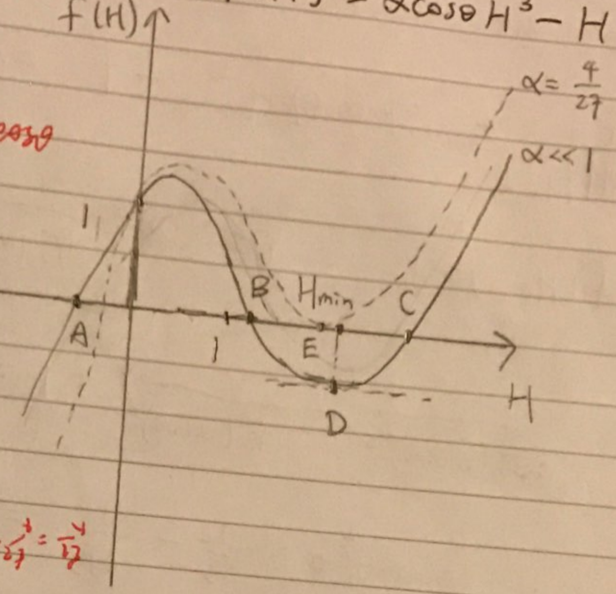
$$= 1 + \alpha\cos\theta + O(\alpha^2, x^2)$$

$$\therefore H \simeq 1 + \alpha\cos\theta$$

$$\therefore h(\theta) \simeq \frac{Q}{U}(1 + \alpha\cos\theta)$$

Consider function $f(H) = \alpha\cos\theta\, H^3 - H + 1$





$y = x^2 - x^3$

~~$\alpha\cos\theta$~~

$x \geq \frac{2}{3} = x_{min}$

$\frac{4}{9} - \frac{8}{27} = \frac{12}{27} - \frac{8}{27} = \frac{4}{27}$

Since $f(0) = 1$, one root of f must be negative. ~~We call this~~ This is represented by point A.

Depend on the value of $\alpha\cos\theta$, f can have 2 more roots (B and C), one more root (E), or no more root.

If f has 2 more ~~roots~~ positive roots, then B is the perturbative solution we found earlier.

When $\alpha\cos\theta$ is small, B is very close to 1. ~~So~~ As $\alpha\cos\theta$ increases, B and C get closer to each other. Eventually the local minimum ~~of~~ point D become positive and no more positive root does f(H) have.

The largest $\alpha\cos\theta$ for f(H) to have positive root is when there is only one positive root E. The value of this root is H_{min}.

then at point D, $H = H_{min}$

$$0 = \frac{df}{dH}\Big|_{H_{min}} = 3[\alpha\cos\theta] H_{min}^2 - 1 \Rightarrow H_{min} = \frac{1}{\sqrt{3\alpha\cos\theta}}$$

$\because f(H_{min}) = 0 \quad \therefore 0 = \alpha\cos\theta \dfrac{1}{3\sqrt{3}(\alpha\cos\theta)^{3/2}} - \dfrac{1}{\sqrt{3\alpha\cos\theta}} + 1$

$$\therefore \frac{1}{3\sqrt{3}(\alpha\cos\theta)^{1/2}} - \frac{1}{\sqrt{3}(\alpha\cos\theta)^{1/2}} + 1 = 0$$

$\frac{PVU^2}{h^2} \quad Py$

$\frac{Py}{\frac{PVU}{h^2}} \sim \frac{y_0^2}{\nu U^2} \sim \alpha.$

$$\therefore (\alpha\cos\theta)^{\frac{1}{2}} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

$$\therefore \alpha\cos\theta = \frac{4}{27} \text{ in that limiting case}$$

If the upper limit of α, $\alpha_{max} > \frac{4}{27}$, then in limiting case $\cos\theta < 1$. Then if we change θ we can make $\alpha\cos\theta > \frac{4}{27}$ and that the function f has no positive root.

So we must have $\alpha_{max} = \frac{4}{27}$

$$\rightarrow \quad \underline{\underline{\alpha \leq \frac{4}{27}}}$$

Solution for point C is not continuous if we reduce $\alpha\cos\theta$ as θ goes ~~to~~ from 0 to π (as $\cos\theta$ goes from $0+\epsilon$ to 0, $h(\theta)$ goes from ∞ to 1) So this solution doesn't make sense.

The only solution that works is B.

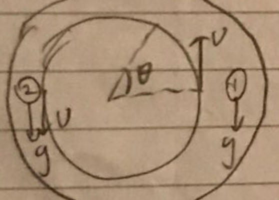
For B, larger $\alpha\cos\theta$ gives larger $h(\theta)$

$\because$ $\alpha\cos\theta$ is largest if $\alpha\cos\theta = \frac{4}{27}$

$\therefore$ $h_{max} = \frac{Q}{U} H_{max}$, H_{max} is the solution of

$$\frac{1}{H_{max}^2} - \frac{1}{H_{max}^3} = \frac{4}{27} \quad \therefore H_{max} = \frac{3}{2}$$

$$\Rightarrow h_{max} = \cancel{\frac{3Q}{2U}} \; \frac{3Q}{2U}$$



This phenomenon is due to the fact that the shear stress balances the ~~gravitational force~~ gravity.

At position ① gravity points down, the cylinder's velocity points up. The velocity gradient has to be negative along z to ~~counter~~ counteract gravity. So velocity is small in ① h can be large to conserve Q.

At position ② gravity points down, cylinder's velocity points down as well. The velocity gradient has to be positive in z to ~~cause~~ counteract gravity. So velocity is large in ②. h should be small to conserve Q

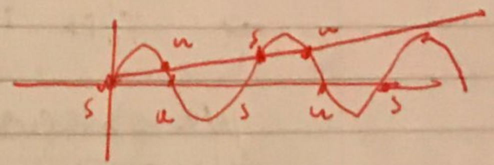
③

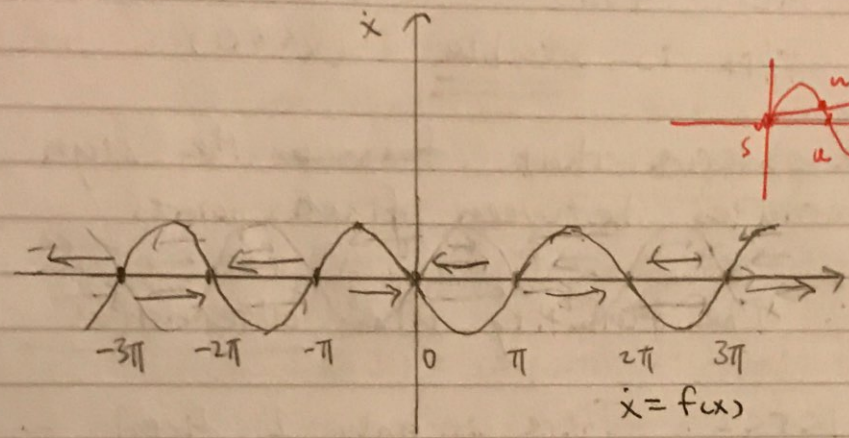
$\dot{x} = rx - \sin x \qquad r > 0 \qquad f(x) = rx - \sin x$

For $r = 0$ $\quad \dot{x} = -\sin x \quad$ fixed points $\dot{x} = 0$ for $x = x^*$

$\Rightarrow -\sin x^* = 0$

$\therefore \quad \underline{\underline{x^* = n\pi}} \quad (n = 0, \pm 1, \pm 2, \dots)$





stability : we calculate $\frac{df}{dx} = \cancel{\cos x} - \cos x$

$\therefore \left.\frac{df}{dx}\right|_{x^*} = -\cos(x^*) = -\cos(n\pi) = \underline{\underline{(-1)^{n+1}}}$

$$\therefore \left.\frac{df}{dx}\right|_{x^*} \begin{cases} > 0 & \text{if } \underline{n \text{ is odd}} \Rightarrow \text{unstable } \cancel{\text{stable}} \\ < 0 & \text{if } \underline{n \text{ is even}} \Rightarrow \underline{\underline{\text{stable}}} \end{cases}$$

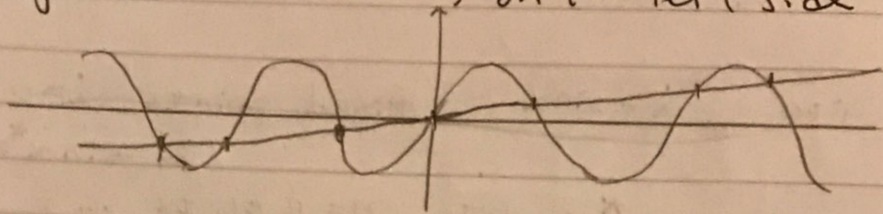
For $0 < r \leq 2$

$\dot{x} = f(x) = rx - \sin x$ (on real axis)

Fixed points $f(x^*) = 0 \quad \therefore \quad rx^* = \sin x^*$

When ~~0 < r < 1~~ $0 < r < 1$, the function $g(x) = rx$ intersects with function $h(x) = \sin x$ multiple times. There are more than 1 fixed points

For the ~~point~~ fixed point $x^* = 0$, on the right side $\sin x > rx$, on the left side $\sin x < rx$



$\therefore$ On RHS $f(x) = rx - \sin x < 0$, on LHS $f(x) > 0$

$\Rightarrow$ This is stable $(x^* = 0)$

We observe that ~~f(x)~~ the sign of $f(x)$ alternates between fixed points

$\therefore$ The stability also alternates

For $1 < r < 2$. There is only 1 fixed point $x^* = 0$ and since $f(x) > 0$ on RHS and < 0 on LHS in this case, this is unstable

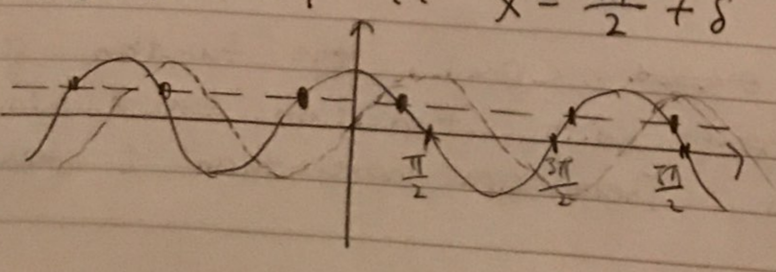
$\rightarrow$ For $0 < r \ll 1$:

Bifurcation points occurs ~~at~~ when $g(x) = rx$ is tangent to $h(x) = \sin x$

in this case: $g(x) = h(x)$ $g'(x) = h'(x)$

$\therefore$ $\sin(x) = rx$, $\cos(x) = r$

$\because$ $\cos(x) = r \ll 1$ $\therefore$ $x = \frac{n\pi}{2} + \delta$



we see that for $x > 0$. $\sin(x) = rx > 0$

since $\cos(x) = r \ll 1$, ~~$x = \frac{n\pi}{2} - \delta$~~

we set $x = \frac{n\pi}{2} - \delta$ $\because \cos(\frac{n\pi}{2}) = 0$

$(n = 1, 3, 5, 7, \cdots)$

to focus on the tangent point $x > 0$

For $n = 3, 7, 11 \cdots = 4m+3$

we see that $\delta < 0$, this corresponds to $\sin(x) < 0$, Not acceptable

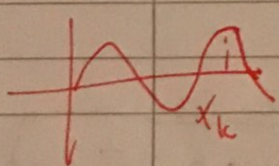
$\therefore \delta > 0$, $n = 1, 5, 9, \cdots = 4m+1$ $(m = 1, 2, 3, \cdots)$

saddle node bifurcation

$\therefore \sin(\frac{4m+1}{2}\pi - \delta) = \sin(2m\pi + \frac{1}{2}\pi - \delta) = \sin(\frac{\pi}{2} - \delta)$
$= \cos(\delta) \simeq 1$

$\cos(\frac{4m+1}{2}\pi - \delta) = \cos(2m\pi + \frac{1}{2}\pi - \delta)$
$= \cos(\frac{\pi}{2} - \delta) = \sin(\delta) \simeq \delta$

$\therefore$ We have $\sin(x) \simeq 1$ $\cos(x) \simeq \delta$



$\therefore 1 \simeq r(\frac{n\pi}{2} - \delta)$, $\delta \simeq r$

$\therefore 1 \simeq r(\frac{n\pi}{2} - r)$

$x_k = 2\pi k + \frac{\pi}{2}$.

To first order in r, we have $1 = \frac{rn\pi}{2}$

$rx_k = 1$

$r = \frac{1}{2\pi k + \frac{\pi}{2}}$

$\therefore r = \frac{2}{n\pi} = \frac{2}{(4m+1)\pi}$ $(m = 1, 2, 3, \cdots)$

If $m=0$. $r = \frac{2}{\pi}$ is an estimate to the tangent of first crest, which should be $r=1$

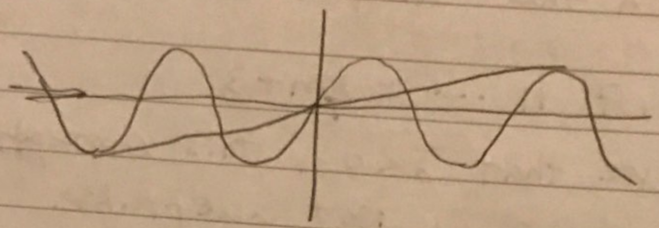
This is poor. But for $m \geq 1$, since r is small

The approximation works well.

$\therefore \ r = \dfrac{2}{(4m+1)\pi} \quad (m = 1, 2, 3, \cdots)$, and $r = 1$

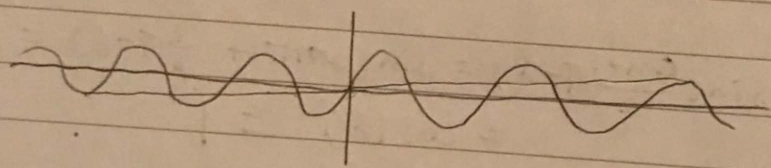
are Bifurcation points

$m=1$



5 fixed points

$m=2$



9 fixed points

$m=m$ $\quad (4m+1)$ fixed points

If $r \geq 1$. only 1 fixed point $x^* = 0$

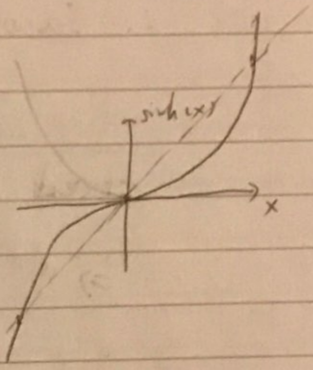
$\therefore$ # of fixed points for $0 < x \leq 2$ is

$$\boxed{F(r) = \begin{cases} 4m+3 & \dfrac{2}{(4m+1)\pi} > r > \dfrac{2}{(4m+5)\pi} \\ 4m+1 & r = \dfrac{2}{(4m+1)\pi} \\ 3 & \dfrac{2}{5\pi} < r < 1 \\ 1 & 1 \leq r < 2 \end{cases} \quad m = (1, 2, 3, \cdots)}$$

If in imaginary axis $x \to ix$

$$i\dot{x} = irx - \sin(ix) = irx - \frac{e^{i(ix)} - e^{-i(ix)}}{2i}$$

$$= i\left(rx + \frac{1}{2}(e^{-x} - e^{x})\right)$$

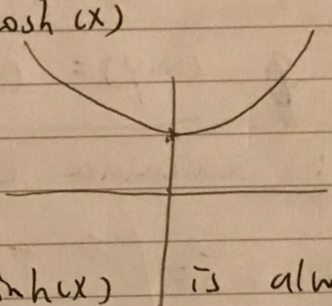


$\therefore \quad \dot{x} = rx + \frac{1}{2}(e^{-x} - e^{x})$

$\therefore \quad \dot{x} = rx - \sinh(x)$

$\because \quad \frac{d}{dx}\sinh(x) = \cosh(x)$

and $\cosh(x)$

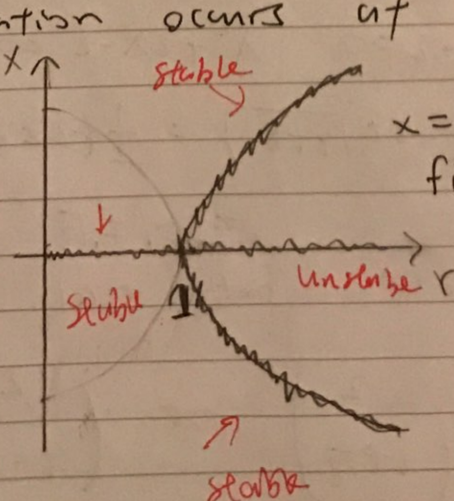


$\therefore$ the slope of $\sinh(x)$ is always greater than 1

$\therefore$ If $r \leq 1$, then rx cannot intersect with $\sinh(x)$ more than once

If $r > 1$, rx intersects with $\sinh(x)$ 3 times.

Bifurcation occurs at $\underline{r = 1}$ only



$x = 0$ is always a fixed point

supercritical pitchfork bifurcation

The system

$$\dot{x} = y^3 - y = f(x,y)$$
$$\dot{y} = x - y^2 = g(x,y)$$

$\therefore$ Jacobian $J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 3y^2-1 \\ 1 & -2y \end{pmatrix}$

fixed points $\dot{x}=0$, $\dot{y}=0$

$\Rightarrow$ $\cancel{y+y^2}$ $y(y+1)(y-1)=0$, $\cancel{x=y^2}$

$x = y^2$

$\therefore$ $(x,y) = (0,0)\ ,\ (1,1)\ ,\ (1,-1)$

Classification: ~~(0,0)~~ calculate eigenvalues $\det(J-\lambda I)=0$

$(0,0)$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\therefore \lambda^2+1=0$ $\lambda = \pm i$

centre

$(1,1)$, $J = \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}$

$\lambda(2+\lambda)-2=0$ $\lambda^2+2\lambda-2=0$

$\lambda = \frac{1}{2}(-2 \pm \sqrt{4+8}) = \frac{1}{2}(-2 \pm \sqrt{12})$

$= -1 \pm \sqrt{3}$

$\lambda_1 > 0$, $\lambda_2 < 0$ both real $\Leftrightarrow$ Saddle node

eigenvectors

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\sqrt{3} \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 1-\sqrt{3} \end{pmatrix}$

$(1,-1)$ $J = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}$

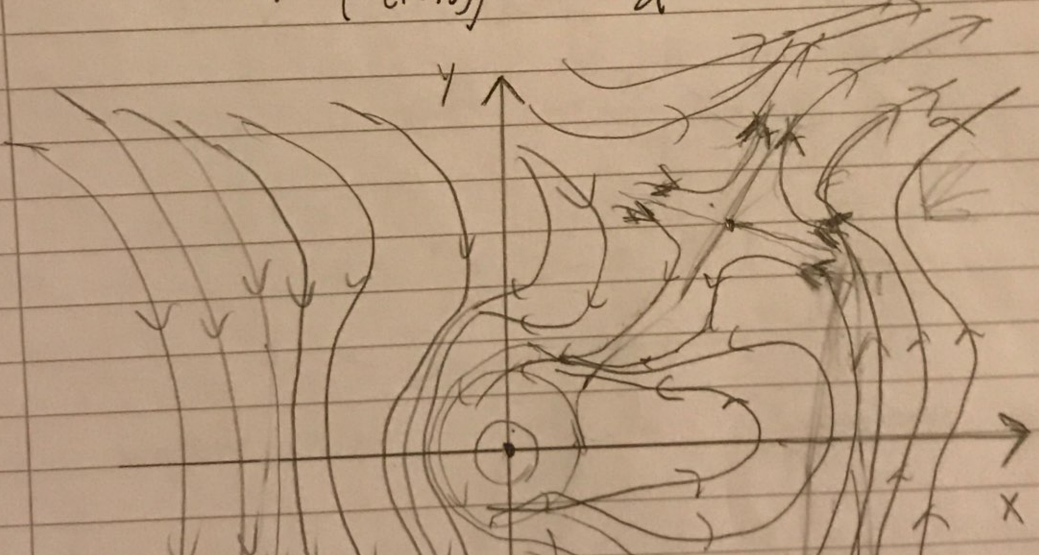
$\lambda(\lambda-2) - 2 = 0 \quad \therefore \lambda^2 - 2\lambda - 2 = 0$

$\lambda = \frac{1}{2}(2 \pm \sqrt{12}) = 1 \pm \sqrt{3}$

$\lambda_1 > 0$, $\lambda_2 < 0$ $\therefore$ saddle node
both real

eigenvectors $\begin{pmatrix} a \\ b \end{pmatrix}$ $(1 \pm \sqrt{3})a + 2b = 0$

$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -(1+\sqrt{3}) \end{pmatrix}$ or $\begin{pmatrix} 2 \\ \sqrt{3}-1 \end{pmatrix}$



4.

$$\frac{dR}{dt} = k_R - \gamma_R R$$

$$\frac{dP}{dt} = k_P R - \cancel{\gamma_P R} \gamma_P P$$

$$\therefore \quad \frac{dR}{k_R - \gamma_R R} = \cancel{k_R k_R}\, dt.$$

$$\int_0^{R(t)} \frac{dR}{1 - \frac{\gamma_R}{k_R} R} = \int_0^t k_R\, dt$$

$$\therefore -\frac{k_R}{\gamma_R} \ln\left(1 - \frac{\gamma_R}{k_R} R\right) = \cancel{k} k_R t.$$

$$\therefore \quad \cancel{R} \quad 1 - \frac{\gamma_R}{k_R} R = e^{-\gamma_R t}$$

$$\therefore R = \frac{k_R}{\gamma_R}\left(1 - e^{-\gamma_R t}\right)$$

steady state $\quad \frac{dR}{dt} = 0, \quad \frac{dP}{dt} = 0$

$$\therefore \quad R = \frac{k_R}{\gamma_R}, \quad P = \frac{k_P}{\gamma_P} R = \frac{k_P k_R}{\gamma_P \gamma_R}$$

single gene transcribed in 1 min

life time of RNA is 10 min $\therefore \frac{1}{\gamma_R} = 10\,\text{min}$

$$\therefore R(1\,min) = \frac{k_R}{\gamma_R}\left(1 - e^{-1/10}\right)$$

$$= 0.095 \frac{k_R}{\gamma_R}$$

$$= \underline{0.0095 k_R}$$

Now with the noise

$$\frac{dR}{dt} = k_R - \gamma_R R + \eta_R(t)$$

let $R = \langle R \rangle + \delta R(t)$ with $\langle R \rangle = \frac{k_R}{\gamma_R}$

$$\therefore \frac{d}{dt}(\langle R \rangle + \delta R) = \cancel{k_R} - \cancel{\gamma_R\left(\frac{k_R}{\gamma_R}\right)} - \gamma_R \delta R + \eta_R(t)$$

$$\therefore \dot{\delta R} = -\gamma_R \delta R + \eta_R(t)$$

Fourier transform

$$\delta R(t) = \int \frac{d\omega}{2\pi} e^{i\omega t} \delta \tilde{R}(\omega)$$

$$\therefore i\omega \delta \tilde{R}(\omega) = -\gamma_R \delta R(\omega) + \eta_R(\omega)$$

$$\delta R(\omega) = \frac{\eta_R(\omega)}{\gamma_R + i\omega}$$

$$\langle \eta_R(\omega) \eta_R^*(\omega') \rangle = \int dt \int dt' e^{i\omega t + i\omega' t'} \underbrace{\langle \eta_R(t) \eta_R^*(t') \rangle}_{q_R(t-t')}$$

$$= q_R \int dt\, e^{i(\omega+\omega')t}$$

$$= q_R \delta(\omega + \omega')$$

$$\langle \delta R(t)\delta R(t')\rangle = \iint \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi} \cancel{\langle \eta_R(\omega)\eta_R(\omega')\rangle}$$

$$\langle \delta\tilde{R}(\omega)\,\delta\tilde{R}(\omega')\rangle$$

$\delta(\omega+\omega')$

$$= \iint \frac{d\omega}{2\pi}\frac{d\omega'}{2\pi}\frac{\langle \eta_R(\omega)\eta_R^*(\omega')\rangle}{(\gamma_R+i\omega)(\gamma_R+i\omega')}\,\cancel{e^{i\omega t}}\,e^{i\omega' t'}$$

$\omega = -\omega'$
$\omega' = -\omega$

$$\therefore \cancel{\langle \delta R(t)\delta R(t')\rangle} = \int \frac{d\omega}{2\pi}\frac{q_R}{\gamma_R^2+\omega^2} = \frac{q_R}{2\gamma_R}$$

$\langle |\delta R(t)|^2\rangle$

$$\cancel{\delta R}\quad \langle \delta P^2\rangle = \langle P\rangle\left(1+\frac{k_p}{\gamma_R+\gamma_P}\right)$$

$$= \langle P\rangle\left(1+\frac{k_p/\gamma_R}{1+\frac{\gamma_P}{\gamma_R}}\right) = \langle P\rangle\left(1+\frac{b}{1+\phi}\right)$$

$$b = \frac{k_p}{\gamma_R} \qquad \phi = \frac{\gamma_P}{\gamma_R}$$

b, the deviation from poisson distribution, is the strength of noise

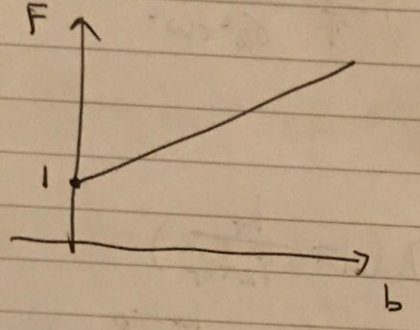
~~ϕ~~ ϕ is the ratio between decay rates of RNA and protein. This ~~res~~ represents the instability of protein production.

If possionian $\underline{F = 1}$

RNA much more unstable than protein

$\therefore \ \phi \simeq 0$

$F = 1 + b$ ~~=~~



~~more~~ higher production rate of protein, higher fluctuations in the protein ~~...~~ concentration

→ Intrinsic noise is caused by ~~...~~ stochastic fluctuations within a cell (translation or transcription)

→ Extrinsic noise is caused by differences between cells (number of ribosomes or copy numbers of key proteins.

→ beneficial : optimize growth rate of cells

detrimental : more prone to virus infection.